Singular discrete second order BVPs with p-Laplacian

Irena Rachůnková and Lukáš Rachůnek

Department of Mathematics, Palacký University, Tomkova 40, 77900 Olomouc, Czech Republic, e-mail: rachunko@inf.upol.cz

Abstract. We study singular discrete boundary value problems with mixed boundary conditions and with the p-Laplacian of the form

$$\Delta\Big(\phi_p(\Delta u(t-1))\Big) + f(t, u(t), \Delta u(t-1)) = 0, \quad t \in [1, T+1],$$

$$\Delta u(0) = u(T+2) = 0,$$

where $[1, T+1] = \{1, 2, \ldots, T+1\}$, $T \in \mathbb{N}$, $\phi_p(y) = |y|^{p-2}y$, $p > 1$. We assume that f is continuous on $[1, T+1] \times (0, \infty) \times \mathbb{R}$ and $f(t, x, y)$ has a singularity at $x = 0$. We prove the existence of a positive solution by means of the lower and upper functions method, the Brouwer fixed point theorem and by a convergence of approximate regular problems.

Keywords. Singular discrete BVP, mixed conditions, lower and upper functions, Brouwer fixed point theorem, approximate regular problems

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1 Introduction

Let $T \in \mathbb{N}$ be fixed. We define the discrete interval $[1, T+1] = \{1, 2, \ldots, T+1\}$ and consider the second order difference equation with the p-Laplacian

$$\Delta\Big(\phi_p(\Delta u(t-1))\Big) + f(t, u(t), \Delta u(t-1)) = 0, \quad t \in [1, T+1] \tag{1.1}$$

subjected to the mixed boundary conditions

$$\Delta u(0) = 0, \quad u(T+2) = 0. \tag{1.2}$$

Here Δ denotes the forward difference operator with the step size 1, i.e. $\Delta u(t-1) = u(t) - u(t-1)$, and $\phi_p(y) = |y|^{p-2}y$, $p > 1$. We will investigate the solvability of problem (1.1), (1.2).

Definition 1.1 By *a solution* u of problem (1.1), (1.2) we mean $u: [0, T+2] \to \mathbb{R}$, u satisfies the difference equation (1.1) on $[1, T+1]$ and the boundary conditions (1.2). If $u(t) > 0$ for $t \in [1, T+1]$, we say that u is *a positive solution* of problem (1.1), (1.2).

Let $\mathcal{D} \subset \mathbb{R}^2$. We say that f is continuous on $[1, T+1] \times \mathcal{D}$, if $f(\cdot, x, y)$ is defined on $[1, T+1]$ for each $(x, y) \in \mathcal{D}$ and if $f(t, \cdot, \cdot)$ is continuous on $\mathcal{D}$ for each $t \in [1, T+1]$.

If $\mathcal{D} = \mathbb{R}^2$, problem (1.1), (1.2) is called regular. If $\mathcal{D} \neq \mathbb{R}^2$ and f has singularities on $\partial\mathcal{D}$, then problem (1.1), (1.2) is singular.

Here we will assume that

$$\left.\begin{array}{l} \mathcal{D} = (0, \infty) \times \mathbb{R},\ f \text{ is continuous on } [1, T+1] \times \mathcal{D} \\ \text{and } f \text{ has a singularity at } x = 0, \text{ i.e.} \\ \limsup\limits_{x \to 0+} |f(t, x, y)| = \infty \text{ for each } t \in [1, T+1] \\ \text{and for some } y \in \mathbb{R}. \end{array}\right\} \qquad (1.3)$$

Discrete second order nonlinear boundary value problems have been investigated in several monographs (e.g. [1], [10], [8], [23]) and papers (e.g. [11], [12], [13], [16], [19], [20], [27], [30]). Most of the above results concern regular problems. Singular discrete problems have received less attention. We refer to [3] and [8] where the solvability of the Dirichlet singular discrete problem was studied. Existence theorems for singular higher order discrete problems can be found in [10]. The paper [19] deals with problem (1.1), (1.2) where f is regular and has the form $f(t, x) = a(t)g(x)$. Here, we extend the existence results of [19] onto the singular problem (1.1), (1.2) where f depends both on u and on Δu. The continuous versions of mixed singular problems for differential equations without p-Laplacian have been investigated e.g. in [6], [9], [15], [21], [24], [26], [28] and for problems with the p-Laplacian in [7] or [18].

