Solvability of Nonlinear Singular Problems for Ordinary Differential Equations

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April 2, 2007

[1]Supported by the Council of Czech Government MSM 6198959214
[2]Supported by the Council of Czech Government MSM 6198959214
[3]Supported by the grant No. A100190703 of the Grant Agency of the Academy of Sciences of the Czech Republic and by the Institutional Research Plan No. AV0Z10190503

Contents

Preface

The topic of singular boundary value problems has been of substantial and rapidly growing interest for many scientists and engineers. This book is devoted to singular boundary value problems for ordinary differential equations. It presents existence theory for a variety of problems having unbounded nonlinearities in regions where their solutions are searched for. The importance of thorough investigation of analytical solvability is emphasized by the fact that numerical simulations of solutions to such problems usually break down near singular points.

The contents of the monograph is mainly based on results obtained by the authors during the last few years. Nevertheless, most of the more advanced results achieved to date in this field can be found here. Besides, some known results are presented in a new way. The selection of topics reflects the particular interests of the authors.

The book is addressed to researchers in related areas, to graduate students or advanced undergraduates and, in particular, to those interested in singular and nonlinear boundary value problems. It can serve as a reference book on the existence theory for singular boundary value problems of ordinary differential equations as well as a textbook for graduate or undergraduate classes. The readers need basic knowledge of real analysis, linear and nonlinear functional analysis, theory of Lebesgue measure and integral, theory of ordinary differential equations (including the Carathéodory theory and boundary value problems) on the graduate level.

The monograph deals with boundary value problems which are considered in the frame of the Carathéodory theory. If nonlinearities in differential equations fulfil the Carathéodory conditions, the boundary value problems are called *regular*, while, if the Carathéodory conditions are not fulfilled on the whole region, the problems are called *singular*. Two types of singularities are distinguished – time and space ones. For singular boundary value problems we introduce notions of a solution and of a w-solution. *Solutions* of n −th order differential equations are understood as functions having absolutely continuous derivatives up to order $n-1$ on the whole basic compact interval. On the other hand, *w-solutions* have these derivatives only locally absolutely continuous on a noncompact subset of the basic interval. The main attention is paid to the existence of solutions of singular problems. The proofs are mostly based on regularization and sequential technique. The impact of our theoretical results is demonstrated by illustrative examples.

Essentially, the book is divided into two parts and an appendices.

Part I consists of 6 chapters and is devoted to scalar higher order singular boundary value problems. In Chapter 1, time and space singularities are defined, three existence principles for problems with time singularities and two for problems with space singularities are formulated and proved. Chapter 2 presents existence results for focal problems with a time singularity and for focal problems having space singularities in all variables. Chapters 3–6 investigate other higher order boundary value problems having only space singularities which appear most frequently in literature. They provide existence results for (n,p) – problems, conjugate problems, Sturm-Liouville problems and for Lidstone problems.

Part II consists of Chapters 7–11 and deals with scalar second order singular boundary value problems with one-dimensional ϕ – Laplacian. The exposition is focused mainly on Dirichlet and periodic problems which are considered in Chapter 7 and 8, respectively. Section 7.1 is fundamental for further investigation. The operator representation of the regular Dirichlet problem with ϕ – Laplacian is derived here and the methods of a priori estimates and lower and upper functions are developed. In Sections 7.2–7.4 three existence principles are presented. These principles together with the principles of Chapter 1 are then specialized to important particular cases and existence theorems and criteria extending and supplementing earlier results are obtained. Section 7.2 deals with time singularities, Section 7.3 with space singularities and Section 7.4 with mixed singularities, i.e. both time and space ones. In Chapter 8 we consider the existence of periodic solutions. We start with the method of lower and upper functions and with its relationship to the Leray-Schauder degree in Section 8.1. Section 8.2 is devoted to problems with a nonlinearity having an attractive singularity in its first space variable. Sections 8.3 and 8.4 deal with problems with strong and weak repulsive space singularities, respectively. An existence theorem for periodic problems with time singularities is given in the last section of Chapter 8. In Chapter 9 we study two singular mixed boundary value problems. The latter arises in the theory of shallow membrane caps and we discuss its solvability in dependence on parameters which appear in the differential equation. In Chapter 10 we treat problems which may have singularities in space variables. Boundary conditions under discussion are generally nonlinear and nonlocal. We present general principles for solvability of regular and singular nonlocal problems and show some of their applications. Chapter 11 is devoted to a class of problems having singularities in space variables. Implementation of a parameter into the equation enables us to prove solvability of problems with three independent (generally nonlocal) boundary conditions. We deliver an existence principle and its specialization to the problem with given maximal values for positive solutions.

Appendices give an overview of some basic classical theorems and assertions which are used in Chapters 1–11. Appendix A presents several criteria for uniform integrability or equicontinuity. Some convergence theorems are given in Appendix B. In particular, we recall the Lebesgue dominated convergence theorem, the Fatou lemma, the Vitali convergence theorem for integrable functions and the Arzelà-Ascoli theorem and the diagonalization theorem for differentiable functions. Appendix C contains the Schauder fixed point theorem, the Leray-Schauder degree theorem, the Borsuk antipodal theorem and the Fredholm type existence theorem. Appendix D collects some useful facts from half-linear analysis which are needed in Chapter 8.

List of notation

Let $J \subset \mathbb{R}$, $[a,b] \subset \mathbb{R}$, $k \in \mathbb{N}$, $p \in (1,\infty)$, $\mathcal{M} \subset \mathbb{R}^k$. Then we will write:

- $L_\infty(J)$ for the set of functions essentially bounded and (Lebesgue) measurable on J; the corresponding norm is

  $$\|u\|_\infty = \text{sup ess}\{|u(t)| : t \in J\}.$$

- $L_1(J)$ for the set of functions (Lebesgue) integrable on J; the corresponding norm is $\|u\|_1 = \int_J |u(t)|\,\mathrm{d}t$.

- $L_{loc}(J)$ for the set of functions (Lebesgue) integrable on each compact interval $I \subset J$.

- $L_p(J)$ for the set of functions whose p-th powers of modulus are integrable on J; the corresponding norm is $\|u\|_p = \left(\int_J |u(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}$.

- $C(J)$ and $C^k(J)$ for the sets of functions continuous on J and having continuous k-th derivatives on J, respectively.

- $AC(J)$ and $AC^k(J)$ for the sets of functions absolutely continuous on J and having absolutely continuous k-th derivatives on J, respectively.

- $AC_{loc}(J)$ and $AC^k_{loc}(J)$ for the sets of functions absolutely continuous on each compact interval $I \subset J$ and having absolutely continuous k-th derivatives on each compact interval $I \subset J$, respectively.

- $Car([a,b] \times \mathcal{M})$ for the set of functions satisfying the Carathéodory conditions on $[a,b] \times \mathcal{M}$. If $J \subset [a,b]$ and $J \neq \overline{J}$, then $f \in Car(J \times \mathcal{M})$ will denote that $f \in Car(I \times \mathcal{M})$ for each compact interval $I \subset J$.

If $J = [a,b]$, we will simply write $C[a,b]$ instead of $C([a,b])$ and similarly for other types of intervals and other functional sets defined above.

If $u \in L_\infty[a,b] \cap C[a,b]$, then $\max\{|u(t)| : t \in [a,b]\} = \sup \operatorname{ess}\{|u(t)| : t \in [a,b]\}$. Therefore the norms in $C[a,b]$ and $C^k[a,b]$ will be denoted by

$$\|u\|_\infty = \max\{|u(t)| : t \in [a,b]\} \quad \text{and} \quad \|u\|_{C^k} = \sum_{i=0}^{k} \|u^{(i)}\|_\infty,$$

respectively.

$\overline{\mathcal{M}}$ will denote the closure of $\mathcal{M}$, $\partial\mathcal{M}$ the boundary of $\mathcal{M}$ and $\operatorname{meas}(\mathcal{M})$ the Lebesgue measure of $\mathcal{M}$.

The symbol $\deg(\mathcal{I} - \mathcal{F}, \Omega)$ stands for the Leray-Schauder degree of $\mathcal{I} - \mathcal{F}$ with respect to Ω, where $\mathcal{I}$ denotes the identity operator.

We will say that some property holds for a.e. $t \in J$ (a.e. on J) if it is fulfilled for each $t \in J \setminus J_0$ where $\operatorname{meas}(J_0) = 0$.

Throughout this text we exploit the following basic theorems listed in Appendices:

Lebesgue dominated convergence theorem	(Theorem B.1)
Fatou lemma	(Theorem B.2)
Vitali convergence theorem	(Theorem B.3)
Arzelà-Ascoli theorem	(Theorem B.5)
Diagonalization theorem	(Theorem B.6)
Schauder fixed point theorem	(Theorem C.1)
Leray-Schauder degree theorem	(Theorem C.2)
Borsuk antipodal theorem	(Theorem C.3)
Fredholm type existence theorem	(Theorem C.5)
Sharp Poincaré inequality	(Lemma D.2)

Part I

Higher order singular problems

Consider the boundary value problem

$$u^{(n)} = f(t, u, \dots, u^{(n-1)}), \quad u \in \mathcal{B}, \tag{BVP}$$

where $n \in \mathbb{N}$, $[0,T] \subset \mathbb{R}$ and $\mathcal{B} \subset C[0,T]$. In what follows, we will investigate the solvability of problem (BVP) on the set $[0,T] \times \mathcal{A}$, where $\mathcal{A}$ is a closed subset of $\mathbb{R}^n$. If we impose some additional conditions on solutions of (BVP), for example if we search for positive or for monotonous solutions, we express this requirement in terms of the set $\mathcal{A} \neq \mathbb{R}^n$ and prove the existence of a solution u such that $(u(t), \dots, u^{(n-1)}(t)) \in \mathcal{A}$ for $t \in [0,T]$. On the other hand, if there are no additional requirements on solutions, we can assume $\mathcal{A} = \mathbb{R}^n$.

Let $\mathcal{M} \subset \mathbb{R}^n$. We say that a function f satisfies *the Carathéodory conditions* on the set $[a,b] \times \mathcal{M}$ ($f \in Car([a,b] \times \mathcal{M})$) if

(i) $f(\cdot, x_0, \dots, x_{n-1})\colon [a,b] \to \mathbb{R}$ is measurable for all $(x_0, \dots, x_{n-1}) \in \mathcal{M}$,

(ii) $f(t, \cdot, \dots, \cdot)\colon \mathcal{M} \to \mathbb{R}$ is continuous for a.e. $t \in [a,b]$,

(iii) for each compact set $\mathcal{K} \subset \mathcal{M}$ there is a function $m_{\mathcal{K}} \in L_1[a,b]$ such that

$$|f(t, x_0, \dots, x_{n-1})| \le m_{\mathcal{K}}(t) \text{ for a.e. } t \in [a,b] \text{ and all } (x_0, \dots, x_{n-1}) \in \mathcal{K}.$$

If $J \subset [a,b]$ and $J \neq \overline{J}$, then $f \in Car(J \times \mathcal{M})$ means that $f \in Car(I \times \mathcal{M})$ for each compact interval $I \subset J$.

The classical existence results are based on the assumption

$$f \in Car([0,T] \times \mathcal{A}).$$

In this case we will say that problem (BVP) is *regular on* $[0,T] \times \mathcal{A}$. If $f \notin Car([0,T] \times \mathcal{A})$ we will say that problem (BVP) is *singular on* $[0,T] \times \mathcal{A}$. The research of singular problems was essentially initiated by Kiguradze in [114] and [115]. For further development see for example the monographs Agarwal [2], Agarwal and O'Regan [12], Agarwal, O'Regan and Wong [21], O'Regan [148], Kiguradze [116], Kiguradze and Shekhter [118], Mawhin [135], Rachůnková, Staněk and Tvrdý [163] and references therein.

EXAMPLE. In certain problems in fluid dynamics and boundary layer theory (see e.g. Callegari and Friedman [53], Callegari and Nachman [54], [55]) the second order differential equation

$$u'' + \frac{\psi(t)}{u^\lambda} = 0$$

arose. Here $\lambda \in (0, \infty)$ and $\psi \in C(0,1)$, $\psi \notin L_1[0,1]$. This equation is known as the generalized Emden-Fowler equation. Its solvability with the Dirichlet boundary conditions

$$u(0) = u(1) = 0$$

was investigated by Taliaferro [190] in 1979 and subsequently by many other authors. Since solutions positive on $(0,1)$ have been searched for, this Dirichlet problem has been studied on the set $[0,1] \times \mathcal{A}$ with $\mathcal{A} = [0, \infty)$. We can see that $f(t,x) = \psi(t)\, x^{-\lambda}$ does not fulfil conditions (ii) and (iii) with $[a,b] = [0,1]$ and $\mathcal{M} = [0,\infty)$. Hence the above problem is singular on $[0,1] \times [0,\infty)$.

EXAMPLE. Consider the fourth order degenerate parabolic equation

$$U_t + (|U|^\mu\, U_{yyy})_y = 0$$

which arises in droplets and thin viscous flows models (see e.g. Bernis, Peletier and Williams [39] and Bertozzi, Brenner, Dupont and Kadanoff [40]). The source-type solutions of this equation have the form

$$U(y,t) = t^{-b}\, u(y\, t^{-b}), \quad b = \frac{1}{\mu + 4},$$

which leads to the study of the third order ordinary differential equation

$$u''' = b\, t\, u^{1-\mu}$$

on $[-1,1]$. We see that $f(t,x) = b\, t\, x^{1-\mu}$ is singular on $[-1,1] \times [0,\infty)$ if $\mu > 1$.

EXAMPLE. Similarly to the previous example, the sixth order degenerate equation

$$U_t - (|U|^\mu\, U_{yyyyy})_y = 0$$

which arises in semiconductor models (Bernis [37], [38]) leads to the fifth order ordinary differential equation

$$-u^{(5)} = \frac{t}{u^{\lambda}}$$

which is singular for $\lambda > 0$.

Example. Consider the nonlinear elliptic partial differential equation

$$\Delta u + g(r, u) = 0 \quad \text{on} \quad \Omega, \quad u|_{\Gamma} = 0,$$

where Δ is the Laplace operator, Ω is the open unit disk in $\mathbb{R}^n$ centered at the origin, Γ is its boundary and r is the radial distance from the origin. When searching for positive radially symmetric solutions to this problem, we get the singular problem of the form

$$u'' + \frac{n-1}{t} u' + g(t, u) = 0, \quad u'(0) = 0, \; u(1) = 0.$$

See Berestycki, Lions and Peletier [36] or Gidas, Ni and Nirenberg [96].

Example. Assume $f \in Car([0, \infty) \times \mathbb{R})$ and consider the regular boundary value problem

$$u'' = f(t, u), \quad u(1) = 0, \; u(\infty) = 0$$

on the infinite interval $[1, \infty)$. We can transform this problem to a finite interval, e.g. on $[0, 1]$. Then we get the singular problem of the form

$$v'' + \frac{2}{t} v' = \frac{1}{t^4} f(\frac{1}{t}, v), \quad v(0) = v(1) = 0.$$

Chapter 1

Existence Principles for Singular Problems

1.1 Formulation of the problem

For $n \in \mathbb{N}$, $[0,T] \subset \mathbb{R}$, $i \in \{0,1,\dots,n-1\}$ and a closed set $\mathcal{B} \subset C^i[0,T]$ consider the boundary value problem

$$u^{(n)} = f(t,u,\dots,u^{(n-1)}), \tag{1.1}$$

$$u \in \mathcal{B}. \tag{1.2}$$

A decision concerning solvability for singular boundary value problems requires an exact definition of a solution to such problems. Here, we will work with the same definition of a solution both for the regular problems and for the singular ones.

Definition 1.1. A function $u \in AC^{n-1}[0,T] \cap \mathcal{B}$ is called a *solution of problem* (1.1), (1.2), if it satisfies the equality

$$u^{(n)}(t) = f(t,u(t),\dots,u^{(n-1)}(t))$$

for a.e. $t \in [0,T]$. If we investigate problem (1.1), (1.2) on $[0,T] \times \mathcal{A}$ where $\mathcal{A} \neq \mathbb{R}^n$, we moreover require $(u(t),\dots,u^{(n-1)}(t)) \in \mathcal{A}$ for $t \in [0,T]$.

