PERIODIC SINGULAR PROBLEM WITH QUASILINEAR DIFFERENTIAL OPERATOR

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Abstract. We study the singular periodic boundary value problem of the form

$$\big(\phi(u')\big)' + h(u)u' = g(u) + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T),$$

where $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing and odd homeomorphism such that $\phi(\mathbb{R}) = \mathbb{R}$, $h \in C[0,\infty)$, $e \in L_1[0,T]$ and $g \in C(0,\infty)$ can have a space singularity at $x = 0$, i.e. $\limsup\limits_{x\to 0+} |g(x)| = \infty$ may hold. We prove new existence results both for the case of an attractive singularity, when $\liminf\limits_{x\to 0+} g(x) = -\infty$, and for the case of a strong repulsive singularity, when $\lim\limits_{x\to 0+} \int_x^1 g(\xi)\,\mathrm{d}\xi = \infty$. In the latter case we assume that $\phi(y) = \phi_p(y) = |y|^{p-2}y$, $p > 1$, is the well-known p-Laplacian. Our results extend and complete those obtained recently by Jebelean and Mawhin and by Liu Bing.

Keywords: singular periodic boundary value problem, positive solution, ϕ-Laplacian, p-Laplacian, attractive singularity, repulsive singularity, strong singularity, lower function, upper function

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1. Introduction

Second order nonlinear differential equations or systems with singularities appear naturally in the description of particles submitted to Newtonian type forces or to

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forces caused by compressed gases. Their mathematical study was started by Forbat and Huaux [10], [13], Derwidué [6] and Faure [8]. The equations they studied may be transformed to the form

$$u'' + h(u)u' = g(t,u) + e(t), \tag{1.1}$$

sometimes also referred to as the generalized Forbat's equations.

In 1987, motivated by the model equation $u'' = \beta u^{-\lambda} + e(t)$ with $\lambda > 0$, $\beta \neq 0$ and $e \in L_1[0,T]$, Lazer and Solimini [17] investigated the existence of positive solutions to the Duffing equation $u'' = g(u) + e(t)$ using topological arguments and the lower and upper functions method. The restoring force g was allowed to have an attractive singularity or a strong repulsive singularity at origin. Starting with this paper, the interest in periodic singular problems considerably increased. The results by Lazer and Solimini have been generalized or extended e.g. by Habets and Sanchez [12], Mawhin [19], del Pino, Manásevich and Montero [4], Fonda [9], Omari and Ye [22], Zhang [30] and [32], Ge and Mawhin [11], Rachůnková, Tvrdý and Vrkoč [26]. Some of these papers (e.g. [12], [22], [30] or [32]) cover also the Liénard equation (1.1) with $g(t,x)$ having an essentially autonomous singularity at $x = 0$. However, all of them, when dealing with the repulsive singularity, supposed that the strong force condition is satisfied. For the case of the weak singularity, first results were established by Rachůnková, Tvrdý and Vrkoč in [25]. Further results were established later also by Bonheure and De Coster [1], Bonheure, Fonda and Smets [2] and Torres [28].

Regular periodic problems with ϕ- or p-Laplacian on the left hand side were considered by several authors, see e.g. del Pino, Manásevich and Murúa [5] or Yan [29]. General existence principles for the regular vector problem, based on the homotopy to the averaged nonlinearity, were presented by Manásevich and Mawhin in [20] (see also Mawhin [21]).

In the well-ordered case, the lower/upper functions method was extended to problems with a ϕ-Laplacian operator on the left hand side by Cabada and Pouso in [3], Jiang and Wang in [16] and Staněk in [27]. The general existence principle valid also when lower/upper functions are non-ordered was presented by Rachůnková and Tvrdý in [24] and, for the case when impulses are admitted, also in [23].

The singular periodic problem for the Liénard type equation

$$\bigl(|u'|^{p-2}u'\bigr)' + h(u)u' = g(u) + e(t) \tag{1.2}$$

with the p-Laplacian on the left hand side was treated by Liu [18] (see Theorem 3.2 below) and Jebelean and Mawhin [14] and [15] (see Theorems 2.1 and 3.1 below). Their main tool was the continuation type existence principle due to Manásevich and Mawhin [20] (see Lemma 3.4 below).

In this paper we present two new existence results for singular periodic problems of the form

$$(\phi(u'))' + h(u)u' = g(u) + e(t), \tag{1.3}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{1.4}$$

where

(1.5) $\phi\colon\ \mathbb{R} \to \mathbb{R}$ is an increasing and odd homeomorphism such that $\phi(\mathbb{R}) = \mathbb{R}$,

(1.6) $0 < T < \infty, \quad h \in C[0,\infty), \quad e \in L_1[0,T]$

and

(1.7) $g \in C(0,\infty)$ can have a space singularity at $x = 0$.

