# Singular mixed boundary value problem

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**Abstract.** We study singular boundary value problems with mixed boundary conditions of the form

$$u'' + f(t, u, u') = 0, \quad u'(0) = 0, \quad u(T) = 0,$$

where $[0, T] \subset \mathbb{R}$, $\mathcal{D} = (0, \infty) \times (-\infty, 0)$, $f$ is a non-negative function and satisfies the Carathéodory conditions on $(0, T) \times \mathcal{D}$. Here, $f$ can have a time singularity at $t = 0$ and/or $t = T$ and a space singularity at $x = 0$ and/or $y = 0$. We present conditions for the existence of solutions positive on $[0, T)$ and having continuous first derivatives on $[0, T]$.

**Keywords.** Singular mixed boundary value problem, positive solution, lower and upper functions, convergence of approximate regular problems

**Mathematics Subject Classification 2000.** 34B16, 34B18

## 1 Introduction

We investigate the solvability of the singular mixed boundary value problem

$$u'' + f(t, u, u') = 0, \quad u'(0) = 0, \quad u(T) = 0, \tag{1.1}$$

where $[0, T] \subset \mathbb{R}$, $\mathcal{D} = (0, \infty) \times (-\infty, 0)$, $f$ satisfies the Carathéodory conditions on $(0, T) \times \mathcal{D}$. Here, $f$ can have a time singularity at $t = 0$ and/or at $t = T$ and a space singularity at $x = 0$ and/or at $y = 0$. We prove the existence of solutions of (1.1) which are positive on $[0, T)$ and have continuous first derivatives on $[0, T]$.

Let $[a, b] \subset \mathbb{R}$, $\mathcal{M} \subset \mathbb{R}^2$. Recall that a real valued function $f$ satisfies the Carathéodory conditions on the set $[a, b] \times \mathcal{M}$ if

(i) $f(\cdot, x, y) : [a, b] \to \mathbb{R}$ is measurable for all $(x, y) \in \mathcal{M}$,

(ii) $f(t,\cdot,\cdot) : \mathcal{M} \to \mathbb{R}$ is continuous for a.e. $t \in [a,b]$,

(iii) for each compact set $K \subset \mathcal{M}$ there is a function $m_K \in L_1[0,T]$ such that $|f(t,x,y)| \le m_K(t)$ for a.e. $t \in [a,b]$ and all $(x,y) \in K$.
We write $f \in Car([a,b] \times \mathcal{M})$. By $f \in Car((0,T) \times \mathcal{D})$ we mean that $f \in Car([a,b] \times \mathcal{D})$ for each $[a,b] \subset (0,T)$ and $f \notin Car([0,T] \times \mathcal{D})$.

**Definition 1.1.** Let $f \in Car((0,T) \times \mathcal{D})$.

We say that $f$ has *a time singularity* at $t = 0$ and/or at $t = T$ if there exists $(x,y) \in \mathcal{D}$ such that

$$\int_0^{\varepsilon} |f(t,x,y)|dt = \infty \quad \text{and/or} \quad \int_{T-\varepsilon}^{T} |f(t,x,y)|dt = \infty$$

for each sufficiently small $\varepsilon > 0$. The point $t = 0$ and/or $t = T$ will be called *a singular point of* $f$.

We say that $f$ has *a space singularity* at $x = 0$ and/or at $y = 0$ if

$$\limsup_{x \to 0+} |f(t,x,y)| = \infty \quad \text{for a.e. } t \in [0,T] \text{ and for some } y \in (-\infty, 0)$$

and/or

$$\limsup_{y \to 0-} |f(t,x,y)| = \infty \quad \text{for a.e. } t \in [0,T] \text{ and for some } x \in (0, \infty).$$