2 Lower and upper function for regular problems

We start our investigation with the equation

$$\Delta\Big(\phi_p(\Delta u(t-1))\Big) + h(t, u(t), \Delta u(t-1)) = 0, \quad t \in [1, T+1], \qquad (2.1)$$

where h is continuous on $[1, T+1] \times \mathbb{R}^2$ and we apply the lower and upper functions method for the regular problem (2.1), (1.2).

Definition 2.1 $\alpha: [0, T+2] \to \mathbb{R}$ is called *a lower function* of problem (2.1), (1.2) if

$$\Delta\Big(\phi_p(\Delta\alpha(t-1))\Big) + h(t, \alpha(t), \Delta\alpha(t-1)) \geq 0 \text{ for } t \in [1, T+1], \qquad (2.2)$$

$$\Delta\alpha(0) \geq 0, \quad \alpha(T+2) \leq 0. \tag{2.3}$$

$\beta\colon [0, T+2] \to \mathbb{R}$ is called *an upper function* of problem (2.1), (1.2) if

$$\Delta\Big(\phi_p(\Delta\beta(t-1))\Big) + h(t, \beta(t), \Delta\beta(t-1)) \leq 0 \text{ for } t \in [1, T+1], \tag{2.4}$$

$$\Delta\beta(0) \leq 0, \quad \beta(T+2) \geq 0. \tag{2.5}$$

Theorem 2.2 (Lower and upper functions method) *Let α and β be a lower and an upper function, respectively, of* (2.1), (1.2) *and $\alpha \leq \beta$ on $[1, T+1]$. Let h be continuous on $[1, T+1] \times \mathbb{R}^2$ and nonincreasing in its third variable. Then problem* (2.1), (1.2) *has a solution u satisfying*

$$\alpha(t) \leq u(t) \leq \beta(t) \text{ for } t \in [0, T+2]. \tag{2.6}$$

Proof. *Step* 1. For $t \in [1, T+1]$, $x, z \in \mathbb{R}$, define functions

$$\sigma(t, z) = \begin{cases} \beta(t-1) & \text{if} \quad z > \beta(t-1) \\ z & \text{if} \quad \alpha(t-1) \leq z \leq \beta(t-1) \\ \alpha(t-1) & \text{if} \quad z < \alpha(t-1), \end{cases}$$

$$\tilde{h}(t, x, x-z) = \begin{cases} h(t, \beta(t), \beta(t) - \sigma(t, z)) - \dfrac{x - \beta(t)}{x - \beta(t) + 1} & \text{if} \quad x > \beta(t) \\ h(t, x, x - \sigma(t, z)) & \text{if} \quad \alpha(t) \leq x \leq \beta(t) \\ h(t, \alpha(t), \alpha(t) - \sigma(t, z)) + \dfrac{\alpha(t) - x}{\alpha(t) - x + 1} & \text{if} \quad x < \alpha(t). \end{cases}$$

Then $\tilde{h}$ is continuous on $[1, T+1] \times \mathbb{R}^2$ and there exists $M > 0$ such that

$$|\tilde{h}(t, x, y)| \leq M \quad \text{for } t \in [1, T+1], \ (x, y) \in \mathbb{R}^2. \tag{2.7}$$

We will study the auxiliary difference equation

$$\Delta\Big(\phi_p(\Delta u(t-1))\Big) + \tilde{h}(t, u(t), \Delta u(t-1)) = 0, \quad t \in [1, T+1], \tag{2.8}$$

and we will prove that problem (2.8), (1.2) has a solution (see Steps 2–3).

