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MULTIPLICITY RESULTS FOR FOUR-POINT BOUNDARY VALUE PROBLEMS

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1. INTRODUCTION

LET $R = (-\infty, +\infty)$, $I = [a, b]$, $-\infty < a < c \leq d < b < +\infty$, $f: I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous functions. This paper proves existence and multiplicity results of Ambrosetti–Prodi type for the four-point resonance problem

$$u'' + f(t, u, u') = s, \tag{1.1}$$

$$u(a) = u(c), \qquad u(d) = u(b), \tag{1.2}$$

where s is a real parameter.

Our results have been motivated by similar ones concerning the number of solutions of periodic problems for first and second order differential equations [1, 3]. Our method of proof is close to that of [1]. It is based on the use of strict upper and lower solutions and on coincidence topological degree arguments.

This four-point problem can be understood as an approximation of the Neumann problem, where derivatives at the points a, b are replaced by differences.

We write $C^k(I)$ for the space of real valued C^k-functions u on I with the norm

$$\|u\|_k = \sum_{i=0}^{k} \max\{|u^{(i)}(t)| : t \in I\}.$$

We recall that $\sigma_1, \sigma_2 \in C^2(I)$ are lower and upper solutions for (1.1), (1.2), respectively, if

$$[\sigma_i'' + f(t, \sigma_i, \sigma_i') - s](-1)^i \leq 0 \qquad \text{for each } t \in I, \tag{1.3}$$

$$[\sigma_i(a) - \sigma_i(c)](-1)^i \geq 0, \qquad [\sigma_i(d) - \sigma_i(b)](-1)^i \leq 0, \qquad i \in \{1, 2\}. \tag{1.4}$$

Similarly, $\sigma_1, \sigma_2 \in C^2(I)$ are strict lower and upper solutions for (1.1), (1.2), respectively, if

$$[\sigma_i'' + f(t, \sigma_i, \sigma_i') - s](-1)^i < 0 \qquad \text{for each } t \in I, \tag{1.5}$$

$$\sigma_i(a) = \sigma_i(c), \qquad \sigma_i(d) = \sigma_i(b), \qquad i \in \{1, 2\}. \tag{1.6}$$

A continuous function $\omega: (0, +\infty) \to (\varepsilon, +\infty)$, with $\varepsilon > 0$, will be called a Nagumo function, if

$$\int_0^{+\infty} \frac{z\,dz}{\omega(z)} = +\infty. \tag{1.7}$$

We say that $f: I \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Bernstein–Nagumo conditions, if for any $r \in (0, +\infty)$ there exists a Nagumo function ω_r such that

$$f(t, x, y)\,\mathrm{sgn}\, y \geq -\omega_r(|y|) \qquad \text{on } I \times [-r, r] \times \mathbb{R} \tag{1.8}$$

and

$$f(t, x, y)\,\mathrm{sgn}\, y \leq \omega_r(|y|) \qquad \text{on } [a, c] \times [-r, r] \times \mathbb{R}. \tag{1.9}$$

In what follows

$$\begin{aligned} D(-r_1) &= \{x \in C^2(I) : x(t) > -r_1 \text{ for each } t \in I\}, \\ D(r_1) &= \{x \in C^2(I) : x(t) < r_1 \text{ for each } t \in I\}, \end{aligned} \tag{1.10}$$

where $r_1 \in (0, +\infty)$.

2. AUXILIARY RESULTS

We shall need some lemmas whose proofs follow the approach proposed in [5]. Let us consider the equation

$$u'' = g(t, u, u') \tag{2.1}$$

where $g \in C^0(I \times R^2)$.

LEMMA 1. Let σ_1 be a lower solution and σ_2 an upper solution of (2.1), (1.2) with $\sigma_1(t) \leq \sigma_2(t)$ for each $t \in I$. Further, let there exist $k \in (0, +\infty)$ such that for each $t \in I$, $x, y \in \mathbb{R}$, where $\sigma_1(t) \leq x \leq \sigma_2(t)$, the inequality

$$|g(t, x, y)| \leq k$$

is fulfilled.

