On a singular boundary value problem arising in the theory of shallow membrane caps

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Abstract. We investigate the following singular boundary value problem which originates from the theory of shallow membrane caps,

$$(t^3u'(t))' + t^3\left(\frac{1}{8u^2(t)} - \frac{a_0}{u(t)} - b_0t^{2\gamma-4}\right) = 0, \quad \lim_{t\to 0+} t^3u'(t) = 0, \quad u(1) = 0,$$

where a_0, b_0, and γ are given constants. We show the existence of a positive solution to the above problem by means of a generalized lower and upper functions method involving limiting processes. We illustrate the theory by numerical experiments, in which we used the new version of the MATLAB code `sbvp` based on polynomial collocation, to approximate the solution of the membrane problem.

Keywords. Singular mixed boundary value problem, positive solution, shallow membrane, collocation method, lower and upper functions.

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1 Introduction

We investigate the solvability of the singular mixed boundary value problem for a scalar implicit ordinary differential equation (ODE) of second order,

$$(t^3u'(t))' + t^3\left(\frac{1}{8u^2(t)} - \frac{a_0}{u(t)} - b_0t^{2\gamma-4}\right) = 0, \quad 0 < t < 1, \tag{1.1}$$

subject to boundary conditions (BCs)

$$\lim_{t\to 0+} t^3u'(t) = 0, \quad u(1) = 0, \tag{1.2}$$

where $a_0 \geq 0, b_0 > 0, \gamma > 1$, arising in the theory of shallow membrane caps, cf. [16], and [22].
Our aim is to prove the existence of a positive solution to (1.1), (1.2), and then to use collocation in order to numerically approximate u, for certain values of parameters

a_0, b_0 and γ. We are especially interested in covering the case of continuous solutions $u \in C[0,1]$.
Note that the problem (1.1), (1.2) is singular and that it exhibits both, time and space singularities. We can see this immediately, by transforming (1.1) into a first order system, by means of the substitution $x_1(t) = u(t),\ x_2(t) = t^3 u'(t)$,

$$x_1' = f_1(t, x_1, x_2) = \frac{1}{t^3} x_2, \quad x_2' = f_2(t, x_1, x_2) = -t^3 \left(\frac{1}{8x_1^2} - \frac{a_0}{x_1} - b_0 t^{2\gamma-4} \right).$$

Because of the $1/t^3$ term in the first equation f_1 is not integrable in t on any right neighborhood of $t = 0$, the function f_1 has an essential time singularity[1] at $t = 0$. Moreover, f_2 is not continuous in x_1 having a space singularity at $x_1 = 0$. For existence results concerning other types of singular mixed problems, we refer the reader to [1]–[5], [12]–[14], [21], [23]–[25], and [32]–[38].
The present investigation of the boundary value problem (1.1), (1.2) is strongly motivated by the results given in [23], where the second boundary condition in (1.2) has the form $u(1) = u_1 > 0$. It turns out that in this case solutions of (1.1), (1.2) are positive on $[0,1]$ and consequently, the problem has no space singularities. As a technical tool in the existence proof, the lower and upper functions method has been used in [23]. In our case, $u_1 = 0$, we need to cope with a space singularity at $u = 0$ and therefore it will be necessary to generalize the approach, see the discussion in Section 2. Analytical results will be presented in Section 3, and results of the numerical simulation can be found in Section 4.

Throughout the paper the following notation and definitions will be used.

Definition 1.1. A function u is called *a positive solution* of the problem (1.1), (1.2), if u satisfies the following requirements:
(i) $u \in C[0,1] \cap C^2(0,1)$ with $t^3 u' \in C[0,1)$,
(ii) $u(t) > 0$ for all $t \in (0,1)$,
(iii) u satisfies the ODE (1.1) for $t \in (0,1)$, and the BCs (1.2).

Let $J \subset \mathbb{R}$, then we denote by $L_\infty(J)$ the set of functions which are essentially bounded and Lebesgue measurable on J. We equip $L_\infty(J)$ with the norm defined by $\|u\| := \sup \mathrm{ess}_{t \in J} |u(t)|$. We use $C^n(J)$ to denote the set of functions which are n-times continuously differentiable on J. For $J = [a,b]$ and $u \in C[a,b]$, the above norm is the maximum norm, $\max_{a \le t \le b} |u(t)| = \sup \mathrm{ess}_{a \le t \le b} |u(t)|$, and therefore we will denote the maximum norm in $C[a,b]$ by $\|\cdot\|$. We define $AC(J)$ to be a set of functions being absolutely continuous on J, and by $AC_{loc}(J)$ the set of functions which are absolutely continuous on each compact subinterval $I \subset J$.
Finally, we require the following definitions:

Definition 1.2. The function $f : J \times \mathbb{R} \to \mathbb{R}$ is said to satisfy the L_∞-Carathéodory conditions on the set $J \times \mathbb{R}$, if
(i) $f(\cdot, x) : J \to \mathbb{R}$ is measurable for all $x \in \mathbb{R}$,
(ii) $f(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for a.e. $t \in J$,
(iii) for each compact set $K \subset \mathbb{R}$, there exists a function $m_K \in L_\infty(J)$ such that $|f(t,x)| \le m_K(t)$, for a.e. $t \in J$ and all $x \in K$.

Definition 1.3. Let $f : (0,1] \times (0,\infty) \to \mathbb{R}$. Then, $f(t,x)$ has *a time singularity* at $t = 0$, if there exists an $x \in (0,\infty)$ such that

$$\int_0^\varepsilon |f(t,x)| dt = \infty$$

[1] or singularity of the second kind

for each sufficiently small $\varepsilon > 0$. We say that $f(t,x)$ has *a space singularity* at $x = 0$, if

$$\limsup_{x\to 0+} |f(t,x)| = \infty,$$

for a.e. $t \in (0,1]$.

2 Generalized lower and upper functions method

The lower and upper functions method combined with fixed point theorems or topological degree arguments is an important and powerful tool for analyzing the solvability of boundary value problems, see e.g. [3], [23], [25] or [33]-[36]. The method is based on the assumption that there exist two functions σ_1 and σ_2 which satisfy certain inequalities. These inequalities are specified in Definition 2.1. Functions σ_1 and σ_2 are called lower and upper functions (or lower and upper solutions) of the boundary value problem in question. Provided that the functions $\sigma_1 \le \sigma_2$ are given, we can construct an auxiliary differential equation which has a solution u satisfying the prescribed boundary conditions. The auxiliary differential equation is constructed in such a way, that if $\sigma_1 \le u \le \sigma_2$ holds, then u solves also the original equation.

Let $[0,T] \subset \mathbb{R}$ and consider the following boundary value problem:

$$(p(t)u'(t))' + p(t)q(t)f(t,u(t)) = 0, \quad 0 < t < T, \tag{2.1}$$

$$\lim_{t\to 0+} p(t)u'(t) = 0, \quad u(T) = 0, \tag{2.2}$$

where p, q, and f are given. We now define the lower and the upper function for the problem (2.1), (2.2).

Definition 2.1. A function $\sigma \in C[0,T]$ is called *a lower function of* (2.1), (2.2), if there exists a finite set $\Sigma \subset (0,T)$ such that $p\sigma' \in AC_{loc}((0,T) \setminus \Sigma)$, and $\sigma'(\tau+), \sigma'(\tau-) \in \mathbb{R}$ for each $\tau \in \Sigma$. Moreover, σ has to satisfy

$$(p(t)\sigma'(t))' + p(t)q(t)f(t,\sigma(t)) \ge 0, \tag{2.3}$$

for a.e. $t \in [0,T]$ and

$$\lim_{t\to 0+} p(t)\sigma'(t) \ge 0, \quad \sigma(T) \le 0, \quad \sigma'(\tau-) < \sigma'(\tau+), \tag{2.4}$$

for each $\tau \in \Sigma$. If the inequalities in (2.3) and (2.4) are reversed, then σ is called *an upper function of* (2.1), (2.2).
Note that the first derivatives of lower and upper functions can be unbounded at the endpoints of the interval of integration, $t = 0$ and $t = T$.

For the subsequent analysis we make the following assumptions:

A1: We assume p and q to be continuous, $p \in C[0,T]$, $q \in C(0,T]$, and positive, $p(t) > 0$, $q(t) > 0$, for $t \in (0,T]$.