Step 2. We denote

$$E = \{u\colon [0, T+2] \to \mathbb{R}, \ \Delta u(0) = 0, \ u(T+2) = 0\} \tag{2.9}$$

and define $\|u\| = \max\{|u(t)|\colon t \in [1, T+1]\}$. Then E is a Banach space with $\dim E = T+1$. Further we put $\sum\limits_{i=b}^{a} = 0$ for each $a, b \in \mathbb{N} \cup \{0\}$, $a < b$, and define an operator $\mathcal{T}\colon E \to E$ by

$$(\mathcal{T}u)(t) = \sum_{s=t}^{T+1} \phi_q\left(\sum_{i=1}^{s} \tilde{h}(i, u(i), \Delta u(i-1))\right), \quad t \in [0, T+2]. \tag{2.10}$$

Here $\phi_q = \phi_p^{-1}$, $\frac{1}{p} + \frac{1}{q} = 1$.

Since $\phi_q: \mathbb{R} \to \mathbb{R}$ and $\tilde{h}: [1, T+1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, we see that $\mathcal{T}$ is a continuous operator. Moreover, (2.7) and (2.10) imply that if $r \geq \sum_{s=1}^{T+1} \phi_q(sM)$, then $\mathcal{T}\left(\overline{B(r)}\right) \subset \overline{B(r)}$, where $B(r) = \{u \in E: \|u\| < r\}$. Therefore the Brouwer fixed point theorem yields the existence of at least one point $u \in \overline{B(r)}$ such that $u = \mathcal{T}u$.

Step 3. We prove that u is a fixed point of $\mathcal{T}$ if and only if u is a solution of problem (2.8), (1.2).

(i) Assume that $u = \mathcal{T}u$. Then $u \in E$ and so u satisfies (1.2). Further we have

$$\Delta u(t-1) = u(t) - u(t-1) = \sum_{s=t}^{T+1} \phi_q \left(\sum_{i=1}^{s} \tilde{h}(i, u(i), \Delta u(i-1)) \right) -$$

$$- \sum_{s=t-1}^{T+1} \phi_q \left(\sum_{i=1}^{s} \tilde{h}(i, u(i), \Delta u(i-1)) \right),$$

$$\phi_p(\Delta u(t-1)) = - \sum_{i=1}^{t-1} \tilde{h}(i, u(i), \Delta u(i-1)),$$

$$\Delta\Big(\phi_p(\Delta u(t-1))\Big) = \phi_p(\Delta u(t)) - \phi_p(\Delta u(t-1)) =$$

$$= -\tilde{h}(t, u(t), \Delta u(t-1)) \quad \text{for } t \in [1, T+1].$$

(ii) Assume that u is a solution of (2.8), (1.2). Then $u \in E$ and $\phi_p(\Delta u(0)) = \phi_p(0) = 0$. Further we have $\Delta\Big(\phi_p(\Delta u(0))\Big) = -\tilde{h}(1, u(1), \Delta u(0))$, which yields $\phi_p(\Delta u(1)) = -\tilde{h}(1, u(1), \Delta u(0))$.

Similarly

$$\Delta\Big(\phi_p(\Delta u(1))\Big) = -\tilde{h}(2, u(2), \Delta u(1)),$$

and hence

$$\phi_p(\Delta u(2)) = -\tilde{h}(1, u(1), \Delta u(0)) - \tilde{h}(2, u(2), \Delta u(1)).$$

By induction we get

$$\phi_p(\Delta u(t)) = - \sum_{i=1}^{t} \tilde{h}(i, u(i), \Delta u(i-1))$$

and

$$\Delta u(t) = -\phi_q \left(\sum_{i=1}^{t} \tilde{h}(i, u(i), \Delta u(i-1)) \right), \quad t \in [1, T+1]. \tag{2.11}$$

Using (1.2) and (2.11) we get

$$u(T+1)=\phi_q\left(\sum_{i=1}^{T+1}\tilde{h}(i,u(i),\Delta u(i-1))\right),$$

$$u(T)=\phi_q\left(\sum_{i=1}^{T}\tilde{h}(i,u(i),\Delta u(i-1))\right)+\phi_q\left(\sum_{i=1}^{T+1}\tilde{h}(i,u(i),\Delta u(i-1))\right),$$

and by induction we get

$$u(t)=\sum_{s=t}^{T+1}\phi_q\left(\sum_{i=1}^{s}\tilde{h}(i,u(i),\Delta u(i-1))\right),\quad t\in[0,T+2].$$

Note that for $t=0$ and $t=T+2$ we use the equalities $\sum\limits_{i=1}^{0}=0$ and $\sum\limits_{s=T+2}^{T+1}=0$.