Then problem (2.1), (1.2) has a solution u fulfilling

$$\sigma_1(t) \leq u(t) \leq \sigma_2(t) \qquad \text{for each } t \in I. \tag{2.2}$$

Proof. Similarly, to the proof of [5, lemma 6], we put

$$w_i(t, x, y) = (-1)^i m(x - \sigma_i)[g(t, \sigma_i, \sigma_i') - g(t, \sigma_i, y) + (-1)^i r_0/m], \qquad i = 1, 2,$$

$$g_m(t, x, y) = \begin{cases} g(t, \sigma_1, \sigma_1') - r_0/m & \text{for } x \leq \sigma_1(t) - 1/m \\ g(t, \sigma_1, y) + w_1 & \text{for } \sigma_1(t) - 1/m < x < \sigma_1(t) \\ g(t, x, y) & \text{for } \sigma_1(t) \leq x \leq \sigma_2(t) \\ g(t, \sigma_2, y) + w_2 & \text{for } \sigma_2(t) < x < \sigma_2(t) + 1/m \\ g(t, \sigma_2, \sigma_2') + r_0/m & \text{for } x \geq \sigma_2(t) + 1/m, \end{cases}$$

where m is a natural number and $(t, x, y) \in I \times R^2$, and consider the equation

$$u'' = (1/m)u + g_m(t, u, u'). \tag{2.3}$$

By the Fredholm nonlinear alternative theorem, problem (2.3), (1.2) has a solution u_m, because g_m is bounded and the linear problem corresponding to (2.3), (1.2) has only the trivial solution.

Similarly to [5, lemma 6], it can be checked that

$$\sigma_1(t) = 1/m \leq u_m(t) \leq \sigma_2(t) + 1/m$$

for each $t \in I$ and any natural m. This implies, by (1.2), (2.3), that the sequences $(u_m)_1^\infty$ and $(u_m')_1^\infty$ are uniformly bounded and equi-continuous on I and thus, by the Arzelo–Ascoli theorem, we conclude that $(u_m)_1^\infty$ contains a subsequence converging in $C^1(I)$. Writing, for every m, equations (2.3) in integral forms, it is easily seen that the limit of that subsequence is a solution of (2.1), (1.2) and satisfies (2.2). The proof is complete.

Lemma 2 (on *a priori* estimate). Let, for $r \in (0, +\infty)$, ω_r be a Nagumo function. Then there exists a number $\rho = \rho(r, \omega_r)$ such that for any function $u \in C^2(I)$ the conditions

$$\|u\|_0 \leq r, \qquad u(a) = u(c), \qquad u(b) = u(d), \tag{2.4}$$

$$u'' \operatorname{sgn} u' \leq \omega_r(|u'|) \qquad \text{for each } t \in I, \tag{2.5}$$

and

$$u'' \operatorname{sgn} u' \geq -\omega_r(|u'|) \qquad \text{for each } t \in [a, c] \tag{2.6}$$

imply the estimate

$$\|u'\|_0 < \rho. \tag{2.7}$$

Proof. In view of (2.4) we can choose $a_1 \in (a, c)$ such that $u'(a_1) = 0$. From (1.7) it follows that there exists $\rho \in (r, +\infty)$ such that

$$\int_0^\rho \frac{z\,dz}{\omega_r(z)} > 2r. \tag{2.8}$$

Now, let us suppose that there exists $t_0 \in (a_1, b]$ such that

$$u'(t_0) \geq \rho. \tag{2.9}$$

Let $[\alpha, \beta] \subset [a, b]$ be the maximal interval containing t_0 with $u'(t) \geq 0$ for $t \in [\alpha, \beta]$. Let $t^* \in (\alpha, \beta]$ be such point that $u'(t^*) = c_1 = \max\{u'(t)\colon \alpha \leq t \leq \beta\}$. Then, from (2.5), it follows

$$\int_0^{c_1} \frac{z\,dz}{\omega(z)} \leq 2r,$$

which implies, by (2.8), $c_1 < \rho$. The latter inequality contradicts (2.9). Similarly, supposing that there exists $t_0 \in (a_1, b]$ with

$$u'(t_0) \leq -\rho \tag{2.10}$$

and choosing the maximal interval $[\alpha, \beta] \subset [a_1, b]$ such that $t_0 \in (\alpha, \beta]$ and $u'(t) \leq 0$ on $[\alpha, \beta]$, we can get the same contradiction. Finally, if we suppose that t_0 satisfying (2.9) or (2.10) can be chosen in $[a, a_1)$, then using (2.6) instead of (2.5) we obtain a contradiction by the same arguments as above. Therefore $\|u'\|_0 < \rho$ and lemma is proved.