In particular, $\limsup\limits_{x\to 0+} |g(x)| = \infty$ may hold. First, in Theorem 2.3, we prove the existence of a positive solution to problem (1.3), (1.4) in the case that the function g can have an attractive singularity at $x = 0$. Our main tool is the lower/upper functions method. Contrary to Jebelean and Mawhin, we need not restrict ourselves to the case that the equation (1.3) reduces to (1.2). Let us recall that a function g is said to have an *attractive* singularity at $x = 0$ if

$$\liminf_{x\to 0+} g(x) = -\infty. \tag{1.8}$$

Alternatively, we say that g has a repulsive singularity at the origin if

$$\limsup_{x\to 0+} g(x) = +\infty. \tag{1.9}$$

Theorem 3.5 is devoted to the case that g has a repulsive singularity at $x = 0$. Similarly to the previous authors, we also rely on the continuation type principle by Manásevich and Mawhin. Unlike Jebelean and Mawhin, we need not restrict ourselves to the case that g is bounded below on $(0,\infty)$ nor to assume any dissipativity conditions, and, in comparison with Liu, we replace her crucial assumption (see (3.4)) by its asymptotic form (3.6). However, when dealing with the repulsive singularity, we also restrict ourselves to the case

$$\phi(y) \equiv \phi_p(y) := |y|^{p-2}y \quad \text{for} \quad y \in \mathbb{R},$$

i.e. we assume that ϕ reduces to the p-Laplacian.

As usual, for an arbitrary subinterval I of $\mathbb{R}$, we denote by $C(I)$ the set of functions $x\colon I \to \mathbb{R}$ which are continuous on I, $C^1[0,T]$ stands for the set of functions $x \in C[0,T]$ with the first derivative continuous on $[0,T]$. Further, $L_1[0,T]$ is the set of functions $x\colon [0,T] \to \mathbb{R}$ which are measurable and Lebesgue integrable on $[0,T]$. For $p > 1$, $L_p[0,T]$ is the set of functions $x \in L_1[0,T]$ for which also $|x|^p$ is Lebesgue integrable on $[0,T]$, $L_\infty[0,T]$ is a subset of $L_1[0,T]$ formed by functions which are essentially bounded on $[0,T]$ and $AC[0,T]$ is the set of functions absolutely continuous on $[0,T]$. For $x \in L_1[0,T]$ and $p \in [1,\infty)$ we put

$$\|x\|_p = \left(\int_0^T |x(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}}, \quad \|x\|_\infty = \sup_{t\in[0,T]}\operatorname{ess}|x(t)| \quad \text{and} \quad \bar{x} = \frac{1}{T}\int_0^T x(s)\,\mathrm{d}s.$$

It is well known that $C^1[0,T]$ becomes a Banach space when equipped with the norm $\|x\|_{C^1} = \|x\|_\infty + \|x'\|_\infty$.

A function $u\colon [0,T] \to \mathbb{R}$ is a *positive solution* to problem (1.3), (1.4) if $\phi(u') \in AC[0,T]$, $u > 0$ on $[0,T]$,

$$(\phi(u'(t)))' + h(u(t))u'(t) = g(u(t)) + e(t) \quad \text{for a.e.} \quad t \in [0,T]$$

and (1.4) is satisfied. (Notice that the requirement $\phi(u') \in AC[0,T]$ implies that $u \in C^1[0,T]$.)

2. Attractive singular forces

First, let us consider the simpler case when the function g can have an attractive singularity at $x = 0$, i.e. we admit that (1.8) can hold. If $\phi = \phi_p$ and $e \in L_\infty$, the following result is available.

2.1 Theorem (Jebelean and Mawhin [14, Theorem 1])**.** *Assume* (1.6) *with* $e \in L_\infty[0,T]$, (1.7), $p > 1$,

$$\limsup_{x\to 0+}[g(x) + \|e\|_\infty] < 0 \tag{2.1}$$

and

$$\liminf_{x\to\infty}[g(x) + \bar{e}] > 0. \tag{2.2}$$

Then problem (1.2), (1.4) *has a positive solution* u.

The proof of Theorem 2.1 is based on the lower/upper functions method and, in particular, on the following existence principle which is an easy consequence of [27, Theorem 1] (see also [24, Theorem 2.1]).