In literature an alternative approach to solvability of singular problems can be found. In that approach, authors search for solutions which are defined as functions whose $(n-1)$-st derivatives can have discontinuities at some points in $[0,T]$. Here, we will call them w-solutions. According to Kiguradze [115] or Agarwal and O'Regan [12] we define them as follows. In contrast to our starting setting, to define w-solutions we assume (in general) that $\mathcal{B}$ is a closed subset in $C^i[0,T]$, where $i \in \{0,1,\dots,n-2\}$.

Definition 1.2. We say that u is a *w-solution of problem* (1.1), (1.2), if there exists a finite number of points $t_\nu \in [0,T]$, $\nu = 1,2,\dots,r$, such that if we denote $J = [0,T] \setminus \{t_\nu\}_{\nu=1}^r$, then $u \in C^{n-2}[0,T] \cap AC_{loc}^{n-1}(J) \cap \mathcal{B}$ satisfies

$$u^{(n)}(t) = f(t, u(t), \dots, u^{(n-1)}(t)) \text{ for a.e. } t \in [0,T].$$

If $\mathcal{A} \neq \mathbb{R}^n$ we require $(u(t), \dots, u^{(n-1)}(t)) \in \mathcal{A}$ for $t \in J$.

Clearly each solution is a w-solution and each w-solution which belongs to $AC^{n-1}[0,T]$ is a solution. While only the existence of w-solutions was proved in the works cited above, our main goal is to prove the existence of solutions. However, in some cases, we first find w-solutions and then prove that they are also solutions.

When studying the singular problem (1.1), (1.2), we will focus our attention on two types of singularities of the function f:

Let $J \subset [0,T]$. We say that $f : J \times \mathcal{A} \to \mathbb{R}$ has singularities in its *time variable* t, if $J \neq \overline{J} = [0,T]$ and

$$f \in Car(J \times \mathcal{A}) \quad \text{and} \quad f \notin Car([0,T] \times \mathcal{A}). \tag{1.3}$$

Let $\mathcal{D} \subset \mathcal{A}$. We say that $f : [0,T] \times \mathcal{D} \to \mathbb{R}$ has singularities in its *space variables* $x_0, x_1, \dots, x_{n-1}$, if $\mathcal{D} \neq \overline{\mathcal{D}} = \mathcal{A}$ and

$$f \in Car([0,T] \times \mathcal{D}) \quad \text{and} \quad f \notin Car([0,T] \times \mathcal{A}). \tag{1.4}$$

We will study particular cases of (1.3) and (1.4), which will be described in Section 1.2 and Section 1.3, respectively.

1.2 Singularities in time variable

A function f has a singularity in its time variable t (in short *a time singularity*) if, roughly speaking, f is not integrable on $[0,T]$. Let us define it more precisely. Let $k \in \mathbb{N}$, $t_i \in [0,T], i = 1, \dots, k$, $J = [0,T] \setminus \{t_1, t_2 \dots, t_k\}$ and let $f \in Car(J \times \mathcal{A})$. Assume that for each $i \in \{1, \dots, k\}$ there exists $(x_0, \dots, x_{n-1}) \in \mathcal{A}$ such that

$$\begin{cases} \displaystyle\int_{t_i}^{t_i+\varepsilon} |f(t, x_0, \dots, x_{n-1})| \, \mathrm{d}t = \infty \qquad \text{or} \\ \displaystyle\int_{t_i-\varepsilon}^{t_i} |f(t, x_0, \dots, x_{n-1})| \, \mathrm{d}t = \infty \end{cases} \tag{1.5}$$

for any sufficiently small $\varepsilon > 0$. Then $f \notin Car[0,T] \times \mathcal{A})$ and f has singularities in its time variable t, namely at the values $t = t_1, \dots, t_k$. We will call $t_1, \dots, t_k$ *singular points* of f.

EXAMPLE. Let $f_i \colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \dots, k$, be continuous. Then the function

$$f(t, x_0, \dots, x_{n-1}) = \sum_{i=1}^{k} \frac{1}{t - t_i} f_i(x_0, \dots, x_{n-1}),$$

has singular points $t_1, t_2, \dots, t_k$.

To establish existence of a solution of a singular problem we usually introduce a sequence of approximate regular problems which are solvable. Solutions of these regular problems are called *approximate solutions.* Then we pass to the limit of the sequence of approximate solutions to get a solution of the original singular problem. Here we provide existence principles which contain main rules for the construction of such sequences to get either w-solutions or solutions.

Consider problem (1.1), (1.2) on $[0,T] \times \mathcal{A}$. For the sake of simplicity assume that f has only one time singularity at $t = t_0$, $t_0 \in [0,T]$. Thus

$$\begin{cases} J = [0,T] \setminus \{t_0\},\ f \in Car(J \times \mathcal{A}) \text{ satisfies one of the conditions:} \\ \text{(i)} \quad \displaystyle\int_{t_0-\varepsilon}^{t_0} |f(t, x_0, \dots, x_{n-1})| \, \mathrm{d}t = \infty, \quad t_0 \in (0,T], \\ \text{(ii)} \quad \displaystyle\int_{t_0}^{t_0+\varepsilon} |f(t, x_0, \dots, x_{n-1})| \, \mathrm{d}t = \infty, \quad t_0 \in [0,T), \\ \text{for some } (x_0, \dots, x_{n-1}) \in \mathcal{A} \text{ and each sufficiently small } \varepsilon > 0. \end{cases} \tag{1.6}$$

Further, consider a sequence of regular problems

$$u^{(n)}(t) = f_k(t, u(t), \dots, u^{(n-1)}(t)), \quad u \in \mathcal{B}, \tag{1.7}$$

where $f_k \in Car([0,T] \times \mathbb{R}^n)$, $k \in \mathbb{N}$. Solutions of problem (1.7) are understood in the sense of Definition 1.1. The following two theorems deal with the case

$$\mathcal{B} \text{ is a closed subset in } C^{n-2}[0,T]. \tag{1.8}$$

Theorem 1.3 (**First principle for time singularities**).
Let (1.6) *and* (1.8) *hold. Assume that the conditions*

$$\begin{cases} \textit{for each } k \in \mathbb{N} \textit{ and each } (x_0, \dots, x_{n-1}) \in \mathcal{A}, \\ f_k(t, x_0, \dots, x_{n-1}) = f(t, x_0, \dots, x_{n-1}) \textit{ a.e. on } [0,T] \setminus \triangle_k, \\ \textit{where } \triangle_k = (t_0 - \frac{1}{k}, t_0 + \frac{1}{k}) \cap [0,T] \end{cases} \tag{1.9}$$

and

$$\begin{cases} \textit{there exists a bounded set } \Omega \subset C^{n-1}[0,T] \textit{ such that} \\ \textit{for each } k \in \mathbb{N}, \textit{ the regular problem (1.7) has a solution} \\ u_k \in \Omega \textit{ and } (u_k(t), \dots, u_k^{(n-1)}(t)) \in \mathcal{A} \textit{ for } t \in [0,T] \end{cases} \tag{1.10}$$

are fulfilled.

Then

$$\begin{cases} \textit{there exist a function } u \in C^{n-2}[0,T] \textit{ and a subsequence} \\ \{u_{k_\ell}\} \subset \{u_k\} \textit{ such that } \lim_{\ell\to\infty} \|u_{k_\ell} - u\|_{C^{n-2}} = 0, \end{cases} \tag{1.11}$$

$$\begin{cases} \lim_{\ell\to\infty} u_{k_\ell}^{(n-1)}(t) = u^{(n-1)}(t) \textit{ locally uniformly on } J \\ \textit{and } (u(t), \dots, u^{(n-1)}(t)) \in \mathcal{A} \textit{ for } t \in J, \end{cases} \tag{1.12}$$

$$u \in AC_{loc}^{n-1}(J) \textit{ and } u \textit{ is a w-solution of problem (1.1), (1.2).} \tag{1.13}$$

Assume, moreover,

$$\begin{cases} \textit{there exist } \psi \in L_1[0,T],\ \eta > 0,\ \ell_0 \in \mathbb{N} \textit{ and } \lambda_1, \lambda_2 \in \{-1, 1\} \\ \textit{such that} \\ \qquad \lambda_1 f_{k_\ell}(t, u_{k_\ell}(t), \dots, u_{k_\ell}^{(n-1)}(t)) \ge \psi(t) \\ \textit{for all } \ell \in \mathbb{N}, \ell \ge \ell_0, \textit{ and for a.e. } t \in [t_0 - \eta, t_0) \subset [0,T] \\ \textit{provided (1.6) (i) holds} \\ \textit{and} \\ \qquad \lambda_2 f_{k_\ell}(t, u_{k_\ell}(t), \dots, u_{k_\ell}^{(n-1)}(t)) \ge \psi(t) \\ \textit{for all } \ell \in \mathbb{N}, \ell \ge \ell_0, \textit{ and for a.e. } t \in (t_0, t_0 + \eta] \subset [0,T] \\ \textit{provided (1.6) (ii) is true.} \end{cases} \tag{1.14}$$

Then $u \in AC^{n-1}[0,T]$, u *is a solution of problem* (1.1), (1.2) *and*

$$(u(t),\dots,u^{(n-1)}(t)) \in \mathcal{A} \quad \textit{for} \quad t \in [0,T].$$

Proof. *Step* 1. *Convergence of the sequence of approximate solutions.*

Condition (1.10) implies that the sequences $\{u_k^{(i)}\}, 0 \le i \le n-2$, are bounded and equicontinuous on $[0,T]$. By the Arzelà-Ascoli theorem, we see that assertion (1.11) is true and $u \in \mathcal{B} \subset C^{n-2}[0,T]$. Let $t_0 \neq 0$. Since $\{u_k^{(n-1)}\}$ is bounded on $[0,T]$, we get, due to (1.9), that for each $\tau \in [0,t_0)$ there exist $k_\tau \in \mathbb{N}$ and $h_\tau \in L_1[0,T]$ such that for each $k \ge k_\tau$

$$|f_k(s,u_k(s),\dots,u_k^{(n-1)}(s))| \le h_\tau(s) \quad \text{for a.e. } s \in [0,\tau]. \tag{1.15}$$

Hence, by virtue of (1.7), for $k \ge k_\tau$, $t_1,t_2 \in [0,\tau]$, we have

$$|u_k^{(n-1)}(t_2) - u_k^{(n-1)}(t_1)| \le \left| \int_{t_1}^{t_2} h_\tau(s)\,\mathrm{d}s \right|,$$

which implies that the sequence $\{u_k^{(n-1)}\}$ is equicontinuous on $[0,\tau]$. The same holds on $[\tau,T]$ if $\tau \in (t_0,T]$ and $t_0 \neq T$. The Arzelà-Ascoli theorem implies that for each compact subset $\mathcal{K} \subset J = [0,T] \setminus \{t_0\}$ a subsequence of $\{u_k^{(n-1)}\}$ uniformly converging to $u^{(n-1)}$ on $\mathcal{K}$ can be chosen. Therefore, using the diagonalization theorem, we can choose a subsequence $\{u_{k_\ell}\}$ satisfying both (1.11) and (1.12).

Step 2. *Convergence of the sequence of approximate nonlinearities.*

Let $\mathcal{V}_1$ be the set of all $t \in [0,T]$ such that $f(t,\cdot,\dots,\cdot) : \mathbb{R}^n \to \mathbb{R}$ is not continuous and let $\mathcal{V}_2$ be the set of all $t \in [0,T]$ such that (1.9) is not satisfied. Then $\operatorname{meas}(\mathcal{V}_1 \cup \mathcal{V}_2) = 0$. Choose an arbitrary $\tau \in [0,T] \setminus (\mathcal{V}_1 \cup \mathcal{V}_2)$. Then there exists $\ell_0 \in \mathbb{N}$ such that for $\ell \ge \ell_0$

$$f_{k_\ell}(\tau,u_{k_\ell}(\tau),\dots,u_{k_\ell}^{(n-1)}(\tau)) = f(\tau,u_{k_\ell}(\tau),\dots,u_{k_\ell}^{(n-1)}(\tau))$$

and, by (1.11) and (1.12),

$$\lim_{\ell\to\infty} f_{k_\ell}(\tau,u_{k_\ell}(\tau),\dots,u_{k_\ell}^{(n-1)}\tau)) = f(\tau,u(\tau),\dots,u^{(n-1)}(\tau)).$$

Hence,

$$\begin{cases} \lim_{\ell\to\infty} f_{k_\ell}(t,u_{k_\ell}(t),\dots,u_{k_\ell}^{(n-1)}(t) = f(t,u(t),\dots,u^{(n-1)}(t)) \\ \hfill \text{for a.e. } t \in [0,T]. \end{cases} \tag{1.16}$$

Step 3. *The function* u *is a w-solution of problem* (1.1), (1.2).

Let $t_0 \neq 0$ and $\ell \in \mathbb{N}$. Choose an arbitrary $\tau \in [0, t_0)$ and integrate the equality

$$u_{k_\ell}^{(n)}(t) = f_{k_\ell}(t, u_{k_\ell}(t), \dots, u_{k_\ell}^{(n-1)}(t)) \quad \text{for a.e. } t \in [0, T].$$

We get

$$u_{k_\ell}^{(n-1)}(\tau) = u_{k_\ell}^{(n-1)}(0) + \int_0^\tau f_{k_\ell}(s, u_{k_\ell}(s), \dots, u_{k_\ell}^{(n-1)}(s))\, \mathrm{d}s.$$

According to (1.15), (1.16) and the Lebesgue dominated convergence theorem on $[0, \tau]$ we can deduce (having in mind that τ is arbitrary) that if $t_0 \neq 0$ the limit u solves the equation

$$\begin{cases} u^{(n-1)}(t) = u^{(n-1)}(0) + \displaystyle\int_0^t f(s, u(s), \dots, u^{(n-1)}(s))\, \mathrm{d}s \\ \hfill \text{for } \ t \in [0, t_0). \end{cases} \tag{1.17}$$

Similarly, if $t_0 \neq T$ the limit u solves the equation

$$\begin{cases} u^{(n-1)}(t) = u^{(n-1)}(T) - \displaystyle\int_t^T f(s, u(s), \dots, u^{(n-1)}(s))\, \mathrm{d}s \\ \hfill \text{for } \ t \in (t_0, T]. \end{cases} \tag{1.18}$$

The equalities (1.17) and (1.18) immediately yield (1.13).

Step 4. *The function* u *is a solution of problem* (1.1), (1.2).