A2: We assume that the following integrals are bounded:

$$\int_0^T p(s)q(s)ds < \infty, \quad \int_0^T \frac{1}{p(t)} \int_0^t p(s)q(s)ds\,dt < \infty.$$

A3: Function f satisfies the L_∞-Carathéodory conditions on $[0,T] \times \mathbb{R}$.

Definition 2.2. A function $u \in C[0,T] \cap C^1(0,T]$ with $pu' \in AC[0,T]$, is called *a solution* of the problem (2.1), (2.2), if it satisfies the ODE (2.1) for a.e. $t \in [0,T]$, and if the BCs (2.2) hold.
It turns out that if $J \subseteq (0,T)$, f is continuous on $J \times \mathbb{R}$, and u is a solution of (2.1), (2.2), then $u \in C^2(J)$, $pu' \in C^1(J)$ and u satisfies (2.1) for all $t \in J$.

To prove the existence of a solution u, we use the generalized lower and upper functions method for the problem (2.1), (2.2). The related fundamental statement is given in Theorem 2.3.

Theorem 2.3. *Let σ_1 and σ_2 be a lower and an upper function of the problem* (2.1), (2.2) *respectively, such that $\sigma_1(t) \le \sigma_2(t)$, $t \in [0,T]$. Let us also assume that A1, A2, and A3 hold. Then, the boundary value problem* (2.1), (2.2) *has a solution u satisfying*

$$\sigma_1(t) \le u(t) \le \sigma_2(t), \quad t \in [0,T]. \tag{2.5}$$

If moreover,

$$\lim_{t\to 0+} \frac{1}{p(t)} \int_0^t p(s)q(s)ds = 0, \tag{2.6}$$

then

$$u \in C^1[0,T], \quad u'(0) = 0. \tag{2.7}$$

Proof. In the first step of the proof, we show the existence of a solution u of the auxiliary problem. For a.e. $t \in [0,T]$ and all $x \in \mathbb{R}$ we define

$$f^*(t,x) = \begin{cases} f(t,\sigma_2(t)) - \frac{x-\sigma_2(t)}{x-\sigma_2(t)+1}, & x > \sigma_2(t), \\ f(t,x), \quad \sigma_1(t) \le x \le \sigma_2(t), \\ f(t,\sigma_1(t)) + \frac{\sigma_1(t)-x}{\sigma_1(t)-x+1}, & x < \sigma_1(t), \end{cases}$$

and consider the equation

$$(p(t)u'(t))' + p(t)q(t)f^*(t,u(t)) = 0. \tag{2.8}$$

Define the operator $\mathcal{F} : C[0,T] \to C[0,T]$ by

$$(\mathcal{F}u)(t) := \int_t^T -\frac{1}{p(\tau)} \int_0^\tau p(s)q(s)f^*(s,u(s))ds\, d\tau. \tag{2.9}$$

Since A3 holds, we can find a function $m^* \in L_\infty[0,T]$ such that

$$|f^*(t,x)| \le m^*(t) \tag{2.10}$$

for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}$. Therefore, due to A2, $\mathcal{F}$ is continuous and compact, and the Schauder Fixed Point Theorem guaranties that a fixed point $u \in C[0,T]$ of $\mathcal{F}$ exists. According to (2.9) we now have

$$u(t) = \int_t^T -\frac{1}{p(\tau)} \int_0^\tau p(s)q(s)f^*(s,u(s))ds\, d\tau, \quad t \in [0,T].$$

Hence, u satisfies (2.8) a.e. in $[0,T]$, BCs (2.2) hold, and $pu' \in AC[0,T]$. The assumptions $p \in C[0,T]$ and $p > 0$ on $(0,T]$ result in $u \in C^1(0,T]$. This means that u is a solution of the problem (2.8), (2.2).

If additionally, (2.6) holds, we can use (2.10) to conclude

$$\begin{aligned}\lim_{t\to 0+}|u'(t)| &= \lim_{t\to 0+}\left|-\frac{1}{p(t)}\int_0^t p(s)q(s)f^*(s,u(s))ds\right| \\ &\le \|m^*\| \lim_{t\to 0+}\frac{1}{p(t)}\int_0^t p(s)q(s)ds = 0.\end{aligned}$$

Finally, we set $u'(0) := \lim_{t\to 0+} u'(t) = 0$, and (2.7) follows.

In the second step, we show that u solves (2.1). To this end we verify that (2.5) holds. Let us set $v(t) := u(t) - \sigma_2(t)$, $t \in [0,T]$, and assume that

$$\max_{0\le t\le T} v(t) = v(t_0) > 0. \tag{2.11}$$

Since $\sigma_2(T) \ge 0$ and $u(T) = 0$, it follows that $t_0 \in [0,T)$. Let $t_0 = 0$, then we have from (2.2) and (2.4) that $\lim_{t\to 0+} p(t)v'(t) \ge 0$. Let $\lim_{t\to 0+} p(t)v'(t) > 0$ then $\lim_{t\to 0+} v'(t) > 0$, which contradicts (2.11). Therefore, $\lim_{t\to 0+} p(t)v'(t) = 0$ holds. Now, let $t_0 \in (0,T)$. Then (2.11) implies $v'(t_0) = 0$.

We summarize: For $t_0 \in [0,T)$ we have $p(t_0)v'(t_0) = 0$ and we can find a $\delta > 0$ such that $v(t) > 0$ on $(t_0, t_0+\delta) \subset (0,T)$, and

$$\begin{aligned}(p(t)v'(t))' &= (p(t)u'(t))' - (p(t)\sigma_2'(t))' \\ &\ge -p(t)q(t)\left(f(t,\sigma_2(t)) - \frac{u(t)-\sigma_2(t)}{u(t)-\sigma_2(t)+1}\right) + p(t)q(t)f(t,\sigma_2(t)) \\ &= p(t)q(t)\frac{v(t)}{v(t)+1} > 0\end{aligned}$$

a.e. in $(t_0, t_0+\delta)$. Therefore

$$0 < \int_{t_0}^t p(s)q(s)\frac{v(s)}{v(s)+1}ds \le \int_{t_0}^t (p(s)v'(s))'ds = p(t)v'(t)$$

for $t \in (t_0, t_0+\delta)$, contradicting (2.11). We have shown that $u(t) \le \sigma_2(t)$, $t \in [0,T]$. The inequality $\sigma_1(t) \le u(t)$, $t \in [0,T]$ follows analogously. The definition of f^* finally implies that u is a solution of (2.1). □

Example. Let $a > 0$, $\varepsilon > 0$, $p(t) = t^a$, $q(t) = t^{\varepsilon-1}$. Then p and q satisfy A1, A2, and (2.6).

The main difficulty in applying Theorem 2.3 is to find a lower function σ_1 and an upper function σ_2 for the problem (2.1), (2.2), which are well ordered, i.e., $\sigma_1(t) \le \sigma_2(t)$ for all $t \in [0,T]$. If $f(\cdot,x)$ in (2.1) changes its sign on $[0,T]$, for instance, then lower and upper functions of (2.1), (2.2) have to be nonconstant and therefore their computation can be difficult. In Lemmas 2.4 and 2.5 we present two pairs of well ordered lower and upper functions for the problem (1.1), (1.2).

Lemma 2.4. *Let $\gamma > 3/2$. Then there exist constants $\nu_*, c_* \in (0,\infty)$ such that for each $\nu \in (0,\nu_*]$ and $c \ge c_*$ the following functions:*

$$\sigma_1(t) = \nu(t+\nu)(1-t), \quad \sigma_2(t) = c\sqrt{1-t^2}, \quad t \in [0,1] \tag{2.12}$$

are lower and upper functions of the problem (1.1), (1.2).