Lemma 3. Let s be a real number. Assume that the function f in equation (1.1) satisfies the Bernstein–Nagumo conditions. Further let σ_1 and σ_2 be lower and upper solutions of problem (1.1), (1.2), respectively, with $\sigma_1(t) \leq \sigma_2(t)$ for each $t \in I$.

Then problem (1.1), (1.2) has at least one solution u satisfying (2.2).

Proof. Let

$$r_i = \max\{|\sigma_1^{(i)}(t)| + |\sigma_2^{(i)}(t)| : t \in I\}, \qquad i = 0, 1.$$

Then for r_0 there exists a Nagumo function ω_{r_0} such that f satisfies (1.8) and (1.9) where $r = r_0$. Put $\tilde{\omega}(z) = |s| + \omega_{r_0}(z)$, $z \in (0, +\infty)$. We can easily verify that $\tilde{\omega}$ is also a Nagumo function.

Further, let $\rho = \rho(r_0, \tilde{\omega})$ be the number found by lemma 2. Put $\mu = \rho + r_0 + r_1$ and

$$\chi(\mu, z) = \begin{cases} 1 & \text{for } \quad |z| \le \mu \\ 2 - z/\mu & \text{for } \mu < |z| < 2\mu \\ 0 & \text{for } \quad |z| \ge 2\mu, \end{cases}$$

$$\tilde{f}(t, x, y) = \chi(\mu, |x| + |y|) f(t, x, y). \tag{2.11}$$

Next, for fixed real s consider the equation

$$u'' + \tilde{f}(t, u, u') = s. \tag{2.12}$$

Since $\|\sigma_i\|_1 < \mu$, $i = 1, 2$, we can see that σ_1 is a lower solution and σ_2 an upper solution for (2.12), (1.2). Moreover

$$|s - \tilde{f}(t, x, y)| \le k + |s| \qquad \text{on } I \times \mathbb{R}^2,$$

where $k = \max\{|f(t, x, y)| : t \in I, |x| + |y| \le 2\mu\}$. Thus, by lemma 1, problem (2.12), (1.2) has a solution u satisfying (2.2). Therefore u fulfills (2.4). Further, by (2.11) and the first part of the proof, u satisfies (2.5) and (2.6) where $r = r_0$ and $\omega_r = \tilde{\omega}$. So, applying lemma 2 we get estimate (2.7). Therefore $\|u\|_1 < \mu$ and u is also a solution of problem (1.1), (1.2). The lemma is proved.

3. EXISTENCE RESULTS

THEOREM 1. Let $f \in C^0(I \times R^2)$ satisfy the Bernstein–Nagumo conditions and let there exist numbers $r_1 > 0$ and s_1 such that for all $t \in I$

$$f(t, -r_1, 0) > s_1 > f(t, 0, 0). \tag{3.1}$$

Then there exists $s_0 < s_1$ (with the possibility $s_0 = -\infty$) such that
(a) for $s < s_0$, (1.1), (1.2) has no solution in $\overline{D(-r_1)}$,
(b) for $s \in (s_0, s_1]$, (1.1), (1.2) has at least one solution $u_s \in D(-r_1)$. [For $D(-r_1)$ see (1.10).]