Assume, moreover, that (1.14) and (1.6) (i) hold. Since

$$u_{k_\ell}^{(n-1)}(t) - u_{k_\ell}^{(n-1)}(t_0 - \eta) = \int_{t_0-\eta}^t f_{k_\ell}(s, u_{k_\ell}(s), \dots, u_{k_\ell}^{(n-1)}(s))\, \mathrm{d}s$$

for $t \in (0, t_0)$, we get due to (1.10) that there is a $c \in (0, \infty)$ such that

$$\lambda_1 \int_{t_0-\eta}^{t_0} f_{k_\ell}(s, u_{k_\ell}(s), \dots, u_{k_\ell}^{(n-1)}(s))\, \mathrm{d}s \le c \tag{1.19}$$

for each $\ell \in \mathbb{N}$. By the Fatou lemma, using conditions (1.16), (1.14) and (1.19), we deduce that

$$f(t, u(t), \dots, u^{(n-1)}(t)) \in L_1[t_0 - \eta, t_0].$$

Similarly, if condition (1.6) (ii) holds, we deduce that

$$f(t,u(t),\dots,u^{(n-1)}(t)) \in L_1[t_0,t_0+\eta].$$

Hence

$$f(t,u(t),\dots,u^{(n-1)}(t)) \in L_1([t_0-\eta,t_0+\eta]\cap[0,T]).$$

Recall that by (1.12) we have $(u(t),\dots,u^{(n-1)}(t)) \in \mathcal{A}$ for $t\in J$ and, by (1.6), $f\in Car(J\times\mathcal{A})$. Further, by virtue of (1.10) and (1.11), the functions $u,u',\dots,u^{(n-2)}$ are bounded on $[0,T]$ and (1.10), (1.12) imply that $u^{(n-1)}$ is bounded on $[0,T]\setminus(t_0-\eta,t_0+\eta)$. Hence

$$f(t,u(t),\dots,u^{(n-1)}(t)) \in L_1\left([0,T]\setminus(t_0-\eta,t_0+\eta)\right),$$

which together with the above arguments yields

$$f(t,u(t),\dots,u^{(n-1)}(t)) \in L_1[0,T].$$

Therefore due to (1.17) and (1.18) we have that $u\in AC^{n-1}[0,T]$, i.e. u is a solution of problem (1.1), (1.2). Finally, since $\mathcal{A}$ is closed, we get

$$\lim_{t\to t_0}(u(t),\dots,u^{(n-1)}(t)) = (u(t_0),\dots,u^{(n-1)}(t_0)) \in \mathcal{A}. \qquad \square$$

Theorem 1.4 (**Second principle for time singularities**).
Let (1.6), (1.8), (1.9) and (1.10) hold. Assume that

$$\begin{cases} \textit{there exist } \psi\in L_1[0,T],\ \eta>0 \textit{ and } \lambda_1,\lambda_2\in\{-1,1\} \textit{ such that} \\ \qquad \lambda_1\, f_{k_\ell}(t,u_{k_\ell}(t),\dots,u_{k_\ell}^{(n-1)}(t)) \operatorname{sign} u_{k_\ell}^{(n-1)}(t) \ge \psi(t) \\ \textit{for all } \ell\in\mathbb{N} \textit{ and for a.e. } t\in[t_0-\eta,t_0)\subset[0,T] \\ \textit{provided } (1.6)\,(\mathrm{i}) \textit{ holds} \\ \textit{and} \\ \qquad \lambda_2\, f_{k_\ell}(t,u_{k_\ell}(t),\dots,u_{k_\ell}^{(n-1)}(t)) \operatorname{sign} u_{k_\ell}^{(n-1)}(t) \ge \psi(t) \\ \textit{for all } \ell\in\mathbb{N} \textit{ and for a.e. } t\in(t_0,t_0+\eta]\subset[0,T] \\ \textit{provided } (1.6)\,(\mathrm{ii}) \textit{ is true.} \end{cases} \tag{1.20}$$

Then there exists a function $u\in AC^{n-1}[0,T]$ satisfying (1.11) and (1.12) which is a solution of problem (1.1), (1.2) and $(u(t),\dots,u^{(n-1)}(t))\in\mathcal{A}$ for $t\in[0,T]$.

Proof. *Steps* 1–3 are the same as in the proof of Theorem 1.3 and guarantee the existence of a w-solution u of problem (1.1), (1.2).

Step 4. Arguing as in *Step* 4 of the proof of Theorem 1.3 we see that to show $u \in AC^{n-1}[0,T]$ it suffices to prove $f(t,u(t),\dots,u^{(n-1)}(t)) \in L_1(I_0)$, where $I_0 = [t_0-\eta, t_0+\eta] \cap [0,T]$. Put $\mathcal{M} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$, where

$$\begin{aligned}
\mathcal{V}_1 &= \{t \in I_0 : f(t,\cdot,\dots,\cdot) : \mathbb{R}^n \to \mathbb{R} \ \text{ is not continuous}\},\\
\mathcal{V}_2 &= \{t \in I_0 : t \ \text{ is an isolated zero of } \ u^{(n-1)}\},\\
\mathcal{V}_3 &= \{t \in I_0 : u^{(n)}(t) \ \text{ does not exist or (1.1) is not fulfilled}\}.
\end{aligned}$$

Then $\operatorname{meas}(\mathcal{M}) = 0$. Choose an arbitrary $s \in I_0 \setminus \mathcal{M}$, $s \neq t_0$.

a) Let $u^{(n-1)}(s) \neq 0$. Assume for example $\operatorname{sign} u^{(n-1)}(s) = 1$. Then there exists $\ell_0 \in \mathbb{N}$ such that for each $\ell \geq \ell_0$ we have $\operatorname{sign} u_{k_\ell}^{(n-1)}(s) = 1$ and so, due to (1.9), (1.11), (1.12) and $s \notin \mathcal{V}_1$,

$$\begin{cases} \lim_{\ell\to\infty} \lambda_1 f_{k_\ell}(s, u_{k_\ell}(s), \dots, u_{k_\ell}^{(n-1)}(s)) \operatorname{sign} u_{k_\ell}^{(n-1)}(s) \\ = \lambda_1 f(s, u(s), \dots, u^{(n-1)}(s)) \operatorname{sign} u^{(n-1)}(s). \end{cases} \tag{1.21}$$

If $\operatorname{sign} u^{(n-1)}(s) = -1$, we get (1.21) in the same way.

b) Let s be an accumulation point of a set of zeros of $u^{(n-1)}$. Then there exists a sequence $\{s_m\} \subset I_0$ such that $u^{(n-1)}(s_m) = 0$ and $\lim_{m\to\infty} s_m = s$. Since $u^{(n-1)}$ is continuous on $I_0 \setminus \{t_0\}$, we get $u^{(n-1)}(s) = 0$. Further,

$$\lim_{m\to\infty} \frac{u^{(n-1)}(s_m) - u^{(n-1)}(s)}{s_m - s} = 0$$

and, by virtue of $s \notin \mathcal{V}_3$, we get $0 = u^{(n)}(s) = f(s, u(s), \dots, u^{(n-1)}(s))$. Since $s \notin \mathcal{V}_1$, we have by (1.9), (1.11) and (1.12)

$$\begin{aligned}
&\lim_{\ell\to\infty} f_{k_\ell}(s, u_{k_\ell}(s), \dots, u_{k_\ell}^{(n-1)}(s)) \operatorname{sign} u_{k_\ell}^{(n-1)}(s)\\
&\quad = f(s, u(s), \dots, u^{(n-1)}(s)) \lim_{\ell\to\infty} \operatorname{sign} u_{k_\ell}^{(n-1)}(s) = 0.
\end{aligned}$$

So, we have proved that (1.21) is valid for a.e. $s \in I_0$.

Assume that (1.6) (i) holds and $t_0 - \eta \geq 0$. Then, by (1.10), there exist $c > 0$ and $\ell_0 \in \mathbb{N}$ such that for each $\ell \geq \ell_0$

$$\int_{t_0-\eta}^{t_0} \lambda_1\, f_{k_\ell}(s, u_{k_\ell}(s), \dots, u_{k_\ell}^{(n-1)}(s))\ \mathrm{sign}\ u_{k_\ell}^{(n-1)}(s)\, \mathrm{d}s$$
$$= \lambda_1 \int_{t_0-\eta}^{t_0} |u_{k_\ell}^{(n-1)}(s)|'\, \mathrm{d}s = \lambda_1\, (|u_{k_\ell}^{(n-1)}(t_0)| - |u_{k_\ell}^{(n-1)}(t_0-\eta)|) \leq c,$$

and hence, due to (1.20) and (1.21), we can use the Fatou lemma to deduce that

$$\lambda_1\, f(t, u(t), \dots, u^{(n-1)}(t))\ \mathrm{sign}\ u^{(n-1)}(t) \in L_1[t_0-\eta, t_0],$$

which yields $f(t, u(t), \dots, u^{(n-1)}(t)) \in L_1[t_0-\eta, t_0]$. Similarly, if (1.6) (ii) holds and $t_0 + \eta \leq T$, we deduce that $f(t, u(t), \dots, u^{(n-1)}(t)) \in L_1[t_0, t_0+\eta]$. □

Now, we will consider the boundary conditions (1.2) which are characterized by the set $\mathcal{B}$, where

$$\mathcal{B} \text{ is a closed subset in } C^{n-1}[0,T]. \tag{1.22}$$

Theorem 1.5 (**Third principle for time singularities**).
Let (1.6), (1.9), (1.10) and (1.22) hold. Assume that

$$\{u_k^{(n-1)}\} \ \ \textit{is equicontinuous at}\ \ t_0. \tag{1.23}$$

Then there exist a function $u \in \overline{\Omega}$ and a subsequence $\{u_{k_\ell}\} \subset \{u_k\}$ such that $\lim_{\ell\to\infty} \|u_{k_\ell} - u\|_{C^{n-1}} = 0$, $(u(t), \dots, u^{(n-1)}(t)) \in \mathcal{A}$ for $t \in [0,T]$ and $u \in C^{n-1}[0,T]$ is a w-solution of problem (1.1), (1.2).

If, in addition, (1.20) holds, then $u \in AC^{n-1}[0,T]$, i.e. u is a solution of problem (1.1), (1.2).

Proof. *Step* 1. *Convergence of the sequence of approximate solutions* $\{u_k\}$.

By (1.10) there is a $c > 0$ such that

$$\|u_k\|_{C^{n-1}} \leq c \quad \text{for each} \ \ k \in \mathbb{N}. \tag{1.24}$$

This implies that sequences $\{u_k^{(i)}\}, 0 \leq i \leq n-2$, are equicontinuous on $[0,T]$. Let us prove that $\{u_k^{n-1}\}$ is also equicontinuous on $[0,T]$. Choose an arbitrary $\varepsilon > 0$. By (1.23) we can find $\delta_0 > 0$ such that for each $k \in \mathbb{N}$ and each $t \in [t_0 - \delta_0, t_0 + \delta_0] \cap [0, T]$ the inequality

$$|u_k^{(n-1)}(t) - u_k^{(n-1)}(t_0)| < \varepsilon$$

holds. Therefore, for each $t_1, t_2 \in [t_0 - \delta_0, t_0 + \delta_0] \cap [0, T]$ we have

$$|u_k^{(n-1)}(t_1) - u_k^{(n-1)}(t_2)| < 2\varepsilon. \tag{1.25}$$

Now, let $t_1, t_2 \in \mathcal{K}$ where $\mathcal{K} = [0, T] \setminus (t_0 - \delta_0, t_0 + \delta_0)$. Put

$$h(t) = \sup\{|f(t, x_0, \dots, x_{n-1})| : |x_i| \le c,\ i = 0, \dots, n-1\}.$$

Then $h \in L_1(\mathcal{K})$ and we can find $\delta_1 > 0$ such that

$$|t_1 - t_2| < \delta_1 \Longrightarrow \left| \int_{t_1}^{t_2} h(t)\,\mathrm{d}t \right| < \varepsilon.$$

By (1.24) we have $|f_k(t, u_k(t), \dots, u_k^{(n-1)}(t))| \le h(t)$ a.e. on $\mathcal{K}$ for each sufficiently large $k \in \mathbb{N}$. Hence we get

$$|t_1 - t_2| < \delta_1 \Longrightarrow |u_k^{(n-1)}(t_1) - u_k^{(n-1)}(t_2)| < \varepsilon. \tag{1.26}$$

Finally, let $t_1 \in (t_0 - \delta_0, t_0 + \delta_0) \cap [0, T]$, $t_2 \in \mathcal{K}$, $t_2 > t_0 + \delta_0$. Put $\delta = \min\{\delta_0, \delta_1\}$ and assume that $|t_1 - t_2| < \delta$. Then, by (1.25) and (1.26), $|u_k^{(n-1)}(t_1) - u_k^{(n-1)}(t_2)| < 3\varepsilon$. For $t_2 < t_0 - \delta_0$ we argue similarly. So, we have proved that $\{u_k^{(n-1)}\}$ is equicontinuous on $[0, T]$. By the Arzelà-Ascoli theorem there exists a function $u \in \overline{\Omega}$ and a subsequence $\{u_{k_\ell}\} \subset \{u_k\}$ such that

$$\lim_{\ell \to \infty} \|u_{k_\ell} - u\|_{C^{n-1}} = 0 \quad \text{and} \quad (u(t), \dots, u^{(n-1)}(t)) \in \mathcal{A} \text{ for } t \in [0, T].$$

Moreover, $u \in \mathcal{B} \subset C^{n-1}[0, T]$ and, by Theorem 1.3, u is a w-solution of problem (1.1), (1.2).

Step 2. If we assume, in addition, that (1.20) holds, then to prove that $u \in AC^{n-1}[0, T]$ we can argue as in *Step* 4 of the proof of Theorem 1.4. □

1.3 Singularities in space variables

A function f has a singularity in one of its *space variables* (in short *a space singularity*) if f is not continuous in this variable on a region where f is studied. Motivated by the equation

$$u'' + \psi(t)\, u^{-\lambda} = 0,$$

where $\lambda \in (0,\infty)$, we will consider the following case of discontinuity. Let $\mathcal{A}_i \subset \mathbb{R}$ be a closed interval and let $c_i \in \mathcal{A}_i, \mathcal{D}_i = \mathcal{A}_i \setminus \{c_i\}, i = 0,1,\ldots,n-1$. Let us choose $j \in \{0,1,\ldots,n-1\}$ and assume that

$$\begin{cases} \limsup\limits_{x_j \to c_j,\, x_j \in \mathcal{D}_j} |f(t,x_0,\ldots,x_j,\ldots,x_{n-1})| = \infty \quad \text{for a.e. } t \in [0,T] \\ \text{and for some } x_i \in \mathcal{D}_i,\ i = 0,1,\ldots,n-1,\ i \neq j. \end{cases} \tag{1.27}$$

If we put $\mathcal{A} = \mathcal{A}_0 \times \cdots \times \mathcal{A}_{n-1}$, we see that f is not continuous on $\mathcal{A}$ (for a.e. $t \in [0,T]$). Consequently, f has a singularity in its space variable x_j, namely at the value c_j. Let u be a solution of (1.1), (1.2) and let a point $t_u \in [0,T]$ be such that $u^{(j)}(t_u) = c_j$. Then t_u is called *a singular point corresponding to the solution* u. Now, let u be a w-solution of (1.1), (1.2). Assume that a point $t_u \in [0,T]$ is such that $u^{(n-1)}(t_u)$ does not exist or $u^{(j)}(t_u) = c_j$. Then t_u is called *a singular point corresponding to the w-solution* u.

Example. Let $\alpha \in (0,\infty), h_1, h_2, h_3 \in L_1[0,T], h_2 \neq 0, h_3 \neq 0$ a.e. on $[0,T]$. Consider the Dirichlet problem

$$u'' + h_1(t) + \frac{h_2(t)}{u(t)} + \frac{h_3(t)}{|u'(t)|^\alpha} = 0, \quad u(0) = u(T) = 0. \tag{1.28}$$

Let u be a solution of (1.28). Then 0 and T are singular points corresponding to u. Moreover, there exists at least one point $t_u \in (0,T)$ satisfying $u'(t_u) = 0$, which means that t_u is also a singular point corresponding to u. Note that (in contrast to the points 0 and T) we do not know the location of t_u in $(0,T)$.

In accordance with this example, we will distinguish two types of singular points corresponding to solutions or to w-solutions: *singular points of type I*

, where we know their location in $[0, T]$, and *singular points of type II* whose location is not known.

Similarly to Section 1.2 we will establish sufficient conditions for approximate sequences of regular problems and of their solutions. Using the properties of those approximate solutions we will pass to a limit thus obtaining a solution or a w-solution of the original singular problem (1.1), (1.2). Let $\mathcal{A}_i \subset \mathbb{R}, i = 0, \ldots, n-1$, be closed intervals and let $\mathcal{A} = \mathcal{A}_0 \times \cdots \times \mathcal{A}_{n-1}$. Consider problem (1.1), (1.2) on $[0, T] \times \mathcal{A}$. Denote

$$\mathcal{D}_i = \mathcal{A}_i \setminus \{c_i\},\ i = 0, \ldots, n-1.$$

First, we will assume that f has one singularity at each x_i, namely at the values $c_i \in \mathcal{A}_i$, $i = 0, \ldots, n-2$. Hence, we assume

$$\begin{cases} \mathcal{D} = \mathcal{D}_0 \times \cdots \times \mathcal{D}_{n-2} \times \mathcal{A}_{n-1}, \\ f \in Car([0,T] \times \mathcal{D}) \text{ satisfies (1.27) for } j = 0, \ldots, n-2. \end{cases} \tag{1.29}$$

In the next two theorems we work with the notion of uniform integrability which can be find in Appendix A.