Proof. It follows from (2.12) that $\sigma_1'(t) = \nu(1-2t-\nu)$ and $\sigma_2'(t) = \frac{-ct}{\sqrt{1-t^2}}$. Thus,

$$\lim_{t\to 0+} t^3\sigma_1'(t) = 0,\ \lim_{t\to 0+} t^3\sigma_2'(t) = 0,\ \sigma_1(1) = \sigma_2(1) = 0. \tag{2.13}$$

By inserting σ_1 into (1.1), we obtain

$$\begin{aligned}(t^3\sigma_1'(t))' + t^3 &\left(\frac{1}{8\sigma_1^2(t)} - \frac{a_0}{\sigma_1(t)} - b_0 t^{2\gamma-4}\right)\\ &= t^2\left(\nu\varphi_1(t,\nu) + \frac{t}{\nu^2(1-t)^2(t+\nu)^2}\varphi_2(t,\nu)\right), \quad 0 < t < 1,\end{aligned}$$

where

$$\varphi_1(t,\nu) = 3 - 3\nu - 8t,\ \ \varphi_2(t,\nu) = \frac{1}{8} - a_0\nu(1-t)(t+\nu) - b_0 t^{2\gamma-4}\nu^2(1-t)^2(t+\nu)^2.$$

Let us choose $\nu_0 \in (0, \frac{3}{11})$ such that

$$a_0\nu_0(1+\nu_0) + b_0\nu_0^2(1+\nu_0)^2 < \frac{1}{16},$$

then for all $\nu \in (0,\nu_0)$, we have

$$\varphi_1(t,\nu) > 0, \quad \varphi_2(t,\nu) > 0, \quad t \in [0,\nu].$$

Moreover, we can find $\nu_* \in (0,\nu_0)$ such that

$$\nu_*^2\varphi_1(t,\nu_*) + \frac{1}{16\nu_*(1+\nu_*)^2} > 0, \quad t \in [\nu_*, 1],$$

and consequently, for all $\nu \in (0,\nu_*]$, it follows

$$(t^3\sigma_1'(t))' + t^3\left(\frac{1}{8\sigma_1^2(t)} - \frac{a_0}{\sigma_1(t)} - b_0 t^{2\gamma-4}\right) \ge 0, \quad t \in [0,1). \tag{2.14}$$

By (2.13) and (2.14), σ_1 is a lower function of the problem (1.1), (1.2).
We now insert σ_2 into (1.1) and obtain

$$(t^3\sigma_2'(t))' + t^3\left(\frac{1}{8\sigma_2^2(t)} - \frac{a_0}{\sigma_2(t)} - b_0 t^{2\gamma-4}\right) \le t^3\varphi_3(t,c), \quad t \in [0,1),$$

where

$$\varphi_3(t,c) = -c(1-t^2)^{-\frac{3}{2}}\left(1 - \frac{\sqrt{1-t^2}}{8c^3}\right).$$

Hence, $\lim_{c\to\infty}\varphi_3(t,\nu) = -\infty$ uniformly in $[0,1)$. Therefore, there exists a constant $c_* > 0$ such that for all $c \in [c_*,\infty)$ in the definition of σ_2, cf. (2.12), we have

$$(t^3\sigma_2'(t))' + t^3\left(\frac{1}{8\sigma_2^2(t)} - \frac{a_0}{\sigma_2(t)} - b_0 t^{2\gamma-4}\right) \le 0, \quad t \in [0,1). \tag{2.15}$$

Finally, we conclude from (2.13) and (2.15) that σ_2 is an upper function of the problem (1.1), (1.2) which completes the proof. $\square$

Lemma 2.5. *Let $\gamma \in (1, 3/2]$. Then there exist constants $\nu_*, c_* \in (0,\infty)$ such that for each $\nu \in (0,\nu_*]$ and $c \ge c_*$ the following functions:*

$$\sigma_1(t) = \nu t^{2-\gamma}(1-t), \quad \sigma_2(t) = c\sqrt{1-t^2}, \quad t \in [0,1], \tag{2.16}$$

are lower and upper functions of the problem (1.1), (1.2).

Proof: We first calculate the derivatives of σ_1 and σ_2,

$$\sigma_1'(t) = \nu t^{1-\gamma}(2-\gamma-(3-\gamma)t), \quad \sigma_2'(t) = \frac{-ct}{\sqrt{1-t^2}}.$$

Clearly, σ_1 and σ_2 satisfy (2.13). By inserting σ_1 into (1.1), we obtain

$$\begin{aligned}
&(t^3\sigma_1'(t))' + t^3\left(\frac{1}{8\sigma_1^2(t)} - \frac{a_0}{\sigma_1(t)} - b_0 t^{2\gamma-4}\right) \\
&\qquad = \nu t^{3-\gamma}[(4-\gamma)(2-\gamma)-(5-\gamma)(3-\gamma)t] + \frac{t^{2\gamma-1}}{\nu^2(1-t)^2}\psi(t,\nu) \\
&\qquad > t^{4-\gamma}\left(-\nu(5-\gamma)(3-\gamma) + \frac{t^{3\gamma-5}}{\nu^2(1-t)^2}\psi(t,\nu)\right), \quad t \in (0,1),
\end{aligned}$$

where

$$\psi(t,\nu) = \frac{1}{8} - a_0\nu(1-t)t^{2-\gamma} - b_0\nu^2(1-t)^2.$$

We now find a constant $\nu_0 > 0$ such that $\psi(t,\nu) > 0$ for $t \in [0,1]$ and $\nu \in (0,\nu_0]$. Furthermore,

$$\lim_{\nu\to 0+} \frac{1}{t^{5-3\gamma}(1-t)^2\nu^2} = \infty$$

uniformly in $(0,1)$ and therefore, we are able to provide another constant $\nu_* \in (0,\nu_0]$ such that for any $\nu \in (0,\nu_*]$ in the definition of σ_1, see (2.16),

$$(t^3\sigma_1'(t))' + t^3\left(\frac{1}{8\sigma_1^2(t)} - \frac{a_0}{\sigma_1(t)} - b_0 t^{2\gamma-4}\right) > 0, \quad t \in (0,1), \tag{2.17}$$

holds. This means that by (2.13) and (2.17), σ_1 is a lower function of the problem (1.1), (1.2). By Lemma 2.4 function σ_2 is an upper function, and the result follows. □

3 Analytical results

In this section we present the main analytical results characterizing the solvability of the problem (1.1), (1.2). We begin with considering the case $\gamma > 3/2$. This study will utilize results provided by Lemma 2.4.

Theorem 3.1. *Let $\gamma > 3/2$. Then there exists a positive solution u of the problem (1.1), (1.2). Moreover, this solution satisfies*

$$u(0) > 0, \quad \lim_{t\to 0+} u'(t) = 0. \tag{3.1}$$

Proof. We first construct auxiliary functions f_k. Let $T = 1$. We set

$$p(t) = t^3,\ q(t) = 1,\ f(t,x) = \frac{1}{8x^2} - \frac{a_0}{x} - b_0 t^{2\gamma-4}. \tag{3.2}$$

It is easily seen that p and q satisfy A1, A2, and (2.6), but A3 does not hold for f. To remedy the situation, we introduce a sequence of functions f_k, $k \in \mathbb{N}$, $k > 3$, $t \in [0,1]$, $x \in \mathbb{R}$,

$$f_k(t,x) := \begin{cases} 0, \quad t \in [0, \frac{1}{k}), \\ f(t,\alpha(t,x)), \quad t \in [\frac{1}{k}, 1-\frac{1}{k}], , \\ 1, \quad t \in (1-\frac{1}{k}, 1], \end{cases} \tag{3.3}$$

where

$$\alpha(t,x) := \begin{cases} \sigma_2(t), & x > \sigma_2(t), \\ x, & \sigma_1(t) \le x \le \sigma_2(t), \\ \sigma_1(t), & x < \sigma_1(t). \end{cases}$$

Note that all functions f_k satisfy A3.