The proof of theorem 1 follows the approach proposed in [1], for periodic solutions. However condition (3.1) is weaker than the corresponding one in [1], where it is assumed

$$f(t, x, 0) > s_1 > f(t, 0, 0) \qquad \text{for all } t \in \mathbb{R} \text{ and all } x \le -r_1. \tag{3.2}$$

Proof. Put

$$h(t, x, y) = \begin{cases} f(t, x, y) & \text{for } x \ge -r_1 \\ f(t, -r_1, y) & \text{for } x < -r_1 \end{cases} \tag{3.3}$$

and consider the equation

$$u'' + h(t, u, u') = s. \tag{3.4}$$

We can see that h satisfies the Bernstein–Nagumo conditions. Let $s^* = \max\{h(t, 0, 0) : t \in I\}$. Then for $s = s^*$, 0 is an upper solution and $-r_1$ is a lower solution for (3.4), (1.2). Then, by lemma 3, problem (3.4), (1.2) has a solution u^* with $-r_1 \le u^*(t) \le 0$ on I. By (3.3), u^* is a solution for (1.1), (1.2) as well.

Now, suppose first that (3.4), (1.2) has a solution u for some $s \le s_1$ and show that $u \in D(-r_1)$. Let, on the contrary, $\min\{u(t) : t \in I\} = u(t_0) \le -r_1$. Then, by (1.2), $u'(t_0) = 0$, $u''(t_0) \ge 0$. On the other hand, from (3.1), (3.3), it follows that

$$u''(t_0) = s - h(t_0, u(t_0), 0) = s - f(t_0, -r_1, 0) \le s_1 - f(t_0, -r_1, 0) < 0,$$

a contradiction.

Next, let us show that if problem (3.4), (1.2) has a solution $\tilde{u}$ for $s = \tilde{s} < s_1$, then it has at least one solution for each $s \in [\tilde{s}, s_1]$. From the above considerations it follows that $\tilde{u} \in D(-r_1)$. Further, $\tilde{u}'' + h(t, \tilde{u}, \tilde{u}') = \tilde{s} \le s$ and so $\tilde{u}$ is an upper solution for (3.4), (1.2), where $s \in [\tilde{s}, s_1]$. Similarly, since $h(t, -r_1, 0) > s_1 \ge s$, $-r_1$ is a lower solution for (3.4), (1.2), where $s \in [\tilde{s}, s_1]$. Hence, we can use lemma 3 again to get that (3.4), (1.2) has at least one solution in $D(-r_1)$ provided $s \in [\tilde{s}, s_1]$. From the latter it is a solution for (1.1), (1.2) as well.

Finally, taking $s_0 = \inf\{s \in \mathbb{R}$: (1.1), (1.2) has at least one solution in $D(-r_1)\}$, we have $s_0 \le s^* < s_1$ and from the above considerations (a) and (b) follow. The theorem is proved.

THEOREM 2. Let $f \in C^0(I \times R^2)$ satisfy the Bernstein–Nagumo conditions and let there exist numbers $r_1 > 0$ and s_1 such that for all $t \in I$

$$f(t, 0, 0) > s_1 > f(t, r_1, 0). \tag{3.5}$$

Then there exists $s_0 > s_1$ (with the possibility $s_0 = +\infty$) such that

(a) for $s > s_0$, (1.1), (1.2) has no solution in $\overline{D(-r_1)}$,

(b) for $s \in [s_1, s_0)$, (1.1), (1.2) has at least one solution in $D(r_1)$.

Proof. Theorem 2 can be obtained from theorem 1 if f is replaced by $-f$ and x by $-x$.

4. MULTIPLICITY RESULTS

THEOREM 3. Let $f \in C^0(I \times R^2)$ satisfy the Bernstein–Nagumo conditions and let there exist $r_1, r_2 \in (0, +\infty)$, $s_1 \in \mathbb{R}$ such that for all $t \in I$ the inequality (3.1) is fulfilled and for all $s \le s_1$ any solution u_s of (1.1), (1.2) belonging to $D(-r_1)$ satisfies

$$u_s(t) < r_2 \qquad \text{for each } t \in I. \tag{4.1}$$

Then the number s_0 in theorem 1 is finite and

(a) for $s < s_0$, problem (1.1), (1.2) has no solution in $\overline{D(-r_1)}$,

(b) for $s = s_0$, problem (1.1), (1.2) has at least one solution in $\overline{D(-r_1)}$,

(c) for $s \in (s_0, s_1]$, problem (1.1), (1.2) has at least two solutions in $D(-r_1)$.

A similar theorem for a periodic problem is proved in [1], where the stronger condition (3.2) is assumed instead of (3.1) and moreover the function $f(., ., 0)$ is required to be bounded below.