Theorem 1.6 (**First principle for space singularities**).
(i) *Let (1.8), (1.10) and (1.29) hold. Assume that*

$$\begin{cases} \text{for each } k \in \mathbb{N}, \text{ for a.e. } t \in [0,T] \text{ and each } (x_0, \ldots, x_{n-1}) \in \mathcal{D} \\ f_k(t, x_0, \ldots, x_{n-1}) = f(t, x_0, \ldots, x_{n-1}) \\ \text{ if } |x_i - c_i| \geq \frac{1}{k},\ 0 \leq i \leq n-1. \end{cases} \tag{1.30}$$

Then assertion (1.11) is valid.

(ii) If, moreover, the set of singular points

$$\mathcal{S} = \left\{ s \in [0,T] \colon u^{(i)}(s) = c_i \ \text{ for } i \in \{0, \ldots, n-2\} \right\} \quad \text{is finite,}$$

then assertion (1.12) is valid for $J = [0,T] \setminus \mathcal{S}$ and if

$$\begin{cases} \text{the sequence } \ \{f_{k_\ell}(t, u_{k_\ell}(t), \ldots, u_{k_\ell}^{(n-1)}(t))\} \\ \text{is uniformly integrable on each interval } [a,b] \subset J, \end{cases} \tag{1.31}$$

then $u \in AC^{n-1}_{loc}(J)$ *is a w-solution of problem* (1.1), (1.2).

(iii) If, in addition, there exists a function $\psi \in L_1[0,T]$ *such that*

$$f_{k_\ell}(t, u_{k_\ell}(t), \dots, u^{(n-1)}_{k_\ell}(t)) \ge \psi(t) \quad \textit{for a.e.} \quad t \in [0,T] \quad \textit{and all} \quad \ell \in \mathbb{N},$$

then $u \in AC^{n-1}[0,T]$ *and* u *is a solution of problem* (1.1), (1.2).

Proof. *Step* 1. *Convergence of the sequence of approximate solutions.*

As in *Step* 1 of the proof of Theorem 1.3 we derive from (1.10) that (1.11) holds and $u \in \mathcal{B} \subset C^{n-2}[0,T]$. Assume that $\mathcal{S}$ is finite and choose an arbitrary $[a,b] \subset J$. Then there exist $k_0 \in \mathbb{N}$ and $h \in L_1[0,T]$ such that for each $k \in \mathbb{N}$, $k \ge k_0$

$$|u^{(i)}_k(t) - c_i| \ge \frac{1}{k} \quad \text{for } t \in [a,b],\ i \in \{0, \dots, n-1\}$$

and, for a.e. $t \in [a,b]$,

$$|f_k(t, u_k(t), \dots, u^{(n-1)}_k(t))| = |f(t, u_k(t), \dots, u^{(n-1)}_k(t))| \le h(t).$$

So, for each $\varepsilon > 0$, there exists $\delta > 0$ such that the implication

$$|t_2 - t_1| < \delta \Longrightarrow |u^{(n-1)}_k(t_2) - u^{(n-2)}_k(t_1)| \le \left| \int_{t_1}^{t_2} h(t)\, \mathrm{d}t \right| < \varepsilon$$

is valid for $t_1, t_2 \in [a,b]$, $k \ge k_0$. Thus the sequence $\{u^{(n-1)}_k\}$ is equicontinuous on $[a,b]$. By (1.10) the sequence $\{u^{(n-1)}_k\}$ is bounded on $[0,T]$. Using the Arzelà-Ascoli theorem and the diagonalization theorem we deduce that the subsequence $\{u_{k_\ell}\}$ in (1.11) can be chosen so that it fulfils (1.12).

Step 2. *Convergence of the sequence of approximate nonlinearities.*

Consider the set

$$\mathcal{V}_1 = \{t \in [0,T] : f(t, \cdot, \dots, \cdot) \colon \mathcal{D} \to \mathbb{R} \text{ is not continuous}\}.$$

We can see that $\operatorname{meas}(\mathcal{V}_1) = 0$. By (1.30), there exists $\mathcal{V}_2 \subset [0,T]$ such that $\operatorname{meas}(\mathcal{V}_2) = 0$ and for each $k \in \mathbb{N}$, each $t \in [0,T] \setminus \mathcal{V}_2$ and each $(x_0, \dots, x_{n-1}) \in \mathcal{D}$, the equality

$$f_k(t, x_0, \dots, x_{n-1}) = f(t, x_0, \dots, x_{n-1})$$

holds if $|x_i - c_i| \ge \frac{1}{k}$, $0 \le i \le n-1$. Denote $\mathcal{U} = \mathcal{S} \cup \mathcal{V}_1 \cup \mathcal{V}_2$ and choose an arbitrary $t \in [0,T] \setminus \mathcal{U}$. By (1.11) and (1.12) there exists $\ell_0 \in \mathbb{N}$ such that for each $\ell \in \mathbb{N}, \ell \ge \ell_0$,

$$|u^{(i)}(t) - c_i| > \frac{1}{k_\ell}, \quad |u_{k_\ell}^{(i)}(t) - c_i| \ge \frac{1}{k_\ell} \quad \text{for } i \in \{0, \dots, n-1\}.$$

According to (1.30) we have

$$f_{k_\ell}(t, u_{k_\ell}(t), \dots, u_{k_\ell}^{(n-1)}(t)) = f(t, u_{k_\ell}(t), \dots, u_{k_\ell}^{(n-1)}(t))$$

and, by (1.11), (1.12),

$$\lim_{\ell \to \infty} f_{k_\ell}(t, u_{k_\ell}(t), \dots, u_{k_\ell}^{(n-1)}(t)) = f(t, u(t), \dots, u^{(n-1)}(t)). \tag{1.32}$$

Since $\operatorname{meas}(\mathcal{U}) = 0$, equality (1.32) holds for a.e. $t \in [0,T]$.

Step 3. *The function u is a w-solution of problem* (1.1), (1.2).

Choose an arbitrary interval $[a,b] \subset J$. By virtue of (1.31) and (1.32) we can use the Vitali convergence theorem to show that

$$f(t, u(t), \dots, u^{(n-1)}(t)) \in L_1[a,b]$$

and that if we pass to the limit in the sequence

$$u_{k_\ell}^{(n-1)}(t) = u_{k_\ell}^{(n-1)}(a) + \int_a^t f_{k_\ell}(s, u_{k_\ell}(s), \dots, u_{k_\ell}^{(n-1)}(s))\, \mathrm{d}s, \quad t \in [a,b],$$

we get

$$u^{(n-1)}(t) = u^{(n-1)}(a) + \int_a^t f(s, u(s), \dots, u^{(n-1)}(s))\, \mathrm{d}s, \qquad t \in [a,b].$$

Since $[a,b] \subset J$ is an arbitrary interval, we conclude that $u \in AC_{loc}^{n-1}(J)$ satisfies equation (1.1) for a.e. $t \in [0,T]$.

Step 4. *The function u is a solution of problem* (1.1), (1.2).

Let, moreover,

$$f_{k_\ell}(t, u_{k_\ell}(t), \dots, u_{k_\ell}^{(n-1)}(t)) \ge \psi(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } \ell \in \mathbb{N}.$$

Assumption (1.10) yields the existence of $c > 0$ such that

$$\int_0^T f_{k_\ell}(t, u_{k_\ell}(t), \ldots, u_{k_\ell}^{(n-1)}(t))\mathrm{d}t = u_{k_\ell}^{(n-1)}(T) - u_{k_\ell}^{(n-1)}(0) \le c.$$

Therefore, by (1.32) and the Fatou lemma, $f(t, u(t), \ldots, u^{(n-1)}(t)) \in L_1[0,T]$ and $u \in AC^{n-1}[0,T]$. □

Now we will consider problem (1.1), (1.2) on $[0,T] \times \mathcal{A}$ provided $\mathcal{A} = \mathcal{A}_0 \times \cdots \times \mathcal{A}_{n-1}$ and f has space singularities at each x_i, namely at the values $c_i \in \mathcal{A}_i$, $i = 0, \ldots, n-1$. So, we assume $\mathcal{D}_i = \mathcal{A}_i \setminus \{c_i\}$, $i = 0, \ldots, n-1$,

$$\begin{cases} f \in Car([0,T] \times \mathcal{D}) \text{ satisfies (1.27) for } j = 0, \ldots, n-1, \\ \text{where } \mathcal{D} = \mathcal{D}_0 \times \cdots \times \mathcal{D}_{n-2} \times \mathcal{D}_{n-1}. \end{cases} \tag{1.33}$$

Theorem 1.7 (**Second principle for space singularities**).
Let (1.10), (1.22), (1.30) *and* (1.33) *hold. Assume that the sequence*

$$\{f_k(t, u_k(t), \ldots, u_k^{(n-1)}(t))\} \quad \textit{is uniformly integrable on } [0,T]. \tag{1.34}$$

Then there exist a function $u \in \overline{\Omega}$ *and a subsequence* $\{u_{k_\ell}\} \subset \{u_k\}$ *such that* $\lim_{\ell\to\infty} \|u_{k_\ell} - u\|_{C^{n-1}} = 0$ *and* $(u(t), \ldots, u^{(n-1)}(t)) \in \mathcal{A}$ *for* $t \in [0,T]$.

If, moreover, the functions $u^{(i)} - c_i$, $0 \le i \le n-1$, *have at most a finite number of zeros in* $[0,T]$, *then* $u \in AC^{n-1}[0,T]$ *is a solution of* (1.1), (1.2).

Proof. *Step* 1. *Convergence of the sequence of approximate solutions.*

Assumption (1.34) yields that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $t_1, t_2 \in [0,T]$ and each $k \in \mathbb{N}$ the implication

$$|t_2 - t_1| < \delta$$

$$\Longrightarrow |u_k^{(n-1)}(t_2) - u_k^{(n-1)}(t_1)| = \left| \int_{t_1}^{t_2} f_k(t, u_k(t), \ldots, u_k^{(n-1)}(t))\,\mathrm{d}t \right| < \varepsilon$$

is valid. Therefore the sequence $\{u_k^{(n-1)}\}$ is equicontinuous on $[0,T]$. This together with (1.10) and the Arzelà-Ascoli theorem guarantees the existence of a subsequence $\{u_{k_\ell}\}$ of $\{u_k\}$ such that

$$\lim_{\ell\to\infty} \|u_{k_\ell} - u_k\|_{C^{n-1}} = 0.$$

Since $\mathcal{A}$ is closed in $\mathbb{R}^n$ and $\mathcal{B}$ is closed in $C^{n-1}[0,T]$, we get

$$(u(t),\dots,u^{(n-1)}(t))\in\mathcal{A} \quad \text{for } \ t\in[0,T] \text{ and } u\in\mathcal{B}.$$

Step 2. As in *Step* 2 in the proof of Theorem 1.5 we get that (1.32) is valid.

Step 3. *The function* u *is a solution of problem* (1.1), (1.2).

By virtue of (1.7) we have for $\ell\in\mathbb{N}$

$$u_{k_\ell}^{(n)}(t)=f(t,u_{k_\ell}(t),\dots,u_{k_\ell}^{(n-1)}(t)) \quad \text{for a.e. } t\in[0,T]$$

and

$$u_{k_\ell}^{(n-1)}(t)=u_{k_\ell}^{(n-1)}(0)+\int_0^t f_{k_\ell}(s,u_{k_\ell}(s),\dots,u_{k_\ell}^{(n-1)}(s))\,\mathrm{d}s \quad \text{for } t\in[0,T].$$

By (1.32), (1.34) and the Vitali convergence theorem we can pass to the limit and get

$$u^{(n-1)}(t)=u^{(n-1)}(0)+\int_0^t f(s,u(s),\dots,u^{(n-1)}(s))\,\mathrm{d}s \quad \text{for } \ t\in[0,T]$$

with $f(t,u(t),\dots,u^{(n-1)}(t))\in L_1[0,T]$. Therefore $u\in AC^{n-1}[0,T]$ satisfies equation (1.1) a.e. on $[0,T]$. $\square$

All the above mentioned existence principles (Theorems 1.3–1.7) require condition (1.10) and so, in order to apply them, we need global a priori estimates for all approximate solutions u_k and for all their derivatives $u_k^{(i)}, 1\le i\le n-1$. We can see in literature that local a priori estimates of $u_k^{(n-1)}$ can be sufficient for the existence of w-solutions (see e.g. Kiguradze and Shekhter [118]). However, such existence results give w-solutions with, in general, unbounded $(n-1)$ st derivative. Here, our main goal is to prove the existence of solutions. To this purpose only w-solutions whose $(n-1)$ st derivatives are bounded on the set where they are defined are useful. Therefore condition (1.10) appears in all our principles.

Bibliographical notes

The proof of Theorem 1.3 is given in Rachůnková, Staněk and Tvrdý [163]. Theorems 1.4, 1.5 and 1.6 are new. Theorem 1.7 was published in [163] and its modifications can be found in Rachůnková and Staněk [159], [160], [161].

Chapter 2

Focal problem

Focal problems have received large attention (see e.g. Agarwal [2]). This is due to the fact that these types of problems are basic, in the sense that the methods employed in their study are extendable to other types of problems. Here we will consider the n-th order differential equation with $(p, n-p)$ right focal conditions

$$u^{(i)}(0) = 0, \quad 0 \le i \le p-1, \qquad u^{(j)}(T) = 0, \quad p \le j \le n-1 \tag{2.1}$$

or with $(n-p, p)$ left focal conditions

$$u^{(i)}(0) = 0, \quad p \le i \le n-1, \qquad u^{(j)}(T) = 0, \quad 0 \le j \le p-1, \tag{2.2}$$

where $n \in \mathbb{N}$, $n \ge 2$ and $p \in \{1, \dots, n-1\}$ is fixed.

Using the existence principles of Chapter 1 we will investigate both the focal problems with time singularities and the focal problems with space singularities.

2.1 Time singularities

First, consider a $(1, n-1)$ left focal problem

$$u^{(n)} = f(t, u, \dots, u^{(n-1)}), \tag{2.3}$$

$$u^{(n-1)}(0) = 0, \qquad u^{(i)}(T) = 0, \; 0 \le i \le n-2. \tag{2.4}$$

We will assume

$$f \in Car([0,T) \times \mathbb{R}^n) \;\; \text{has a time singularity at} \;\; t = T \tag{2.5}$$

and prove the existence result for problem (2.3), (2.4) by means of Theorem 1.5 (Third principle for time singularities). Since we impose no additional conditions on solutions of (2.3), (2.4), we have

$$\mathcal{A} = \mathbb{R}^n, \quad \mathcal{B} = \{u \in C^{n-1}[0,T] : u \text{ satisfies } (2.4)\}.$$

Theorem 2.1. *Assume* (2.5) *and let*

$$\begin{cases} f(t,x_0,\dots,x_{n-1})\ \mathrm{sign}\, x_{n-1} \le -h(t)|x_{n-1}| + \displaystyle\sum_{j=0}^{n-1} h_j(t)|x_j|^{\alpha_j} \\ \text{for a.e. } t\in[0,T] \text{ and all } (x_0,\dots,x_{n-1})\in\mathbb{R}^n, \end{cases} \tag{2.6}$$

where $\alpha_j\in(0,1)$, $h_j\in L_1[0,T]$, $j=0,\dots,n-1$, *are nonnegative and* $h\in L_{loc}[0,T)$ *is nonnegative and satisfies*

$$\int_{T-\varepsilon}^{T} h(s)\,\mathrm{d}s = +\infty \quad \text{for each sufficiently small } \varepsilon>0. \tag{2.7}$$

Then problem (2.3), (2.4) *has a solution* $u\in AC^{n-1}[0,T]$.