Step 4. We prove that the solution u of (2.8), (1.2) satisfies (2.6). Put $v(t)=u(t)-\beta(t)$ for $t\in[0,T+2]$ and assume that $\max\{v(t)\colon t\in[0,T+2]\}=v(\ell)>0$. Conditions (1.2) and (2.5) imply $\ell\in[1,T+1]$. Thus we have $v(\ell+1)\le v(\ell)$, $v(\ell-1)\le v(\ell)$, and consequently $\Delta u(\ell)\le\Delta\beta(\ell)$, $\Delta u(\ell-1)\ge\Delta\beta(\ell-1)$. This leads to $\phi_p(\Delta u(\ell))\le\phi_p(\Delta\beta(\ell))$, $\phi_p(\Delta u(\ell-1))\ge\phi_p(\Delta\beta(\ell-1))$ and

$$\Delta\Big(\phi_p(\Delta u(\ell-1))\Big)\le\Delta\Big(\phi_p(\Delta\beta(\ell-1))\Big).\tag{2.12}$$

On the other hand, since h is nonincreasing in its third variable, we get by (2.8)

$$\begin{aligned}&\Delta\Big(\phi_p(\Delta u(\ell-1))\Big)-\Delta\Big(\phi_p(\Delta\beta(\ell-1))\Big)=\\&=-\tilde{h}(\ell,u(\ell),\Delta u(\ell-1))-\Delta\Big(\phi_p(\Delta\beta(\ell-1))\Big)=\\&=-h\Big(\ell,\beta(\ell),\beta(\ell)-\sigma(\ell,u(\ell-1))\Big)+\frac{v(\ell)}{v(\ell)+1}-\Delta\Big(\phi_p(\Delta\beta(\ell-1))\Big)\ge\\&\ge-h(\ell,\beta(\ell),\Delta\beta(\ell-1))+\frac{v(\ell)}{v(\ell)+1}-\Delta\Big(\phi_p(\Delta\beta(\ell-1))\Big)\ge\\&\ge\frac{v(\ell)}{v(\ell)+1}>0,\end{aligned}$$

which contradicts (2.12). So, we have proved $u(t)\le\beta(t)$ for $t\in[0,T+2]$. The inequality $\alpha(t)\le u(t)$ for $t\in[0,T+2]$ can be proved similarly. Therefore u satisfies (2.6) and hence u is a solution of problem (2.1), (1.2). □

3 Main result and example

The next theorem provides sufficient conditions for the solvability of the singular problem (1.1), (1.2). The proof is based on the construction of a sequence of approximating auxiliary regular problems and on the lower and upper functions method from Theorem 2.2.

Theorem 3.1 *Assume* (1.3) *and let the following conditions hold:*

$$\text{there exists } c \in (0,\infty) \text{ such that } f(t,c,0) \le 0 \text{ for } t \in [1,T+1], \tag{3.1}$$

$$f \text{ is nonincreasing in } y \text{ for } t \in [1,T+1],\ x \in (0,c], \tag{3.2}$$

$$\lim_{x\to 0+} f(t,x,y) = \infty \text{ for } t \in [1,T+1],\ y \in [-c,c]. \tag{3.3}$$

Then problem (1.1), (1.2) *has a solution* u *satisfying*

$$0 < u(t) \le c \text{ for } t \in [0,T+1]. \tag{3.4}$$

Proof. *Step* 1. For $k \in \mathbb{N}$, $t \in [1,T+1]$, $(x,y) \in \mathbb{R}^2$ define

$$f_k(t,x,y) = \begin{cases} f(t,|x|,y) & \text{if} \quad |x| \ge \frac{1}{k} \\ f\left(t,\frac{1}{k},y\right) & \text{if} \quad |x| < \frac{1}{k}. \end{cases}$$