The importance of singular mixed problems is derived, in part, from the fact that they arised when searching for positive, radially symmetric solutions to the nonlinear elliptic partial differential equations

$$\Delta u + g(r,u) = 0 \text{ on } \Omega, \quad u|_{\Gamma} = 0, \tag{1.2}$$

where $\Omega$ is the open unit disk in $\mathbb{R}^n$ (centered at the origin), $\Gamma$ is its boundary, and $r$ is the radial distance from the origin. Radially symmetric solutions to this problem are solutions of the following ordinary differential equation with the mixed boundary conditions (see e.g. [9] or [11])

$$u'' + \frac{n-1}{t}u' + g(t,u) = 0, \quad u'(0) = 0, \; u(1) = 0, \tag{1.3}$$

where $f(t,x,y) = \frac{n-1}{t}y + g(t,x)$ has a time singularity at $t = 0$.

Particularly, Gatica, Oliker and Waltman [10] investigated problem (1.3) with $g(t,x) = \psi(t)x^{-\alpha}$, $\alpha \in (0,1)$, $\psi \in C[0,1)$. Since $\alpha > 0$, we see that $g$ has a space singularity at $x = 0$. In [10], moreover, $\psi$ is allowed to have a time singularity at $t = 1$ and the authors have found conditions for the existence of a solution positive on $[0,1)$.

In the mathematical literature there are two approaches to solvability of singular problems which depend on different definitions of a solution. Here, we work with the following one:

**Definition 1.2.** By *a solution* of problem (1.1) we understand a function $u \in AC^1[0,T]$ satisfying

$$u''(t) + f(t, u(t), u'(t)) = 0 \quad \text{for a.e. } t \in [0,T], \quad u'(0) = u(T) = 0. \tag{1.4}$$

We see that our solution has continuous first derivatives on $[0,T]$, particularly at the singular point $t = T$. Note that such solution of (1.3) is important for the associated problem (1.2). The alternative approach is based on the following definition of a "solution", which we will call a w-solution.

**Definition 1.3.** By *a w-solution* of problem (1.1) we understand a function $u \in AC^1_{loc}[0,T)$ satisfying (1.4).

Hence having a w-solution we do not know a behaviour of its derivative near the singular point $t = T$. For the existence of w-solutions of (1.1) we refer to [1] - [3], [6], [14] - [16], while the existence of solutions of (1.1) can be found e.g. in [4], [5], [7], [8], [12], [13], [17], [18]. Note that the papers [2], [4], [13], [17], [18] deal with problem (1.1) allowing just space singularities but not time ones and the papers [5], [7], [8] consider both time and space (at $x = 0$ ) singularities. Motivated by the existence results in [2] and [5] which are based on a lot of rather complicated conditions, we offer simple conditions which guarantee the existence of solutions for (1.1) provided both time and space (at $x = 0$ and moreover at $y = 0$) are allowed.

# 2 Lower and upper functions

In the investigation of singular problems lower and upper functions of corresponding regular problems can be a fruitfull tool. See [5], [12] or [15]. Therefore we first consider an auxiliar regular mixed problem

$$u'' + h(t, u, u') = 0, \quad u'(0) = 0, \; u(T) = 0, \tag{2.1}$$

where $h \in Car([0,T] \times \mathbb{R}^2)$.

**Definition 2.1.** A function $\sigma \in C[0,T]$ is called *a lower function of* (2.1) if there exists a finite set $\Sigma \subset (0,T)$ such that $\sigma \in AC^1_{loc}([0,T] \setminus \Sigma)$, $\sigma'(\tau+), \sigma'(\tau-) \in \mathbb{R}$ for each $\tau \in \Sigma$,

$$\sigma''(t) + h(t, \sigma(t), \sigma'(t)) \geq 0 \quad \text{for a.e. } t \in [0,T] \tag{2.2}$$

and

$$\sigma'(0) \geq 0, \quad \sigma(T) \leq 0, \quad \sigma'(\tau-) < \sigma'(\tau+) \quad \text{for each } \tau \in \Sigma. \tag{2.3}$$

If the inequalities in (2.2) and (2.3) are reversed, then $\sigma$ is called *an upper function of* (2.1).