Proof. *Step* 1. *Approximate regular problems.*

For $s,\rho\in(0,\infty)$ put

$$\chi(s,\rho) = \begin{cases} 1 & \text{if } s\in[0,\rho], \\ \dfrac{2\rho-s}{\rho} & \text{if } s\in(\rho,2\rho), \\ 0 & \text{if } s\ge 2\rho \end{cases}$$

Further, for $k\in\mathbb{N}$, $(x_0,\dots,x_{n-1})\in\mathbb{R}^n$ and for a.e. $t\in[0,T]$, define

$$f_k(t,x_0,\dots,x_{n-1}) = \begin{cases} f(t,x_0,\dots,x_{n-1}) & \text{if } t\in[0,T-\frac{1}{k}], \\ 0 & \text{if } t\in(T-\frac{1}{k},T] \end{cases} \tag{2.8}$$

and

$$g_k(t,x_0,\dots,x_{n-1}) = \chi\left(\sum_{i=0}^{n-1}|x_i|,\rho\right) f_k(t,x_0,\dots,x_{n-1}). \tag{2.9}$$

Choose a $k\in\mathbb{N}$ and consider auxiliary approximate regular problems

$$u^{(n)} = f_k(t,u,\dots,u^{(n-1)}), \quad (2.4) \tag{2.10}$$

and

$$u^{(n)} = g_k(t,u,\dots,u^{(n-1)}), \quad (2.4). \tag{2.11}$$

For a.e. $t \in [0,T]$ define

$$m_k(t) = \begin{cases} \sup\{|f(t,x_0,\dots,x_{n-1})| : \sum_{i=0}^{n-1}|x_i| \le 2\rho\} & \text{if } t \le T - \frac{1}{k}, \\ 0 & \text{if } t > T - \frac{1}{k}. \end{cases}$$

Then $m_k \in L_1[0,T]$ and $g_k(t,x_0,\dots,x_{n-1})| \le m_k(t)$ for a.e. $t \in [0,T]$. Since the homogeneous problem $u^{(n)} = 0$, (2.4) has only the trivial solution, we get by the Fredholm type existence theorem that problem (2.11) has a solution $u_k \in AC^{(n-1)}[0,T]$.

Step 2. Estimates of approximate solutions u_k .

Let us fix $k \in \mathbb{N}$ and assume

$$\max\{|u_k^{(n-1)}(t)| : t \in [0,T]\} = |u_k^{(n-1)}(b)| = r > 0.$$

By condition (2.4), we have $b \in (0,T]$ and we can find $a \in [0,b)$ such that

$$|u_k^{(n-1)}(a)| = 0 \text{ and } |u_k^{(n-1)}(t)| > 0 \text{ for } t \in (a,b].$$

Since $u_k^{(n-1)}(t) = u_k^{(n-1)}(T - \frac{1}{k})$ for $t \in [T - \frac{1}{k}, T]$, we can assume that $b \le T - \frac{1}{k}$. By virtue of assumption (2.6) we get for a.e. $t \in [a,b]$

$$\begin{aligned} &u_k^{(n)}(t) \operatorname{sign} u_k^{(n-1)}(t) \\ &\quad = \chi\left(\sum_{i=0}^{n-1}|u_k^{(i)}(t)|, \rho\right) f(t,u_k(t),\dots,u_k^{(n-1)}(t)) \operatorname{sign} u_k^{(n-1)}(t) \\ &\quad \le \chi\left(\sum_{i=0}^{n-1}|u_k^{(i)}(t)|, \rho\right) \sum_{j=0}^{n-1} h_j(t)\,|u_k^{(j)}(t)|^{\alpha_j} \le \sum_{j=0}^{n-1} h_j(t)\,|u_k^{(j)}(t)|^{\alpha_j}, \end{aligned}$$

and hence

$$|u_k^{(n-1)}(t)|' \le \sum_{j=0}^{n-1} h_j(t)\,|u_k^{(j)}(t)|^{\alpha_j}. \tag{2.12}$$

Conditions (2.4) yield $\|u_k^{(j)}\|_\infty \le r\,T^{n-j-1}$, $j = 0,\dots,n-2$. Integrating inequality (2.12) over $[a,b]$ we obtain

$$r = |u_k^{(n-1)}(b)| \le \sum_{j=0}^{n-1} T^{\alpha_j(n-j-1)}\, r^{\alpha_j} \int_0^T h_j(t)\,\mathrm{d}t$$

and

$$1 \le \sum_{j=0}^{n-1} T^{\alpha_j(n-j-1)}\, r^{\alpha_j - 1} \|h_j\|_1 =: F(r). \tag{2.13}$$

We have $\lim_{x\to\infty} F(x) = 0$, which implies the existence of $r^* > 0$ such that $F(x) < 1$ for all $x \ge r^*$. Therefore, by (2.13), the estimate $r < r^*$ must be true. Since r^* does not depend on u_k (but just on T, h_j, α_j), we get

$$\|u_k\|_{C^{n-1}} < r^* \sum_{j=0}^{n-1} T^{n-j-1} \quad \text{for each} \quad k \in \mathbb{N}.$$

If we define

$$\rho = r^* \sum_{j=0}^{n-1} T^{n-j-1} \quad \text{and} \quad \Omega = \{x \in C^{n-1}[0,T] : \|x\|_{C^{n-1}} \le \rho\},$$

we see that u_k is a solution of (2.10) and $u_k \in \Omega$ for each $k \in \mathbb{N}$. We have proved that conditions (1.9) and (1.10) of Theorem 1.5 are valid.

Step 3. Properties of approximate solutions.

According to (2.6) and (2.8) we get for a.e. $t \in [0, T - \frac{1}{k}]$

$$\begin{aligned} &f_k(t, u_k(t), \dots, u_k^{(n-1)}(t)) \operatorname{sign} u_k^{(n-1)}(t) \\ &\qquad \le \sum_{j=0}^{n-1} h_j(t)\, |u_k^{(j)}(t)|^{\alpha_j} < (\rho+1) \sum_{j=0}^{n-1} h_j(t). \end{aligned}$$

Put

$$\psi(t) = -(\rho+1) \sum_{j=1}^{n-1} h_j(t) \quad \text{for a.e.} \quad t \in [0,T].$$

Then $\psi \in L_1[0,T]$, $\psi \le 0$ a.e. on $[0,T]$, and

$$\begin{cases} -f_k(t, u_k(t), \dots, u_k^{(n-1)}(t)) \operatorname{sign} u_k^{(n-1)}(t) \ge \psi(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for a.e. } t \in [0,T]. \end{cases} \tag{2.14}$$

Due to (2.7), condition (1.6)(i) with $t_0 = T$ is satisfied.

Put $\lambda_1 = -1$ and choose an arbitrary $\eta \in (0,T)$. Then, by (2.14), we get (1.20). Moreover condition (2.4) yields (1.22).

Now, let us put $v_k(t) = u_k^{(n-1)}(t)$ for $t \in [0,T]$. Then for each $k \in \mathbb{N}$, $k > \frac{1}{\eta}$, the function v_k satisfies (A.19) with $h^* = 0$ a.e. on $[T - \frac{1}{k}, T]$. Since $u_k \in \Omega$, we can find $\beta_0 \in (0,\rho)$ such that v_k fulfils condition (A.17). By (2.6) we get (A.18), where $g^*(t) = (\rho+1)\sum_{j=0}^{n-1} h_j(t)$. Hence, by Criterion A.11, the sequence $\{v_k\}$ is equicontinuous at T from the left. Therefore $\{u_k^{(n-1)}\}$ satisfies (1.23) with $t_0 = T$ and, by Theorem 1.5, there exists a solution $u \in AC^{n-1}[0,T]$ of problem (2.3), (2.4). □

Example. Let $c \in \mathbb{R}, \alpha \in [1,\infty)$. Then the function

$$f(t, x_0, \dots, x_{n-1}) = -\frac{x_{n-1}}{t^\alpha} + \frac{c}{\sqrt{t}} \sum_{j=0}^{n-1} x_j^{\frac{2}{3}}$$

satisfies (2.5) and (2.6), where $h_j(t) = \frac{|c|}{\sqrt{t}}$, $h(t) = \frac{1}{t^\alpha}$, $\alpha_j = \frac{2}{3}$ for $j = 0, \dots, n-1$. Therefore the corresponding problem (2.3), (2.4) has a solution $u \in AC^{n-1}[0,T]$.

2.2 Space singularities

Let $\mathbb{R}_- = (-\infty, 0)$ and $\mathbb{R}_+ = (0,\infty)$. We study the singular $(p, n-p)$ right focal problem

$$(-1)^{n-p}\, u^{(n)} = f(t, u, \dots, u^{(n-1)}), \tag{2.15}$$

$$u^{(i)}(0) = 0,\ 0 \le i \le p-1, \quad u^{(j)}(T) = 0,\ p \le j \le n-1, \tag{2.16}$$

where $f \in Car([0,T] \times \mathcal{D})$ with

$$\mathcal{D} = \begin{cases} \underbrace{\mathbb{R}_+^{p+1} \times \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_- \times \cdots \times \mathbb{R}_+}_{n} & \text{if } n-p \text{ is odd,} \\ \underbrace{\mathbb{R}_+^{p+1} \times \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_- \times \cdots \times \mathbb{R}_-}_{n} & \text{if } n-p \text{ is even} \end{cases}$$

and f may be singular at the value 0 of any of its space variables. Notice that if f is positive then the singular points corresponding to the solutions of problem (2.15), (2.16) are of type I. The Green function of problem $u^{(n)}=0$, (2.16) is presented in Agarwal [1], Agarwal, O'Regan and Usmani [23], [24] and Agarwal, O'Regan and Wong [21].

We introduce the following assumptions:

$$\begin{cases} f\in Car([0,T]\times\mathcal{D}) \text{ and there exist positive constants } a,\, r \\ \text{such that} \\ \qquad a\,(T-t)^r \le f(t,x_0,\dots,x_{n-1}) \\ \text{for a.e. } t\in[0,T] \text{ and each } (x_0,\dots,x_{n-1})\in\mathcal{D}, \end{cases} \tag{2.17}$$

$$\begin{cases} \qquad f(t,x_0,\dots,x_{n-1}) \le h\Big(t,\sum_{j=0}^{n-1}|x_j|\Big)+\sum_{j=0}^{n-1}\omega_j(|x_j|) \\ \text{for a.e. } t\in[0,T] \text{ and each } (x_0,\dots,x_{n-1})\in\mathcal{D}, \quad \text{where} \\ h\in Car([0,T]\times[0,\infty)) \text{ is positive and nondecreasing} \\ \text{in the second variable,} \\ \omega_j:\mathbb{R}_+\to\mathbb{R}_+ \text{ is nonincreasing for } 0\le j\le n-1, \\ \qquad \limsup_{v\to\infty}\dfrac{1}{v}\displaystyle\int_0^T h(t,V\,v)\,\mathrm{d}t<1, \text{ where } V=\begin{cases}\frac{T^n-1}{T-1} & \text{if } T\ne 1,\\ n & \text{if } T=1,\end{cases} \\ \text{and} \\ \qquad \displaystyle\int_0^1 \omega_j(t^{r+n-j})\,\mathrm{d}t<\infty \quad \text{for } 0\le j\le n-1. \end{cases} \tag{2.18}$$

Substituting $t=T-s$ in (2.15), (2.16), we get the singular $(n-p,p)$ left focal problem

$$(-1)^p\,u^{(n)}=\widetilde{f}(s,u,\dots,u^{(n-1)}), \tag{2.19}$$

$$u^{(i)}(0)=0,\ p\le i\le n-1, \quad u^{(j)}(T)=0,\ 0\le j\le p-1, \tag{2.20}$$

where $\widetilde{f} \in Car([0,T] \times \mathcal{D}_*)$ fulfils

$$\widetilde{f}(t, x_0, x_1, \dots, x_{n-1}) = f(T-t, x_0, -x_1, \dots, (-1)^{n-1} x_{n-1})$$

for a.e. $t \in [0,T]$ and all $(x_0, \dots, x_{n-1}) \in \mathcal{D}_*$. Here

$$\mathcal{D}_* = \begin{cases} \underbrace{\mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_- \times \mathbb{R}_+^{n-p}}_{n} & \text{if } p \text{ is even,} \\ \underbrace{\mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \times \mathbb{R}_-^{n-p}}_{n} & \text{if } p \text{ is odd.} \end{cases}$$

The corresponding assumptions for problem (2.19), (2.20) have the form:

$$\begin{cases} \widetilde{f} \in Car([0,T] \times \mathcal{D}_*) \text{ and there exist positive constants } a, r \\ \text{such that} \\ \quad a\, t^r \le \widetilde{f}(t, x_0, \dots, x_{n-1}) \\ \text{for a.e. } t \in [0,T] \text{ and each } (x_0, \dots, x_{n-1}) \in \mathcal{D}_*, \end{cases} \tag{2.21}$$

$$\begin{cases} \quad \widetilde{f}(t, x_0, \dots, x_{n-1}) \le h\Big(t, \sum_{j=0}^{n-1} |x_j|\Big) + \sum_{j=0}^{n-1} \omega_j(|x_j|) \\ \text{for a.e. } t \in [0,T] \text{ and each } (x_0, \dots, x_{n-1}) \in \mathcal{D}_*, \\ \text{where the functions } h \text{ and } \omega_j,\ 0 \le j \le n-1, \text{ have} \\ \text{the properties given in (2.18).} \end{cases} \tag{2.22}$$

A priori estimates

Let us choose positive constants a and r and define the set

$$\mathcal{B}(r,a) = \{u \in AC^{n-1}[0,T] : u \text{ fulfils (2.16) and (2.24)}\} \tag{2.23}$$

where

$$(-1)^{n-p}\, u^{(n)}(t) \ge a\,(T-t)^r \quad \text{for a.e. } t \in [0,T]. \tag{2.24}$$

The next two lemmas are devoted to the study of the set $\mathcal{B}(r,a)$. The results obtained in this part will be used in the proofs of existence results for auxiliary regular problems.

Lemma 2.2. *There exists* $c > 0$ *such that the inequalities*

$$u^{(j)}(t) \geq c\, t^{r+n-j} \quad \textit{for} \;\; 0 \leq j \leq p-1, \tag{2.25}$$

$$(-1)^{j-p}\, u^{(j)}(t) \geq c\,(T-t)^{r+n-j} \quad \textit{for} \;\; p \leq j \leq n-1, \tag{2.26}$$

are true for $t \in [0, T]$ *and each* $u \in \mathcal{B}(r, a)$.

Proof. Put

$$c = \frac{a}{(r+1)(r+2)\dots(r+n)}.$$

Then, integrating inequality (2.24) and using condition (2.16), we get step by step that (2.26) holds on $[0, T]$ and that

$$u^{(p-1)}(t) \geq c\,(T^{r+n-p+1} - (T-t)^{r+n-p+1}) \quad \text{for} \;\; t \in [0, T]. \tag{2.27}$$

Set $\nu = r+n-p+1$ and consider the function $\varphi(t) = T^\nu - (T-t)^\nu - t^\nu$ on $[0, T]$. Since $\nu > 2$, $\varphi(0) = \varphi(T) = 0$ and φ is concave on $[0, T]$, we have $\varphi > 0$ on $(0, T)$ and thus $T^{r+n-p+1} - (T-t)^{r+n-p+1} > t^{r+n-p+1}$ holds on $(0, T)$, which together with inequality (2.27) yields

$$u^{(p-1)}(t) \geq c\, t^{r+n-p+1} \quad \text{for} \;\; t \in [0, T]. \tag{2.28}$$

Now, using (2.16) again and integrating (2.28), we successively obtain inequality (2.25) for $t \in [0, T]$. ☐

Lemma 2.3. *Let functions* h *and* ω_j, $0 \leq j \leq n-1$, *have the properties given in condition* (2.18). *Then there exists a positive constant* S *such that for each function* $u \in \mathcal{B}(r, a)$ *satisfying*

$$(-1)^{n-p}\, u^{(n)}(t) \leq h\Big(t, n + \sum_{j=0}^{n-1} |u^{(j)}(t)|\Big) + \sum_{j=0}^{n-1} [\omega_j(1) + \omega_j(|u^{(j)}(t)|)] \tag{2.29}$$

for a.e. $t \in [0, T]$, *the estimate*

$$\|u^{(n-1)}\|_\infty < S \tag{2.30}$$

is valid.

Proof. Given a function $u \in \mathcal{B}(r,a)$ which satisfies (2.29) a.e. on $[0,T]$, we put $\rho = \|u^{(n-1)}\|_\infty$. Then we integrate the inequality

$$|u^{(n-1)}(t)| \le \rho \quad \text{for} \quad t \in [0,T],$$

and due to condition (2.16) we successively get

$$\|u^{(j)}\| \le \rho T^{n-j-1}, \quad 0 \le j \le n-2. \tag{2.31}$$

Further, we integrate (2.29) over $[t,T] \subset [0,T]$ and in view of (2.31) we see that the inequality

$$\begin{cases} \rho \le \displaystyle\int_0^T h\Big(t, n + \rho \sum_{j=0}^{n-1} T^{n-j-1}\Big)\,\mathrm{d}t \\ \qquad\qquad\qquad + \displaystyle\sum_{j=0}^{n-1} \int_0^T \omega_j(|u^{(j)}(t)|)\,\mathrm{d}t + T \sum_{j=0}^{n-1} \omega_j(1) \end{cases} \tag{2.32}$$

holds. In order to find S fulfilling inequality (2.30) we need to estimate the integrals

$$\int_0^T \omega_j(|u^{(j)}(t)|)\,\mathrm{d}t, \quad 0 \le j \le n-1.$$

For this purpose we distinguish two cases.