Theorem 3 is valid not only for problem (1.1), (1.2) but also for Neumann and periodic problems.

Proof. Let us consider the equation (3.4) where h satisfies (3.3). Then h fulfills the Bernstein–Nagumo conditions and, according to the proof of theorem 1, each solution of problem (3.4), (1.2) belongs to $D(-r_1)$ provided $s \le s_1$.

Now, proving theorem 3, we shall need several auxiliary propositions.

PROPOSITION 1. There exist numbers σ, M, $\sigma < s_1 < M$, such that for any $s \le s_1$ and any solution u_s of (3.4), (1.2)

$$\sigma \le h(t, u_s, 0) \le M \qquad \text{for each } t \in I. \tag{4.2}$$

Proof of proposition 1. Let $s \le s_1$. Then, by (4.1), any solution u_s of (3.4), (1.2) fulfills

$$-r_1 < u_s(t) < r_2 \qquad \text{for each } t \in I. \tag{4.3}$$

Therefore we can put

$$\sigma = \min\{h(t, x, 0) : t \in I, x \in [-r_1, r_2]\} \quad \text{and} \quad M = \max\{h(t, x, 0) : t \in I, x \in [-r_1, r_2]\}.$$

From (3.1), (3.3) it follows that

$$\sigma < s_1 < M. \tag{4.4}$$

PROPOSITION 2. There exists $s_0 \in [\sigma, s_1)$ such that for $s < s_0$, problem (3.4), (1.2) has no solution and for $s \in (s_0, s_1]$ it has at least one solution in $D(-r_1)$.

Proof of proposition 2. Suppose on the contrary that for $s < \sigma$ problem (3.4), (1.2) has a solution. Then, by (4.3), $\min\{u(t) : t \in I\} = u(t_0) \in (-r_1, r_2)$, $u'(t_0) = 0$, $u''(t_0) \ge 0$. On the other hand, by (4.2), $u''(t_0) < 0$, which is impossible. Hence there exists $s_0 \ge \sigma$ such that (3.4), (1.2) has no solution for $s < s_0$. By (4.4) and theorem 1 we can deduce $s_0 < s_1$ and (3.4), (1.2) has at least one solution in $D(-r_1)$ for each $s \in (s_0, s_1]$.

From now on, let $\tilde{s} \in (s_0, s_1)$ be fixed and $\tilde{u}$ denote a solution of (3.4), (1.2) for $s = \tilde{s}$. Then $\tilde{u} \in D(-r_1)$. Further, let for $t \in I$, $x, y \in \mathbb{R}$

$$\alpha(x) = \begin{cases} -r_1 & \text{for } x < -r_1 \\ x & \text{for } -r_1 \le x \le \tilde{u}(t) \\ \tilde{u}(t) & \text{for } x > \tilde{u}(t) \end{cases}$$

and

$$g(t, x, y) = f(t, \alpha(x), y) - x + \alpha(x). \tag{4.5}$$

We shall consider the equation

$$u'' + g(t, u, u') = s. \tag{4.6}$$

PROPOSITION 3. For each $s \in (\tilde{s}, s_1]$ any solution u of problem (4.6), (1.2) satisfies

$$-r_1 < u < \tilde{u} \qquad \text{on } I.$$

Proof of proposition 3. Let u be a solution of (4.6), (1.2) where $s \in (\tilde{s}, s_1]$. Suppose that for some $t \in I$ $u(t) \ge \tilde{u}(t)$. Then there exists $t_0 \in (a, b)$ such that $u(t_0) \ge \tilde{u}(t_0)$, $u'(t_0) = \tilde{u}'(t_0)$, $u''(t_0) \le \tilde{u}''(t_0)$. But from (4.5) we can get $u''(t_0) > \tilde{u}''(t_0)$, which is a contradiction. The inequality $-r_1 < u$ can be proved by similar arguments.

Now, for an arbitrary fixed $s \le s_1$, let us consider the class of equations

$$u'' - (1 - \lambda)u + \lambda[g(t, u, u') - s] = 0, \tag{4.7λ}$$

where a real parameter λ varies from 0 to 1.

