# Existence of oscillatory solutions of singular nonlinear differential equations*

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**Abstract**

Asymptotic properties of solutions of the singular differential equation $(p(t)u'(t))' = p(t)f(u(t))$ are described. Here $f$ is Lipschitz continuous on $\mathbb{R}$ and has at least two zeros 0 and $L > 0$. The function $p$ is continuous on $[0, \infty)$, has a positive continuous derivative on $(0, \infty)$ and $p(0) = 0$. Further conditions for $f$ and $p$ under which the equation has oscillatory solutions converging to 0 are given.

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## 1 Introduction

For $k \in \mathbb{N}$, $k > 1$, and $L \in (0, \infty)$, consider the equation

$$u'' + \frac{k-1}{t}u' = f(u), \quad t \in (0, \infty), \tag{1}$$

where

$$f \in Lip_{\rm loc}(\mathbb{R}),\ f(0) = f(L) = 0,\ f(x) < 0,\ x \in (0, L), \tag{2}$$

$$\exists \bar{B} \in (-\infty, 0):\ f(x) > 0,\ x \in [\bar{B}, 0). \tag{3}$$

Let us put

$$F(x) = -\int_0^x f(z)\,\mathrm{d}z \quad \text{for } x \in \mathbb{R}. \tag{4}$$

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Moreover we assume that $f$ fulfils

$$F(\bar{B}) = F(L) \tag{5}$$

and denote

$$L_0 = \inf\{x < \bar{B} : f(x) > 0\} \geq -\infty.$$

Due to (2)-(4), we see that $F \in C^1(\mathbb{R})$ is decreasing and positive on $(L_0, 0)$, increasing and positive on $(0, L]$.

Equation (1) arises in many areas. For example: In the study of phase transitions of Van der Waals fluids [4], [11], [27], in population genetics, where it serves as a model for the spatial distribution of the genetic composition of a population [9], [10], in the homogenenous nucleation theory [1], in relativistic cosmology for description of particles which can be treated as domains in the universe [20], in the nonlinear field theory, in particular, when describing bubbles generated by scalar fields of the Higgs type in the Minkowski spaces [8]. Numerical simulations of solutions of (1), where $f$ is a polynomial with three zeros, have been presented in [7], [15], [19]. Close problems about the existence of positive solutions can be found in [3], [5], [6].

In this paper we investigate a generalization of equation (1) of the form

$$(p(t)u')' = p(t)f(u), \quad t \in (0, \infty), \tag{6}$$

where $f$ satisfies (2)-(5) and $p$ fulfils

$$p \in C[0, \infty) \cap C^1(0, \infty), \quad p(0) = 0, \tag{7}$$

$$p'(t) > 0, \ t \in (0, \infty), \ \lim_{t \to \infty} \frac{p'(t)}{p(t)} = 0. \tag{8}$$

Equation (6) is singular in the sense that $p(0) = 0$. If $p(t) = t^{k-1}$, with $k > 1$, then $p$ satisfies (7), (8) and equation (6) is equal to (1).

**Definition 1** A function $u \in C^1[0, \infty) \cap C^2(0, \infty)$ which satisfies equation (6) for all $t \in (0, \infty)$ is called a *solution* of (6).

Consider a solution $u$ of equation (6). Since $u \in C^1[0, \infty)$, we have $u(0), u'(0) \in \mathbb{R}$ and the assumption $p(0) = 0$ yields $p(0)u'(0) = 0$. We can find $M > 0$ and $\delta > 0$ such that $|f(u(t))| \leq M$ for $t \in (0, \delta)$. Integrating equation (6) we get

$$|u'(t)| = \left| \frac{1}{p(t)} \int_0^t p(s) f(u(s)) \, \mathrm{d}s \right| \leq \frac{M}{p(t)} \int_0^t p(s) \, \mathrm{d}s \leq Mt, \quad t \in (0, \delta).$$

Consequently the condition

$$u'(0) = 0 \tag{9}$$

is necessary for each solution of equation (6). Denote

$$u_{\sup} = \sup\{u(t) : t \in [0, \infty)\}.$$

**Definition 2** Let $u$ be a solution of equation (6). If $u_{\sup} < L$, then $u$ is called a *damped* solution.

If a solution $u$ of equation (6) satisfies $u_{\sup} = L$ or $u_{\sup} > L$, then we call $u$ a bounding homoclinic solution or an escape solution. These three types of solutions have been investigated in [23]-[26]. Here we continue the investigation of the existence and asymptotic properties of damped solutions. Due to (9) and Definition 2, it is reasonable to study solutions of equation (6) satisfying the initial conditions

$$u(0) = u_0 \in (L_0, L], \quad u'(0) = 0. \tag{10}$$

Note that if $u_0 > L$, then a solution $u$ of problem (6), (10) satisfies $u_{\sup} > L$ and consequently $u$ is not a damped solution. Assume that $L_0 > -\infty$. Then $f(L_0) = 0$ and if we put $u_0 = L_0$, a solution $u$ of (6), (10) is a constant function equal to $L_0$ on $[0, \infty)$. Since we impose no sign assumption on $f(x)$ for $x < L_0$, we do not consider the case $u_0 < L_0$. In fact, the choice of $u_0$ between two zeros $L_0$ and 0 of $f$ has been motivated by some hydrodynamical model in [19].

A lot of papers is devoted to oscillatory solutions of nonlinear differential equations. J.S.W. Wong [28] published an account on a nonlinear oscillation problem originated from earlier works of F. V. Atkinson and Z. Nehari. Wong's paper is concerned with the study of oscillatory behaviour of second–order Emden–Fowler equations

$$y''(x) + a(x)|y(x)|^{\gamma-1}y(x) = 0, \quad \gamma > 0, \tag{11}$$

where $a$ is nonnegative and absolutely continuous on $(0, \infty)$. Both superlinear case ($\gamma > 1$) and sublinear case ($\gamma \in (0, 1)$) are discussed and conditions for the function $a$ giving oscillatory or non–oscillatory solutions of (11) are presented, see also [21]. Further extensions of these results have been proved for more general differential equations. For example P.J.Y. Wong and R.P. Agarwal [29] or W.T. Li [18] worked with an equation

$$(a(t)(y'(t))^{\sigma})' + q(t)f(y(t)) = 0, \tag{12}$$

where $\sigma > 0$ is a positive quotient of odd integers, $a \in C^1(\mathbb{R})$ is positive, $q \in C(\mathbb{R})$, $f \in C^1(\mathbb{R})$, $xf(x) > 0$, $f'(x) \ge 0$ for all $x \ne 0$. M.R.S. Kulenović and Ć. Ljubović [17] investigated an equation

$$(r(t)g(y'(t)))' + p(t)f(y(t)) = 0, \tag{13}$$

where $g(u)/u \le m$, $f(u)/u \ge k > 0$ or $f'(u) \ge k$ for all $u \ne 0$. The investigation of oscillatory and nonoscillatory solutions has been also realized in the class of quasilinear equations. We refer to the paper [12] by L.F. Ho, dealing with an equation

$$(t^{n-1}\Phi_p(u'))' + t^{n-1}\sum_{i=1}^{N} \alpha_i t^{\beta_i}\Phi_{q_i}(u) = 0, \tag{14}$$

where $1 < p < n$, $\alpha_i > 0$, $\beta_i \geq -p$, $q_i > p-1$, $i = 1, \dots, N$, $\Phi_p(y) = |y|^{p-2}y$.