2.2 Theorem. *Assume* (1.6) *and* (1.7). *Moreover, let there exist lower and upper functions* σ_1 *and* σ_2 *of problem* (1.3), (1.4) *such that*

$$0 < \sigma_1 \leqslant \sigma_2 \quad \text{on} \quad [0,T]. \tag{2.3}$$

Then problem (1.3), (1.4) *has a positive solution* u *such that*

$$\sigma_1 \leqslant u \leqslant \sigma_2 \quad \text{on } [0,T]. \tag{2.4}$$

Let us recall that a function $\sigma \in C^1[0,T]$ is a *lower function* of problem (1.3), (1.4) if

$$(\phi(\sigma'(t)))' + h(\sigma(t))\sigma'(t) \geqslant g(\sigma(t)) + e(t) \quad \text{for a.e. } t \in [0,T], \tag{2.5}$$

and

$$\sigma(0) = \sigma(T), \quad \sigma'(0) \geqslant \sigma'(T). \tag{2.6}$$

If the inequalities in (2.5), (2.6) are reversed, σ is called an *upper function* of problem (1.3), (1.4).

The next theorem is a generalization of Theorem 2.1 to the case of a more general Laplacian operator on the left hand side of equation (1.3).

2.3 Theorem. *Assume* (1.5)–(1.7) *and* (2.2). *Moreover, let the following conditions be fulfilled:*

$$\text{there exists } \alpha > 0 \text{ such that } \liminf_{|y|\to\infty} \frac{|\phi(y)|}{|y|^\alpha} > 0, \tag{2.7}$$

$$\text{there exists } r > 0 \text{ such that } g(r) + e(t) \leqslant 0 \quad \text{for a.e. } t \in [0,T]. \tag{2.8}$$

Then problem (1.3), (1.4) *has a positive solution* u *such that* $u \geqslant r$ *on* $[0,T]$.

P r o o f. S t e p 1. Notice that, due to (2.8), $\sigma_1(t) \equiv r$ is a lower function of (1.3), (1.4).

S t e p 2. We will construct an upper function σ_2 of (1.3), (1.4). By (2.2), there is an $R > r$ such that

$$g(x) + \bar{e} > 0 \quad \text{for} \quad x \geqslant R. \tag{2.9}$$

Take an arbitrary $C \in \mathbb{R}$ and consider an auxiliary problem

$$(\phi(v'))' + \lambda h(v + C)v' = \lambda b(t), \quad v(0) = v(T) = 0, \tag{2.10}$$

where $b(t) = g_0 + e(t)$ for a.e. $t \in [0,T]$, $g_0 = \inf\{g(x)\colon\ x \in [R,\infty)\}$ and $\lambda \in [0,1]$ is a parameter. For a given $\lambda \in [0,1]$, define an operator $\mathcal{F}_\lambda\colon\ C^1[0,T] \times \mathbb{R} \to C^1[0,T] \times \mathbb{R}$ by

$$\mathcal{F}_\lambda\colon\ (v,a) \to \Bigg(\int_0^t \phi^{-1}\Big(a + \lambda \int_0^s [b(\tau) - h(v(\tau)+C)v'(\tau)]\,\mathrm{d}\tau\Big)\,\mathrm{d}s,$$
$$a - \int_0^T \phi^{-1}\Big(a + \lambda \int_0^s [b(\tau) - h(v(\tau)+C)v'(\tau)]\,\mathrm{d}\tau\Big)\,\mathrm{d}s\Bigg).$$

Due to (1.5) and (1.7), the operator $\mathcal{F}_\lambda$ is completely continuous for each $\lambda \in [0,1]$ and v is a solution of (2.10) satisfying $\phi(v'(0)) = a$ if and only if $\mathcal{F}_\lambda(v,a) = (v,a)$. Now, let $\mathcal{F}_\lambda(v,a) = (v,a)$ for some $\lambda \in (0,1]$ and some $(v,a) \in C^1[0,T] \times \mathbb{R}$. We have

$$(\phi(v'(t)))' + \lambda h(v(t)+C)v'(t) = \lambda b(t) \quad \text{for a.e.} \quad t \in [0,T], \tag{2.11}$$

$v(0) = v(T) = 0$ and $\phi(v'(0)) = a$. Multiplying the equality (2.11) by $v(t)$ and integrating over $[0,T]$, we get

$$-\int_0^T \phi(v'(t))v'(t)\,\mathrm{d}t = \lambda \int_0^T b(t)v(t)\,\mathrm{d}t. \tag{2.12}$$

By (2.7), there are $k > 0$ and $y_0 > 0$ such that

$$\frac{\phi(|y|)}{|y|^\alpha} > \frac{k}{2} \quad \text{for } |y| \geqslant y_0. \tag{2.13}$$