Then f_k is continuous on $[1,T+1]\times\mathbb{R}^2$ and nonincreasing in y for $t \in [1,T+1]$, $x \in [-c,c]$. Assumption (3.3) implies the existence of $k_0 \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \ge k_0$

$$f_k(t,0,0) = f\left(t,\frac{1}{k},0\right) > 0 \quad \text{for } t \in [1,T+1].$$

Consider an auxiliary sequence of equations

$$\Delta\Big(\phi_p(\Delta u(t-1))\Big) + f_k(t,u(t),\Delta u(t-1)) = 0, \quad t \in [1,T+1], \tag{3.5}$$

$k \in \mathbb{N}$, $k \ge k_0$. Put $\alpha(t) = 0$, $\beta(t) = c$ for $t \in [0,T+2]$. Then α and β are a lower and an upper function of each problem (3.5), (1.2) and $\alpha(t) < \beta(t)$ for $t \in [1,T+1]$. By Theorem 2.2, there exists a solution u_k of problem (3.5), (1.2) satisfying

$$0 \le u_k(t) \le c \quad \text{for } t \in [0,T+2],\ k \in \mathbb{N},\ k \ge k_0. \tag{3.6}$$

Consequently

$$|\Delta u_k(t)| \le c \quad \text{for } t \in [0,T+1],\ k \in \mathbb{N},\ k \ge k_0. \tag{3.7}$$

Step 2. Let $k \in \mathbb{N}$, $k \ge k_0$. Since u_k satisfies (3.5), we get by (2.11)

$$\Delta u_k(t) = \phi_q\left(-\sum_{i=1}^{t} f_k(i,u_k(i),\Delta u_k(i-1))\right), \quad t \in [1,T+1]. \tag{3.8}$$

By (3.3) there exists $\varepsilon_1 \in \left(0, \frac{1}{k_0}\right)$ such that if $k \geq \frac{1}{\varepsilon_1}$, then

$$f_k(1, x, y) > \phi_p(c) \quad x \in (0, \varepsilon_1],\ y \in [-c, c]. \tag{3.9}$$

Assume that $k \geq \frac{1}{\varepsilon_1}$ and $u_k(1) < \varepsilon_1$. Then, by (3.8) and (3.9), we get

$$\Delta u_k(1) = \phi_q\Big(-f_k(1, u_k(1), \Delta u_k(0))\Big) < \phi_q(-\phi_p(c)) = -c,$$

which contradicts (3.7). Therefore

$$u_k(1) \geq \varepsilon_1 \quad \text{for each } k \in \mathbb{N},\ k \geq \frac{1}{\varepsilon_1}. \tag{3.10}$$

Denote

$$m_1 = \max\{|f_k(1, x, y)|\colon\ x \in [\varepsilon_1, c],\ y \in [-c, c]\}.$$

By (3.3) there exists $\varepsilon_2 \in (0, \varepsilon_1]$ such that if $k \geq \frac{1}{\varepsilon_2}$, then

$$f_k(2, x, y) > \phi_p(c) + m_1 \quad \text{for } x \in (0, \varepsilon_2],\ y \in [-c, c]. \tag{3.11}$$

Assume that $k \geq \frac{1}{\varepsilon_2}$ and $u_k(2) < \varepsilon_2$. Then, by (3.8), (3.10) and (3.11), we get

$$\Delta u_k(2) = \phi_q\Big(-f_k(1, u_k(1), \Delta u_k(0)) - f_k(2, u_k(2), \Delta u_k(1))\Big) <$$

$$< \phi_q\Big(m_1 - f_k(2, u_k(2), \Delta u_k(1))\Big) < \phi_q(-\phi_p(c)) = -c,$$

which contradicts (3.7). Therefore

$$u_k(2) \geq \varepsilon_2 \quad \text{for each } k \in \mathbb{N},\ k \geq \frac{1}{\varepsilon_2}.$$