Oscillation results for the equation

$$(a(t)\Phi_p(x'))' + b(t)\Phi_q(x) = 0, \tag{15}$$

where $a, b \in C([0, \infty))$ are positive, can be found in [2]. We can see that the nonlinearity $f(y) = |y|^{\gamma-1}y$ in equation (11) is an increasing function on $\mathbb{R}$ having a unique zero at $y = 0$.

Nonlinearities in all other equations (12)–(15) have similar globally monotonous behaviour. We want to emphasize that, in contrast to the above papers, the nonlinearity $f$ in our equation (6) need not be globally monotonous. Moreover, we deal with solutions of equation (6) starting at a singular point $t = 0$ and we provide an interval for starting values $u_0$ giving oscillatory solutions (see Theorems 5, 12, 18). We specify a behaviour of oscillatory solutions in more details (decreasing amplitudes – see Theorems 12, 18) and we show conditions which guarantee that oscillatory solutions converge to 0 (Theorem 20).

The paper is organized in this manner: Section 2 contains results about existence, uniqueness and other basic properties of solutions of problem (6), (10). These results which mainly concern damped solutions are taken from [26] and extended or modified a little. We also provide here new conditions for the existence of oscillatory solutions in Theorem 18. Section 3 is devoted to asymptotic properties of oscillatory solutions and the main result is contained in Theorem 20.

## 2 Solutions of the initial problem (6), (10)

Let us give an account of this section in more details. The main objective of this paper is to characterize asymptotic properties of oscillatory solutions of problem (6), (10). In order to present more complete results about the solutions, we start this section with the unique solvability of problem (6), (10) on $[0, \infty)$ (Theorem 3). Having such global solutions, we have proved (see papers [23]–[26]) that oscillatory solutions of problem (6), (10) can be found just in the class of damped solutions of this problem. Therefore we give here one result about the existence of damped solutions (Theorem 5). Example 7 shows that there are damped solutions which are not oscillatory. Consequently, we bring results about the existence of oscillatory solutions in the class of damped solutions. This can be found in Theorem 12, which is an extension of Theorem 3.4 of [26] and in Theorem 18, which is new. Theorems 12 and Theorem 18 cover different classes of equations which is illustrated by examples.

**Theorem 3** *(Existence and uniqueness) Assume that* (2)-(5), (7), (8) *hold and that there exists* $C_L \in (0, \infty)$ *such that*

$$0 \leq f(x) \leq C_L \quad \text{for } x \geq L. \tag{16}$$

*Then the initial problem* (6), (10) *has a unique solution* $u$. *The solution* $u$ *satisfies*

$$\begin{array}{lll} u(t) \geq u_0 & \textit{if} & u_0 < 0 \\ u(t) > \bar{B} & \textit{if} & u_0 \geq 0 \end{array} \quad \textit{for } t \in [0,\infty). \tag{17}$$

*Proof.* Let $u_0 < 0$. Then the assertion is contained in Theorem 2.1 of [26]. Now, assume that $u_0 \in [0, L]$. Then the proof of Theorem 2.1 in [26] can be slightly modified. □

For close existence results see also Chapters 13 and 14 of [22], where this kind of equations is studied.

**Remark 4** Clearly, for $u_0 = 0$ and $u_0 = L$, problem (6), (10) has a unique solution $u \equiv 0$ and $u \equiv L$, respectively. Since $f \in Lip_{\rm loc}(\mathbb{R})$, no solution of problem (6), (10) with $u_0 < 0$ or $u_0 \in (0, L)$ can touch the constant solutions $u \equiv 0$ and $u \equiv L$.

In particular, assume that $C \in \{0, L\}$, $a > 0$, $u$ is a solution of problem (6), (10) with $u_0 < L$, $u_0 \neq 0$ and (2), (7), (8) hold. If $u(a) = C$, then $u'(a) \neq 0$ and if $u'(a) = 0$, then $u(a) \neq C$.

The next theorem provides an extension of Theorem 2.4 in [26].

**Theorem 5** *(Existence of damped solutions) Assume that* (2)-(5), (7), (8) *hold. Then for each* $u_0 \in [\bar{B}, L)$ *problem* (6), (10) *has a unique solution. This solution is damped.*

*Proof.* First assume that there exists $C_L > 0$ such that $f$ satisfies (16). Then, by Theorem 3, problem (6), (10) has a unique solution $u$ satisfying (17). Assume that $u$ is not damped, that is

$$\sup\{u(t) : t \in [0,\infty)\} \geq L. \tag{18}$$

By (3)-(5), the inequality $F(u_0) \leq F(L)$ holds. Since $u$ fulfils equation (6), we have

$$u''(t) + \frac{p'(t)}{p(t)} u'(t) = f(u(t)) \quad \text{for } t \in (0,\infty). \tag{19}$$

Multiplying (19) by $u'$ and integrating between 0 and $t > 0$ we get

$$0 < \frac{u'^2(t)}{2} + \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s = F(u_0) - F(u(t)), \quad t \in (0,\infty), \tag{20}$$

and consequently

$$0 < \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s \leq F(u_0) - F(u(t)), \quad t \in (0,\infty).$$

By (18) we can find $b \in (0,\infty]$ such that $u(b) \geq L$, ( $u(\infty) = \limsup_{t\to\infty} u(t)$ ), and hence, according to (5),

$$0 < \int_0^b \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s \leq F(u_0) - F(u(b)) \leq F(B) - F(L) \leq 0,$$

a contradiction. We have proved that $\sup\{u(t) : t \in [0,\infty)\} < L$, that is $u$ is damped. Consequently assumption (16) can be omitted. $\square$

