Neumann problems with time singularities

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Abstract: In this paper we study the existence and uniqueness of solutions to a nonlinear Neumann problem for a scalar second order ordinary differential equation

$$u'' = \frac{a}{t}u' + f(t, u, u'),$$

where $a < 0$, and $f(t, x, y)$ satisfies the local Carathéodory conditions on $[0, T] \times \mathbb{R} \times \mathbb{R}$.

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1 Motivation

The aim of this work is to show the existence and uniqueness of solutions to a nonlinear Neumann problem exhibiting a singularity of the first kind in time. In many applications, second order singular models, cf. [4], [5], [10], [19], and [30], assume the forms

$$u'' = \frac{a_1}{t^\alpha}u' + \frac{a_0}{t^{\alpha+1}}u + f(t, u, u'), \quad u'' = \frac{a}{t^\alpha}u' + f(t, u, u'), \quad t > 0, \tag{1.1}$$

where a_1, a_0, a and f are given. We say that for $\alpha = 1$ the problem exhibits a singularity of the first kind at $t = 0$, while for $\alpha > 1$, the singularity is essential or of the second kind. In [30], the existence and uniqueness results in case of smooth data function f has been developed. This analysis is based on techniques proposed in [12]. However, in applications mentioned below this smoothness assumption does not hold and therefore, there is a need for covering the case of unsmooth inhomogenities f.

Here, we consider differential equations with a singularity of the first kind, $\alpha = 1$, of the form

$$u'' = \frac{a}{t}u' + f(t, u, u'), \tag{1.2}$$

where $a \in \mathbb{R} \setminus \{0\}$, and the function $f(t, x, y)$ is defined for a.e. $t \in [0, T]$ and for all $(x, y) \in \mathcal{D} \subset \mathbb{R} \times \mathbb{R}$. Clearly, the above equation is singular at $t = 0$ because of the first term in the right-hand side, which is in general unbounded for $t \to 0$. Moreover, we also alow the function f to be unbounded or bounded but discontinuous for certain values of the time variable $t \in [0, T]$. This form of f is motivated by a variety of initial and boundary value problems known from applications and having nonlinear, discontinuous forcing terms, such as electronic devices which are often driven by square waves or more complicated discontinuous inputs. Typically, such problems are modelled by differential equations where f has jump discontinuities at a discrete set of points in $(0, T)$, cf. [23]. Many other applications, cf. [1]–[11], [16], [19], [24]–[29] also show these structural difficulties.

In this paper we extend results from [21] and [30] based on ideas presented in [12], where, as already mentioned, problems of the above form but with *appropriately smooth data function* f have been discussed.

2 Introduction

The following notation will be used throughout the paper. Let $J \subset \mathbb{R}$ be an interval. Then, we denote by $L_1(J)$ the set of functions which are (Lebesgue) integrable on J. The corresponding norm is $\|u\|_1 := \int_J |u(t)| \mathrm{d}t$.

Moreover, let us by $C(J)$ and $C^1(J)$ denote the sets of functions being continuous on J, and having continuous first derivatives on J, respectively. The norm on $C[0, T]$ is defined as $\|u\|_\infty := \max_{t \in [0,T]} |u(t)|$.

Finally, we denote by $AC(J)$ and $AC^1(J)$ the sets of functions which are absolutely continuous on J, and which have absolutely continuous first derivatives on J, respectively. Analogously, $AC_{loc}(J)$ and $AC^1_{loc}(J)$ are the sets of functions being absolutely continuous on each compact subinterval $I \subset J$, and having absolutely continuous first derivatives on each compact subinterval $I \subset J$, respectively.

As already said, we investigate differential equations of the form

$$u''(t) = \frac{a}{t}u'(t) + f(t, u(t), u'(t)) \text{ a.e. on } [0, T], \tag{2.1}$$

where $a \in \mathbb{R} \setminus \{0\}$. For the subsequent analysis we assume that

$$f \text{ satisfies the local Carathéodory conditions on } [0, T] \times \mathbb{R} \times \mathbb{R}, \tag{2.2}$$

specified in the following definition.

Definition 2.1. A function f satisfies *the local Carathéodory conditions* on the set $[0, T] \times \mathbb{R} \times \mathbb{R}$ if
(i) $f(\cdot, x, y) : [0, T] \to \mathbb{R}$ is measurable for all $(x, y) \in \mathbb{R} \times \mathbb{R}$,
(ii) $f(t, \cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous for a.e. $t \in [0, T]$,
(iii) for each compact set $\mathcal{K} \subset \mathbb{R} \times \mathbb{R}$ there exists a function $m_{\mathcal{K}}(t) \in L_1[0, T]$ such that $|f(t, x, y)| \le m_{\mathcal{K}}(t)$ for a.e. $t \in [0, T]$ and all $(x, y) \in \mathcal{K}$.

Definition 2.2. A function $u : [0, T] \to \mathbb{R}$ is called *a solution of equation* (2.1) if $u \in AC^1[0, T]$ and

$$u''(t) = \frac{a}{t}u'(t) + f(t, u(t), u'(t))$$

holds a.e. on $[0, T]$.

We will provide existence and/or uniqueness results for solutions of equation (2.1) for $a < 0$ subject to the Neumann boundary conditions $u'(0) = u'(T) = 0$. The paper is organized as follows. In Section 3 we generalize some results from [28] and give a description of an asymptotical behavior for $t \to 0+$ of functions u satisfying (2.1) a.e. on $[0, T]$ for a both positive and negative. The Neumann problem is then analyzed in Section 4 by means of the results of Section 3. Finally, in Section 5, we illustrate the theoretical findings by means of numerical experiments.

3 Limit properties of functions satisfying singular equations

We consider the nonlinear equation (2.1), where f satisfies the global Carathéodory conditions on $[0, T] \times \mathbb{R} \times \mathbb{R}$, specified in the following definition.