Define

$$\beta(y) = \phi(y) - ky^\alpha \quad \text{for} \quad y \geqslant 0. \tag{2.14}$$

Then $\beta \in C[0,\infty)$ and, due to (2.13),

$$-\frac{\beta(y)}{y^\alpha} < \frac{k}{2} \quad \text{for} \quad y \geqslant y_0. \tag{2.15}$$

Next, since ϕ is odd, we have $\phi(y)y \geqslant 0$ and $|\phi(y)| = \phi(|y|)$ for each $y \in \mathbb{R}$. In particular,

$$\phi(|y|)|y| = \phi(y)y \quad \text{for all} \quad y \in \mathbb{R}.$$

The relation (2.12) can be rewritten as

$$-k\|v'\|_{\alpha+1}^{\alpha+1} - \int_0^T \beta(|v'(t)|)|v'(t)|\,\mathrm{d}t = \lambda \int_0^T b(t)v(t)\,\mathrm{d}t. \tag{2.16}$$

Now, denote $J = \{t \in [0,T]\colon\ |v'(t)| \geqslant y_0\}$ and $M = \max\{\beta(y)\colon\ y \in [0, y_0]\}$ and assume that $\|v\|_\infty \geqslant 1$. Then, the relations (2.15) and (2.16) imply that

$$\begin{aligned} k\|v'\|_{\alpha+1}^{\alpha+1} &\leqslant \|b\|_1\|v\|_\infty + My_0T - \int_J \frac{\beta(|v'(t)|)}{|v'(t)|^\alpha}|v'(t)|^{\alpha+1}\,\mathrm{d}t \\ &\leqslant (\|b\|_1 + My_0T)\,\|v\|_\infty + \frac{k}{2}\|v'\|_{\alpha+1}^{\alpha+1} \end{aligned}$$

holds. It follows that

$$\frac{k}{2}\|v'\|_{\alpha+1}^{\alpha+1} \leqslant (\|b\|_1 + My_0T)\,\|v\|_\infty. \tag{2.17}$$

Further, by Hölder's inequality we have

$$\|v\|_\infty \leqslant \int_0^T |v'(s)|\,\mathrm{d}s \leqslant T^{\frac{\alpha}{\alpha+1}}\|v'\|_{\alpha+1}. \tag{2.18}$$

Inserting (2.18) into (2.17) we conclude that

$$\|v'\|_{\alpha+1} \leqslant \Big(\frac{2}{k}\big(TMy_0 + \|b\|_1\big)\Big)^{\frac{1}{\alpha}} T^{\frac{1}{\alpha+1}}.$$

Thus, including into our consideration also the case $\|v\|_\infty < 1$, we deduce from (2.18) that the estimate

$$\|v\|_\infty < D := T\Big(\frac{2}{k}\big(TMy_0 + \|b\|_1\big)\Big)^{\frac{1}{\alpha}} + 1 \tag{2.19}$$

holds.

Furthermore, as $v(0) = v(T)$, there is a $\tau_0 \in (0,T)$ such that $v'(\tau_0) = 0$. Therefore, the integration of the equality (2.11) yields

$$\phi(v'(t)) + \lambda \int_{v(\tau_0)}^{v(t)} h(x+C)\,\mathrm{d}x = \lambda \int_{\tau_0}^t b(s)\,\mathrm{d}s \quad \text{for } t \in [0,T],$$

wherefrom the estimate

$$|\phi(v'(t))| \leqslant 2D\max\{|h(x)|\colon\ |x| \leqslant C + D\} + \|b\|_1 \quad \text{for all}\ \ t \in [0,T]$$

follows. Hence,

$$\|v'\|_\infty \leqslant \phi^{-1}(H) \quad \text{and} \quad |a| = |\phi(v'(0))| \leqslant H,$$

where $H = 2D \max\{|h(x)|\colon |x| \leqslant C + D\} + \|b\|_1$. This, together with (2.19), means that we can choose $\varrho > D + \phi^{-1}(H) + H$ so that

$$(v, a) \neq \mathcal{F}_\lambda(v, a) \quad \text{for each } \lambda \in (0, 1] \text{ and each } \ (v, a) \in \partial\mathcal{B}(\varrho),$$

where $\mathcal{B}(\varrho) = \{(v, a) \in C^1[0, T] \times \mathbb{R}\colon \|v\|_\infty + \|v'\|_\infty + |a| < \varrho\}$. Moreover, it is easy to see that $\mathcal{F}_0(v, a) = (v, a)$ if and only if $(v, a) = (0, 0)$. So, $I - \mathcal{F}_\lambda$ generates a homotopy on $\mathrm{cl}(\mathcal{B}(\varrho)) \times [0, 1]$. Hence, as $\mathcal{F}_0$ is an odd mapping, we have

$$\deg(I - \mathcal{F}_1, \mathcal{B}(\varrho)) \equiv \deg(I - \mathcal{F}_0, \mathcal{B}(\varrho)) \neq 0.$$