**Example 6** Consider the equation

$$u'' + \frac{2}{t}u' = u(u-1)(u+2), \tag{21}$$

which is relevant to applications in [7],[15] and [19]. Here $p(t) = t^2$, $f(x) = x(x-1)(x+2)$, $L_0 = -2$, $L = 1$. Hence $f(x) < 0$ for $x \in (0,1)$, $f(x) > 0$ for $x \in (-2,0)$, and

$$F(x) = -\int_0^x f(z)\,\mathrm{d}z = -\frac{x^4}{4} - \frac{x^3}{3} + x^2.$$

Consequently, $F$ is decreasing and positive on $[-2,0)$ and increasing and positive on $(0,1]$. Since $F(1) = 5/12$ and $F(-1) = 13/12$, there exists a unique $\bar{B} \in (-1,0)$ such that $F(\bar{B}) = 5/12 = F(1)$. We can see that all assumptions of Theorem 5 are fulfilled an so, for each $u_0 \in [\bar{B},1)$, problem (21), (10) has a unique solution which is damped. We will show later (see Example 13), that each damped solution of problem (21), (10) is oscillatory.

In the next example we will show that damped solutions can be nonzero and monotonous on $[0,\infty)$ with a limit equal to zero at $\infty$. Clearly, such solutions are not oscillatory.

**Example 7** Consider the equation

$$u'' + \frac{3}{t}u' = f(u), \tag{22}$$

where

$$f(x) = \begin{cases} -x^3 & \text{for} \quad x \le 1 \\ x-2 & \text{for} \quad x \in (1,3) \\ 1 & \text{for} \quad x \ge 3. \end{cases}$$

We see that $p(t) = t^3$ in (22) and the functions $f$ and $p$ satisfy conditions (2)-(5), (7), (8) with $L = 2$. Clearly $L_0 = -\infty$. Further,

$$F(x) = -\int_0^x f(z)\,\mathrm{d}z = \begin{cases} x^4/4 & \text{for} \quad x \le 1 \\ -x^2/2 + 2x - 5/4 & \text{for} \quad x \in (1,3) \\ -x + 13/4 & \text{for} \quad x \ge 3. \end{cases}$$

Since $F(L) = F(2) = 3/4$, assumption (5) yields $F(\bar{B}) = \bar{B}^4/4 = 3/4$ and $\bar{B} = -3^{1/4}$. By Theorem 5, for each $u_0 \in [-3^{1/4}, 2)$ problem (22), (10) has a unique solution $u$ which is damped. On the other hand, we can check by a direct computation that for each $u_0 \le 1$ the function

$$u(t) = \frac{8u_0}{8 + u_0^2 t^2}, \quad t \in [0,\infty),$$

is a solution of equation (22) and satifies conditions (10). If $u_0 < 0$, then $u < 0$, $u' > 0$ on $(0, \infty)$, if $u_0 \in (0, 1]$, then $u > 0$, $u' < 0$ on $(0, \infty)$. In both cases $\lim_{t\to\infty} u(t) = 0$.

In Example 7 we also demonstrate that there are equations fulfilling Theorem 5 for which all solutions with $u_0 < L$, not only those with $u_0 \in [\bar{B}, L)$, are damped. Some additional conditions giving moreover bounding homoclinic solutions and escape solutions are presented in [23]-[25].

In our further investigation of asymptotic properties of damped solutions the following lemmas are useful.

**Lemma 8** *Assume* (2)*,* (7) *and* (8)*. Let $u$ be a damped solution of problem* (6)*,* (10) *with $u_0 \in (L_0, L)$ which is eventually positive or eventually negative. Then*

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0. \tag{23}$$

*Proof.* Let $u$ be eventually positive, that is there exists $t_0 \geq 0$ such that

$$u(t) > 0 \quad \text{for } t \in [t_0, \infty). \tag{24}$$

Denote $\theta = \inf\{t_0 \geq 0 : u(t) > 0,\ t \in [t_0, \infty)\}$.

Let $\theta > 0$. Then $u(\theta) = 0$ and, by Remark 4, $u'(\theta) > 0$. Assume that $u' > 0$ on $(\theta, \infty)$. Then $u$ is increasing on $(\theta, \infty)$ and there exists $\lim_{t\to\infty} u(t) = \ell \in (0, L)$. Multiplying (19) by $u'$, integrating between $\theta$ and $t$ and using notation (4), we obtain

$$\frac{u'^2(t)}{2} + \int_\theta^t \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s = F(u_0) - F(u(t)), \quad t \in (\theta, \infty). \tag{25}$$

Letting $t \to \infty$, we get

$$\lim_{t\to\infty} \frac{u'^2(t)}{2} = -\lim_{t\to\infty} \int_\theta^t \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s + F(u_0) - F(\ell).$$

Since the function $\int_\theta^t (p'(s)/p(s)) u'^2(s)\,\mathrm{d}s$ is positive and increasing, it follows that it has a limit at $\infty$ and hence there exists also $\lim_{t\to\infty} u'(t) \geq 0$. If $\lim_{t\to\infty} u'(t) > 0$, then $L > \ell = \lim_{t\to\infty} u(t) = \infty$, a contradiction. Consequently

$$\lim_{t\to\infty} u'(t) = 0. \tag{26}$$

Letting $t \to \infty$ in (19) and using (2), (8) and $\ell \in (0, L)$, we get $\lim_{t\to\infty} u''(t) = f(\ell) < 0$, and so $\lim_{t\to\infty} u'(t) = -\infty$, contrary to (26). This contradiction implies that the inequality $u' > 0$ on $(\theta, \infty)$ cannot be satisfied and that there exists $a > \theta$ such that $u'(a) = 0$. Since $u > 0$ on $(a, \infty)$, we get by (2), (6), and (10) that $(pu')' < 0$ on $(a, \infty)$. Due to $p(a)u'(a) = 0$, we see that $u' < 0$ on $(a, \infty)$. Therefore $u$ is decreasing on $(a, \infty)$ and $\lim_{t\to\infty} u(t) = \ell_0 \in [0, L)$. Using (25) with $a$ in place of $\theta$ we deduce as above that (26) holds and that

$\lim_{t\to\infty} u''(t) = f(\ell_0) = 0$. Consequently $\ell_0 = 0$. We have proved that (23) holds provided $\theta > 0$.