We continue similarly for $t = 3, \ldots, T$ and get $0 < \varepsilon_T \leq \varepsilon_{T-1} \leq \cdots \leq \varepsilon_1$ such that

$$u_k(t) \geq \varepsilon_t \quad \text{for } t \in [1, T],\ k \in \mathbb{N},\ k \geq \frac{1}{\varepsilon_T}. \tag{3.12}$$

If we denote

$$m_i = \max\{|f_k(i, x, y)|\colon\ x \in [\varepsilon_i, c],\ y \in [-c, c]\}, \quad i \in [1, T]$$

then by virtue of (3.3) there exists $\varepsilon_{T+1} \in (0, \varepsilon_T]$ such that if $k \geq \frac{1}{\varepsilon_{T+1}}$, then

$$f_k(T+1, x, y) > \phi_p(c) + \sum_{i=1}^{T} m_i \quad \text{for } x \in (0, \varepsilon_{T+1}],\ y \in [-c, c]. \tag{3.13}$$

Assume that $k \geq \frac{1}{\varepsilon_{T+1}}$ and $u_k(T+1) < \varepsilon_{T+1}$. Then, by (3.8), (3.12) and (3.13), we get

$$\Delta u_k(T+1) =$$

$$= \phi_q\left(-\sum_{i=1}^{T} f_k(i, u_k(i), \Delta u_k(i-1)) - f_k(T+1, u_k(T+1), \Delta u_k(T))\right) <$$

$$< \phi_q\left(\sum_{i=1}^{T} m_i - f_k(T+1, u_k(T+1), \Delta u_k(T))\right) < \phi_q(-\phi_p(c)) = -c,$$

which contradicts (3.7). Therefore, if we put $\varepsilon = \varepsilon_{T+1}$, we get

$$0 < \varepsilon \le u_k(t) \le c \quad \text{for } t \in [0, T+1],\ k \in \mathbb{N},\ k \ge \frac{1}{\varepsilon}. \tag{3.14}$$

Since u_k satisfies (3.14) and (1.2) we can choose a subsequence $\{u_{k_n}\} \subset \{u_k\}$ such that $\lim_{n\to\infty} u_{k_n}(t) = u(t)$, $t \in [0, T+2]$, where $u \in E$ (see (2.9)). Moreover, (3.8) yields for each sufficiently large $n \in \mathbb{N}$

$$\Delta u_{k_n}(t) = -\phi_q\left(\sum_{i=1}^{t} f(i, u_{k_n}(i), \Delta u_{k_n}(i-1))\right), \quad t \in [1, T+1].$$

Letting $n \to \infty$ and using the continuity of ϕ_q on $\mathbb{R}$ and f on $[1, T+1] \times \mathcal{D}_\varepsilon$, where $\mathcal{D}_\varepsilon = [\varepsilon, \infty) \times \mathbb{R}$, we get

$$\Delta u(t) = -\phi_q\left(\sum_{i=1}^{t} f(i, u(i), \Delta u(i-1))\right), \quad t \in [1, T+1],$$

from which the equality

$$\Delta\phi_p(\Delta u(t-1)) = -f(t, u(t), \Delta u(t-1)), \quad t \in [1, T+1]$$

follows. Therefore u is a solution of (1.1) and, by (3.14), u satisfies (3.4). The theorem is proved. □

Example. Let $T \in \mathbb{N}$, $\alpha \in [0, \infty)$, $c, \beta \in (0, \infty)$, $p \in (1, \infty)$, $a\colon [1, T+1] \to \mathbb{R}$. By Theorem 3.1 the problem

$$\Delta\Big(\phi_p(\Delta u(t-1))\Big) + \Big(a(t) + (u(t))^\alpha + (u(t))^{-\beta}\Big)(c - u(t)) - (\Delta u(t-1))^3,$$

$$t \in [1, T+1], \quad \Delta u(0) = u(T+2) = 0$$

has a solution u satisfying (3.4).

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