Definition 3.1. A function f satisfies *the global Carathéodory conditions* on the set $[0, T] \times \mathbb{R} \times \mathbb{R}$ if f satisfies conditions (i) and (ii) of Definition 2.1 and if there exists a function $g \in L_1[0, T]$ such that

$$|f(t, x, y)| \le g(t) \quad \text{for a.e. } t \in [0, T] \text{ and all } (x, y) \in \mathbb{R} \times \mathbb{R}. \tag{3.1}$$

Example 3.2. Let $v \in C(\mathbb{R}^2)$ be bounded and let $r \in L_1[0,T]$. Then the functions

$$f_1(t,x,y) = r(t)v(x,y), \quad f_2(t,x,y) = r(t) + v(x,y)$$

satisfy Definition 3.1.

Theorem 3.3. *Let us assume that f satisfies the global Carathéodory conditions on the set $[0,T] \times \mathbb{R} \times \mathbb{R}$. Let $a > 0$ and let $u \in AC^1_{loc}(0,T]$ satisfy equation (2.1) a.e. on $(0,T]$. Then*

$$\lim_{t \to 0+} u'(t) = 0, \tag{3.2}$$

and u can be extended on $[0,T]$ in such a way that $u \in AC^1[0,T]$.

Proof. Integrating (2.1) we get

$$u'(t) = \left(\frac{t}{T}\right)^a u'(T) - t^a \int_t^T s^{-a} f(s, u(s)u'(s))\mathrm{d}s, \quad t \in (0,T],$$

and, using (3.1), we obtain

$$|u'(t)| \le |u'(T)| \left(\frac{t}{T}\right)^a + t^a \int_t^T s^{-a} g(s)\mathrm{d}s, \quad t \in (0,T]. \tag{3.3}$$

By virtue of the inequality

$$t^a \int_t^T s^{-a} g(s)\mathrm{d}s \le \int_t^\tau g(s)\mathrm{d}s + \left(\frac{t}{\tau}\right)^a \int_\tau^T g(s)\mathrm{d}s, \quad 0 < t \le \tau < T,$$

we conclude that

$$\limsup_{t \to 0+} \left(t^a \int_t^T s^{-a} g(s)\mathrm{d}s\right) \le \int_0^\tau g(s)\mathrm{d}s, \quad \tau \in (0,T).$$

If we pass to the limit in this inequality as $\tau \to 0+$, we get

$$\lim_{t \to 0+} \left(t^a \int_t^T s^{-a} g(s)\mathrm{d}s\right) = 0,$$

which tohether with (3.3) give (3.2). Clearly (3.2) implies that there exists a finite limit $\lim_{t \to 0+} u(t)$.

In order to prove that u can be extended on $[0,T]$ as a function in $AC^1[0,T]$, we have to show that

$$\int_0^T |u''(t)|\mathrm{d}t < \infty.$$

Equality (2.1) and condition (3.1) yield

$$\left|u''(t) - \frac{a}{t}u'(t)\right| \le g(t) \quad \text{for a.e. } t \in (0,T]. \tag{3.4}$$

Integrating (3.4) and using (3.3) we get

$$\int_0^T |u''(t)|\mathrm{d}t \le a\int_0^T \frac{|u'(t)|}{t}\mathrm{d}t + \int_0^T g(t)\mathrm{d}t$$

$$\le a|u'(T)|T^{-a}\int_0^T t^{a-1}\mathrm{d}t + a\int_0^T t^{a-1}\left(\int_t^T s^{-a}g(s)\mathrm{d}s\right)\mathrm{d}t + \int_0^T g(t)\mathrm{d}t$$

$$= |u'(T)| + 2\int_0^T g(t)\mathrm{d}t < \infty.$$

□

Theorem 3.4. *Let us assume that f satisfies the global Carathéodory conditions on the set $[0,T]\times\mathbb{R}\times\mathbb{R}$. Let $a<0$ and let $u \in AC^1_{loc}(0,T]$ satisfy equation (2.1) a.e. on $(0,T]$. Then either $\lim_{t\to 0+} u'(t) = 0$ or $\lim_{t\to 0+} u'(t) = \pm\infty$.*

In particular, u can be extended on $[0,T]$ with $u \in AC^1[0,T]$ if and only if $\lim_{t\to 0+} u'(t) = 0$.

Proof. Keeping in mind that u is fixed, consider the linear equation

$$v'(t) - \frac{a}{t}v(t) = f(t,u(t),u'(t)). \tag{3.5}$$

Each function $v \in AC_{loc}(0,T]$ satisfying (3.5) for a.e. $t \in (0,T]$ has the form

$$v(t) = ct^a + t^a\int_0^t s^{-a}f(s,u(s),u'(s))\mathrm{d}s, \quad t \in (0,T],$$

where $c \in \mathbb{R}$. Hence we get

$$\lim_{t\to 0+} v(t) = 0 \quad \text{for } c = 0, \quad \lim_{t\to 0+} v(t) = \infty \cdot \operatorname{sign} c \quad \text{for } c \neq 0. \tag{3.6}$$

Since $u \in AC^1_{loc}(0,T]$ satisfies equation (2.1) for a.e. $t \in (0,T]$, there exists $c_0 \in \mathbb{R}$ such that $v = u'$ on $(0,T]$ for $c = c_0$. Therefore, by (3.6), either $\lim_{t\to 0+} u'(t) = 0$ or $\lim_{t\to 0+} u'(t) = \pm\infty$.