If $\theta = 0$, then we take $a = 0$ and use the above arguments. If $u$ is eventually negative we argue similarly. $\square$

**Lemma 9** *Assume* (2)-(5), (7), (8) *and*

$$p \in C^2(0,\infty), \quad \limsup_{t\to\infty} \left| \frac{p''(t)}{p'(t)} \right| < \infty, \tag{27}$$

$$\lim_{x\to 0+} \frac{f(x)}{x} < 0. \tag{28}$$

*Let $u$ be a solution of problem* (6), (10) *with $u_0 \in (0, L)$. Then there exists $\delta_1 > 0$ such that*

$$u(\delta_1) = 0, \quad u'(t) < 0 \text{ for } t \in (0, \delta_1]. \tag{29}$$

*Proof.* Assume that such $\delta_1$ does not exist. Then $u$ is positive on $[0,\infty)$ and, by Lemma 8, $u$ satisfies (23). We define a function

$$v(t) = \sqrt{p(t)}\,u(t), \quad t \in [0,\infty). \tag{30}$$

By (27), we have $v \in C^2(0,\infty)$ and

$$v'(t) = \frac{p'(t)u(t)}{2\sqrt{p(t)}} + \sqrt{p(t)}\,u'(t),$$

$$v''(t) = v(t)\left[\frac{1}{2}\frac{p''(t)}{p(t)} - \frac{1}{4}\left(\frac{p'(t)}{p(t)}\right)^2 + \frac{f(u(t))}{u(t)}\right], \quad t \in (0,\infty). \tag{31}$$

By (8) and (27) we get

$$\lim_{t\to\infty}\left[\frac{1}{2}\frac{p''(t)}{p(t)} - \frac{1}{4}\left(\frac{p'(t)}{p(t)}\right)^2\right] = \frac{1}{2}\lim_{t\to\infty}\frac{p''(t)}{p'(t)}\cdot\frac{p'(t)}{p(t)} = 0.$$

Since $u$ is positive on $(0,\infty)$, conditins (23) and (28) yield

$$\lim_{t\to\infty}\frac{f(u(t))}{u(t)} = \lim_{x\to 0+}\frac{f(x)}{x} < 0.$$

Consequently there exist $\omega > 0$ and $R > 0$ such that

$$\frac{1}{2}\frac{p''(t)}{p(t)} - \frac{1}{4}\left(\frac{p'(t)}{p(t)}\right)^2 + \frac{f(u(t))}{u(t)} < -\omega \quad \text{for } t \geq R. \tag{32}$$

By (30), $v$ is positive on $(0,\infty)$ and, due to (31) and (32), we get

$$v''(t) < -\omega v(t) < 0 \quad \text{for } t \geq R. \tag{33}$$

Thus, $v'$ is decreasing on $[R, \infty)$ and $\lim_{t\to\infty} v'(t) = V$. If $V < 0$, then $\lim_{t\to\infty} v(t) = -\infty$ contrary to the positivity of $v$. If $V \geq 0$, then $v' > 0$ on $[R, \infty)$ and $v(t) \geq v(R) > 0$ for $t \in [R, \infty)$. Then (33) yields $0 > -\omega v(R) \geq -\omega v(t) > v''(t)$ for $t \in [R, \infty)$. We get $\lim_{t\to\infty} v'(t) = -\infty$ which contradicts $V \geq 0$. The obtained contradictions imply that $u$ has at least one zero in $(0, \infty)$. Let $\delta_1 > 0$ be the first zero of $u$. Then $u > 0$ on $[0, \delta_1)$ and, by (2) and (6), $u' < 0$ on $(0, \delta_1)$. Due to Lemma 4, we have also $u'(\delta_1) < 0$. $\square$

For negative starting value we can prove a dual lemma by similar arguments.

**Lemma 10** *Assume* (2)-(5), (7), (8), (27) *and*

$$\lim_{x\to 0-} \frac{f(x)}{x} < 0. \tag{34}$$

*Let $u$ be a solution of problem* (6), (10) *with $u_0 \in (L_0, 0)$. Then there exists $\theta_1 > 0$ such that*

$$u(\theta_1) = 0, \quad u'(t) > 0 \text{ for } t \in (0, \theta_1]. \tag{35}$$

The arguments of the proof of Lemma 10 can be also found in the proof of Lemma 3.1 in [26], where both (28) and (34) were assumed. If we argue as in the proofs of Lemma 9 and Lemma 10 working with $a_1$, $A_1$ and $b_1$, $B_1$ in place of 0, $u_0$, we get the next corollary.

**Corollary 11** *Assume* (2)-(5), (7), (8), (27), (28) *and* (34). *Let $u$ be a solution of problem* (6), (10) *with $u_0 \in (L_0, 0) \cup (0, L)$.*

*I. Assume that there exist $b_1 > 0$ and $B_1 \in (L_0, 0)$ such that*

$$u(b_1) = B_1, \quad u'(b_1) = 0. \tag{36}$$

*Then there exists $\theta > b_1$ such that*

$$u(\theta) = 0, \quad u'(t) > 0 \text{ for } t \in (b_1, \theta]. \tag{37}$$

*II. Assume that there exist $a_1 > 0$ and $A_1 \in (0, L)$ such that*

$$u(a_1) = A_1, \quad u'(a_1) = 0. \tag{38}$$

*Then there exists $\delta > a_1$ such that*

$$u(\delta) = 0, \quad u'(t) < 0 \quad \text{for } t \in (a_1, \delta]. \tag{39}$$

Note that if all conditions of Lemma 9 and Lemma 10 are satisfied, then each solution of problem (6), (10) with $u_0 \in (L_0, 0) \cup (0, L)$ has at least one simple zero in $(0, \infty)$. Corollary 11 makes possible to construct an unbounded sequence of all zeros of any damped solution $u$. In addition, these zeros are simple (see the proof of Theorem 12). In such a case, $u$ has either a positive maximum or a negative minimum between each two neighbouring zeros. If we denote sequences of these maxima and minima by $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$, respectively, then we call the numbers $|A_n - B_n|$, $n \in \mathbb{N}$ amplitudes of $u$.

In [26] we give conditions implying that each damped solution of problem (6), (10) with $u_0 < 0$ has an unbounded set of zeros and decreasing sequence of amplitudes. Here, there is an extension of this result for $u_0 \in (0, L)$.