Therefore, for each $C \in \mathbb{R}$, problem (2.10) has a solution (v_C, a_C). It follows from the construction of the operator $\mathcal{F}_1$ that v_C is a solution of (2.10) with $\lambda = 1$ and $a_C = \phi(v_C'(0))$. Moreover, in view of (2.19), we have also $\|v_C\|_\infty < D$ on $[0, T]$.

Now, if we put $C = R + D$, we can see that $\sigma_2 = v_C + C$ is an upper function of (1.3), (1.4). Indeed, we have $\sigma_2(0) = \sigma_2(T) = C$ and, due to (2.9),

$$\phi(\sigma_2'(T)) - \phi(\sigma_2'(0)) = T\bar{b} = T(g_0 + \bar{e}) \geqslant 0.$$

Moreover, $\sigma_2(t) > C - D = R > r \equiv \sigma_1(t)$ on $[0, T]$. Therefore, by (2.9),

$$\begin{aligned}(\phi(\sigma_2'(t)))' &= -h(\sigma_2(t))\sigma_2'(t) + g_0 + e(t)\\ &\leqslant -h(\sigma_2(t))\sigma_2'(t) + g(\sigma_2(t)) + e(t) \quad \text{for a.e.} \quad t \in [0, T].\end{aligned}$$

S t e p 3. By Theorem 2.2, problem (1.3), (1.4) has a positive solution u such that $u \in [r, \sigma_2(t)]$ for each $t \in [0, T]$. ☐

E x a m p l e. If $\phi = \phi_p$, $p > 1$, or $\phi(y) = (|y|y + y)\ln(1 + 1/|y|)$ or $\phi(y) = y(\exp(y^2) - 1)$ and $h \in C[0, \infty)$, $\beta > 0$, $\lambda > 0$, $e \in L_\infty[0, T]$ and $g(x) = -\beta x^{-\lambda}$ for $x \in (0, \infty)$, then, by Theorem 2.3, problem (1.3), (1.4) has a positive solution if and only if $\bar{e} < 0$.

3. Repulsive singular forces

The case when (1.9) holds is considerably more complicated since it does not usually lead to the existence of a well ordered couple of lower and upper functions.

Under the assumptions ensuring either that g is bounded below on $(0, \infty)$ or that h satisfies the dissipativity condition, the existence of a positive solution to problem (1.2), (1.4) was established by Jebelean and Mawhin.

3.1 Theorem ([14, Theorem 2] and [15, Theorem 3]). *Assume* $p > 1$ *and* (1.6). *Furthermore, let*

$$\text{(3.1)}\quad g \in C(0,\infty) \quad \textit{and} \quad \lim_{x\to 0+} \int_x^1 g(\xi)\,\mathrm{d}\xi = +\infty$$

$$\text{(3.2)}\quad \liminf_{x\to 0+} [g(x) + \bar{e}] > 0 > \limsup_{x\to\infty} [g(x) + \bar{e}],$$

$$\text{(3.3)}\quad \begin{cases} \liminf\limits_{x\to\infty} g(x) > -\infty \\ \textit{or} \\ e \in L_2[0,T] \ \textit{and} \ \ h(x) \geqslant h_* > 0 \ (\textit{or } h(x) \leqslant -h_* < 0) \ \textit{for all } x \in [0,\infty). \end{cases}$$

Then problem (1.2), (1.4) *has a positive solution.*

Another related result concerning both the non-dissipative case and the case where g need not be bounded below on $(0,\infty)$ was obtained by Liu.

For $p > 1$, define

$$\pi_p = \frac{2\pi(p-1)^{\frac{1}{p}}}{p\sin(\frac{\pi}{p})}.$$

3.2 Theorem (Liu, [18, Theorem 1]). *Assume* $p > 1$, (1.6), (3.1) *and* (3.2). *Furthermore, let*

$$\text{(3.4)}\quad \begin{cases} \textit{there exist } a, \ 0 \leqslant a < (\pi_p/T)^p, \ \textit{and} \ \gamma \geqslant 0 \ \textit{such} \\ \textit{that } g(x)x \geqslant -(ax^p + \gamma) \quad \textit{for all } x > 0. \end{cases}$$

Then problem (1.2), (1.4) *has a positive solution.*

3.3 Remark. Let us notice that $(\pi_p/T)^p$ is the first eigenvalue of the Dirichlet problem

$$(\phi_p(u'))' + \lambda\phi_p(u) = 0, \quad u(0) = u(T) = 0$$

(see [7]).