We now design the lower and upper functions for auxiliary problems, where f has been replaced by f_k,

$$(t^3 u'(t))' + t^3 f_k(t, u(t)) = 0. \tag{3.4}$$

Let σ_1 and σ_2 be specified by (2.12), where $\nu \le \nu_* < \frac{1}{9}$ and $c \ge c_* > 1$. Then, by Lemma 2.4, σ_1 is a lower function and σ_2 is an upper function of the problem (1.1), (1.2). Since $k > 3$, we have

$$(t^3 \sigma_1'(t))' = t^2 \nu(3 - 3\nu - 8t) > 0, \quad t \in \left[0, \frac{1}{k}\right),$$

and

$$(t^3 \sigma_1'(t))' + t^3 = t^2(\nu(3 - 3\nu - 8t) + t) > 0, \quad t \in \left(1 - \frac{1}{k}, 1\right].$$

Similarly,

$$(t^3 \sigma_2'(t))' = -ct^3(1-t^2)^{-\frac{3}{2}}(4 - 3t^2) < 0, \quad t \in \left[0, \frac{1}{k}\right),$$

and

$$(t^3 \sigma_2'(t))' + t^3 = t^3(-c(1-t^2)^{-\frac{3}{2}}(4 - 3t^2) + 1) < 0, \quad t \in \left(1 - \frac{1}{k}, 1\right).$$

Therefore σ_1 and σ_2 are also lower and upper functions of the problem (3.4), (1.2). With no loss of generality, we can choose $\nu \in (0, \nu_*)$ and $c \ge c_*$ in such a way that $\nu(1+\nu) < c$ holds. Then $\sigma_1 \le \sigma_2$ on $[0,1]$ and, by Theorem 2.3, the problem (3.4), (1.2) has a solution $u_k \in C^1[0,1]$, $k > 3$, satisfying

$$\sigma_1(t) \le u_k(t) \le \sigma_2(t), \quad t \in [0,1], \quad u_k'(0) = 0. \tag{3.5}$$

We regard the sequence u_k, $k \in \mathbb{N}$, $k > 3$, of solutions to the problem (3.4), (1.2), as a sequence of approximations to u, and first discuss the convergence properties of $\{u_k\}$. Let us choose an interval $[0,b] \subset [0,1)$. Then, there exists an index $k_1 \in \mathbb{N}$ such that $[0,b] \subset [0, 1 - \frac{1}{k}]$ for $k \ge k_1$ and due to (3.4), (1.2) and (3.5), we have

$$t^3 u_k'(t) + \int_0^t s^3 \left(\frac{1}{8u_k^2(s)} - \frac{a_0}{u_k(s)} - b_0 s^{2\gamma-4} \right) ds = 0, \quad t \in [0,b], \quad k \ge k_1. \tag{3.6}$$

Let

$$r_b := \min_{0 \le t \le b} \sigma_1(t), \quad m_b := \frac{1}{8r_b^2} + \frac{a_0}{r_b}. \tag{3.7}$$

It follows from (2.12) that $r_b > 0$ and hence, (3.2), (3.3) and (3.6) yield

$$|t^3 f_k(t, u_k(t))| \le m_b t^3 + b_0 t^{2\gamma-1}, \quad |t^3 u_k'(t)| \le \frac{m_b}{4} t^4 + \frac{b_0}{2\gamma} t^{2\gamma}, \quad t \in [0,b], \tag{3.8}$$

provided that $k \geq k_1$. Due to (3.5) and (3.8), the sequences $\{u_k\}$ and $\{u_k'\}$ are bounded on $[0,b]$, which implies that $\{u_k\}$ is equicontinuous on $[0,b]$. Furthermore, for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $t_1, t_2 \in [0,b]$ and $k \geq k_1$,

$$|t_1 - t_2| < \delta \;\Rightarrow\; |t_1^3 u_k'(t_1) - t_2^3 u_k'(t_2)| \leq m_b \left| \int_{t_1}^{t_2} s^3 ds \right| + b_0 \left| \int_{t_1}^{t_2} s^{2\gamma-1} ds \right| < \varepsilon.$$

Hence the sequence $\{t^3 u_k'\}$ is equicontinuous on $[0,b]$ and, by (3.8), it is bounded on $[0,b]$. The Ascoli-Arzelà Theorem now implies that there exists a subsequence $\{u_{k_\ell}\} \subset \{u_k\}$ such that

$$\lim_{\ell\to\infty} u_{k_\ell} = u, \quad \lim_{\ell\to\infty} t^3 u_{k_\ell}' = t^3 u'$$

uniformly in $[0,b]$. Finally, by diagonalization, we find a subsequence[2] satisfying

$$\lim_{k\to\infty} u_k = u, \quad \lim_{k\to\infty} t^3 u_k' = t^3 u' \tag{3.9}$$

locally uniformly in $[0,1)$.

We conclude the proof by describing the properties of the limiting function u. Due to (3.5) and (3.9) we have

$$\sigma_1(t) \leq u(t) \leq \sigma_2(t), \quad t \in [0,1), \quad u \in C[0,1), \quad t^3 u' \in C[0,1). \tag{3.10}$$

Since $\sigma_1(1) = \sigma_2(1) = 0$ and $\lim_{t\to 0+} t^3 u_k'(t) = 0$, it follows that

$$\lim_{t\to 1-} u(t) = 0, \quad \lim_{t\to 0+} t^3 u'(t) = 0. \tag{3.11}$$

Moreover, (3.3) and (3.9) imply

$$\lim_{k\to\infty} t^3 f_k(t, u_k(t)) = t^3 f(t, u(t)), \quad t \in (0,1),$$

and from (3.4) and (3.8) we obtain

$$|t^3 f_k(t, u_k(t))| \leq m_b t^3 + b_0 t^{2\gamma-1},$$

for $t \in [0,b]$ and $k \geq k_1$. Consequently, we can use the Lebesgue Theorem on $[0,b]$, and having in mind that $b \in (0,1)$ is arbitrary, we conclude by letting $k \to \infty$ in (3.6),

$$t^3 u'(t) + \int_0^t s^3 \left(\frac{1}{8u^2(s)} - \frac{a_0}{u(s)} - b_0 s^{2\gamma-4} \right) ds = 0, \quad t \in (0,1). \tag{3.12}$$

Thus $u \in C^2(0,1)$ and u satisfies (1.1) for $t \in (0,1)$. Setting $u(1) := \lim_{t\to 1-} u(t)$, we obtain $u(1) = 0$ and $u \in C[0,1]$. These smoothness properties of u together with (3.10), and (3.11), guarantee that u is a positive solution of (1.1), (1.2). It remains to show that (3.1) holds. From $\sigma_1(0) > 0$, the first condition in (3.1) follows. The second condition results on noting that

$$\lim_{t\to 0+} |u'(t)| \leq \lim_{t\to 0+} \frac{m_b}{4} t + \lim_{t\to 0+} \frac{b_0}{2\gamma} t^{2\gamma-3} = 0,$$

due to (3.12), (3.7) and (3.10). □

Now, we apply results from Lemma 2.5, in order to cover the case $\gamma \in (1, 3/2]$.

[2] for simplicity we do not change the notation here

Theorem 3.2. *Let $\gamma \in (1, 3/2]$. Then there exists a positive solution u of the problem* (1.1), (1.2). *For $\gamma = 3/2$, the solution u satisfies*

$$u(0) > 0, \quad \lim_{t\to 0+} u'(t) = \frac{b_0}{3}. \tag{3.13}$$

Proof. The arguments are similar to those given in the proof of Theorem 3.1 and the beginning of the proof is fully analogous. The main difference is the definition of the lower function σ_1 which is now specified by (2.16), with $\nu \le \nu_* < \frac{1}{15}$. By Lemma 2.5, σ_1 is a lower function of (1.1), (1.2). For $k > 3$, we again have

$$(t^3\sigma_1'(t))' = \nu t^{4-\gamma}((5-\gamma)(3-\gamma) - (6-\gamma)(4-\gamma)t) > 0, \quad t \in \left[0, \frac{1}{k}\right),$$

and

$$(t^3\sigma_1'(t))' + t^3 = \nu t^{4-\gamma}((5-\gamma)(3-\gamma) - (6-\gamma)(4-\gamma)t) + t^3 > 0, \quad t \in \left(1 - \frac{1}{k}, 1\right],$$

which implies that σ_1 is also a lower function of (3.4), (1.2). Since σ_2 is the same as in the previous proof, it is an upper function of (3.4), (1.2). Now, for $k \in \mathbb{N}, k > 3$, the sequence $\{u_k\}$ defined in the proof of Theorem 3.1 is a sequence of solutions to the problems (3.4), (1.2). Also, $u_k \in C^1[0,1]$ and it satisfies (3.5).