Let (3.2) hold. Then $c_0 = 0$ and

$$u'(t) = t^a\int_0^t s^{-a}f(s,u(s),u'(s))\mathrm{d}s, \quad t \in (0,T]. \tag{3.7}$$

Clearly (3.2) implies that there exists a finite limit $\lim_{t\to 0+} u(t)$. In order to prove that u can be extended on $[0,T]$ as a function in $AC^1[0,T]$, we have to show that

$$\int_0^T |u''(t)|\mathrm{d}t < \infty.$$

By (3.7) and (3.1),

$$\frac{|u'(t)|}{t} \le t^{a-1} \int_0^t s^{-a} g(s) \mathrm{d}s, \quad t \in (0, T].$$

Choose an arbitrary $\varepsilon > 0$. Then, by integration of the last inequality, we get

$$\int_\varepsilon^T \frac{|u'(t)|}{t} \mathrm{d}t \le \int_\varepsilon^T t^{a-1} \left(\int_0^t s^{-a} g(s) \mathrm{d}s \right) \mathrm{d}t$$

$$= -\frac{1}{|a|} \left(T^a \int_0^T t^{-a} g(t) \mathrm{d}t - \varepsilon^a \int_0^\varepsilon t^{-a} g(t) \mathrm{d}t - \int_\varepsilon^T g(t) \mathrm{d}t \right).$$

If we pass to the limit as $\varepsilon \to 0+$, we obtain

$$\int_0^T \frac{|u'(t)|}{t} \mathrm{d}t \le \frac{1}{|a|} \int_0^T g(t) \mathrm{d}t. \tag{3.8}$$

Finally, integrating (3.4) and using (3.8), we find that

$$\int_0^T |u''(t) \mathrm{d}t \le |a| \int_0^T \frac{|u'(t)|}{t} \mathrm{d}t + \int_0^T g(t) \mathrm{d}t \le 2 \int_0^T g(t) \mathrm{d}t < \infty.$$

□

The following corollary will be used in the next section.

Corollary 3.5. *Let us assume that condition* (2.2) *holds. Let* $a \neq 0$ *and let* $u \in AC^1_{loc}(0, T]$ *satisfy equation* (2.1) *a.e. on* $[0, T]$. *Let us also assume that*

$$S := \sup\{|u(t)| + |u'(t)| : t \in (0, T]\} < \infty \tag{3.9}$$

is fulfilled. Then (3.2) *holds and* u *can be extended on* $[0, T]$ *in such a way that* $u \in AC^1[0, T]$.

Proof. Let

$$\chi(z) := \begin{cases} S & \text{if} \quad z > S \\ z & \text{if} \quad |z| \le S \\ -S & \text{if} \quad z < -S \end{cases}$$

and let $\tilde{f}(t, x, y) = f(t, \chi(x), \chi(y))$ for a.e. $t \in [0, T]$ and all $x, y \in \mathbb{R}$. Clearly

$$u''(t) = \frac{a}{t} u'(t) + \tilde{f}(t, u(t), u'(t))$$

holds for a.e. $t \in [0, T]$. By (2.2), there exists a funtion $g \in L_1[0, T]$ such that $|\tilde{f}(t, x, y)| \le g(t)$ for a.e. $t \in [0, T]$ and all $(x, y) \in \mathbb{R} \times \mathbb{R}$. The results now follow from Theorems 3.3 and 3.4, where f is replaced by $\tilde{f}$ in equation (2.1). □

4 Neumann Problem

Using results formulated in Corollary 3.5 and the Fredholm-type existence theorem (see e.g. [22], [26], [27]), we are now in the position to show the existence and/or uniqueness of solutions of the nonlinear singular Neumann boundary value problem

$$u''(t) = \frac{a}{t}u'(t) + f(t, u(t), u'(t)), \tag{4.1a}$$

$$u'(0) = 0, \quad u'(T) = 0. \tag{4.1b}$$

Definition 4.1. A function $u \in AC^1[0,T]$ is called *a solution of the boundary value problem* (4.1), if u satisfies equation (4.1a) a.e. on $[0,T]$, and the Neumann conditions (4.1b).

First, we consider the uniqueness.

Theorem 4.2. (Uniqueness) *Let $a < 0$ and let us assume that condition* (2.2) *holds. Moreover, let us assume that for each compact set $\mathcal{K} \subset \mathbb{R} \times \mathbb{R}$ there exists a nonnegative function $h_{\mathcal{K}} \in L_1[0,T]$ such that*

$$x_1 > x_2 \;\Rightarrow\; f(t, x_1, y_1) - f(t, x_2, y_2) > -h_{\mathcal{K}}(t)|y_1 - y_2| \tag{4.2}$$

for a.e. $t \in [0,T]$ and all $(x_1, y_1), (x_2, y_2) \in \mathcal{K}$. Then problem (4.1) *has at most one solution.*

Proof. Let u_1 and u_2 be different solutions of problem (4.1). Since $u_1, u_2 \in AC^1[0,T]$, there exists a compact set $\mathcal{K} \subset \mathbb{R} \times \mathbb{R}$ such that $(u_i(t), u_i'(t)) \in \mathcal{K}$ for $t \in [0,T]$. Again, $v(t) := u_1(t) - u_2(t)$ for $t \in [0,T]$. Then

$$v'(0) = 0, \quad v'(T) = 0. \tag{4.3}$$

We consider two cases.

Case 1. Assume that $u_1(t_0) = u_2(t_0)$ for some $t_0 \in [0,T]$, that is $v(t_0) = 0$. Since u_1 and u_2 are different, there exists $t_1 \in [0,T]$, $t_1 \neq t_0$, such that $v(t_1) \neq 0$.
(i) Let $t_1 > t_0$. We can assume that $v(t_1) > 0$ and define $v := u_2 - u_1$ otherwise. Then we can find $a_0 \in (t_0, t_1)$ satisfying $v(t) > 0$ for $t \in [a_0, t_1]$ and $v'(a_0) > 0$. Let $b_0 \in (a_0, T]$ be the first zero of v'. Then, if we set $[\alpha, \beta] := [a_0, b_0]$, we see that

$$v(t) > 0 \text{ for } t \in [\alpha, \beta],\; v'(t) > 0 \text{ for } t \in [\alpha, \beta),\; v'(\beta) = 0. \tag{4.4}$$