Proposition 4. There exist positive numbers R, ρ such that for any $s \in [s_0, s_1]$ and any parameter $\lambda \in [0, 1]$, every solution u of (4.7λ), (1.2) satisfies

$$\|u\|_0 < R, \qquad \|u'\|_0 < \rho.$$

Proof of proposition 4. Let us choose an arbitrary fixed $s \in [\tilde{s}_0, s_1]$ and a number R with

$$R > \max\{r_1 + s_1 - \sigma, r_2 + M - s_0\}. \tag{4.8}$$

Suppose that for some $\lambda \in [0, 1]$ and for a corresponding solution u of (4.7λ), (1.2) we can find $t_0 \in I$ such that $\max\{|u(t)| : t \in I\} = |u(t_0)| \ge R$. Let $u(t_0) \ge R$. Then, in view of (1.2), $u'(t_0) = 0$, $u''(t_0) \le 0$ and by (4.7λ), (1.2), (4.8),

$$u''(t_0) \ge (1 - \lambda)R + \lambda[s_0 + M + R - r_2] > 0,$$

a contradiction. Similarly, if $u(t_0) \le -R$, then we get

$$0 \le u''(t_0) \le -(1 - \lambda)R + \lambda(s_1 - \sigma - R + r_1) < 0,$$

a contradiction. Therefore $\|u\|_0 < R$.

Further, since f satisfies the Bernstein–Nagumo conditions, there exists a Nagumo function ω_R and $u'' \operatorname{sgn} u' < \omega_R(|u'|) + R + S_2$ on I and $u'' \operatorname{sgn} u' > -\omega_R(|u'|) - R - S_2$ on $[a, c]$, where $S_2 = \max\{|s_0 - 1|, |s_1|\}$. We can easily check that $\tilde{\omega} = \omega_R + R + S_2$ is a Nagumo function, and so, using lemma 2 for $r = R$ and $\omega_r = \tilde{\omega}$ we can find a number $\rho = \rho(R, \tilde{\omega})$ such that $\|u'\|_0 < \rho$.

Let us put

$$\operatorname{dom} L = \{u \in C^2(I) : u(a) = u(c),\ u(b) = u(d)\},$$

$$L\colon \operatorname{dom} L \to C^0(I), \qquad u \mapsto u'',$$

$$N_s\colon C^1(I) \to C^0(I), \qquad u \mapsto h(\cdot, u(\cdot), u'(\cdot)) - s,$$

$$G_s\colon C^1(I) \to C^0(I), \qquad u \mapsto g(\cdot, u(\cdot), u'(\cdot)) - s,$$

$$I\colon C^1(I) \to C^1(I), \qquad u \mapsto u.$$

Then problems (3.4), (1.2) or (4.6), (1.2) or (4.7λ), (1.2) can be written in the forms

$$(L + N_s)u = 0, \tag{4.9}$$

or

$$(L + G_s)u = 0, \tag{4.10}$$

or

$$(L - (1 - \lambda)I + \lambda G_s)u = 0. \tag{4.11}$$

Similarly to the periodic case, it can be proved (see [6]), that N_s and G_s are L-compact on $C^1(I)$, so that the coincidence degree method (see [2]) can be applied to problems (4.9)–(4.11).

Let us consider two open bounded sets in $C^1(I)$:

$$\Omega = \{u \in C^1(I)\colon -r_1 < u(t) < \tilde{u}(t) \text{ for each } t \in I,\ \|u'\|_0 < \rho\}$$

and

$$\Omega_1 = \{u \in C^1(I) : \|u\|_0 < R,\ \|u'\|_0 < \rho\},$$

where R and ρ are numbers in proposition 4.