For the proofs of Theorems 3.1 and 3.2 the following corollary of the continuation type principle due to Manásevich and Mawhin is essential.

3.4 Lemma ([20, Theorem 3.1] and [14, Lemma 3]). *Assume* $p > 1$, (1.6) *and* $g \in C(0,\infty)$. *Furthermore, let there exist* $r > 0$, $R > r$ *and* $R' > 0$ *such that*

(i) *the inequalities* $r < v < R$ *on* $[0,T]$ *and* $\|v'\|_\infty < R'$ *hold for each* $\lambda \in (0,1]$ *and for each positive solution* v *of the problem*

$$\text{(3.5)}\qquad \big(|v'|^{p-2}v'\big)' = \lambda(-h(v)v' + g(v) + e(t)), \quad v(0) = v(T), \quad v'(0) = v'(T),$$

(ii) $(g(x)+\bar{e}=0) \Longrightarrow r<x<R$,

(iii) $(g(r)+\bar{e})(g(R)+\bar{e})<0$.

Then problem (1.2), (1.4) *has at least one solution* u *such that* $r<u<R$ *on the interval* $[0,T]$.

Next, we will show that the condition (3.4) can be replaced by a related asymptotic condition.

3.5 Theorem. *Assume* $p>1$, (1.6) (3.1) *and* (3.2). *Moreover, let* $1/p+1/q=1$, $e \in L_q[0,T]$ *and*

$$\liminf_{x\to\infty} \frac{g(x)}{x^{p-1}} > -\Big(\frac{\pi_p}{T}\Big)^p. \tag{3.6}$$

Then the problem (1.2), (1.4) *has a positive solution.*

Proof. Similarly to Jebelean's and Mawhin's proof of Theorem 2 in [14], we will verify that the assumptions of Lemma 3.4 are satisfied.

Step 1. One-point estimates of solutions to (3.5). First, we will show that

(3.7) there are $R_0>0$ and $R_1>R_0$ such that $v(t_v) \in (R_0,R_1)$ for some $t_v \in [0,T]$ holds for each $\lambda \in (0,1]$ and each positive solution v of (3.5).

To this aim, choose $\lambda \in (0,1]$ and assume that v is a positive solution to (3.5). Integrating the equality

$$\big(|v'(t)|^{p-2}v'(t)\big)' = \lambda(-h(v(t))v'(t)+g(v(t))+e(t)) \quad \text{for a.e. } \ t \in [0,T] \tag{3.8}$$

over $[0,T]$ and taking into account the periodic conditions (1.4), we get

$$0=\int_0^T (g(v(t))+e(t))\mathrm{d}t = \int_0^T (g(v(t)+\bar{e})\mathrm{d}t. \tag{3.9}$$

On the other hand, by (3.2), there are $R_0>0$ and $R_1>R_0$ such that

$$g(x)+\bar{e}>0 \quad \text{for } \ x \in (0,R_0] \tag{3.10}$$

and

$$g(x)+\bar{e}<0 \quad \text{for } \ x \in [R_1,\infty). \tag{3.11}$$

Consequently, there is a $t_v \in [0,T]$ such that $v(t_v) \in (R_0,R_1)$, otherwise we would meet a contradiction to (3.9). This proves the assertion (3.7).

Step 2. Upper estimates of solutions to (3.5). We will prove that there is $R \in (0,\infty)$ such that

(3.12) $\|v\|_\infty \leqslant R$ holds for each $\lambda \in (0,1]$ and each positive solution v of (3.5).

Assume, on the contrary, that there is a sequence of parameters $\{\lambda_k\} \subset (0,1]$ and a sequence of corresponding positive solutions $\{v_k\}$ of (3.5) such that

$$\lim_{k\to\infty} \max\{v_k(t)\colon\ t \in [0,T]\} = \infty. \tag{3.13}$$

By (3.7), the assertion

$$v_k(t_k) \in (R_0, R_1) \quad \text{for some} \quad t_k \in [0,T] \tag{3.14}$$

is true for each $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and extend the functions v_k and e to functions which are T-periodic on the whole $\mathbb{R}$. Then

$$\bigl(|v_k'(t)|^{p-2}v_k'(t)\bigr)' + \lambda_k(h(v_k(t))v_k'(t) = \lambda_k(g(v_k(t)) + e(t)) \quad \text{for a.e.} \quad t \in \mathbb{R}.$$