Now, by (4.1a), (4.2) and (4.4), we obtain

$$v''(t) > \left(\frac{a}{t} - h_{\mathcal{K}}(t)\right) v'(t) \text{ for a.e. } t \in [\alpha, \beta].$$

Denote by $h^*(t) := \frac{a}{t} - h_{\mathcal{K}}(t)$. Then $h^* \in L_1[\alpha, \beta]$ and $v''(t) - h^*(t)v'(t) > 0$ for a.e. $t \in [\alpha, \beta]$. Consequently,

$$\left(v'(t) \exp\left(-\int_\alpha^t h^*(s)ds\right)\right)' > 0 \text{ for a.e. } t \in [\alpha, \beta].$$

Integrating the last inequality in $[\alpha, \beta]$, we obtain

$$v'(\beta) \exp\left(-\int_\alpha^\beta h^*(s)ds\right) > v'(\alpha) > 0$$

which contradicts $v'(\beta) = 0$.
(ii) Let $v = 0$ on $[t_0, T]$. Since u_1 and u_2 are different, we can find $\beta \in (0, t_0)$ such that (without loss of generality) $v(\beta) > 0$, and $v'(\beta) < 0$. Due to (4.3) it is possible to find $\alpha \in [0, \beta)$ such that

$$v(t) > 0 \text{ for } t \in [\alpha, \beta],\ v'(t) < 0 \text{ for } t \in (\alpha, \beta],\ v'(\alpha) = 0. \tag{4.5}$$

Now, we conclude from (4.1a), (4.2) and $a < 0$,

$$v''(t) > \frac{a}{t} v'(t) - h_{\mathcal{K}}(t)|v'(t)| \geq h_{\mathcal{K}}(t)v'(t) \text{ for a.e. } t \in [\alpha, \beta].$$

As above, we modify the last inequality, integrate it and obtain

$$v'(\beta) \exp\left(-\int_\alpha^\beta h_{\mathcal{K}}(s)ds\right) > v'(\alpha) = 0,$$

which contradicts $v'(\beta) < 0$.

Case 2. Assume that $u_1 \neq u_2$ on $[0, T]$, that is $v \neq 0$ on $[0, T]$. We may assume that $v > 0$ on $[0, T]$.
(i) Let $v' = 0$ on $[0, T]$. Then, by (4.1a) and (4.2),

$$v''(t) > \frac{a}{t} v'(t) - h_{\mathcal{K}}(t)|v'(t)| = 0 \text{ for a.e. } t \in [0, T],$$

in contradiction to $v'' = 0$ on $[0, T]$.
(ii) Let $v'(t_1) \neq 0$ for some $t_1 \in (0, T)$. If $v'(t_1) > 0$, then we can find an interval $[\alpha, \beta] \subset (t_1, T]$ satisfying (4.4). If $v'(t_1) < 0$, then we can find an interval $[\alpha, \beta] \subset [0, t_1)$ satisfying (4.5).

The above discussion shows that the existence of $[\alpha, \beta]$ satisfying either (4.4) or (4.5) leads to a contradiction. Hence, $u_1 = u_2$ on $[0, T]$ which completes the proof. □

Theorem 4.3. (Existence) *Assume* (2.2) *and let* $a < 0$. *Moreover, let there are* $A, B \in \mathbb{R}$, $A \leq B$, $c > 0$, $\omega \in C[0, \infty)$, *and* $\psi \in L_1[0, T]$ *such that the following conditions hold:*

$$f(t, A, 0) \leq 0, \quad f(t, B, 0) \geq 0 \tag{4.6}$$

for a.e. $t \in [0,T]$,

$$f(t,x,y)\,\operatorname{sign} y \le \omega(|y|)(|y| + \psi(t)) \tag{4.7}$$

for a.e. $t \in [0,T]$ *and all* $x \in [A,B]$, $y \in \mathbb{R}$, *where*

$$\omega(x) \ge c, \quad x \in [0,\infty), \qquad \int_0^\infty \frac{\mathrm{d}s}{\omega(s)} = \infty. \tag{4.8}$$

Then problem (4.1) *has a solution* u *such that*

$$A \le u(t) \le B, \quad t \in [0,T]. \tag{4.9}$$

Proof. *Step 1. Existence of auxiliary solutions* u_n.

Let

$$r := \|\psi\|_1 + \left(1 + \frac{T}{c}\right)(B - A).$$

Then, by (4.8), there exists $\rho^* > 0$ such that

$$\int_0^{\rho^*} \frac{\mathrm{d}s}{\omega(s)} > r.$$

For $y \in \mathbb{R}$, let

$$\chi(y) = \begin{cases} 1 & \text{if } |y| \le \rho^*, \\ 2 - \frac{|y|}{\rho^*} & \text{if } \rho^* < |y| < 2\rho^*, \\ 0 & \text{if } |y| \ge 2\rho^*. \end{cases}$$

Without loss of generality we can assume that $\frac{1}{n} < T$ for each $n \in \mathbb{N}$. Otherwise $\mathbb{N}$ is replaced by $\mathbb{N}' = \{n \in \mathbb{N} : \frac{1}{n} < T\}$. Motivated by [17], we choose $n \in \mathbb{N}$ and, for a.e. $t \in [0,T]$, all $x, y \in \mathbb{R}$, $\varepsilon \in [0,1]$, we define

$$h_n(t,x,y) := \begin{cases} \chi(y)\Big(\frac{a}{t}y + f(t,x,y)\Big) - \frac{A}{n} & \text{if } t \in (\frac{1}{n}, T], \\ -\frac{A}{n} & \text{if } t \in [0, \frac{1}{n}], \end{cases}$$

$$w_A(t,\varepsilon) := \sup\{|h_n(t,A,0) - h_n(t,A,y)| : |y| \le \varepsilon\},$$

$$w_B(t,\varepsilon) := \sup\{|h_n(t,B,0) - h_n(t,B,y)| : |y| \le \varepsilon\},$$

$$f_n(t,x,y) := \begin{cases} h_n(t,B,y) + w_B\Big(t, \frac{x-B}{x-B+1}\Big) & \text{if } x > B, \\ h_n(t,x,y) & \text{if } A \le x \le B, \\ h_n(t,A,y) - w_A\Big(t, \frac{A-x}{A-x+1}\Big) & \text{if } x < A. \end{cases}$$