PROPOSITION 5. Let $s \in (\tilde{s}, s_1]$. Then

$$d_L(L + N_s, \Omega) = \pm 1. \tag{4.12}$$

Proof of proposition 5. Suppose that $s \in (\tilde{s}, s_1]$. Then, by proposition 4, for any $\lambda \in [0, 1]$, each solution u of (4.11) belongs to Ω_1 and so $u \notin \partial\Omega_1$. Further, for $\lambda = 0$, (4.11) has the form $(L - I)u = 0$ and since $\operatorname{Ker}(L - I) = \{0\}$, we get

$$d_L(L - I, \Omega_1) = \pm 1.$$

(See [2, proposition II.16].) Next, for $\lambda = 1$, (4.11) is equal to (4.10) and so, by the property of invariance under homotopy (see [2, p. 15]) we have

$$d_L(L + G_s, \Omega_1) = \pm 1.$$

Now, using propositions 3 and 4, we get for each solution u of (4.10) that $u \in \Omega$. Therefore, by the excession property [2, p. 15],

$$d_L(L + G_s, \Omega) = \pm 1.$$

Since, $N_s = G_s$ on Ω, we get

$$d_L(L + N_s, \Omega) = \pm 1.$$

PROPOSITION 6. Let $s \in (\tilde{s}, s_1]$. Then

$$d_L(L + N_s, \Omega_1 \backslash \bar{\Omega}) = \pm 1. \tag{4.13}$$

Proof of proposition 6. Clearly $\Omega_1 \backslash \bar{\Omega}$ is a nonempty open bounded set in $C^1(I)$. Since problem (4.9) has no solution for $s < s_0$ (see proposition 2), it is an immediate consequence of the existence property (see [2, p. 16]) that, for $s < s_0$

$$d_L(L + N_s, \Omega_1) = 0. \tag{4.14}$$

On the other hand, by (4.3), for $s_0 - 1 < s \le s_1$ any solution u of (4.9) belongs to Ω_1 and so $u \notin \partial\Omega_1$ (see proposition 4). Letting s vary from $s_0 - 1$ to s_1 we can deduce by the property of invariance under homotopy that (4.14) holds for each $s \in (s_0 - 1, s_1]$. Now, for $s \in (\tilde{s}, s_1]$, using (4.12) and (4.14), it follows from the additivity property of degree (see [2, p. 15]) that

$$d_L(L + N_s, \Omega_1 \backslash \bar{\Omega}) = \pm 1.$$

Now, by means of the above propositions, we can complete the proof of theorem 3 as follows.

Proposition 2 and relation (3.3), together with the fact that any solution of (3.4), (1.2) belongs to $D(-r_1)$, imply assertion (a).

The relations (4.12) and (4.13) imply that, for $s \in (\tilde{s}, s_1]$, equation (4.9) has at least one solution in Ω and at least another one in $\Omega_1 \backslash \bar{\Omega}$. Since any solution of (4.9) belongs to $D(-r_1)$ and $\tilde{s}$ is arbitrary in (s_0, s_1), conclusion (c) is proved.

Finally, to prove (b), let $(t_n)_1^\infty$ be a sequence in (s_0, s_1) which converges to s_0 and let u_n be a solution of (4.9) with $s = t_n$. Using proposition 4, one gets $(u_n)_1^\infty$ bounded in $C^1(I)$ and hence in $C^2(I)$ by the equation. By Ascoli's theorem and the integrated form of the equation, one gets the existence of a converging subsequence of $(u_n)_1^\infty$ whose limit is a solution u_0 of (4.9) with $s = s_0$. Clearly, $u_0 \in \overline{D(-r_1)}$ is a solution of (1.1), (1.2). Theorem 3 is proved.

Similarly we can prove theorem 4.

THEOREM 4. Let $f \in C^0(I \times \mathbb{R}^2)$ satisfy the Bernstein–Negumo conditions and let there exist $r_1, r_2 \in (0, +\infty)$, $s_1 \in \mathbb{R}$ such that for all $t \in I$ the inequality (3.5) is fulfilled and for all $s \geq s_1$ any solution u_s of (1.1), (1.2) belonging to $D(r_1)$ satisfies

$$-r_1 < u_s(t) \qquad \text{for each } t \in I.$$

Then the number s_0 in theorem 2 is finite and

(a) for $s > s_0$, problem (1.1), (1.2) has no solution in $\overline{D(r_1)}$,

(b) for $s = s_0$, problem (1.1), (1.2) has at least one solution in $\overline{D(r_1)}$,

(c) for $s \in [s_1, s_0)$, problem (1.1), (1.2) has at least two solutions in $D(r_1)$.

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