Case 1. Let $0 \le j \le p-1$. Then, by Lemma 2.2, there exists $c > 0$ such that

$$\int_0^T \omega_j(|x^{(j)}(t)|)\,\mathrm{d}t \le \int_0^T \omega_j(c\,t^{r+n-j})\,\mathrm{d}t = \int_0^T \omega_j((c_j\,t)^{r+n-j})\,\mathrm{d}t \tag{2.33}$$

where $c_j^{r+n-j} = c$. Therefore

$$\int_0^T \omega_j(|u^{(j)}(t)|)\,\mathrm{d}t \le \frac{1}{c_j}\int_0^{c_j T} \omega_j(t^{r+n-j})\,\mathrm{d}t =: C_j.$$

Case 2. Let $p \le j \le n-1$. Then, by Lemma 2.2 and inequality (2.26),

$$\int_0^T \omega_j(|u^{(j)}(t)|)\,\mathrm{d}t \le \int_0^T \omega_j(c\,(T-t)^{r+n-j})\,\mathrm{d}t = \int_0^T \omega_j(c\,t^{r+n-j})\,\mathrm{d}t = C_j,$$

that is, (2.33) holds for $p \le j \le n-1$, too.

After inserting (2.33) into (2.32) we obtain

$$\rho \le \int_0^T h(t, n + \rho V)\,\mathrm{d}t + \sum_{j=0}^{n-1} [C_j + T\,\omega_j(1)], \tag{2.34}$$

where V is given in assumption (2.18). Since

$$\limsup_{v\to\infty} \frac{1}{v} \int_0^T h(t, Vv)\,\mathrm{d}v < 1$$

by our assumption, there exists a positive constant S such that

$$\int_0^T h(t, n + V\,v)\,\mathrm{d}t + \sum_{j=0}^{n-1} [C_j + T\,\omega_j(1)] < v$$

whenever $v \ge S$. This together with (2.34) shows that $\rho < S$, which proves inequality (2.30). □

Approximate regular problems

Let S be the positive constant from the assertion of Lemma 2.3. For $m \in \mathbb{N}$, $0 \le j \le n-1$ and $v \in \mathbb{R}$, put

$$\rho_j = 1 + S\,T^{n-j-1} \tag{2.35}$$

and

$$\sigma_j(\tfrac{1}{m}, v) = \begin{cases} \frac{1}{m}\operatorname{sign} v & \text{if } |v| < \frac{1}{m}, \\ v & \text{if } \frac{1}{m} \le |v| \le \rho_j, \\ \rho_j \operatorname{sign} v & \text{if } \rho_j < |v|. \end{cases}$$

Let f^* denote the extension of f onto $[0,T] \times (\mathbb{R} \setminus \{0\})^n$ as an even function in each its space variable x_j, $0 \le j \le n-1$, and for a.e. $t \in [0,T]$ and for all $(x_0, \ldots, x_{n-1}) \in \mathbb{R}^n$, $m \in \mathbb{N}$, define an auxiliary function

$$f_m(t, x_0, \ldots, x_{n-1}) = f^*(t, \sigma_0(\tfrac{1}{m}, x_0), \ldots, \sigma_{n-1}(\tfrac{1}{m}, x_{n-1})). \tag{2.36}$$

Consider the sequence of regular differential equations

$$(-1)^{n-p}\, u^{(n)} = f_m(t, u, \dots, u^{(n-1)}) \tag{2.37}$$

depending on $m \in \mathbb{N}$.

Lemma 2.4. *Let assumptions* (2.17) *and* (2.18) *hold, let* $\mathcal{B}(r,a)$ *be given in* (2.23) *and let* S *be from Lemma* 2.3. *Then, for each* $m \in \mathbb{N}$, *problem* (2.37), (2.16) *has a solution* $u_m \in \mathcal{B}(r,a)$ *and*

$$\|u_m^{(n-1)}\|_\infty < S. \tag{2.38}$$

Proof. Fix an arbitrary $m \in \mathbb{N}$. Assumption (2.17) and formula (2.36) yield $f_m \in Car([0,T] \times \mathbb{R}^n)$. Put

$$g_m(t) = \sup\{|f^*(t, x_0, \dots, x_{n-1})| : \tfrac{1}{m} \le |x_j| \le \rho_j,\ 0 \le j \le n-1\},$$

where ρ_j, $0 \le j \le n-1$, are given by (2.35). Then $g_m \in L_1[0,T]$ and

$$|f_m(t, x_0, \dots, x_{n-1})| \le g_m(t) \text{ for a.e. } t \in [0,T] \text{ and all } (x_0, \dots, x_{n-1}) \in \mathbb{R}^n.$$

Since the problem $(-1)^{n-p}\, u^{(n)} = 0$, (2.16) has only the trivial solution, the Fredholm type existence theorem implies that problem (2.37), (2.16) has a solution $u_m \in AC^{n-1}[0,T]$. Further, by assumptions (2.17) and (2.18), we see that the inequalities

$$a\,(T-t)^r \le f_m(t, x_0, \dots, x_{n-1}), \tag{2.39}$$

$$f_m(t, x_0, \dots, x_{n-1}) \le h\Big(t, n + \sum_{j=0}^{n-1} |x_j|\Big) + \sum_{j=0}^{n-1} [\omega_j(1) + \omega_j(|x_j|)] \tag{2.40}$$

are satisfied for a.e. $t \in [0,T]$ and all $(x_0, \dots, x_{n-1}) \in \mathbb{R}^n$. Notice that inequality (2.40) follows from the relations

$$|\sigma_j(\tfrac{1}{m}, x_j)| \le 1 + |x_j|, \quad \omega_j(|\sigma_j(\tfrac{1}{m}, x_j)|) \le \omega_j(1) + \omega_j(|x_j|),\ 0 \le j \le n-1,$$

and the facts that h is nondecreasing in the second variable and ω_j is nonincreasing. In view of (2.39) we have $u_m \in \mathcal{B}(r,a)$ and therefore from (2.40) and Lemma 2.3, we conclude (2.38). $\square$

Existence results

First, we consider the singular $(p, n-p)$ right focal problem (2.15), (2.16) with $1 \le p \le n-1$.

Theorem 2.5. *Let assumptions* (2.17) *and* (2.18) *hold. Then there exists a solution* $u \in AC^{n-1}[0,T]$ *of problem* (2.15), (2.16) *such that*

$$\begin{cases} u^{(j)} > 0 & \text{on } (0,T] \text{ for } 0 \le j \le p-1, \\ (-1)^{j-p} u^{(j)} > 0 & \text{on } [0,T) \text{ for } p \le j \le n-1. \end{cases} \tag{2.41}$$

Proof. According to Lemma 2.4, for each $m \in \mathbb{N}$ problem (2.37), (2.16) has a solution $u_m \in \mathcal{B}(r,a)$ satisfying inequality (2.38) where S is a positive constant independent of m. By Lemma 2.2 there exists $c > 0$ such that for $m \in \mathbb{N}$ and $t \in [0,T]$ we have

$$u_m^{(j)}(t) \ge c\, t^{r+n-j} \qquad \text{for } 0 \le j \le p-1, \tag{2.42}$$

$$(-1)^{j-p} u_m^{(j)}(t) \ge c\,(T-t)^{r+n-j} \quad \text{for } p \le j \le n-1. \tag{2.43}$$

Condition (2.16) and inequality (2.30) yield

$$\|u_m^{(j)}\|_\infty < S\,T^{n-j-1} < \rho_j, \quad 0 \le j \le n-1. \tag{2.44}$$

Here ρ_j is defined in formula (2.35). We now show that the sequence $\{f_m(t, u_m(t), \dots, u_m^{(n-1)}(t))\}$ is uniformly integrable on $[0,T]$. By assumption (2.17) and inequalities (2.40), (2.42)–(2.44) we have

$$0 \le f_m(t, u_m(t), \dots, u_m^{(n-1)}(t)) \le h(t, n + SV) + q(t) + \sum_{j=0}^{n-1} \omega_j(1) \tag{2.45}$$

for a.e. $t \in [0,T]$ and all $m \in \mathbb{N}$, where

$$q(t) = \sum_{j=0}^{p-1} \omega_j(c\, t^{r+n-j}) + \sum_{j=p}^{n-1} \omega_j(c\,(T-t)^{r+n-j}).$$

Put $c_j = \sqrt[r+n-j]{c}$ for $0 \le j \le n-1$. Then

$$\int_0^T q(t)\,\mathrm{d}t = \sum_{j=0}^{p-1} \frac{1}{c_j} \int_0^{c_j T} \omega_j(t^{r+n-j})\,\mathrm{d}t + \sum_{j=p}^{n-1} \frac{1}{c_j} \int_0^{c_j T} \omega_j(t^{r+n-j})\,\mathrm{d}t.$$

By assumption (2.18), the functions $h(t, n + V S)$ and $\omega_j(t^{r+n-j})$, $0 \le j \le n-1$, belong to $L_1[0,T]$. Therefore $h(t, n+SV) + q(t) \in L_1[0,t]$ and from (2.45) and Criterion A.1 it follows that $\{f_m(t, u_m(t), \ldots, u_m^{(n-1)}(t))\}$ is uniformly integrable on $[0, T]$. Hence the first assertion in Theorem 1.7 guarantees the existence of a subsequence $\{u_{m'}\}$ of $\{u_m\}$ which converges in $C^{n-1}[0, T]$ to a function $u \in C^{n-1}[0, T]$. Letting $m' \to \infty$ in inequalities (2.42) and (2.43) (with m' instead of m) yields

$$\begin{aligned} u^{(j)}(t) &\geq c\, t^{r+n-j} && \text{for } 0 \leq j \leq p-1, \\ (-1)^{j-p}\, u^{(j)}(t) &\geq c\, (T-t)^{r+n-j} && \text{for } p \leq j \leq n-1 \end{aligned}$$

for $t \in [0, T]$ and so u satisfies inequality (2.41). We see that $u^{(j)}$ has exactly one zero on $[0, T]$ for $0 \leq j \leq n-1$. Hence $u \in AC^{n-1}[0, T]$ and u is a solution of problem (2.15), (2.16) by Theorem 1.7. □

Substituting $t = T - s$ in (2.15), (2.16) and using Theorem 2.5 we obtain the following existence result for the singular $(n-p, p)$ left focal problem (2.19), (2.20) with $1 \leq p \leq n-1$.

Theorem 2.6. *Let assumptions (2.21) and (2.22) hold. Then problem (2.19), (2.20) has a solution $u \in AC^{n-1}[0, T]$ and*

$$\begin{cases} (-1)^j\, u^{(j)} > 0 & \text{on } [0, T) \text{ for } 0 \leq j \leq p-1, \\ (-1)^p\, u^{(j)} > 0 & \text{on } (0, T] \text{ for } p \leq j \leq n-1. \end{cases}$$

□

Example. Let $r > 0$, $\alpha_j \in (0, \frac{1}{r+n-j})$ for $0 \leq j \leq n-1$. Let $c \in L_\infty[0, T]$, $a_j \in L_\infty[0, T]$, $b_j \in L_1[0, T]$ be nonnegative for $0 \leq j \leq n-1$, $0 < a < c(t)$ for a.e. $t \in [0, T]$ and

$$\int_0^T \gamma(t)\mathrm{d}t < \frac{1}{V},$$

where $\gamma(t) = \max\{b_j(t) : 0 \leq j \leq n-1\}$ for a.e. $t \in [0, T]$ and V is given in (2.18). Then the differential equation

$$(-1)^{n-p}\, u^{(n)} = c(t)\, (T-t)^r + \sum_{j=0}^{n-1} \left(\frac{a_j(t)}{|u^{(j)}|^{\alpha_j}} + b_j(t)\, |u^{(j)}| \right) \tag{2.46}$$

satisfies all assumptions of Theorem 2.5. Hence, for each $p \in \{1, \ldots, n-1\}$, problem (2.46), (2.16) has a solution $u \in AC^{n-1}[0, T]$ satisfying inequality (2.41).

Bibliographical notes

Theorem 2.1 is new and represents the first result in literature for the existence of solutions of $(1, n-j)$ focal problems with time singularities. Theorem 2.5 was adapted from Rachůnková and Staněk [159], also see Rachůnková and Staněk [163]. Existence results for positive solutions to singular $(p, n-p)$ focal problems are available in Agarwal [2], Agarwal and O'Regan [8], [9], [10] and Agarwal, O'Regan and Lakshmikantham [15]. The paper [9] is the first to establish the existence of two solutions. Further multiplicity results solutions are established in [10]. The technique presented in [9] and [10] to guarantee the existence of twin solutions to singular $(p, n-p)$ focal problems combines (i) a nonlinear alternative of Leray-Schauder type, (ii) Krasnoselskii's fixed point theorem in a cone, and (iii) lower type inequalities.

Chapter 3

(n,p) problem

Now we are concerned with the singular (n,p) problem

$$-u^{(n)} = f(t, u, , \dots, u^{(n-1)}), \tag{3.1}$$

$$u^{(j)}(0) = 0,\ 0 \le j \le n-2,\ u^{(p)}(T) = 0,\ p \text{ fixed},\ 0 \le p \le n-1, \tag{3.2}$$

where $n \ge 2$, $f \in Car([0,T] \times \mathcal{D})$, $\mathcal{D} \subset \mathbb{R}^n$ and $f(t, x_0, \dots, x_{n-1})$ may be singular at the value 0 of its space variables $x_0, \dots, x_{n-2}$. Notice that the $(n,0)$ problem is simultaneously the $(1, n-1)$ conjugate problem discussed in Chapter 4. For f positive, solutions of problem (3.1), (3.2) have singular points of type I at $t = 0, T$ and also singular points of type II. We will work with the following assumptions on the function f in (3.1):

$$\begin{cases} f \in Car([0,T] \times \mathcal{D}) \quad \text{where } \mathcal{D} = (0,\infty) \times (\mathbb{R} \setminus \{0\})^{n-2} \times \mathbb{R} \\ \text{and there exist a positive function } \psi \in L_1[0,T] \text{ and } K > 0 \\ \text{such that} \\ \quad \psi(t) \le f(t, x_0, \dots, x_{n-1}) \quad \text{for a.e. } t \in [0,T] \\ \text{and each } (x_0, \dots, x_{n-1}) \in (0,K] \times (\mathbb{R} \setminus \{0\})^{n-2} \times \mathbb{R}, \end{cases} \tag{3.3}$$

$$\begin{cases} \quad 0 < f(t, x_0, \dots, x_{n-1}) \le h\Big(t, \displaystyle\sum_{j=0}^{n-1} |x_j|\Big) + \sum_{j=0}^{n-2} \omega_j(|x_j|) \\ \text{for a.e. } t \in [0,T] \text{ and each } (x_0, \dots, x_{n-1}) \in \mathcal{D}, \\ \text{where } h \in Car([0,T] \times [0,\infty)) \text{ is positive and nondecreasing} \\ \text{in the second variable} \\ \omega_j : (0,\infty) \to (0,\infty) \text{ is nonincreasing,} \\ \quad \displaystyle\limsup_{\varrho \to \infty} \frac{1}{\varrho} \int_0^T h(t, V(t)\varrho)\, \mathrm{d}t < 1 \text{ with } V(t) = \sum_{j=0}^{n-1} \frac{t^j}{j!} \\ \text{and} \\ \quad \displaystyle\int_0^1 \omega_j(s^{n-j-1})\, \mathrm{d}s < \infty \text{ for } 0 \le j \le n-2. \end{cases} \tag{3.4}$$

Auxiliary results Put

$$G(t,s) = \frac{1}{(n-1)!}\begin{cases} t^{n-1}\left(1-\dfrac{s}{T}\right)^{n-p-1} - (t-s)^{n-1} & \text{for } 0 \le s \le t \le T, \\ t^{n-1}\left(1-\dfrac{s}{T}\right)^{n-p-1} & \text{for } 0 \le t < s \le T. \end{cases}$$

Then $G(t,s)$ is the Green function of the problem

$$-u^{(n)} = 0, \quad (3.2) \tag{3.5}$$

(see e.g. Agarwal [1] or Agarwal, O'Regan and Wong [21]).