It can be shown that w_A and w_B satisfy the local Carathéodory conditions on $[0,T] \times [0,1]$, are nondecreasing in their second argument and $w_A(t,0) =$

$w_B(t,0) = 0$ a.e. on $[0,T]$, see [17]. Therefore, f_n also satisfies the local Carathéodory conditions on $[0,T] \times \mathbb{R} \times \mathbb{R}$ and there exists a function $m_n \in L_1[0,T]$ such that

$$|f_n(t,x,y)| \le m_n(t) \text{ for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathbb{R} \times \mathbb{R}. \tag{4.10}$$

Note that h_n can be written in the form

$$h_n(t,x,y) = \mu_n(t)\chi(y)\Big(\frac{a}{t}y + f(t,x,y)\Big) - \frac{A}{n}, \tag{4.11}$$

where

$$\mu_n(t) = \begin{cases} 0 & \text{if } t \in [0, \frac{1}{n}], \\ 1 & \text{if } t \in (\frac{1}{n}, T]. \end{cases}$$

We can see that

$$\frac{A}{n} + h_n(t,A,0) \le 0, \quad \frac{B}{n} + h_n(t,B,0) \ge 0 \quad \text{for a.e. } t \in [0,T].$$

Consider the auxiliary regular Neumann problem (4.12), (4.1b), where

$$u'' = \frac{u}{n} + f_n(t,u,u'). \tag{4.12}$$

It is easy to verify that the homogeneous problem

$$u'' = \frac{u}{n}, \quad u'(0) = 0, \quad u'(T) = 0$$

has only the trivial solution. Hence, by (4.10) and the Fredholm-like existence theorem (see e.g. [27, Theorem C.5] or [22]), there exists a solution $u_n \in AC^1[0,T]$ of problem (4.12), (4.1b) for all $n \in \mathbb{N}$.

Step 2. Estimates of u_n.

Now, we prove that

$$A \le u_n(t) \le B, \; t \in [0,T], \; n \in \mathbb{N}. \tag{4.13}$$

Let us define $v := A - u_n$ and assume

$$\max\{v(t) : t \in [0,T]\} = v(t_0) > 0. \tag{4.14}$$

Then $v'(t_0) = 0$, which is clear if $t_0 \in (0,T)$, or it follows from (4.1b) if $t_0 \in \{0,T\}$. First, assume that $t_0 \in [0,T)$. Then we can find $\delta > 0$ such that

$$v(t) > 0, \quad |v'(t)| = |u_n'(t)| < \frac{v(t)}{v(t)+1} < 1 \text{ for } t \in [t_0, t_0+\delta] \subset [0,T].$$

Consequently,

$$\begin{aligned}
u_n''(t) &= f_n(t,u_n(t),u_n'(t)) + \frac{u_n(t)}{n} = h_n(t,A,u_n'(t)) - w_A\Big(t,\frac{v(t)}{v(t)+1}\Big) + \frac{u_n(t)}{n} \\
&\le h_n(t,A,0) + h_n(t,A,u_n'(t)) - h_n(t,A,0) - w_A(t,|u_n'(t)|) + \frac{u_n(t)}{n} \\
&\le h_n(t,A,0) + \frac{A-v(t)}{n} \\
&< 0 \quad \text{for } t\in[t_0,t_0+\delta].
\end{aligned}$$

Hence

$$0 > \int_{t_0}^{t} u_n''(s)\,\mathrm{d}s = u_n'(t) - u_n'(t_0) = u_n'(t) = -v'(t) \quad \text{for } t\in(t_0,t_0+\delta],$$

which contradicts (4.14). Assume that $t_0 = T$. Then we can find $\delta > 0$ such that

$$v(t) > 0, \quad |v'(t)| = |u_n'(t)| < \frac{v(t)}{v(t)+1} < 1 \quad \text{for } t\in[T-\delta,T]\subset[0,T].$$

Then, as above, we have

$$u_n''(t) \le h_n(t,A,0) + \frac{A-v(t)}{n} < 0 \quad \text{for } t\in[T-\delta,T].$$

Hence

$$0 > \int_{t}^{T} u_n''(s)\,\mathrm{d}s = -u_n'(t) = v'(t) \quad \text{for } t\in[T-\delta,T),$$

in a contrary to (4.14).
Now, let $z := u_n - B$ and assume

$$\max\{z(t) : t\in[0,T]\} = z(\xi) > 0. \tag{4.15}$$

Then $z'(\xi) = 0$, where $\xi\in[0,T]$. Assume that $\xi\in[0,T)$. Then we can find $\delta > 0$ such that

$$z(t) > 0, \quad |z'(t)| = |u_n'(t)| < \frac{z(t)}{z(t)+1} < 1 \quad \text{for } t\in[\xi,\xi+\delta]\subset[0,T].$$

Consequently,

$$\begin{aligned}
u_n''(t) &= f_n(t,u_n(t),u_n'(t)) + \frac{u_n(t)}{n} = h_n(t,B,u_n'(t)) + w_B\Big(t,\frac{z(t)}{z(t)+1}\Big) + \frac{u_n(t)}{n} \\
&\ge h_n(t,B,0) + h_n(t,B,u_n'(t)) - h_n(t,B,0) + w_B(t,|u_n'(t)|) + \frac{u_n(t)}{n} \\
&\ge h_n(t,B,0) + \frac{B+z(t)}{n} \\
&> 0 \quad \text{for } t\in[\xi,\xi+\delta].
\end{aligned}$$