**Theorem 12** *(Existence of oscillatory solutions I) Assume that* (2)-(5), (7), (8), (27), (28) *and* (34) *hold. Then each damped solution of problem* (6), (10) *with* $u_0 \in (L_0, 0) \cup (0, L)$ *is oscillatory and its amplitudes are decreasing.*

*Proof.* For $u_0 < 0$ the assertion is contained in Theorem 3.4 of [26]. Let $u$ be a damped solution of problem (6), (10) with $u_0 \in (0, L)$. By (17) and Definition 2, we can find $L_1 \in (0, L)$ such that

$$\bar{B} < u(t) \le L_1 \quad \text{for } t \in [0, \infty). \tag{40}$$

Step 1. Lemma 9 yields $\delta_1 > 0$ satisfying (29). Hence there exists a maximal interval $(\delta_1, b_1)$ such that $u' < 0$ on $(\delta_1, b_1)$. If $b_1 = \infty$, then $u$ is eventually negative and decreasing. On the other hand, by Lemma 8, $u$ satisfies (23). But this is not possible. Therefore $b_1 < \infty$ and there exists $B_1 \in (\bar{B}, 0)$ such that (36) holds. Corollary 11 yields $\theta_1 > b_1$ satisfying (37) with $\theta = \theta_1$. Therefore $u$ has just one negative local minimm $B_1 = u(b_1)$ between its first zero $\delta_1$ and second zero $\theta_1$.

Step 2. By (37) there exists a maximal interval $(\theta_1, a_1)$, where $u' > 0$. If $a_1 = \infty$, then $u$ is eventually positive and increasing. On the other hand, by Lemma 8, $u$ satisfies (23). We get a contradiction. Therefore $a_1 < \infty$ and there exists $A_1 \in (0, L)$ such that (38) holds. Corollary 11 yields $\delta_2 > a_1$ satisfying (39) with $\delta = \delta_2$. Therefore $u$ has just one positive maximum $A_1 = u(a_1)$ between its second zero $\theta_1$ and third zero $\delta_2$.

Step 3. We can continue as in Step 1 and Step 2 and get the sequences $\{A_n\}_{n=1}^{\infty} \subset (0, L)$ and $\{B_n\}_{n=1}^{\infty} \subset [u_0, 0)$ of positive local maxima and negative local minima of $u$, respectively. Therefore $u$ is oscillatory. Using arguments of the proof of Theorem 3.4 of [26], we get that the sequence $\{A_n\}_{n=1}^{\infty}$ is decreasing and the sequence $\{B_n\}_{n=1}^{\infty}$ is increasing. In particular, we use (20) and define a Lyapunov function $V_u$ by

$$V_u(t) = \frac{u'^2(t)}{2} + F(u(t)) = F(u_0) - \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\, \mathrm{d}s, \quad t \in (0, \infty). \tag{41}$$

Then

$$V_u(t) > 0, \quad V_u'(t) = -\frac{p'(t)}{p(t)} u'^2(t) \le 0 \quad \text{for } t \in (0, \infty), \tag{42}$$

and

$$V_u'(t) < 0 \quad \text{for } t \in (0, \infty), \quad t \ne a_n, b_n,\ n \in \mathbb{N}.$$

Consequently

$$c_u := \lim_{t \to \infty} V_u(t) \ge 0. \tag{43}$$

So, sequences $\{V_u(a_n)\}_{n=1}^{\infty} = \{F(A_n)\}_{n=1}^{\infty}$ and $\{V_u(b_n)\}_{n=1}^{\infty} = \{F(B_n)\}_{n=1}^{\infty}$ are decreasing and

$$\lim_{n\to\infty} F(A_n) = \lim_{n\to\infty} F(B_n) = c_u. \tag{44}$$

Finally, due to (4), the sequence $\{A_n\}_{n=1}^{\infty}$ is decreasing and the sequence $\{B_n\}_{n=1}^{\infty}$ is increasing. Hence, the sequence of amplitudes $\{A_n - B_n\}_{n=1}^{\infty}$ is decreasing, as well. □

**Example 13** Consider problem (6), (10), where $p(t) = t^2$ and $f(x) = x(x-1)(x+2)$. In Example 6, we have shown that (2)–(5), (7), (8) with $L_0 = -2$, $L = 1$ are valid. Since

$$\lim_{t\to\infty} \frac{p''(t)}{p'(t)} = \lim_{t\to\infty} \frac{1}{t} = 0$$

and

$$\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} (x-1)(x+2) = -2 < 0,$$

we see that (27), (28) and (34) are satisfied. Therefore, by Theorem 12, each damped solution of (21), (10) with $u_0 \in (-2,0) \cup (0,1)$ is oscillatory and its amplitudes are decreasing.

**Example 14** Consider problem (6), (10), where

$$p(t) = \frac{t^k}{1+t^\ell}, \quad k > \ell \geq 0,$$

$$f(x) = \begin{cases} x(x-1)(x+3) & \text{for } x \leq 0, \\ x(x-1)(x+4) & \text{for } x > 0. \end{cases}$$

Then $L_0 = -3$, $L = 1$,

$$\lim_{t\to\infty} \frac{p''(t)}{p'(t)} = 0, \quad \lim_{x\to 0-} \frac{f(x)}{x} = -3, \quad \lim_{x\to 0+} \frac{f(x)}{x} = -4.$$

We can check that also all remaining assumptions of Theorem 12 are satisfied and this theorem is applicable here.

Assume that $f$ does not fulfil (28) and (34). It occurs for example if $f(x) = -|x|^\alpha \operatorname{sign} x$ with $\alpha > 1$ for $x$ in some neighbourhood of 0. Then Theorem 12 cannot be applied. Now, we will give another sufficient conditions for the existence of oscillatory solutions. For this purpose we introduce the following lemmas.

**Lemma 15** *Assume* (2)-(5), (7), (8) *and*

$$\int_1^\infty \frac{1}{p(s)} \, \mathrm{d}s = \infty \tag{45}$$

*and*

$$\exists \epsilon > 0 : f \in C^1(0,\epsilon) \quad \textit{and} \quad f' \le 0 \ \textit{on} \ (0,\epsilon). \tag{46}$$

*Let $u$ be a solution of problem* (6), (10) *with $u_0 \in (0, L)$. Then there exists $\delta_1 > 0$ such that*

$$u(\delta_1) = 0, \quad u'(t) < 0 \ \textit{for} \ t \in (0, \delta_1].$$

*Proof.* Assume that such $\delta_1$ does not exist. Then $u$ is positive on $[0,\infty)$ and, by Lemma 8, $u$ satisfies (23). In view of (6) and (2) we have $u' < 0$ on $(0,\infty)$. From (46) it follows that there exists $t_0 > 0$ such that

$$0 < u(t) < \epsilon, \quad \text{for } t \in [t_0, \infty).$$

Motivated by arguments of [13] we divide equation (6) by $f(u)$ and integrate it over interval $[t_0, t]$. We get

$$\int_{t_0}^{t} \frac{(p(s)u'(s))'}{f(u(s))} \, \mathrm{d}s = \int_{t_0}^{t} p(s) \, \mathrm{d}s \quad \text{for } t \in [t_0, \infty).$$