Lemma 3.1. *The Green function $G(t,s)$ of problem (3.5) fulfils*

$$G(T,s) > 0 \quad \textit{for} \ \ s \in (0,T) \ \textit{and for} \ \ p > 0, \tag{3.6}$$

$$\frac{\partial^j G(t,s)}{\partial t^j} > 0 \quad \textit{for} \ \ (t,s) \in (0,T) \times (0,T), \tag{3.7}$$

$$\textit{and for} \ \ 0 \le j \le \min\{p, n-2\}, \ \ p \ge 0.$$

Proof. Property (3.6) of G follows from the inequality

$$\left(1-\frac{s}{T}\right)^{n-p-1} > \left(1-\frac{s}{T}\right)^{n-1}$$

which is true for $s \in (0,T)$ and for $p > 0$. Further, let us suppose

$$0 \le j \le \min\{p, n-2\}$$

and prove inequality (3.7). We have

$$\frac{\partial^j G(t,s)}{\partial t^j} = \frac{1}{(n-j-1)!}\begin{cases} t^{n-j-1}\left(1-\frac{s}{T}\right)^{n-p-1} - (t-s)^{n-j-1} \\ \qquad\qquad \text{for } \ 0 \le s \le t \le T \\ t^{n-j-1}\left(1-\frac{s}{T}\right)^{n-p-1} \\ \qquad\qquad \text{for } \ 0 \le t < s \le T \end{cases}$$

and therefore it is sufficient to show that

$$\left(1-\frac{s}{T}\right)^{n-p-1} > \left(1-\frac{s}{t}\right)^{n-j-1} \quad \text{for } \ 0 < s \le t < T. \tag{3.8}$$

Since the inequalities

$$\left(1-\frac{s}{T}\right)^{n-p-1} > \left(1-\frac{s}{t}\right)^{n-p-1} \geq \left(1-\frac{s}{t}\right)^{n-j-1}$$

are valid for $0 < s \leq t < T$, inequality (3.8) is true. □

Lemma 3.2. *Let* $u \in AC^{n-1}[0,T]$ *satisfy condition* (3.2) *and let*

$$-u^{(n)}(t) > 0 \quad \textit{for a.e.} \quad t \in [0,T]. \tag{3.9}$$

Then, if $p > 0$, *we have*

$$\begin{cases} u^{(j)}(t) > 0 \ \textit{for} \ \ t \in (0,T], \quad 0 \leq j \leq p-1, \\ u^{(p)}(t) > 0 \ \textit{for} \ \ t \in (0,T) \end{cases} \tag{3.10}$$

and if $p = 0$, *we have*

$$u(t) > 0 \quad \textit{for}\ t \in (0,T). \tag{3.11}$$

Proof. We will consider two cases, namely (i) $p=n-1$ and (ii) $0 \leq p \leq n-2$.
Case (i). Let $p = n-1$. Then, by conditions (3.2) and (3.9), we have

$$0 < -\int_t^T u^{(n)}(s)\,\mathrm{d}s = u^{(n-1)}(t) \quad \text{for} \quad t \in [0,T). \tag{3.12}$$

Thus, integrating (3.12) from 0 to t and using (3.2), we get step by step

$$u^{(j)}(t) > 0 \quad \text{for} \quad t \in (0,T], \ 0 \leq j \leq n-2. \tag{3.13}$$

Inequalities (3.12) and (3.13) give the assertion of Lemma 3.2.

Case (ii). Let $0 \leq p \leq n-2$. Then, using the formula

$$u(t) = -\int_0^T G(t,s)\,u^{(n)}(s)\,\mathrm{d}s, \tag{3.14}$$

we can see that the assertion of Lemma 3.2 follows from (3.9) and from Lemma 3.1. □

A priori estimates

The following three lemmas give *a priori* estimates from below for functions satisfying conditions (3.2) and (3.9). We consider the cases $p = n-1$, $p = 0$ and $1 \le p \le n-2$ separately.

Lemma 3.3. *Let $p = n-1$ and let $u \in AC^{n-1}[0,T]$ satisfy conditions (3.2), (3.9). Then the inequalities*

$$u^{(j)}(t) \ge \frac{\|u\|_\infty}{T^{n-1}}\, t^{n-j-1} \quad \textit{for} \ \ t \in [0,T], \quad 0 \le j \le n-2, \tag{3.15}$$

are fulfilled.

Proof. Put

$$p_0(t) = \|u\|_\infty \left(\frac{t}{T}\right)^{n-1} \qquad \text{for} \ \ t \in [0,T]. \tag{3.16}$$

Then $p_0(0) = \cdots = p_0^{(n-2)}(0) = 0$, $p_0(T) = \|u\|_\infty$. By virtue of inequality (3.10) we have $\|u\|_\infty = u(T)$. So, if $h(t) = u(t) - p_0(t)$ for $t \in [0,T]$, then h satisfies the boundary conditions $h(0) = \cdots = h^{(n-2)}(0) = 0$, $h(T) = 0$, and moreover

$$h^{(n)}(t) = u^{(n)}(t) - p_0^{(n)}(t) = u^{(n)}(t) < 0 \qquad \text{for a.e. } t \in [0,T].$$

Therefore Lemma 3.2 (with h instead of u) gives $h > 0$ on $(0,T)$, that is

$$u(t) \ge p_0(t) \quad \text{for} \ \ t \in [0,T]. \tag{3.17}$$

Further, put

$$p_1(t) = \|u'\|_\infty \left(\frac{t}{T}\right)^{n-2} \quad \text{for} \ \ t \in [0,T]. \tag{3.18}$$

Then $p_1(0) = \cdots = p_1^{(n-3)}(0) = 0$, $p_1(T) = \|u'\|_\infty$. Since $\|u'\|_\infty = u'(T)$, the function $h_1 = u' - p_1$ satisfies $h_1(0) = \cdots = h_1^{(n-3)}(0) = 0$, $h_1(T) = 0$, and moreover

$$h_1^{(n-1)} = u^{(n)} - p_1^{(n-1)} = u^{(n)} < 0 \quad \text{a.e. on} \ \ [0,T].$$

Thus, by Lemma 3.2 where we use h_1 and $n-1$ instead of u and n, respectively, we have $h_1>0$ on $(0,T)$, that is

$$u'(t)\ge p_1(t) \quad \text{for} \quad t\in[0,T]. \tag{3.19}$$

Similarly, for $2\le j\le n-2$ we put

$$p_j(t)=\|u^{(j)}\|_\infty\left(\frac{t}{T}\right)^{n-j-1} \quad \text{and} \quad h_j(t)=u^{(j)}(t)-p_j(t) \ \text{ for } t\in[0,T].$$

Using Lemma 3.2 (with h_j and $n-j$ instead of u and n), we get $h_j>0$ on $(0,T)$ and therefore

$$u^{(j)}(t)\ge p_j(t) \quad \text{for} \quad t\in[0,T],\ 2\le j\le n-2. \tag{3.20}$$

Now (3.16)–(3.20) together with the inequalities

$$\|u^{(j)}\|_\infty\ge\frac{\|u\|_\infty}{T^j}, \quad 1\le j\le n-2, \tag{3.21}$$

give (3.15). ☐

Lemma 3.4. *Let $p=0$ and let $u\in AC^{n-1}[0,T]$ satisfy assumptions (3.2), (3.9). Then for $0\le j\le n-2$ we have*

$$u^{(j)}(t)\begin{cases}\ge \dfrac{\|u\|_\infty}{T^{n-1}}\,t^{n-j-1} & \textit{for } \ 0\le t\le \xi_{j+1},\\[2mm] \ge \dfrac{\|u\|_\infty}{T^{j+1}}\,(\xi_j-t) & \textit{for } \ \xi_{j+1}\le t\le \xi_j,\\[2mm] \le \dfrac{\|u\|_\infty}{T^{j+1}}\,(\xi_j-t) & \textit{for } \ \xi_j\le t\le T\end{cases} \tag{3.22}$$

with

$$\begin{cases}0<\xi_{n-1}<\xi_{n-2}<\cdots<\xi_2<\xi_1<\xi_0=T\\ \textit{where } \xi_i \ \textit{ is a unique zero of } \ u^{(i)} \ \textit{in} \ \ (0,T),\ 1\le i\le n-1.\end{cases} \tag{3.23}$$

Proof. In view of (3.2) and (3.11) we have $u(0)=u(T)=0$, $u>0$ on $(0,T)$. Further, there is a unique $\xi_1\in(0,T)$ such that $u'(\xi_1)=0$ (otherwise we would get a contradiction to inequality (3.9)). Similarly, in $(0,T)$ there is a unique $\xi_i < \xi_{i-1}$ such that $u^{(i)}(\xi_i) = 0, \quad 2 \le i \le n-1$. According to (3.9) we get

$$u^{(i)} > 0 \text{ on } (0,\xi_i), \quad u^{(i)} < 0 \text{ on } (\xi_i, T], \quad 1 \le i \le n-1. \tag{3.24}$$

Hence

$$u^{(i)} \text{ is concave on } [\xi_{i+2}, T] \text{ and convex on } [0,\xi_{i+2}],\ 0 \le i \le n-2, \tag{3.25}$$

where $\xi_n = 0$. Let us prove inequality (3.22) for $j = 0$. Put

$$p_0(t) = \|u\|_\infty \left(\frac{t}{\xi_1}\right)^{n-1} \qquad \text{for } t \in [0,\xi_1].$$

Then $p_0(0) = \cdots = p_0^{(n-2)}(0) = 0$, $p_0(\xi_1) = \|u\|_\infty$. Since $\|u\|_\infty = u(\xi_1)$, the function $h = u - p_0$ fulfils the boundary conditions $h(0) = \cdots = h^{(n-2)}(0) = 0$, $h(\xi_1) = 0$, and $h^{(n)}(t) < 0$ for a.e. $t \in [0,\xi_1]$. Therefore, by Lemma 3.2 (where we use h and ξ_1 instead of u and T), we deduce that the inequality $h > 0$ holds on $(0,\xi_1)$, which gives

$$u(t) \ge \frac{\|u\|_\infty}{T^{n-1}}\, t^{n-1} \quad \text{for } t \in [0,\xi_1]. \tag{3.26}$$

By property (3.25), u is concave on $[\xi_1, T] \subset [\xi_2, T]$. Thus $u(t) \ge u(\xi_1)\frac{T-t}{T-\xi_1}$ for $t \in [\xi_1, T]$ and therefore

$$u(t) \ge \frac{\|u\|_\infty}{T}\,(T-t) \quad \text{for } t \in [\xi_1, T]. \tag{3.27}$$

Estimates (3.26) and (3.27) lead to inequality (3.22) for $j = 0$.

For $1 \le j \le n-2$, we put

$$p_j(t) = u^{(j)}(\xi_{j+1})\left(\frac{t}{\xi_{j+1}}\right)^{n-j-1} \quad \text{and} \quad h(t) = u^{(j)}(t) - p_j(t)$$

on $[0,\xi_{j+1}]$. Since

$$u^{(j)}(\xi_{j+1}) = \|u^{(j)}\|_\infty \ge \frac{\|u\|_\infty}{T^j}, \quad 1 \le j \le n-2, \tag{3.28}$$

we get as before

$$u^{(j)}(t) \ge \frac{\|u\|_\infty}{T^{n-1}}\, t^{n-j-1} \quad \text{for } t \in [0,\xi_{j+1}]. \tag{3.29}$$

Further, using (3.25), we see that $u^{(j)}$ is concave on $[\xi_{j+1},T]\subset[\xi_{j+2},T]$. Hence

$$\begin{cases} u^{(j)}(t)\geq u^{(j)}\left(\xi_{j+1}\right)\dfrac{\xi_j-t}{\xi_j-\xi_{j+1}}\geq 0 \quad \text{for} \ \ t\in[\xi_{j+1},\xi_j], \\[2ex] u^{(j)}(t)\leq u^{(j)}\left(\xi_{j+1}\right)\dfrac{\xi_j-t}{\xi_j-\xi_{j+1}}\leq 0 \quad \text{for} \ \ t\in[\xi_j,T]. \end{cases} \tag{3.30}$$

Due to estimate (3.28) the above inequalities yield

$$|u^{(j)}(t)|\geq\frac{\|u\|_\infty}{T^{j+1}}\,|\xi_j-t| \quad \text{for} \ \ t\in[\xi_{j+1},T]. \tag{3.31}$$

Estimates (3.29)–(3.31) imply (3.22) for $1\leq j\leq n-2$. ☐

Lemma 3.5. *Let $1\leq p\leq n-2$ and let $u\in AC^{n-1}[0,T]$ satisfy (3.2), (3.9). Then, for $0\leq j\leq p-1$, inequality (3.15) is true and for $p\leq j\leq n-2$, inequalities (3.22) are valid on $[0,T]$ with $0<\xi_{n-1}<\xi_{n-2}<\ldots<\xi_{p+1}<\xi_p=T$, where ξ_i is a unique zero of $u^{(i)}$ in $(0,T)$, $p+1\leq i\leq n-1$.*

Proof. For $0\leq j\leq p-1$ we use the arguments of the proof of Lemma 3.3 and for $p\leq j\leq n-2$ we argue as in the proof of Lemma 3.4. ☐

For the proof of solvability of problem (3.1), (3.2) we will need the following results.

Lemma 3.6. *Let $\psi\in L_1[0,T]$ be positive. Then there is a positive constant $c=c(\psi)$ such that for each function $u\in AC^{n-1}[0,T]$ satisfying (3.2) and*

$$\psi(t)\leq -u^{(n)}(t) \quad \textit{for a.e.} \ \ t\in[0,T] \tag{3.32}$$

the estimate $\|u\|_\infty\geq c$ holds.

Proof. Let G be the Green function of problem (3.5). There are two cases to consider, namely (i) $1\leq p\leq n-1$ and (ii) $p=0$.

Case (i). Suppose $1\leq p\leq n-1$ and define a function Φ by the formula

$$\Phi(t,s)=\frac{G(t,s)}{t^{n-1}} \quad \text{for} \ \ (t,s)\in(0,T]\times(0,T].$$

By Lemma 3.1, the function Φ is continuous and positive on $(0,T]\times(0,T)$. Further, for any $s\in(0,T)$ we have

$$\frac{\partial^{n-1}G(t,s)}{\partial t^{n-1}}\Big|_{(t,s)=(0,s)} = \left(1-\frac{s}{T}\right)^{n-p-1} > 0.$$

Choose an arbitrary $s\in(0,T)$. Then

$$\lim_{t\to 0^+}\Phi(t,s)=\frac{1}{(n-1)!}\frac{\partial^{n-1}G(t,s)}{\partial t^{n-1}}\Big|_{(t,s)=(0,s)}=\frac{1}{(n-1)!}\left(1-\frac{s}{T}\right)^{n-p-1}>0,$$

which means that for any $s\in(0,T)$ we can extend $\Phi(\cdot,s)$ at $t=0$ as a continuous and positive function on $[0,T]$. Thus the function

$$F(t)=\int_0^T \Phi(t,s)\,\psi(s)\,\mathrm{d}s$$

is continuous and positive on $[0,T]$, too. Therefore we can find $d>0$ such that $F(t)\ge d$ on $[0,T]$. Then

$$\begin{aligned} u(t) &= -\int_0^T G(t,s)\,u^{(n)}(s)\,\mathrm{d}s \ge \int_0^T G(t,s)\,\psi(s)\,\mathrm{d}s \\ &= t^{n-1}\int_0^T \frac{G(t,s)}{t^{n-1}}\,\psi(s)\,\mathrm{d}s = t^{n-1}F(t)\ge t^{n-1}d \quad \text{for } t\in[0,T]. \end{aligned}$$

This implies $\|u\|_\infty = u(T)\ge T^{n-1}d=c$.