In what follows we will need the classical lower and upper functions result for regular mixed problem (2.1):

**Lemma 2.2.** [15], Lemma 3.7. *Let $\sigma_1$ and $\sigma_2$ be lower and upper functions for problem* (2.1) *such that $\sigma_1 \leq \sigma_2$ on $[0,T]$. Assume also that there is a function $\psi \in L_1[0,T]$ such that*

$$|h(t,x,y| \leq \psi(t) \quad \textit{for a.e. } t \in [0,T], \ \textit{all } x \in [\sigma_1(t), \sigma_2(t)], \ y \in \mathbb{R}. \tag{2.4}$$

*Then problem* (2.1) *has a solution $u \in AC^1[0,T]$ satisfying*

$$\sigma_1(t) \leq u(t) \leq \sigma_2(t) \quad \textit{for } t \in [0,T]. \tag{2.5}$$

# 3 Main result and example

**Theorem 3.1.** *Let $f \in Car((0,T) \times \mathcal{D})$ can have time singularities at $t = 0, t = T$ and space singularities at $x = 0, y = 0$. Assume that there exist $\varepsilon \in (0,1)$, $\nu \in (0,T)$, $c \in (\nu, \infty)$ such that*

$$f(t, c(T-t), -c) = 0 \quad \textit{for a.e. } t \in [0,T], \tag{3.1}$$

$$0 \leq f(t,x,y) \quad \textit{for a.e. } t \in [0,T], \ \textit{all } x \in (0, c(T-t)], y \in [-c, 0), \tag{3.2}$$

$$\varepsilon \leq f(t,x,y) \quad \textit{for a.e. } \ t \in [0,\nu], \ \textit{all } x \in (0, c(T-t)], \ y \in [-\nu, 0). \tag{3.3}$$

*Then problem* (1.1) *has a solution $u \in AC^1[0,T]$ satisfying*

$$0 < u(t) < c(T-t), \quad -c < u'(t) < 0 \ \textit{for } t \in (0,T). \tag{3.4}$$

Proof. Let $k \in \mathbb{N}$, $k \geq 3/T$.

*Step 1. Approximate solutions.* For $t \in [1/k, T - 1/k]$, $x \in \mathbb{R}$ put

$$\alpha_k(t,x) = \begin{cases} c(T-t) & \text{if} \quad x > c(T-t) \\ x & \text{if} \quad c/k \leq x \leq c(T-t) \\ c/k & \text{if} \quad x < c/k \end{cases},$$

and for $y \in \mathbb{R}$ denote

$$\beta_k(y) = \begin{cases} -\varepsilon/k & \text{if} \quad y > -\varepsilon/k \\ y & \text{if} \quad -c \leq y \leq -\varepsilon/k \\ -c & \text{if} \quad y < -c \end{cases},$$

$$\gamma(y) = \begin{cases} \varepsilon & \text{if} \quad y \ge -\nu \\ \varepsilon(c+y)/(c-\nu) & \text{if} \quad -c < y < -\nu \\ 0 & \text{if} \quad y \le -c \end{cases} .$$

For a.e. $t \in [0,T]$ and $x, y \in \mathbb{R}$ define

$$f_k(t,x,y) = \begin{cases} \gamma(y) & \text{if} \quad t \in [0, 1/k) \\ f(t, \alpha_k(t,x), \beta_k(y)) & \text{if} \quad t \in [1/k, T-1/k] \\ 0 & \text{if} \quad t \in (T-1/k, T] \end{cases} .$$

Then $f_k \in Car([0,T] \times \mathbb{R}^2)$ and there is $\psi_k \in L_1[0,T]$ such that

$$|f_k(t,x,y)| \le \psi_k(t) \quad \text{for a.e. } t \in [0,T], \text{ all } x, y \in \mathbb{R}. \tag{3.5}$$