Consider an interval $[0,b] \subset [0,1)$ and the sequence $\{u_k\}$, $k \in \mathbb{N}, k > 3$. Then (3.6) holds. Let

$$a_1 := \frac{a_0}{\nu(1-b)}, \quad b_1 := \frac{1}{8\nu^2(1-b)^2} + b_0,$$

then

$$\frac{t^3}{8\sigma_1^2(t)} + \frac{a_0 t^3}{\sigma_1(t)} + b_0 t^{2\gamma-1} \le a_1 t^{\gamma+1} + b_1 t^{2\gamma-1}, \quad t \in [0,b]. \tag{3.14}$$

Thus, (3.5), (3.6), and (3.14), yield

$$|t^3 f_k(t, u_k(t))| \le a_1 t^{\gamma+1} + b_1 t^{2\gamma-1}, \quad |t^3 u_k'(t)| \le \frac{a_1}{\gamma+2} t^{\gamma+2} + \frac{b_1}{2\gamma} t^{2\gamma}, \quad t \in [0,b],$$

provided that $k \ge k_1$. Hence, for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $t_1, t_2 \in [0,b]$ and $k \ge k_1$,

$$|t_1 - t_2| < \delta \Rightarrow |t_1^3 u_k'(t_1) - t_2^3 u_k'(t_2)| \le \left| \int_{t_1}^{t_2} (a_1 t^{\gamma+1} + b_1 t^{2\gamma-1}) dt \right| < \varepsilon,$$

and

$$|t_1 - t_2| < \delta \Rightarrow |u_k(t_1) - u_k(t_2)| \le \left| \int_{t_1}^{t_2} \left(\frac{a_1}{\gamma+2} t^{\gamma-1} + \frac{b_1}{2\gamma} t^{2\gamma-3} \right) dt \right| < \varepsilon.$$

Therefore the sequences $\{u_k\}$ and $\{t^3 u_k'\}$ are bounded and equicontinuous on $[0,b]$ and (3.9) results due to the arguments given in the proof of Theorem 3.1.

Smoothness of u and the properties (3.10), (3.11) can be shown as in the proof of Theorem 3.1. Since $\sigma_1(0) = 0$ and $\lim_{t\to 0+} \sigma_1'(t) = \infty$, we conclude

$$u(0) = 0 \Rightarrow \lim_{t\to 0+} u'(t) = \infty \tag{3.15}$$

by virtue of (3.10). Furthermore, because $\gamma > 1$ and $\lim_{t\to 0+} t^3 u'(t) = 0$, relation (3.12) yields

$$\lim_{t\to 0+} \int_0^t s^3 \left(\frac{a_0}{u(s)} - \frac{1}{8u^2(s)} \right) ds = 0,$$

and hence, by the l'Hospital rule,

$$\lim_{t\to 0+} u'(t) = \lim_{t\to 0+} \frac{\int_0^t s^3 \left(\frac{a_0}{u(s)} - \frac{1}{8u^2(s)}\right) ds}{t^3} + \lim_{t\to 0+} \frac{b_0}{2\gamma} t^{2\gamma-3}$$

$$= \frac{1}{3} \lim_{t\to 0+} \frac{t}{u^2(t)} \left(a_0 u(t) - \frac{1}{8}\right) + \frac{b_0}{2\gamma} \lim_{t\to 0+} t^{2\gamma-3}. \tag{3.16}$$

Let $\gamma = 3/2$ and assume that $u(0) = 0$ holds. Then

$$\lim_{t\to 0+} u'(t) = \frac{1}{3} \lim_{t\to 0+} \frac{t}{u^2(t)} \left(-\frac{1}{8}\right) + \frac{b_0}{3} \le \frac{b_0}{3},$$

in contradiction to (3.15). Thus $u(0) > 0$ and, due to (3.16), $\lim_{t\to 0+} u'(t) = \frac{b_0}{3}$. This completes the proof. □

Remark. Consider a positive solution u of (1.1), (1.2) for $\gamma > 1$. We first recapitulate the behavior of u' at the singular point $t = 0$.
Let $\gamma > 3/2$. Then, by (3.1), we know that $u'(0+) = 0$ holds.
Let $\gamma = 3/2$. Then, by (3.13), the derivative satisfies $u'(0+) = b_0/3$.
Let $\gamma \in (1, 3/2)$. Then $u'(0+) = \infty$. This follows from (3.15) for $u(0) = 0$ and from (3.16) for $u(0) > 0$.

Finally, let us consider the singular point $t = 1$. Since $u(1) = 0$, there exists a $\xi \in (0,1)$ such that $a_0 u(t) \le \frac{1}{16}$ for $t \in [\xi, 1]$. Let σ_2 be an upper function given by (2.12) and satisfying (3.10). Then, it follows

$$-\int_\xi^t \frac{ds}{u^2(s)} \le -\int_\xi^t \frac{ds}{\sigma_2^2(s)} \le -\frac{1}{2c^2} \int_\xi^t \frac{ds}{1-s} = \frac{1}{2c^2} \ln \frac{1-t}{1-\xi}, \quad t \in (\xi, 1).$$

Integrating (1.1) yields

$$t^3 u'(t) = \xi^3 u'(\xi) + \int_\xi^t \frac{s^3}{u^2(s)} \left(a_0 u(s) - \frac{1}{8}\right) ds + b_0 \int_\xi^t s^{2\gamma-1} ds$$

$$\le \xi^3 u'(\xi) + \frac{\xi^3}{32c^2} \ln \frac{1-t}{1-\xi} + \frac{b_0}{2\gamma}, \quad t \in (\xi, 1),$$

and therefore $\lim_{t\to 1-} t^3 u'(t) = u'(1-) = -\infty$.

4 Numerical Experiments

In this section we solve the boundary value problem (1.1), (1.2) numerically, in order to illustrate the analytical results formulated in Theorems 3.1 and 3.2. To this end we consider two different formulations of the analytical problem. We first rewrite equation (1.1) and obtain an implicit second order equation,

$$t^3 u''(t) u^2(t) + 3t^2 u'(t) u(t) + t^3 \left(\frac{1}{8} - a_0 u(t) - b_0 t^{2\gamma-4} u^2(t)\right) = 0, \quad 0 < t < 1, \tag{4.1}$$

subject to boundary conditions

$$u'(0) = 0, \quad u(1) = 0. \tag{4.2}$$

Moreover, after applying the transformation $v(t) = t^3 u'(t)$ to (1.1) we have a system of two implicit differential equations of order one,

$$t^3 u'(t) - v(t) = 0, \quad 0 < t < 1, \tag{4.3}$$

$$v'(t)u^2(t) + t^3 \left(\frac{1}{8} - a_0 u(t) - b_0 t^{2\gamma-4} u^2(t) \right) = 0, \quad 0 < t < 1, \tag{4.4}$$

subject to boundary conditions

$$v(0) = 0, \quad u(1) = 0. \tag{4.5}$$

The numerical solution of singular boundary value problems has been extensively discussed in the literature, see for example [17]–[19], and [29]–[31]. The convergence theory for most of the standard numerical methods is often provided for the following model problem,

$$t^\alpha z'(t) - M(t)z(t) = f(t, z(t)), \quad 0 < t \le 1, \tag{4.6}$$

$$b(z(0), z(1)) = 0, \tag{4.7}$$

where the matrix $M(t) \in \mathbb{R}^{n\times n}$ and the functions $f(t, z) : [0, 1]\times\mathbb{R}^n \to \mathbb{R}^n$, $b(z_0, z_1) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are given. For $\alpha = 1$, problem (4.6), (4.7) is said to be singular with a singularity of the first kind, for $\alpha > 1$, to be essentially singular. We assume (4.6), (4.7) to be well-posed and to have an appropriately smooth, locally unique solution.