Multiplying the last equality by $v_k(t)$ and integrating over $[t_k, t_k+T]$ we obtain

$$\|v_k'\|_p^p = -\lambda_k \int_{t_k}^{t_k+T} g(v_k(t))v_k(t)\,\mathrm{d}t - \lambda_k \int_{t_k}^{t_k+T} e(t)v_k(t)\,\mathrm{d}t. \tag{3.15}$$

By (3.6), there are $\eta \in (0, (\pi_p/T)^p)$ and $B > 0$ such that

$$g(x)x \geqslant -\Bigl(\Bigl(\frac{\pi_p}{T}\Bigr)^p - \eta\Bigr)x^p \quad \text{for} \quad x \geqslant B.$$

From the first inequality in (3.2) we deduce that there is an $A \in (0,B)$ such that

$$g(x)x \geqslant -\bar{e}x \quad \text{for} \quad x \in (0,A].$$

Thus, if we put

$$\gamma = |\max\{\bar{e}, -\min\{g(x)\colon\ x \in [A,B]\}\}|,$$

we get

$$g(x)x \geqslant -\Bigl(\Bigl(\frac{\pi_p}{T}\Bigr)^p - \eta\Bigr)x^p - \gamma x \quad \text{for all} \quad x > 0,$$

which, when inserted into (3.15), yields

$$\|v_k'\|_p^p \leqslant \Bigl(\Bigl(\frac{\pi_p}{T}\Bigr)^p - \eta\Bigr)\|v_k\|_p^p + \gamma T^{\frac{1}{q}}\|v_k\|_p + \|e\|_q\|v_k\|_p. \tag{3.16}$$

Put $w_k(t) = v_k(t) - v_k(t_k)$ for $t \in [0,T]$. Then

$$\|v_k\|_p^p \leqslant \Big(\|w_k\|_p + T^{\frac{1}{p}} v_k(t_k)\Big)^p.$$

Due to (3.14), the inequality (3.16) can be rewritten as

$$\|w_k'\|_p^p \leqslant \Big(\Big(\frac{\pi_p}{T}\Big)^p - \eta\Big)\|w_k\|_p^p\Big(1 + \frac{a_1}{\|w_k\|_p}\Big)^p + a_2\|w_k\|_p + a_3, \tag{3.17}$$

where a_i, $i = 1,2,3$, are positive constants not depending on w_k. Moreover, having $w_k(t_k) = w_k(t_k+T) = 0$, we can apply the generalized Poincaré-Wirtinger inequality (see e.g. [31, Lemma 3]) and obtain

$$\|w_k\|_p^p \leqslant \Big(\frac{T}{\pi_p}\Big)^p \|w_k'\|_p^p, \tag{3.18}$$

which together with (3.17) gives

$$\Big(\frac{\pi_p}{T}\Big)^p \leqslant \frac{\|w_k'\|_p^p}{\|w_k\|_p^p} \leqslant \Big(\Big(\frac{\pi_p}{T}\Big)^p - \eta\Big)\Big(1 + \frac{a_1}{\|w_k\|_p}\Big)^p + \frac{a_2}{\|w_k\|_p^{p-1}} + \frac{a_3}{\|w_k\|_p^p}. \tag{3.19}$$

By the Hölder inequality and by (3.13) we have $\lim\limits_{k\to\infty} \|w_k'\|_p = \infty$ wherefrom, by virtue of (3.17), we get $\lim\limits_{k\to\infty} \|w_k\|_p = \infty$. Thus, the inequalities (3.19) lead to a contradiction

$$\Big(\frac{\pi_p}{T}\Big)^p \leqslant \Big(\frac{\pi_p}{T}\Big)^p - \eta.$$

As a consequence, we can conclude that there is $R \in (0,\infty)$ such that (3.12) is true.

Step 3. Estimates of the derivatives of solutions to (3.5). We will find $R' \in (0,\infty)$ such that

(3.20) $\|v'\|_\infty \leqslant R'$ holds for each $\lambda \in (0,1]$ and each positive solution v of (3.5).