Using the per partes integration we obtain

$$\begin{aligned} \frac{p(t)u'(t)}{f(u(t))} &+ \int_{t_0}^{t} \frac{p(s)f'(u(s))u'^2(s)}{f^2(u(s))} \, \mathrm{d}s \\ &= \frac{p(t_0)u'(t_0)}{f(u(t_0))} + \int_{t_0}^{t} p(s) \, \mathrm{d}s, \quad t \in [t_0, \infty). \end{aligned}$$

From (7) and (8) it follows that there exists $t_1 \in (t_0, \infty)$ such that

$$\frac{p(t_0)u'(t_0)}{f(u(t_0))} + \int_{t_0}^{t} p(s) \, \mathrm{d}s \ge 1, \quad t \in [t_1, \infty),$$

and therefore

$$\frac{p(t)u'(t)}{f(u(t))} + \int_{t_0}^{t} \frac{p(s)f'(u(s))u'^2(s)}{f^2(u(s))} \, \mathrm{d}s \ge 1, \quad t \in [t_1, \infty).$$

From the fact that $f'(u(s)) \le 0$ for $s > t_0$ (see (46)) we have

$$\frac{p(t)u'(t)}{f(u(t))} + \int_{t_1}^{t} \frac{p(s)f'(u(s))u'^2(s)}{f^2(u(s))} \, \mathrm{d}s \ge 1, \quad t \in [t_1, \infty).$$

Then

$$\frac{p(t)u'(t)}{f(u(t))} \ge 1 - \int_{t_1}^{t} \frac{p(s)f'(u(s))u'^2(s)}{f^2(u(s))} \, \mathrm{d}s > 0, \quad t \in [t_1, \infty) \tag{47}$$

and

$$\frac{p(t)u'(t)}{f(u(t))\big(1 - \int_{t_1}^{t} p(s)f'(u(s))u'^2(s)f^{-2}(u(s)) \, \mathrm{d}s\big)} \ge 1, \quad t \in [t_1, \infty).$$

Multiplying this inequality by $-f'(u(t))u'(t)/f(u(t)) \geq 0$ we get

$$\Big(\ln(1 - \int_{t_1}^{t} \frac{p(s)f'(u(s))u'^2(s)}{f^2(u(s))}\,\mathrm{d}s)\Big)' \geq -\Big(\ln|f(u(t))|\Big)', \quad t \in [t_1, \infty)$$

and integrating it over $[t_1, t]$ we obtain

$$\ln\Big(1 - \int_{t_1}^{t} \frac{p(s)f'(u(s))u'^2(s)}{f^2(u(s))}\,\mathrm{d}s\Big) \geq \ln\Big(\frac{f(u(t_1))}{f(u(t))}\Big),$$

and therefore

$$1 - \int_{t_1}^{t} \frac{p(s)f'(u(s))u'^2(s)}{f^2(u(s))}\,\mathrm{d}s \geq \frac{f(u(t_1))}{f(u(t))}, \quad t \in [t_1, \infty).$$

According to (47) we have

$$\frac{p(t)u'(t)}{f(u(t))} \geq \frac{f(u(t_1))}{f(u(t))}, \quad t \in [t_1, \infty),$$

and consequently

$$u'(t) \leq f(u(t_1))\frac{1}{p(t)}, \quad t \in [t_1, \infty).$$

Integrating it over $[t_1, t]$ we get

$$u(t) \leq u(t_1) + f(u(t_1)) \int_{t_1}^{t} \frac{1}{p(s)}\,\mathrm{d}s, \quad t \in [t_1, \infty).$$

From (45) it follows that

$$\lim_{t\to\infty} u(t) = -\infty,$$

which is a contradiction. □

By similar arguments we can prove a dual lemma.

**Lemma 16** *Assume* (2)-(5), (7), (8), (45) *and*

$$\exists \epsilon > 0 : f \in C^1(-\epsilon, 0) \quad \textit{and} \quad f' \leq 0 \textit{ on } (-\epsilon, 0). \tag{48}$$

*Let $u$ be a solution of problem* (6), (10) *with $u_0 \in (L_0, 0)$. Then there exists $\theta_1 > 0$ such that*

$$u(\theta_1) = 0, \quad u'(t) > 0 \textit{ for } t \in (0, \theta_1].$$

Following ideas before Corollary 11 we get the next corollary.

**Corollary 17** *Assume* (2)-(5), (7), (8), (45), (46) *and* (48). *Let $u$ be a solution of problem* (6), (10) *with $u_0 \in (L_0, 0) \cup (0, L)$. Then the assertions I and II of Corollary* 11 *are valid.*

Now we are able to formulate another existence result for oscillatory solutions. Its proof is almost the same as the proof of Theorem 12 for $u_0 \in (L_0, 0)$ and the proof of Theorem 3.4 in [26] for $u_0 \in (0, L)$. The only difference is that we use Lemma 15, Lemma 16 and Corollary 17, in place of Lemma 9, Lemma 10 and Corollary 11, respectively.

**Theorem 18** *(Existence of oscillatory solutions II) Assume that* (2)-(5), (7), (8), (45), (46) *and* (48) *hold. Then each damped solution of problem* (6), (10) *with* $u_0 \in (L_0, 0) \cup (0, L)$ *is oscillatory and its amplitudes are decreasing.*

**Example 19** Let us consider equation (6) with

$$p(t) = t^\alpha, \quad t \in [0, \infty),$$

$$f(x) = \begin{cases} -|x|^\lambda \operatorname{sgn} x, & x \le 1, \\ x - 2, & x \in (1, 3), \\ 1, & x \ge 3, \end{cases}$$

where $\lambda$ and $\alpha$ are real parameters.
Case 1. Let $\lambda \in (1, \infty)$ and $\alpha \in (0, 1]$. Then all assumptions of Theorem 18 are satisfied. Note that $f$ satifies neither (28) nor (34) and hence Theorem 12 cannot be applied.
Case 2. Let $\lambda = 1$ and $\alpha \in (0, \infty)$. Then all assumptions of Theorem 12 are satisfied. If $\alpha \in (0, 1]$, then also all assumptions of Theorem 18 are fulfilled but for $\alpha \in (1, \infty)$, the function $p$ does not satisfy (45) and hence Theorem 18 cannot be applied.

# 3 Asymptotic properties of oscillatory solutions

In Lemma 8 we show that if $u$ is a damped solution of problem (6), (10) which is not oscillatory then $u$ converges to 0 for $t \to \infty$. In this section we give conditions under which also oscillatory solutions converge to 0.