We apply polynomial collocation to solve the problems (4.1), (4.2), and (4.3), (4.4), (4.5). Collocation is a widely used and well-studied standard solution method for two-point boundary value problems, see for example [6] and the references therein. Moreover, for singular problems, many popular discretization methods like finite differences, Runge–Kutta or multistep methods show order reductions, thus making computations inefficient and prohibiting asymptotically correct error estimation and reliable mesh adaptation. Therefore, in our code development, we have chosen collocation as a high-order, robust, general-purpose numerical method. It is a new version of the general purpose MATLAB code `sbvp`, cf. [8] and [26], which has already been successfully applied to a variety of problems, see for example [7], [15], [26], and [27]. The code is designed to solve systems of differential equations whose order may vary between four and zero, which means that algebraic constrains which do not involve derivatives are also admitted. Moreover, the problem can be given in a fully implicit form,

$$F(t, z^{(4)}(t), z^{(3)}(t), z''(t), z'(t), z(t)) = 0, \quad 0 < t \le 1, \tag{4.8}$$

$$b(z^{(3)}(0), z''(0), z'(0), z(0), z^{(3)}(1), z''(1), z'(1), z(1)) = 0. \tag{4.9}$$

The numerical approximation defined by collocation is computed as follows: On a mesh

$$\Delta := \{\tau_i : i = 0, \ldots, N\}, \quad 0 = \tau_0 < \tau_1 \cdots < \tau_N = 1$$

we approximate the analytical solution by a piecewise defined collocating function

$$p(s) := p_i(s), \quad s \in [\tau_i, \tau_{i+1}], \quad i = 0, \ldots, N-1,$$

where we require $p \in C^{q-1}[0, 1]$ if the order of the underlying differential equation is q. Here p_i are polynomials of maximal degree $m - 1 + q$ which satisfy the system (4.8) at the *collocation points*

$$\{t_{i,j} = \tau_i + \rho_j(\tau_{i+1} - \tau_i),\ i = 0, \ldots, N-1,\ j = 1, \ldots, m\}, \quad 0 < \rho_1 < \cdots < \rho_m < 1,$$

and the associated boundary conditions (4.9) are also prescribed for p. Classical theory, cf. [6], predicts that the convergence order is at least $O(h^m)$, where h is the maximal stepsize, $h := \max_i |\tau_{i+1} - \tau_i|$. The same could be shown in [20], [39], [10], and [28] for first order problems with a singularity of the first kind. Quite often, even the superconvergence order, in case of Gaussian points $O(h^{2m})$, can be observed in practice. For problems with an essential singularity, extensive numerical evidence and partial theoretical support indicate that the methods retain their convergence order if the collocation points are symmetric, see [11].

To make the computations more efficient, we additionally use an adaptive mesh selection strategy based on an a posteriori estimate for the global error of the collocation solution. We use a classical error estimate based on mesh halving. In this approach, we compute the collocation solution $p_\Delta(s)$ on a mesh Δ. Subsequently, we choose a second mesh Δ_2 where in every interval $[\tau_i, \tau_{i+1}]$ of Δ we insert two subintervals of equal length. On this new mesh, we compute the numerical solution based on the same collocation scheme to obtain the collocating function $p_{\Delta_2}(s)$. Using these two quantities, we define

$$\mathcal{E}(s) := \frac{2^m}{1-2^m}(p_{\Delta_2}(s) - p_\Delta(s)) \tag{4.10}$$

as an error estimate for the approximation $p_\Delta(s)$. Assume that the global error $\delta(s) := p_\Delta(s) - z(s)$ of the collocation solution can be expressed in terms of the principal error function $e(s)$,

$$\delta(s) = e(s)|\tau_{i+1} - \tau_i|^m + O(|\tau_{i+1} - \tau_i|^{m+1}), \quad s \in [\tau_i, \tau_{i+1}], \tag{4.11}$$

where $e(s)$ is independent of Δ. Then obviously the quantity $\mathcal{E}(s)$ satisfies $\mathcal{E}(s) - \delta(s) = O(h^{m+1})$ and the error estimate is asymptotically correct. Our mesh adaptation is based on the equidistribution of the global error of the numerical solution. Thus, we define a *monitor function* $\Theta(s) := \sqrt[m]{\mathcal{E}(s)}/h(s)$, where $h(s) := |\tau_{i+1} - \tau_i|$ for $s \in [\tau_i, \tau_{i+1}]$. Now, the mesh selection strategy aims at the equidistribution of

$$\int_{\tilde{\tau}_i}^{\tilde{\tau}_{i+1}} \Theta(s)\, ds$$

on the mesh consisting of the points $\tilde{\tau}_i$ to be determined accordingly, where at the same time measures are taken to ensure that the variation of the stepsizes is restricted and tolerance requirements are satisfied with small computational effort. Details of the mesh selection algorithm and a proof of the fact that our strategy implies that the global error of the numerical solution is *asymptotically equidistributed* are given in [9].

It is clear that the analytical problems we are studying, do not belong to the class which is covered by the convergence theory, but nevertheless, our code is applicable and we can use it to approximate their solutions. For the first test run we set the parameters to $a_0 = 2$, $b_0 = 3$ and $\gamma = 1.8$, and display the numerical solution $p(t)$ which approximates the unknown analytical solution $u(t)$ computed from the boundary value problems (4.1), (4.2), and (4.3), (4.4), (4.5), in Figures 1 and 2, respectively.

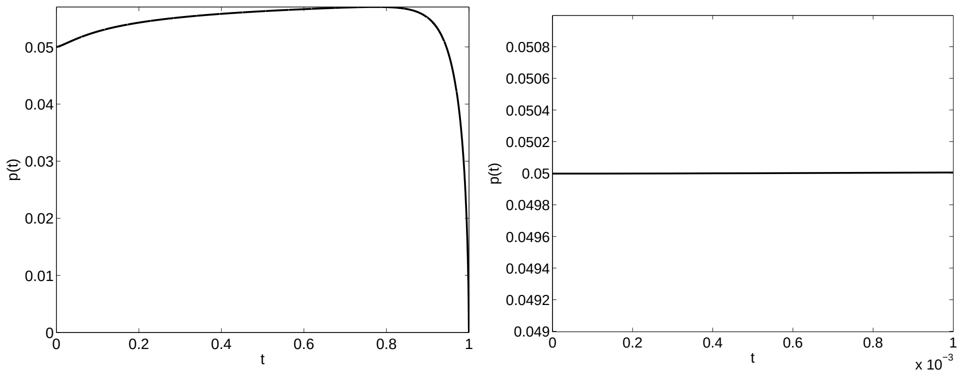

Figure 1: Numerical solution of (4.1), (4.2) for $a_0 = 2$, $b_0 = 3$, $\gamma = 1.8$. Left graph: $p(t)$, $t \in [0, 1]$; Right graph: $p(t)$, $t \in [0, 10^{-3}]$.

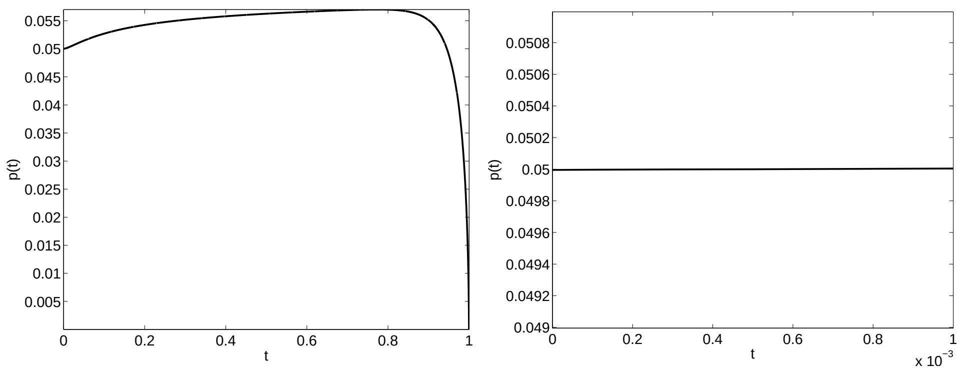

Figure 2: Numerical solution of (4.3), (4.4), (4.5) for $a_0 = 2$, $b_0 = 3$, $\gamma = 1.8$. Left graph: $p(t)$, $t \in [0, 1]$; Right graph: $p(t)$, $t \in [0, 10^{-3}]$.