To this aim choose $\lambda \in (0,1]$ and assume that v is a positive solution of (3.5). Let us put

$$b = -\min\{0, -\bar{e}, \min\{g(x)\colon\ x \in [R_0, R]\}\}.$$

Then, by (3.10) and (3.12), we have $g(v(t)) \geqslant -b$ for $t \in [0,T]$. So, it is easy to see that

$$|g(v(t))| \leqslant g(v(t)) + 2b \quad \text{for} \quad t \in [0,T].$$

Therefore, owing to (3.9), we have

$$\int_0^T |g(v(t))|\,\mathrm{d}t \leqslant \int_0^T (g(v(t)) + 2b + \bar{e} - \bar{e})\,\mathrm{d}t = \int_0^T (2b - \bar{e})\,\mathrm{d}t = T(2b - \bar{e}).$$

Now, due to (1.4), there is $t_0 \in [0,T]$ such that $v'(t_0) = 0$. So, integrating the equality (3.8) from t_0 to $t \in [0,T]$, we obtain that the assertion (3.20) is true with

$$R' = \left(\int_0^R |h(x)|\,\mathrm{d}x + \|e\|_1 + T(2b - \bar{e}) \right)^{\frac{1}{p-1}}.$$

S t e p 4. L o w e r e s t i m a t e s o f s o l u t i o n s t o (3.5). Choose $\lambda \in (0,1]$ and let v be a positive solution (3.5). Put

$$H = \max\{|h(x)| \colon x \in [0,R]\} \quad \text{and} \quad K = R'^2 TH + \int_{R_0}^R |g(x)|\,\mathrm{d}x + R'\|e\|_1.$$

By (3.1), there is $r \in (0, R_0)$ such that

$$\int_x^{R_0} g(x)\,\mathrm{d}x > K \quad \text{for all} \quad x \in (0,r]. \tag{3.21}$$

Let t_1, $t_2 \in [0,T]$ be such that

$$v(t_1) = \min\{v(t) \colon t \in [0,T]\} \quad \text{and} \quad v(t_2) = \max\{v(t) \colon t \in [0,T]\}.$$

In view of (1.4), we have $v'(t_1) = v'(t_2) = 0$. Denote $w(t) = \phi_p(v'(t))$ for $t \in [0,T]$. Then $v'(t) = \phi_p^{-1}(w(t))$ on $[0,T]$ and $w(t_1) = w(t_2) = \phi_p(0) = 0$. Consequently, as $\phi_p^{-1} = \phi_q$, we have also

$$\int_{t_1}^{t_2} (\phi_p(v'(t)))' v'(t)\,\mathrm{d}t = \int_{t_1}^{t_2} w'(t)\phi_q(w(t))\,\mathrm{d}t = \int_{w(t_1)}^{w(t_2)} \phi_q(x)\,\mathrm{d}x = 0.$$

Thus multiplying (3.8) by $v'(t)$ and integrating from t_1 to t_2 yields

$$0 = -\int_{t_1}^{t_2} h(v(t))v'^2(t)\,\mathrm{d}t + \int_{v(t_1)}^{R_0} g(x)\,\mathrm{d}x + \int_{R_0}^{v(t_2)} g(x)\,\mathrm{d}x + \int_{t_1}^{t_2} e(t)v'(t)\,\mathrm{d}t.$$

It follows that

$$\int_{v(t_1)}^{R_0} g(x)\,\mathrm{d}x \leqslant R'^2 TH + \int_{R_0}^R |g(x)|\,\mathrm{d}x + R'\|e\|_1,$$

which is, owing to (3.21), possible only when $v(t_1) > r$.

S t e p 5. F i n a l c o n c l u s i o n. By (3.12) and (3.20), the condition (i) of Lemma 3.4 is satisfied. Furthermore, as $r \in (0, R_0)$ and $R \in (R_1, \infty)$, the conditions (ii) and (iii) are satisfied due to (3.10) and (3.11). Thus, by Lemma 3.4, problem (1.2), (1.4) has a positive solution. □

E x a m p l e. Let $p>1$, $h\in C[0,\infty)$, $0<a<(\pi_p/T)^p$, $\beta>0$, $\alpha\geqslant 1$ and

$$g(x)=-ax^{p-1}+\sin x+\frac{\beta}{x^\alpha}\quad\text{for}\quad x>0.$$

Then, by Theorem 3.5, problem (1.2), (1.4) has a positive solution for each $e\in L_q[0,T]$ with $q=p/(p-1)$, while neither Theorem 3.1 nor Theorem 3.2 can guarantee its existence.

Similarly, if, in addition $p>2$, m is the integer part of $p-2$ and

$$g(x)=-ax^{p-1}+\sum_{i=0}^{m}c_ix^i+\frac{\beta}{x^\alpha}\quad\text{for}\quad x>0,$$

then, by Theorem 3.5, problem (1.2), (1.4) has a positive solution for arbitrary coefficients $c_i\in\mathbb{R}$, $i=0,1,\dots,m$, and each $e\in L_q[0,T]$ with $q=p/(p-1)$.

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