**Theorem 20** *Assume that* (2)-(5), (7), (8) *hold and that there exists* $k_0 > 0$ *such that*

$$\liminf_{t \to \infty} \frac{p(t)}{t^{k_0}} > 0. \tag{49}$$

*Then each damped oscillatory solution* $u$ *of problem* (6), (10) *with* $u_0 \in (L_0, 0) \cup (0, L)$ *satisfies*

$$\lim_{t \to \infty} u(t) = 0, \quad \lim_{t \to \infty} u'(t) = 0. \tag{50}$$

*Proof.* Consider an oscillatory solution $u$ of problem (6), (10) with $u_0 \in (0, L)$.
Step 1. Using the notation and some arguments of the proof of Theorem 12, we have the unbounded sequences $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$, $\{\theta_n\}_{n=1}^\infty$, $\{\delta_n\}_{n=1}^\infty$, such that

$$0 < \delta_1 < b_1 < \theta_1 < a_1 < \delta_2 < \cdots < \delta_n < b_n < \theta_n < a_n < \delta_{n+1} < \dots$$

where $u(\theta_n) = u(\delta_n) = 0$, $u(a_n) = A_n > 0$ is a unique local maximum of $u$ in $(\theta_n, \delta_{n+1})$, $u(b_n) = B_n < 0$ is a unique local minimum of $u$ in $(\delta_n, \theta_n)$, $n \in \mathbb{N}$. Let $V_u$ be given by (41). Then (43) and (44) hold and, by (2)-(4), we see that

$$\lim_{t\to\infty} u(t) = 0 \quad \iff \quad c_u = 0. \tag{51}$$

Assume that (50) does not hold. Then $c_u > 0$. Motivated by arguments of [14] we derive a contradiction in the following steps.

Step 2. *Estimates of $u$.* By (41) and (43), we have

$$\lim_{n\to\infty} \frac{u'^2(\delta_n)}{2} = \lim_{n\to\infty} \frac{u'^2(\theta_n)}{2} = c_u > 0, \tag{52}$$

and the sequences $\{u'^2(\delta_n)\}_{n=1}^{\infty}$ and $\{u'^2(\theta_n)\}_{n=1}^{\infty}$ are decreasing. Consider $n \in \mathbb{N}$. Then $u'^2(\delta_n)/2 > c_u$ and there are $\alpha_n, \beta_n$ satisfying $a_n < \alpha_n < \delta_n < \beta_n < b_n$ and such that

$$u'^2(\alpha_n) = u'^2(\beta_n) = c_u, \quad u'^2(t) > c_u, \quad t \in (\alpha_n, \beta_n). \tag{53}$$

Since $V_u(t) > c_u$ for $t > 0$ (see (43)), we get by (41) and (53) the inequalities $c_u/2 + F(u(\alpha_n)) > c_u$ and $c_u/2 + F(u(\beta_n)) > c_u$ and consequently $F(u(\alpha_n)) > c_u/2$ and $F(u(\beta_n)) > c_u/2$. Therefore, due to (4), there exists $\tilde{c} > 0$ such that

$$u(\alpha_n) > \tilde{c}, \quad u(\beta_n) < -\tilde{c}, \quad n \in \mathbb{N}. \tag{54}$$

Similarly we deduce that there are $\tilde{\alpha}_n$, $\tilde{\beta}_n$ satisfying $b_n < \tilde{\alpha}_n < \theta_n < \tilde{\beta}_n < a_{n+1}$ and such that

$$u(\tilde{\alpha}_n) < -\tilde{c}, \quad u(\tilde{\beta}_n) > \tilde{c}, \quad n \in \mathbb{N}. \tag{55}$$

The behaviour of $u$ and inequalities (54) and (55) yield

$$|u(t)| > \tilde{c}, \quad t \in [\beta_n, \tilde{\alpha}_n] \cup [\tilde{\beta}_n, \alpha_{n+1}], \ n \in \mathbb{N}. \tag{56}$$

Step 3. *Estimates of $\beta_n - \alpha_n$.* We prove that there exist $c_0, c_1 \in (0, \infty)$ such that

$$c_0 < \beta_n - \alpha_n < c_1, \quad n \in \mathbb{N}. \tag{57}$$

Assume on the contrary that there exists a subsequence satisfying $\lim_{\ell\to\infty}(\beta_\ell - \alpha_\ell) = 0$. By the mean value theorem and (54) there is $\xi_\ell \in (\alpha_\ell, \beta_\ell)$ such that $0 < 2\tilde{c} < u(\alpha_\ell) - u(\beta_\ell) = |u'(\xi_\ell)|(\beta_\ell - \alpha_\ell)$. Since $F(u(t)) \geq 0$ for $t \in [0, \infty)$, we get by (25) the inequality

$$|u'(t)| < \sqrt{2F(u_0)}, \quad t \in [0, \infty), \tag{58}$$

and consequently

$$0 < 2\tilde{c} \leq \sqrt{2F(u_0)} \lim_{\ell\to\infty}(\beta_\ell - \alpha_\ell) = 0,$$

a contradiction. So, $c_0$ satisfying (57) exists. Using the mean value theorem again we can find $\tau_n \in (\alpha_n, \delta_n)$ such that $u(\delta_n) - u(\alpha_n) = u'(\tau_n)(\delta_n - \alpha_n)$ and, by (53),

$$\delta_n - \alpha_n = \frac{-u(\alpha_n)}{u'(\tau_n)} = \frac{u(\alpha_n)}{|u'(\tau_n)|} < \frac{A_1}{\sqrt{c_u}}.$$

Similarly we can find $\eta_n \in (\delta_n, \beta_n)$ such that

$$\beta_n - \delta_n = \frac{u(\beta_n)}{u'(\eta_n)} = \frac{|u(\beta_n)|}{|u'(\eta_n)|} < \frac{|B_1|}{\sqrt{c_u}}.$$

If we put $c_1 = (A_1 + |B_1|)/\sqrt{c_u}$, then (57) is fulfilled. Similarly we can prove

$$c_0 < \tilde{\beta}_n - \tilde{\alpha}_n < c_1, \quad n \in \mathbb{N}. \tag{59}$$