Then

$$0 < \int_{t_0}^{t} u_n''(s)\,\mathrm{d}s = u_n'(t) = z'(t) \ \text{ for } t \in (\xi, \xi+\delta],$$

which contradicts (4.15). If $\xi = T$, then there exists $\delta > 0$ such that

$$z(t) > 0, \quad |z'(t)| = |u_n'(t)| < \frac{z(t)}{z(t)+1} < 1 \ \text{ for } t \in [T-\delta, T] \subset [0,T].$$

Arguing as above we have

$$u_n''(t) > 0 \ \text{ for } t \in [T-\delta, T].$$

Then

$$0 < \int_{t}^{T} u_n''(s)\,\mathrm{d}s = -u_n'(t) = -z'(t) \ \text{ for } t \in [T-\delta, T),$$

contradicting (4.15). Consequently, we have shown that (4.14) holds.
Step 3. Estimates of u_n'.
We now show that

$$|u_n'(t)| \le \rho^*, \ t \in [0,T], \ n \in \mathbb{N}. \tag{4.16}$$

Due to (4.11) and (4.12),

$$\begin{aligned} &u_n''(t)\operatorname{sign} u_n'(t) \\ &\quad = \Big\{\mu_n(t)\chi(u_n'(t))\Big(\frac{a}{t}u_n'(t) + f(t, u_n(t), u_n'(t))\Big) + \frac{u_n(t)-A}{n}\Big\}\operatorname{sign} u_n'(t) \end{aligned} \tag{4.17}$$

for a.e. $t \in [0,T]$ and all $n \in \mathbb{N}$. Denote $\rho := \|u_n'\|_\infty = |u_n'(t_0)|$ and assume $\rho > 0$. Then $t_0 \in (0,T)$. In the following part of the proof, we discuss two cases, $u_n'(t_0) = \rho$ and $u_n'(t_0) = -\rho$.
Case 1. Let $u_n'(t_0) = \rho$. Then there exists $t_1 \in [0, t_0)$ such that $u_n'(t) > 0$ on $(t_1, t_0]$ and $u_n'(t_1) = 0$. By (4.7) and (4.17), we obtain

$$\begin{aligned} u_n''(t) &= \mu_n(t)\chi(u_n'(t))\Big(\frac{a}{t}u_n'(t) + f(t, u_n(t), u_n'(t))\Big) + \frac{u_n(t)-A}{n} \\ &\le \mu_n(t)\chi(u_n'(t))f(t, u_n(t), u_n'(t)) + \frac{u_n(t)-A}{n} \\ &\le \omega(u_n'(t))(u_n'(t) + \psi(t)) + \frac{B-A}{n} \\ &\le \omega(u_n'(t)\Big(u_n'(t) + \psi(t) + \frac{B-A}{c}\Big) \end{aligned}$$

for a.e. $t \in [t_1, t_0]$. In particular,

$$\frac{u_n''(t)}{\omega(u_n'(t))} \le u_n'(t) + \psi(t) + \frac{B-A}{c} \ \text{ for a.e. } t \in [t_1, t_0].$$

Hence

$$\int_{t_1}^{t_0} \frac{u_n''(t)}{\omega(u_n'(t))}\,\mathrm{d}t \le \int_{t_0}^{t_1} \Big(u_n'(t) + \psi(t) + \frac{B-A}{c}\Big)\mathrm{d}t,$$

and

$$\int_0^{\rho} \frac{\mathrm{d}t}{\omega(t)} < u_n(t_1) - u_n(t_0) + \|\psi\|_1 + \frac{(B-A)T}{c} \le \Big(1+\frac{T}{c}\Big)(B-A) + \|\psi\|_1 = r.$$

Therefore

$$\int_0^{\rho} \frac{\mathrm{d}t}{\omega(s)} < r,$$

and $\rho < \rho^*$ follows.

Case 2. Let $u_n'(t_0) = -\rho$. Then there exists $t_1 \in [0, t_0)$ such that $u_n'(t) < 0$ on $(t_1, t_0]$ and $u_n'(t_1) = 0$. By (4.7) and (4.17), we deduce

$$\begin{aligned} -u_n''(t) &= -\mu_n(t)\chi(u_n'(t))\Big(\frac{a}{t}u_n'(t) + f(t, u_n(t), u_n'(t))\Big) - \frac{u_n(t)-A}{n} \\ &\le -\mu_n(t)\chi(u_n'(t)) f(t, u_n(t), u_n'(t)) \\ &\le \omega(-u_n'(t))(-u_n'(t) + \psi(t)) \end{aligned}$$

for a.e. $t \in [t_1, t_0]$. In particular,

$$-\frac{u_n''(t)}{\omega(-u_n'(t))} \le -u_n'(t) + \psi(t) \quad \text{for a.e. } t \in [t_1, t_0]$$

and

$$-\int_{t_1}^{t_0} \frac{u_n''(t)}{\omega(-u_n'(t))}\,\mathrm{d}t \le \int_{t_0}^{t_1} (\psi(t) - u_n'(t))\,\mathrm{d}t.$$

Hence

$$\int_0^{\rho} \frac{\mathrm{d}t}{\omega(s)} \le \|\psi\|_1 + u_n(t_0) - u_n(t_1) \le \|\psi\|_1 + B - A < r.$$

Consequently,

$$\int_0^{\rho} \frac{\mathrm{d}t}{\omega(s)} < r,$$

which implies $\rho < \rho^*$. Hence (4.16) holds.

Step 4. Convergence of $\{u_n\}$.