We have got an auxiliary regular problem

$$u'' + f_k(t,u,u') = 0, \quad u'(0) = 0, \; u(T) = 0. \tag{3.6}$$

Conditions (3.2) and (3.1) yield

$$f_k(t,0,0) \ge 0, \quad f_k(t, c(T-t), -c) = 0 \quad \text{for a.e. } t \in [0,T].$$

Put $\sigma_1(t) = 0$, $\sigma_2(t) = c(T-t)$ on $[0,T]$. Then $\sigma_1$ and $\sigma_2$ are lower and upper functions of (3.6). Hence, by Lemma 2.2, problem (3.6) has a solution $u_k$ and

$$0 \le u_k(t) \le c(T-t) \quad \text{on } [0,T]. \tag{3.7}$$

*Step 2. A priori estimates of approximate solutions.* Since $u_k'(0) = 0$ and $f_k(t,x,y) \ge 0$ for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$, we get $u_k'(t) \le 0$ on $[0,T]$. Condition (3.7) and $u_k(T) = 0$ give $u_k(T) - u_k(t) \ge -c(T-t)$ which yields $u_k'(T) \ge -c$. Since $u_k'$ is non-increasing on $[0,T]$, we have proved

$$-c \le u_k'(t) \le 0 \quad \text{on } [0,T]. \tag{3.8}$$

Due to $u_k'(0) = 0$, there is $t_k \in (0,T]$ such that

$$-\nu \le u_k'(t) \le 0 \quad \text{for } t \in [0, t_k].$$

If $t_k \ge \nu$, we get by (3.3)

$$u_k'(t) \le -\varepsilon t \quad \text{for } t \in [0, \nu]. \tag{3.9}$$

Assume that $t_k < \nu$ and $u_k'(t) < -\nu$ for $t \in (t_k, \nu]$. Then

$$u_k'(t) \le -\varepsilon t \quad \text{for } t \in [0, t_k].$$

Since $-\nu < -\varepsilon t$ for $t \in (t_k, \nu]$, we get (3.9) again. Integrating (3.9) on $[0,\nu]$ and using the concavity of $u_k$ on $[0,T]$ we deduce that

$$\frac{\varepsilon \nu^2}{2T}(T-t) \le u_k(t) \quad \text{on } [0,T]. \tag{3.10}$$

*Step 3. Convergence of a sequence of approximate solutions.* Consider the sequence $\{u_k\}$. Choose an arbitrary compact interval $J \subset (0,T)$. By virtue of (3.7)-(3.10) there is $k_0 \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq k_0$

$$\frac{c}{k_0} \leq u_k(t) \leq c(T-t), \quad -c \leq u_k'(t) \leq -\frac{\varepsilon}{k_0} \quad \text{on } J, \tag{3.11}$$

and hence there is $\psi \in L_1(J)$ such that

$$|f_k(t, u_k(t), u_k'(t))| \leq \psi(t) \quad \text{a.e. on } J. \tag{3.12}$$

Using conditions (3.7), (3.8), (3.12), the Arzelà-Ascoli theorem and the diagonalization principle, we can choose $u \in C[0,T] \cap C^1(0,T)$ and a subsequence of $\{u_k\}$ which we denote for the simplicity in the same way such that

$$\begin{aligned} &\lim_{k\to\infty} u_k = u \quad \text{uniformly on } [0,T], \\ &\lim_{k\to\infty} u_k' = u' \quad \text{locally uniformly on } (0,T). \end{aligned} \tag{3.13}$$

Therefore we have $u(T) = 0$.