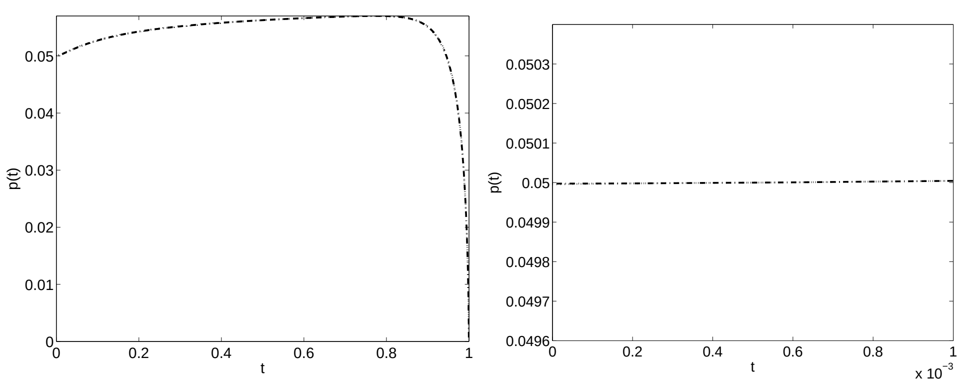

Figure 3: Figures (1) and (2) shown together in one graph. Dashed-dotted line: Solution of the first formulation (4.1), (4.2). Dotted line: Solution of the second formulation (4.3), (4.4), (4.5).

The numerical results are in a good agreement with the statement shown in Theorem 3.1. The value of $p(0)$ is positive and its derivative satisfies $\lim_{t\to 0} p'(t) = 0$, as predicted by the theory.

According to Theorem 3.2, the asymptotical behavior of $u'(0)$ is specified by

$$\lim_{t\to 0} t^3 u'(t) = 0$$

and u' is not bounded at $t = 0$, $u'(0) = \infty$, for $\gamma \in (1, \frac{3}{2})$. Moreover, $\lim_{t\to 0} u'(t) = \frac{b_0}{3}$ for $\gamma = \frac{3}{2}$. Thus, the formulation (4.1), (4.2) is no longer adequate for the numerical treatment and we are therefore limited to (4.3), (4.4), (4.5). Figure 4 shows the related result for the parameter values $a_0 = 1$, $b_0 = 2$, $\gamma = \frac{3}{2}$.

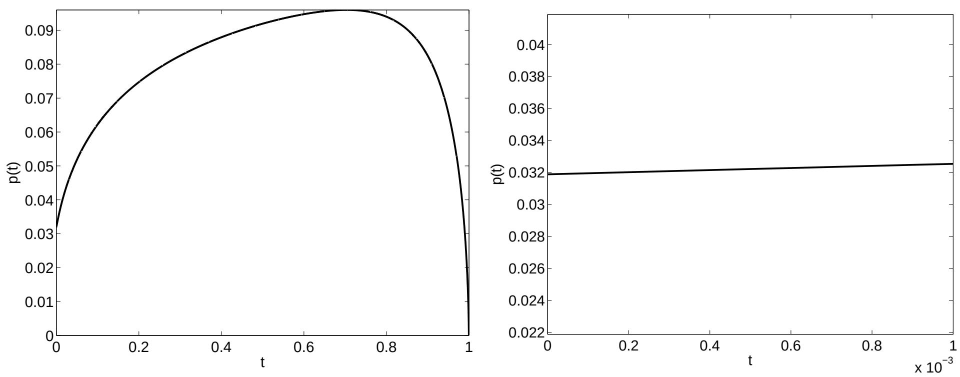

Figure 4: Numerical solution of (4.3), (4.4), (4.5) for $a_0 = 1$, $b_0 = 2$, $\gamma = \frac{3}{2}$. Left graph: $p(t)$, $t \in [0, 1]$; Right graph: $p(t)$, $t \in [0, 10^{-3}]$.

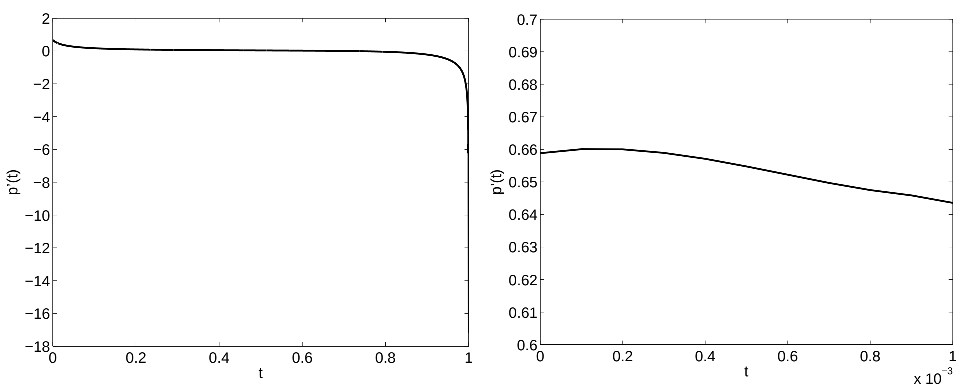

Figure 5: Derivative of the numerical solution of (4.3), (4.4), (4.5) for $a_0 = 1$, $b_0 = 2$, $\gamma = \frac{3}{2}$. Left graph: $p'(t)$, $t \in [0, 1]$; Right graph: $p'(t)$, $t \in [0, 10^{-3}]$.

Again, the experiments reflect correctly the analytical properties of u. It is clear from Figure (4) that $u(0) > 0$ and Figure (5) shows that $\lim_{t\to 0} u'(t) = \frac{b_0}{3} \approx 0.66$ holds.

In the final experiment, we would like to illustrate how the value of $u'(0)$ depends on γ. We expect $u'(0)$ to increase when γ decreases. Therefore we set $a_0 = 1$, $b_0 = 2$

and $\gamma = 1.4$, cf. Figures 6 and 7. Again we observe $u(0) > 0$ and we see that, as expected, $u'(0)$ is now larger than it was for $\gamma = 1.5$.

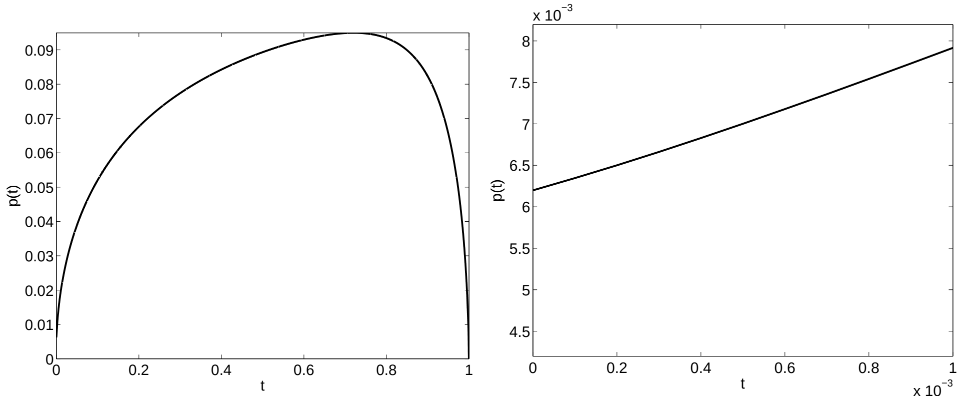

Figure 6: Numerical solution of (4.3), (4.4), (4.5) for $a_0 = 1$, $b_0 = 2$, $\gamma = 1.4$. Left graph: $p(t)$, $t \in [0, 1]$; Right graph: $p(t)$, $t \in [0, 10^{-3}]$.

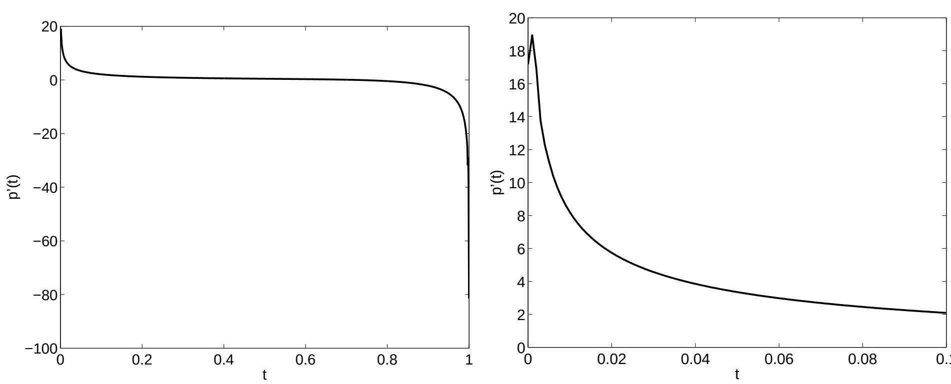

Figure 7: Derivative of the numerical solution of (4.3), (4.4), (4.5) for $a_0 = 1$, $b_0 = 2$, $\gamma = 1.4$. Left graph: $p'(t)$, $t \in [0, 1]$; Right graph: $p'(t)$, $t \in [0, 10^{-1}]$.