By (4.13) and (4.16), $\{u_n\}$ is bounded in $C^1[0,T]$. Since f satisfies the local Carathéodory conditions on $[0,T] \times \mathbb{R}^2$, there exists $m \in L_1[0,T]$ such that

$$|f(t, u_n(t), u_n'(t))| \le m(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } n \in \mathbb{N}. \tag{4.18}$$

Choose $b \in (0,T]$. Then there exists $n_0 \in \mathbb{N}$ such that $[b,T] \subset [\frac{1}{n}, T]$ for all $n \ge n_0$. Hence

$$\begin{aligned} u_n'(t) &= -\int_t^T \Big(f_n(t, s, u_n(s), u_n'(s)) + \frac{u_n(s)}{n}\Big)\mathrm{d}s \\ &= -\int_t^T \Big(f(s, u_n(s), u_n'(s)) + \frac{a}{s}u_n'(s) + \frac{u_n(s)-A}{n}\Big)\mathrm{d}s \end{aligned} \tag{4.19}$$

for $t \in [b, T]$ and $n \geq n_0$. Let $b \leq t_1 < t_2 \leq T$. Then, by (4.18) and (4.19),

$$\begin{aligned} |u_n'(t_2) - u_n'(t_1)| &= \Big| \int_{t_1}^{t_2} \Big(f(t, u_n(t), u_n'(t)) + \frac{a}{t} u_n'(t) + \frac{u_n(t) - A}{n} \Big) \mathrm{d}t \Big| \\ &\leq \int_{t_1}^{t_2} m(t)\, \mathrm{d}t + \Big(\frac{|a|\rho^*}{b} + B - A \Big)(t_2 - t_1) \end{aligned}$$

for $n \geq n_0$. Hence $\{u_n'\}_{n \geq n_0}$ is equicontinuous on $[b, T]$ and since $\{u_n\}$ is bounded in $C^1[0, T]$, the Arzelà-Ascoli theorem and the diagonalization theorem (see e.g. [27, Theorems B.5 and B.6]) guarantee that there exist a subsequence $\{u_\ell\}$ of $\{u_n\}$ and $u \in C[0, T] \cap C^1(0, T]$ such that

$$\begin{aligned} &\lim_{\ell \to \infty} u_\ell(t) = u(t) \ \text{ uniformly on } [0, T], \\ &\lim_{\ell \to \infty} u_\ell'(t) = u'(t) \ \text{ locally uniformly on } (0, T]. \end{aligned}$$

Clearly $u'(T) = 0$. By (4.13) and (4.16)

$$A \leq u(t) \leq B \text{ for } t \in [0, T], \quad |u'(t)| \leq \rho^* \text{ for } t \in (0, T]. \tag{4.20}$$

Hence (3.9) holds. Passing to the limit as $\ell \to \infty$ in (4.19), where u_n is replaced by u_ℓ, we obtain

$$u'(t) = -\int_t^T \Big(f(s, u(s), u'(s)) + \frac{a}{s} u'(s) \Big) \mathrm{d}s \ \text{ for } t \in (0, T]$$

by the Lebesgue dominated convergence theorem. Hence the limit function u belongs to $AC^1_{loc}(0, T]$ and solves equation (4.1a) a.e. on $[0, T]$. The local uniform convergence of $\{u_\ell'\}$ on $(0, T]$ does not guarantee $u'(0) = 0$. However, we can apply Corollary 3.5 to find out that $u \in AC^1[0, T]$ and $u'(0) = 0$. Therefore, u satisfies the Neumann conditions (4.1b). We see that u is a solution of problem (4.1) such that $A \leq u \leq B$ on $[0, T]$ which completes the proof. □

Example 4.4. Let $T = 1$. For $t \in (0, 1]$, $x, y \in \mathbb{R}$, choose

$$f(t, x, y) = \frac{1}{\sqrt{1 - t}} \big(3x(x^2 - 1) + e^x y \big) - \cos(3\pi t). \tag{4.21}$$

Then Theorem 4.3 can be applied to f and for both $A = -5/4$, $B = -1/2$ and $A = 1/2$, $B = 5/4$. Consequently, problem (4.1) with f given by (4.21), has two solutions u_1 and u_2 satisfying

$$-\frac{5}{4} \leq u_1(t) \leq -\frac{1}{2}, \quad \frac{1}{2} \leq u_2(t) \leq \frac{5}{4}, \ t \in [0, 1].$$

The existence of two different solutions u_1 and u_2 corresponds to the fact that f does not satisfy condition (4.2) of Theorem 4.2.

The next theorem for the existence of a unique solution of the Neumann problem (4.1) follows immediately from Theorems 4.2 and 4.3.

Theorem 4.5. (Existence and uniqueness) *Let all assumptions of Theorem* 4.2 *and Theorem* 4.3 *be satisfied. Then problem* (4.1) *has a unique solution* u. *This solution satisfies* (4.9).

Example 4.6. The following function,

$$f(t,x,y) = \frac{1}{\sqrt{1-t}}(x^3 - y^5) - 10\sin(4\pi t), \tag{4.22}$$

$t \in (0,1],\ x, y \in \mathbb{R}$, satisfies the assumptions of Theorem 4.5 for $A = -10^{1/3}$ and $B = 10^{1/3}$. Therefore problem (4.1) with f given by (4.22) has a unique solution.

5 Numerical Simulation

To illustrate the solution behavior, described by Theorems 4.3 and 4.5 we carried out a series of numerical calculations using a MATLAB™ software package `bvpsuite` designed to solve boundary value problems in ordinary differential equations and differential algebraic equations. The solver is based on a class of collocation method (including methods of different orders). The code also provides the asymptotically correct estimate for the error of the numerical approximation. To enhance the efficiency the code attempts to solve the problem on a mesh adapted to the solution behavior, in such a way that the tolerance is satisfied with the least possible effort. Error estimate procedure and the mesh adaptation work dependably provided that the solution of the problem is appropriately smooth[1]. The software and the manual with a short description of the code can be downloaded from `http://www.math.tuwien.ac.at/∼ewa`. Further information can be found in [18] and [20]. This software has already been used for a variety of singular boundary value problems relevant for applications, see e.g. [25].