*Step 4. Convergence of a sequence of approximate problems.* Choose an arbitrary $\xi \in (0,T)$ such that

$$f(\xi, \cdot, \cdot) \quad \text{is continuous on } (0,\infty) \times (-\infty, 0).$$

By (3.11) there exist a compact interval $J^* \subset (0,T)$ and $k_* \in \mathbb{N}$ such that $\xi \in J^*$ and for each $k \geq k_*$

$$u_k(\xi) > \frac{c}{k_*}, \quad u_k'(\xi) < -\frac{\varepsilon}{k_*}, \quad J^* \subset [1/k, T-1/k].$$

Therefore

$$f_k(\xi, u_k(\xi), u_k'(\xi)) = f(\xi, u_k(\xi), u_k'(\xi))$$

and, due to (3.13),

$$\lim_{k\to\infty} f_k(t, u_k(t), u_k'(t)) = f(t, u(t), u'(t)) \quad \text{for a.e. } t \in (0,T). \tag{3.14}$$

Choose an arbitrary $t \in (0,T)$. Then there exists a compact interval $J \subset (0,T)$ such that (3.12) holds for all sufficiently large $k$. By virtue of (3.6) we get

$$u_k'(T/2) - u_k'(t) = \int_{T/2}^{t} f_k(s, u_k(s), u_k'(s))ds.$$

Letting $k \to \infty$ and using (3.12), (3.13), (3.14) and the Lebesgue convergence theorem on $J$, we get

$$u'(T/2) - u'(t) = \int_{T/2}^{t} f(s, u(s), u'(s))ds \quad \text{for each } t \in (0,T). \tag{3.15}$$

Therefore $u \in AC^1_{loc}(0,T)$ satisfies

$$u''(t) + f(t, u(t), u'(t)) = 0 \quad \text{a.e. on } (0,T). \tag{3.16}$$

Further, according to (3.6) and (3.8) we have for each $k \geq 3/T$

$$\int_0^T f_k(s, u_k(s), u_k'(s))ds = -u_k'(T) \leq c,$$

which together with (3.2), (3.7), (3.8) and (3.14) yield, by the Fatou lemma, that $f(t, u(t), u'(t)) \in L_1[0,T]$. Therefore, by (3.16), $u \in AC^1[0,T]$. Moreover for each $k \geq 3/T$ and $t \in (0,T)$

$$|u_k'(t)| \leq \int_0^t |f_k(s, u_k(s), u_k'(s)) - f(s, u(s), u'(s))|ds + \int_0^t |f(s, u(s), u'(s)|ds.$$

Hence, by (3.13) and (3.14), for each $\varepsilon > 0$ there exists $\delta > 0$ and for each $t \in (0, \delta)$ there exists $k_0 = k_0(\varepsilon, t) \in \mathbb{N}$ such that

$$|u'(t)| \leq |u'(t) - u_{k_0}'(t)| + |u_{k_0}'(t)| < \varepsilon.$$

It means that $u'(0) = \lim_{t \to 0+} u'(t) = 0$. We have proved that $u$ is a solution of problem (1.1).

*Example.* Let $\alpha, \gamma \in (0, \infty)$, $k, \beta \in [0, \infty)$. By Theorem 3.1 problem

$$u'' + (u^{-\alpha} + u^\beta + k(-u')^{-\gamma} + 1)(1 + (u')^3) = 0, \quad u'(0) = 0, \quad u(1) = 0 \tag{3.17}$$

has a solution $u \in AC^1[0,1]$ satisfying

$$0 < u(t) < 1 - t, \quad -1 < u'(t) < 0 \quad \text{for } t \in (0,1). \tag{3.18}$$

Note that Theorem 2.2 in [2] yields the existence of a solution of problem (3.17) positive on $[0,1)$ provided the nonlinearity $f(t,x,y) = (x^{-\alpha} + x^\beta + 1)(1 + y^3)$ has a weak space singularity (i.e. $\alpha \in (0,1)$) at $x = 0$ and no singularity (i.e. $k = 0$) at $y$. On the other hand, due to Theorem 3.1, we get a solution $u$ of problem (3.17) satisfying (3.18) even for $f(t,x,y) = (x^{-\alpha} + x^\beta + k(-y)^{-\gamma} + 1)(1 + y^3)$ having a strong space singularity ($\alpha \geq 1$) at $x = 0$ and moreover a space singularity ($k > 0$) at $y = 0$.

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