Step 4. *Estimates of* $\alpha_{n+1} - \alpha_n$. We prove that there exist $c_2 \in (0, \infty)$ such that

$$\alpha_{n+1} - \alpha_n < c_2, \quad n \in \mathbb{N}. \tag{60}$$

Put $m_1 = \min\{f(x) : B_1 \le x \le -\tilde{c}\} > 0$. By (56), $B_1 \le u(t) < -\tilde{c}$ for $t \in [\beta_n, \tilde{\alpha}_n]$, $n \in \mathbb{N}$. Therefore

$$f(u(t)) \ge m_1, \quad t \in [\beta_n, \tilde{\alpha}_n], \quad n \in \mathbb{N}. \tag{61}$$

Due to (8) we can find $t_1 > 0$ such that

$$\frac{p'(t)}{p(t)}\sqrt{2F(u_0)} < \frac{m_1}{2}, \quad t \in [t_1, \infty). \tag{62}$$

Let $n_1 \in \mathbb{N}$ fulfil $\alpha_{n_1} \ge t_1$. Then, according to (19), (58), (61) and (62), we have

$$u''(t) > -\frac{m_1}{2} + m_1 = \frac{m_1}{2}, \quad t \in [\beta_n, \tilde{\alpha}_n],\ n \ge n_1. \tag{63}$$

Integrating (63) from $b_n$ to $\beta_n$ and using (53) we get $2\sqrt{c_u} > m_1(b_n - \beta_n)$ for $n \ge n_1$. Similarly we get $2\sqrt{c_u} > m_1(\tilde{\alpha}_n - b_n)$ for $n \ge n_1$. Therefore

$$\frac{4}{m_1}\sqrt{c_u} > \tilde{\alpha}_n - \beta_n, \quad n \ge n_1. \tag{64}$$

By analogy we put $m_2 = \min\{-f(x) : \tilde{c} \le x \le A_1\} > 0$ and prove that there exists $n_2 \in \mathbb{N}$ such that

$$\frac{4}{m_2}\sqrt{c_u} > \alpha_{n+1} - \tilde{\beta}_n, \quad n \ge n_2. \tag{65}$$

Inequalities (57), (59), (64) and (65) imply the existence of $c_2$ fulfilling (60).
Step 5. *Construction of a contradiction.* Choose $t_0 > c_1$ and integrate the equality in (42) from $t_0$ to $t > t_0$. We have

$$V_u(t) = V_u(t_0) - \int_{t_0}^{t} \frac{p'(\tau)}{p(\tau)} u'^2(\tau)\,\mathrm{d}\tau, \quad t \ge t_0. \tag{66}$$

Choose $n_0 \in \mathbb{N}$ such that $\alpha_{n_0} > t_0$. Further choose $n \in \mathbb{N}$, $n > n_0$ and assume that $t > \beta_n$. Then, by (53),

$$\begin{aligned}\int_{t_0}^{t} \frac{p'(\tau)}{p(\tau)} u'^2(\tau)\,\mathrm{d}\tau &> \sum_{j=n_0}^{n} \int_{\alpha_j}^{\beta_j} \frac{p'(\tau)}{p(\tau)} u'^2(\tau)\,\mathrm{d}\tau \\ &> c_u \sum_{j=n_0}^{n} \int_{\alpha_j}^{\beta_j} \frac{p'(\tau)}{p(\tau)}\,\mathrm{d}\tau = c_u \sum_{j=n_0}^{n} \Big[\ln p(\tau)\Big]_{\alpha_j}^{\beta_j}.\end{aligned}$$

By virtue of (49) there exists $c_3 > 0$ such that $p(t)/t^{k_0} > c_3$ for $t \in [t_0, \infty)$. Thus $\ln p(t) > \ln c_3 + k_0 \ln t$ and

$$\int_{t_0}^{t} \frac{p'(\tau)}{p(\tau)} u'^2(\tau)\,\mathrm{d}\tau > c_u \sum_{j=n_0}^{n} \Big[\ln c_3 + k_0 \ln t\Big]_{\alpha_j}^{\beta_j} = c_u k_0 \sum_{j=n_0}^{n} \ln \frac{\beta_j}{\alpha_j}. \tag{67}$$

Due to (57) and $c_1 < \alpha_{n_0}$ we have

$$1 < \frac{\beta_j}{\alpha_j} < 1 + \frac{c_1}{\alpha_j} < 2, \quad j = n_0, \dots, n,$$

and the mean value theorem yields $\xi_j \in (1, 2)$ such that

$$\ln \frac{\beta_j}{\alpha_j} = \left(\frac{\beta_j}{\alpha_j} - 1\right) \frac{1}{\xi_j} > \frac{\beta_j - \alpha_j}{2\alpha_j}, \quad j = n_0, \dots, n. \tag{68}$$

By (57) and (60), we deduce

$$\frac{\beta_j - \alpha_j}{\alpha_j} > \frac{c_0}{\alpha_j}, \quad \alpha_j < jc_2 + \alpha_1, \quad j = n_0, \dots, n.$$

Thus

$$\frac{\beta_j - \alpha_j}{\alpha_j} > \frac{c_0}{jc_2 + \alpha_1}, \quad j = n_0, \dots, n. \tag{69}$$

Using (67)-(69) and letting $t$ to $\infty$ we obtain

$$\begin{aligned}\int_{t_0}^{\infty} \frac{p'(\tau)}{p(\tau)} u'^2(\tau)\,\mathrm{d}\tau &\geq c_u k_0 \sum_{n=n_0}^{\infty} \ln \frac{\beta_n}{\alpha_n} \geq \frac{1}{2} c_u k_0 \sum_{n=n_0}^{\infty} \frac{\beta_n - \alpha_n}{\alpha_n} \\ &\geq \frac{1}{2} c_u k_0 \sum_{n=n_0}^{\infty} \frac{c_0}{nc_2 + \alpha_1} = \infty.\end{aligned}$$

Using it in (66) we get $\lim_{t\to\infty} V_u(t) = -\infty$, a contradiction. So, we have proved that $c_u = 0$.
Using (19) and (51) we have

$$\lim_{t\to\infty} \left( \frac{u'^2(t)}{2} + \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s \right) = F(u_0) - F(0) = F(u_0).$$

Since the function $\int_0^t (p'(s)/p(s))u'^2(s)\,\mathrm{d}s$ is increasing, there exists

$$\lim_{t\to\infty}\int_0^t \frac{p'(s)}{p(s)}u'^2(s)\,\mathrm{d}s \le F(u_0).$$

Therefore there exists

$$\lim_{t\to\infty} u'^2(t) = \ell^2.$$

If $\ell > 0$, then $\lim_{t\to\infty}|u'(t)| = \ell$, which contradicts (51). Therefore $\ell = 0$ and (50) is proved.
If $u_0 \in (L_0, 0)$, we argue analogously. □


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