We discuss Neumann problems of the form,

$$u''(t) = \frac{a}{t}u'(t) + \frac{1}{\sqrt{1-t}}(3u(t)(u^2(t)-1) + e^{u(t)}u'(t)) - \cos(3\pi t), \tag{5.1a}$$

$$u'(0) = u'(1) = 0, \tag{5.1b}$$

and

$$u''(t) = \frac{a}{t}u'(t) + \frac{1}{\sqrt{1-t}}(u^3(t) - u'^5(t)) - 10\sin(4\pi t), \tag{5.2a}$$

$$u'(0) = u'(1) = 0, \tag{5.2b}$$

[1] The required smoothness of higher derivatives is related to the order of the used collocation method.

cf. Examples 4.4 and 4.6, respectively. All solutions were computed on the unit interval $[0, 1]$.

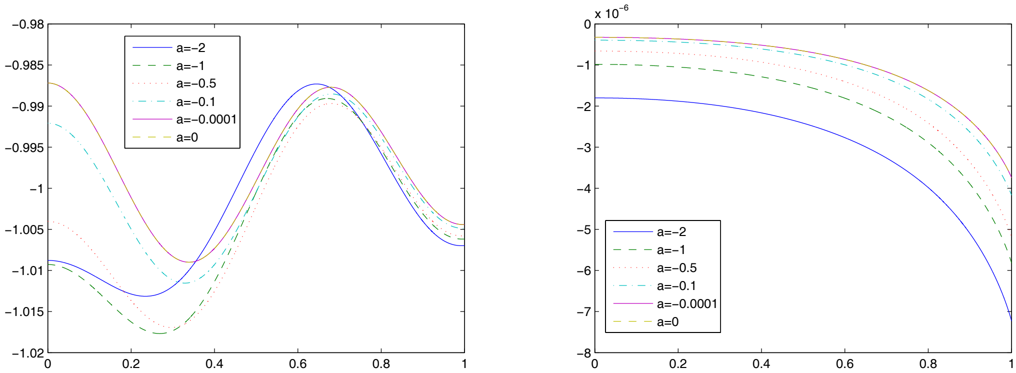

Figure 1: Illustrating Theorem 4.3: Solutions $-5/4 \leq u_1(t) \leq -1/2$ of problem (5.1) for different values of a (left), and the related error estimates (right). The initial solution for the Newton iteration was $u_0(t) \equiv -1$; the number of mesh points used $N = 1000$.

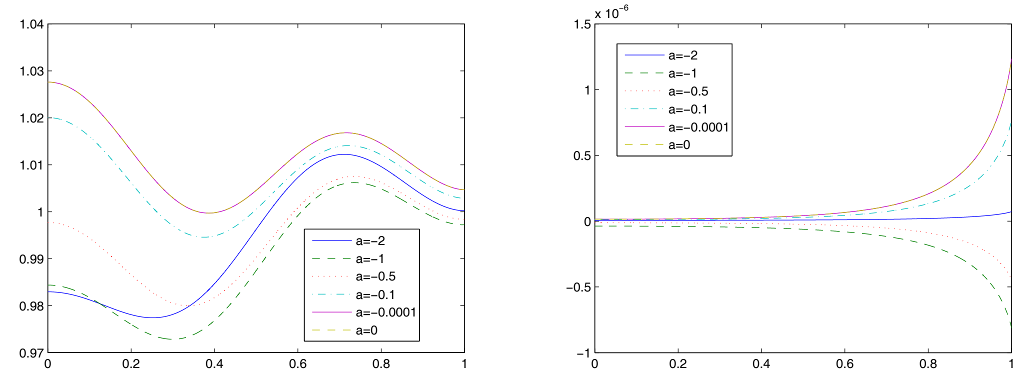

Figure 2: Illustrating Theorem 4.3: Solutions $1/2 \leq u_1(t) \leq 5/4$ of problem (5.1) for different values of a (left), and the related error estimates (right). The initial solution for the Newton iteration was $u_0(t) \equiv 1$; the number of mesh points used $N = 1000$.

As shown in Figures 1 and 2, we could find two different solutions u_1 and u_2 lying in regions indicated in Example 4.4. Recall that Theorem 4.3 guaranties the existence of a solution to a Neumann problem but not its uniqueness. Since in this case the solution is very unsmooth the mesh adaptation strategy does not work properly and therefore the calculations have been carried out on an equidistant mesh containing 1000 mesh points. We doubled the number of mesh points to provide a rough error estimate for the global error of the approximation. According to Theorem 4.5, the solution of problem (5.2a) is unique, cf. Figure 3.

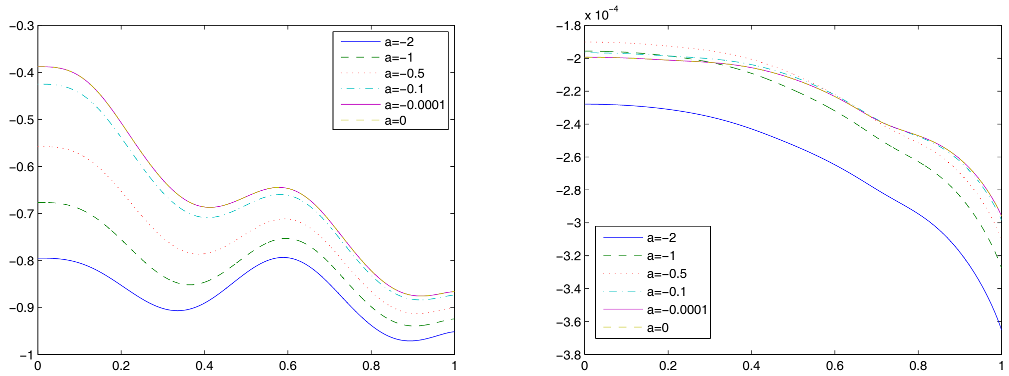

Figure 3: Illustrating Theorem 4.5: Solutions of problem (5.2a) for different values of a (left), and the related error estimates (right). The initial solution for the Newton iteration was $u_0(t) \equiv 1$; the number of mesh points used $N = 1000$.

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