We recapitulate: The numerical experiments are in a good agreement with the analytical results shown in Theorems 3.1 and 3.2. It should be noticed however, that the problem under consideration is very difficult to solve numerically, due to the lack of smoothness in the solution u. The controlling mechanisms, error estimate and procedure for the grid adaptation, require a dependable numerical solution of a high order to work properly. When the code tries to follow a very unsmooth solution on an adaptive grid, it naturally, may become very inefficient. For the above experiments, we therefore decided to use an equidistant grid with 1000 gridpoints, in order to avoid unnecessarily long run times.

Acknowledgments

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References

[1] R. P. Agarwal, and D. O'Regan. Singular boundary value problems for superlinear second order ordinary and delay differential equations. *J. Differential Equations* **130** (1996), 333–355.

[2] R. P. Agarwal, and D. O'Regan. Nonlinear superlinear singular and nonsingular second order boundary value problems. *J. Differential Equations* **143** (1998), 60–95.

[3] R. P. Agarwal, and D. O'Regan. Singular problems motivated from classical upper and lower solutions. *Acta Math. Hungar.* **100** (3) (2003), 245–256.

[4] R. P. Agarwal, D. O'Regan, and S. Staněk. Existence of positive solutions for boundary-value problems with singularities in phase variables. *Proc. Edinb. Math. Soc.* **47** (2004), 1–13.

[5] R. P. Agarwal, and S. Staněk. Nonnegative solutions of singular boundary value problems with sign changing nonlinearities. *Comp. Math. Appl.* **46** (2003), 1827–1837.

[6] U. Ascher, R. M. M. Mattheij, and R. D. Russell. Numerical solution of boundary value problems for ordinary differential equations. Prentice-Hall, Englewood Cliffs, New York, 1988.

[7] W. Auzinger, O. Koch, and E. Weinmüller. Efficient collocation schemes for singular boundary value problems. *Numer. Algorithms* **31** (2002), 5–25.

[8] W. Auzinger, G. Kneisl, O. Koch, and E. Weinmüller. A collocation code for boundary value problems in ordinary differential equations, Numer. Algorithms **33** (2003), 27–39.

[9] W. Auzinger, O. Koch, and E. Weinmüller. Efficient mesh selection for collocation methods applied to singular BVPs. *J. Comput. Appl. Math.* **180** (2005), 213–227.

[10] W. Auzinger, O. Koch, and E. Weinmüller. Analysis of a new error estimate for collocation methods applied to singular boundary value problems. *S*IAM J. Numer. Anal. **42** (2005), 2366–2386.

[11] W. Auzinger, O. Koch, and E. Weinmüller. Collocation methods for singular boundary value problems with an essential singularity. Large-Scale Scientific Computing (I. Lirkov, S. Margenov, J. Wasniewski, and P. Yalamov, eds.). *L*ecture Notes in Computer Science, Springer Verlag **2907** (2004), pp. 347–354.

[12] J. V. Baxley. Some singular nonlinear boundary value problems. *SIAM J. Math. Anal.* **22** (1991), 463–479.

[13] J. V. Baxley, and G. S. Gersdorff. Singular reaction-diffusion boundary value problem. *J. Differential Equations* **115** (1995), 441–457.

[14] J. V. Baxley, and K. P. Sorrells. A class of singular nonlinear boundary value problems. *Math. Comp. Modelling* **32** (2000), 631–641.

[15] C. J. Budd, O. Koch, and E. Weinmüller. Self-similar blow-up in nonlinear PDEs. AURORA TR-2004-15, Inst. for Anal. and Sci. Comput., Vienna Univ. of Technology, Austria, 2004. Available at `http://www.vcpc.univie.ac.at/aurora/publications/`.

[16] R. W. Dickey. Rotationally symmetric solutions for shallow membrane caps. *Quart. Appl. Math.* **47** (1989), 571–581.

[17] F. de Hoog, and R. Weiss. Difference methods for boundary value problems with a singularity of the first kind. *SIAM J. Numer. Anal.* **13** (1976), 775–813.

[18] F. de Hoog, and R. Weiss. The numerical solution of boundary value problems with an essential singularity. *SIAM J. Numer. Anal.* **16** (1979), 637–669.

[19] F. de Hoog, and R. Weiss. The application of Runge–Kutta schems to singular initial value problems. *Math. Comp.* **44** (1985), 93–103.

[20] F. de Hoog, and R. Weiss. Collocation methods for singular boundary value problems. *SIAM J. Numer. Anal.* **15** (1978), 198–217.

[21] J. A. Gatica, V. Oliker, and P. Waltman. Singular nonlinear boundary value problems for second-order ordinary differential equations. *J. Differential Equations* **79** (1989), 62–78.

[22] K. N. Johnson. Circularly symmetric deformation of shallow elastic membrane caps. *Quart. Appl. Math.* **55** (1997), 537–550.

[23] R. Kannan, and D. O'Regan. Singular and nonsingular boundary value problems with sign changing nonlinearities. *J. Inequal. Appl.* **5** (2000), 621–637.

[24] P. Kelevedjiev. Nonnegative solutions to some second-order boundary value problems. *Nonlinear Analysis* **36** (1999), 481–494.

[25] I. T. Kiguradze, and B. L. Shekhter. Singular boundary value problems for second order ordinary differential equations. *Itogi Nauki Tekh. Ser. Sovrm. Probl. Mat. Viniti* **30** (1987), 105–201 (in Russian).

[26] G. Kitzhofer. Numerical treatment of implicit singular BVPs. Ph.D. Thesis, Inst. for Anal. and Sci. Comput., Vienna Univ. of Technology, Austria, 2005. In preparation.

[27] G. Kitzhofer, O. Koch, and E. Weinmüller. Collocation methods for the computation of bubble-type solutions of a singular boundary value problem in hydrodynamics. *J. Sci. Comp.* . To appear. Available at `http://www.math.tuwien.ac.at/~ewa`.

[28] O. Koch. Asymptotically correct error estimation for collocation methods applied to singular boundary value problems. *N*umer. Math. **101** (2005), 143–164.

[29] O. Koch, and E. Weinmüller. Iterated Defect Correction for the solution of singular initial value problems. *SIAM J. Numer. Anal.* **38** (2001), 1784–1799.

[30] M. Lentinni, and H. Keller. Boundary value problems on semi-infinite intervals and their numerical solution. *SIAM J. Numer. Anal.* **17** (1980), 577–604.

[31] P. Markowich, and C. Ringhofer. Collocation methods for boundary value problems on long intervals. *Math. Comp.* **40** (1983), 123–150.

[32] D. O'Regan. Theory of singular boundary value problems. *World Scientific*, Singapore 1994.

[33] I. Rachůnková. Singular mixed boundary value problem. *J. Math. Anal. Appl.* **320** (2006), 611–618.

[34] I. Rachůnková. Superlinear mixed BVP with time and space singularities. Conf. Proc. CDDEA 2005, *Hindawi*, to appear.

[35] I. Rachůnková. Strong singularities in mixed boundary value problems. *Math. Bohemica*, to appear.

[36] I. Rachůnková, S. Staněk, and M. Tvrdý. Singularities and Laplacians in Boundary Value Problems for Nonlinear Ordinary Differential Equations. Handbook of Differential Equations. Ordinary Differential Equations, Ed. by A. Caňada, P. Drábek, A. Fonda, Vol. 3. *Elsevier* , to appear.

[37] J. Y. Shin. A singular nonlinear differential equation arising in the Homann flow. *J. Math. Anal. Appl.* **212** (1997), 443–451.

[38] A. Tineo. On a class of singular boundary value problems which contains the boundary conditions $x'(0) = x(1) = 0$. *J. Differential Equations* **113** (1994), 1–16.

[39] E. Weinmüller. Collocation for singular boundary value problems of second order. *SIAM J. Numer. Anal.* **23** (1986), 1062–1095.