Case (ii). Let $p=0$. Define the function

$$\Phi(t,s)=\frac{G(t,s)}{t^{n-1}(T-t)} \quad \text{for } (t,s)\in(0,T)\times(0,T).$$

In view of Lemma 3.1, Φ is continuous and positive on $(0,T)\times(0,T)$. For any $s\in(0,T)$ we get

$$\lim_{t\to 0+}\Phi(t,s)=\frac{1}{T(n-1)!}\left(1-\frac{s}{T}\right)^{n-1}>0$$

and

$$\begin{aligned} \lim_{t\to T^-}\Phi(t,s) &= -\frac{1}{T^{n-1}}\frac{\partial G(t,s)}{\partial t}\Big|_{(t,s)=(T,s)} \\ &= -\frac{1}{T(n-2)!}\left[\left(1-\frac{s}{T}\right)^{n-1}-\left(1-\frac{s}{T}\right)^{n-2}\right]>0, \end{aligned}$$

which means that for any $s \in (0,T)$ we can extend $\Phi(\cdot, s)$ to $[0,T]$ as a continuous and positive function. Further we can argue as in *Case* (i). $\square$

Lemma 3.7. *Let $a>0$, $K>0$ and let the function $\psi \in L_1[0,T]$ be positive. Furthermore, let the functions h, ω_j, $(0 \le j \le n-2)$ have the properties given in assumption (3.4). Then there exist constants $r>0$ and $\alpha \in (0,K]$ such that for each function $u \in AC^{n-1}[0,T]$ satisfying (3.2),*

$$\begin{cases} -u^{(n)}(t) \le a + h\Big(t, n + \sum_{j=0}^{n-1} |u^{(j)}(t)|\Big) + \sum_{j=0}^{n-2} \omega_j(|u^{(j)}(t)|) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.e. } \ t \in [0,T] \end{cases} \tag{3.33}$$

and

$$\|u\|_\infty \le K \quad \Longrightarrow \quad \psi(t) \le -u^{(n)}(t) \ \text{ for a.e. } \ t \in [0,T], \tag{3.34}$$

the estimates

$$\|u^{(n-1)}\|_\infty < r \quad \text{and} \quad \|u\|_\infty \ge \alpha \tag{3.35}$$

are valid.

Proof. Let $u \in AC^{n-1}[0,T]$ satisfy conditions (3.2), (3.33) and (3.34). Let $\|u\|_\infty \le K$. Then, by (3.34) and Lemma 3.6, there is a positive constant $c = c(\psi)$ such that $\|u\|_\infty \ge c$. Otherwise we would have $\|u\|_\infty > K$. If we put $\alpha = \min\{c, K\}$, then the second inequality in (3.35) is satisfied.

In order to prove the first estimate in (3.35) we put $\|u^{(n-1)}\|_\infty = \rho$. Then $-\rho \le u^{(n-1)}(t) \le \rho$ on $[0,T]$ and if we integrate this inequality from 0 to $t \in (0,T]$ and use (3.2), we get step by step

$$|u^{(j)}(t)| \le \rho \frac{t^{n-j-1}}{(n-j-1)!} \quad \text{for } \ t \in [0,T], \quad 0 \le j \le n-1. \tag{3.36}$$

Lemmas 3.4 and 3.5 guarantee the existence of a unique zero ξ_{n-1} of $u^{(n-1)}$ with $\xi_{n-1} \in (0,T)$ for $0 \le p \le n-2$ and $\xi_{n-1} = T$ for $p = n-1$. Integrating inequality (3.33) from t to ξ_{n-1} gives

$$0 < u^{(n-1)}(t) \le a\,(\xi_{n-1} - t) + \int_t^{\xi_{n-1}} h\Big(s, n + \sum_{j=0}^{n-1} |u^{(j)}(s)|\Big)\,\mathrm{d}s$$

$$+ \sum_{j=0}^{n-2} \int_t^{\xi_{n-1}} \omega_j(|u^{(j)}(s)|)\,\mathrm{d}s$$

for $t \in [0, \xi_{n-1})$. If $p < n-1$ and thus $\xi_{n-1} < T$, we integrate (3.33) from ξ_{n-1} to t and get

$$0 < -u^{(n-1)}(t) \le a\,(t - \xi_{n-1}) + \int_{\xi_{n-1}}^{t} h\Big(s, n + \sum_{j=0}^{n-1} |u^{(j)}(s)|\Big)\, \mathrm{d}s$$

$$+ \sum_{j=0}^{n-2} \int_{\xi_{n-1}}^{t} \omega_j(|u^{(j)}(s)|)\, \mathrm{d}s$$

for $t \in (\xi_{n-1}, T]$. Hence the inequality

$$|u^{(n-1)}(t))| \le a\,T + \Big| \int_{\xi_{n-1}}^{t} h\Big(s, n + \sum_{j=0}^{n-1} |u^{(j)}(s)|\Big)\, \mathrm{d}s \Big|$$

$$+ \sum_{j=0}^{n-2} \Big| \int_{\xi_{n-1}}^{t} \omega_j(|u^{(j)}(s)|)\, \mathrm{d}s \Big|$$

is true for $t \in [0, T]$, and consequently (see (3.36))

$$\rho \le a\,T + \int_0^T h(t, n + V(t)\,\rho)\, \mathrm{d}t + \sum_{j=0}^{n-2} \int_0^T \omega_j(|u^{(j)}(t)|)\, \mathrm{d}t \tag{3.37}$$

where V is given in (3.4). We now estimate the integrals

$$\int_0^T \omega_j(|u^{(j)}(t)|)\, \mathrm{d}t, \quad 0 \le j \le n-2.$$

We will consider three cases.

Case (i). Let $p = n-1$. Then, by Lemma 3.3, for $0 \le j \le n-2$ we have

$$\omega_j(|u^{(j)}(t)|) \le \omega_j\Big(\frac{\|u\|_\infty}{T^{n-1}} t^{n-j-1}\Big) \quad \text{for } t \in (0, T].$$

Thus

$$\omega_j(|u^{(j)}(t)|) \le \omega_j((c_j t)^{n-j-1}) \quad \text{for } t \in (0, T],\ 0 \le j \le n-2, \tag{3.38}$$

where $c_j^{n-j-1} = \alpha T^{1-n}$. Inequality (3.38) implies

$$\int_0^T \omega_j(|u^{(j)}(t)|)\,\mathrm{d}t \le \frac{1}{c_j}\int_0^{c_j T} \omega_j(s^{n-j-1})\,\mathrm{d}s =: B_j$$

and therefore we have

$$\int_0^T \omega_j(|u^{(j)}(t)|)\,\mathrm{d}t \le B_j, \quad 0 \le j \le n-2. \tag{3.39}$$

Case (ii). Let $p = 0$. Then, by Lemma 3.4,

$$\omega_j(|u^{(j)}(t)|) \le \begin{cases} \omega_j((c_j t)^{n-j-1}) & \text{for } 0 \le t \le \xi_{j+1} \\ \omega_j(k_j|\xi_j - t|) & \text{for } \xi_{j+1} \le t \le T \end{cases} \tag{3.40}$$

for $0 \le j \le n-2$, where

$$c_j^{n-j-1} = \alpha\, T^{1-n}, \quad k_j = \alpha T^{-j-1} \tag{3.41}$$

and ξ_j fulfils relation (3.23). Therefore

$$\int_0^T \omega_j(|u^{(j)}(t)|)\,\mathrm{d}t$$

$$\le \int_0^{\xi_{j+1}} \omega_j((c_j\,t)^{n-j-1})\,\mathrm{d}t + \int_{\xi_{j+1}}^{\xi_j} \omega_j(k_j\,(\xi_j - s))\,\mathrm{d}t + \int_{\xi_j}^{T} \omega_j(k_j\,(t-\xi_j))\,\mathrm{d}t$$

$$\le B_j + \frac{1}{k_j}\int_0^{k_j\,(\xi_j-\xi_{j+1})} \omega_j(s)\,\mathrm{d}s + \frac{1}{k_j}\int_0^{k_j\,(T-\xi_j)} \omega_j(s)\,\mathrm{d}s \le B_j + C_j.$$

with $C_j = \frac{2}{k_j}\int_0^{k_j\,T} \omega_j(s)\,\mathrm{d}s$. Consequently, for $0 \le j \le n-2$ we have

$$\int_0^T \omega_j(|u^{(j)}(t)|)\,\mathrm{d}t \le B_j + C_j. \tag{3.42}$$

Case (iii). Let $1 \le p \le n-2$. Then, for $0 \le j \le p-1$, we have estimate (3.39) and, for $p \le j \le n-2$, estimate (3.42) holds where ξ_j $(p+1 \le j \le n-1)$ are from Lemma 3.5.

In view of (3.37), (3.39) and (3.42) we deduce that in all the above three cases

$$\rho \le \int_0^T h(t, n + V(t)\rho)\,\mathrm{d}t + D \tag{3.43}$$

where $D = a\,T + \sum_{j=0}^{n-2}(B_j + C_j)$. Since, by our assumption,

$$\limsup_{\rho\to\infty} \frac{1}{\rho} \int_0^T h(t, V(t)\rho)\,\mathrm{d}t < 1,$$

we have

$$\limsup_{\rho\to\infty} \frac{1}{\rho} \int_0^T h(t, n + V(t)\,\rho)\,\mathrm{d}t < 1$$

and consequently there exists $r > 0$ such that

$$\int_0^T h(t, n + V(t)\,\eta)\,\mathrm{d}t + D < \eta$$

whenever $\eta \ge r$. Then inequality (3.43) gives $\rho < r$, which proves the first inequality in (3.35) since $\rho = \|u^{(n-1)}\|_\infty$. □

Approximate regular problems

The main result on the existence of a solution of problem (3.1), (3.2) will be proved by Theorem 1.7. To this end we consider a sequence of regular problems constructed by the following procedure. Let $K > 0, \psi, h$ and ω_j, $0 \le j \le n-2$, have the properties given in assumption (3.3) and (3.4), $a = \sum_{j=0}^{n-2}\omega_j(1)$ and let positive constants r and α be taken from Lemma 3.7. Put

$$\rho_0 = 1 + r\,T^{n-1} + K, \quad \rho_i = 1 + r\,T^{n-i-1}, \quad 1 \le i \le n-1,$$

$$\sigma_i(x) = \begin{cases} x & \text{for } |x| \le \varrho_i, \\ \rho_i \operatorname{sign} x & \text{for } |x| > \rho_i, \end{cases} \quad 0 \le i \le n-1$$

and, for $0 < c < \rho_0$,

$$\sigma_0^*(c, x) = \begin{cases} c & \text{for } x < c, \\ x & \text{for } c \le x \le \rho_0 \\ \rho_0 & \text{for } \rho_0 < x. \end{cases}$$

Choose $m\in\mathbb{N}$ and use the function f from (3.1) to define an auxiliary function h_m by means of the following recurrent formulas for a.e. $t\in[0,T]$ and all $(x_0,\dots,x_{n-1})\in\mathcal{D}$:

$$h_{m,0}(t,x_0,\dots,x_{n-1}) = f(t,x_0,\dots,x_{n-1}),$$

$$\begin{aligned} &h_{m,i}(t,x_0,\dots,x_{n-1})\\ &= \begin{cases} h_{m,i-1}(t,x_0\dots,x_{n-1}) \qquad \text{if } |x_i|\ge \frac{1}{m},\\ \frac{m}{2}\Big[h_{m,i-1}(t,x_0,\dots,x_{i-1},\frac{1}{m},x_{i+1},\dots,x_{n-1})\,(x_i+\frac{1}{m})\\ \quad -h_{m,i-1}(t,x_0,\dots,x_{i-1},-\frac{1}{m},x_{i+1},\dots,x_{n-1})\,(x_i-\frac{1}{m})\Big]\\ \qquad\qquad\qquad \text{if } |x_i|<\frac{1}{m}, \end{cases} \end{aligned}$$

for $1\le i\le n-2$, and

$$h_m(t,x_0,\dots,x_{n-1}) = h_{m,n-2}(t,x_0,\dots,x_{n-1}).$$

Now, for a.e. $t\in[0,T]$ and all $(x_0,\dots,x_{n-1})\in\mathbb{R}^n$ put

$$f_m(t,x_0,\dots,x_{n-1}) = h_m(t,\sigma_0^*(\tfrac{1}{m},x_0),\sigma_1(x_1),\dots,\sigma_{n-1}(x_{n-1})). \tag{3.44}$$

Then, by conditions (3.3) and (3.4), $f_m\in Car([0,T]\times\mathbb{R}^n)$ and we have

$$\begin{cases} \psi(t)\le f_m(t,x_0,\dots,x_{n-1})\\ \text{for a.e. } t\in[0,T] \text{ and each } (x_0,\dots,x_{n-1})\in\mathbb{R}^n, \quad x_0\le K \end{cases} \tag{3.45}$$

and

$$\begin{cases} 0< f_m(t,x_0,\dots,x_{n-1})\\ \quad \le \displaystyle\sum_{j=0}^{n-2}\omega_j(1) + h\Big(t,n+\sum_{j=0}^{n-1}|x_j|\Big) + \sum_{j=0}^{n-2}\omega_j(|x_j|)\\ \text{for a.e. } t\in[0,T] \text{ and each } (x_0,\dots,x_{n-1})\in(\mathbb{R}\setminus\{0\})^{n-1}\times\mathbb{R}. \end{cases} \tag{3.46}$$

for $m\ge m_0\ge\frac{1}{K}$. Inequality (3.46) follows from the fact that

$$|\sigma_i(x_i)|\le|x_i| \quad \text{for } 1\le i\le n-1,$$
$$|\sigma_0^*(\tfrac{1}{m},x_0)|\le 1+|x_0|, \quad \sigma_0^*(\tfrac{1}{m},x_0)\ge\sigma_0(x_0)$$

and

$$\omega_i(|\sigma_i(x_i)|)\le\omega_i(|x_i|)+\omega_i(1),\ 0\le i\le n-2.$$

Consider auxiliary regular equations

$$-u^{(n)} = f_m(t,u,\dots,u^{(n-1)}) \tag{3.47}$$

where $m \ge m_0$.

Lemma 3.8. *Let assumptions* (3.3) *and* (3.4) *hold. Then for each* $m \in \mathbb{N}$, $m \ge m_0$, *problem* (3.47), (3.2) *has a solution* $u_m \in AC^{n-1}[0,T]$, *the sequence*

$$\{f_m(t,u_m(t),\dots,u_m^{(n-1)}(t))\}_{m\ge m_0} \tag{3.48}$$

is uniformly integrable on $[0,T]$ *and there exists a positive constants* r *such that*

$$\|u_m^{(n-1)}\|_\infty < r \quad \textit{for} \quad m \ge m_0. \tag{3.49}$$

Proof. Choose $m \in \mathbb{N}$, $m \ge m_0$ and put

$$g_m(t) = \sup\Big\{f(t,x_0,\dots,x_{n-1}) : \\ \tfrac{1}{m} \le x_0 \le \rho_0,\ \tfrac{1}{m} \le |x_i| \le \rho_i\ (0 \le i \le n-2),\ |x_{n-1}| \le \rho_{n-1}\Big\}.$$

Since $f \in Car([0,T]\times\mathcal{D})$, we have $g_m \in L_1[0,T]$ and

$$f_m(t,x_0,\dots,x_{n-1}) \le g_m(t) \text{ for a.e. } t \in [0,T] \text{ and all } (x_0,\dots,x_{n-1}) \in \mathbb{R}^n.$$

Since the homogeneous problem $-u^{(n)} = 0$, (3.2) has only the trivial solution, the Fredholm type existence theorem guarantees the existence of a solution $u_m \in AC^{n-1}[0,T]$ of problem (3.47), (3.2). By virtue of (3.45) and (3.46), Lemma 3.7 gives

$$\|u_m^{(n-1)}\|_\infty < r, \quad \|u_m\|_\infty \ge \alpha, \quad m \ge m_0, \tag{3.50}$$

where r and α are positive constants taken from Lemma 3.7. Condition (3.2) and the first inequality in (3.50) yield

$$\|u_m^{(n-j-1)}\|_\infty < r\,T^j < \rho_{n-j-1}, \quad 0 \le j \le n-1. \tag{3.51}$$

It remains to verify that the sequence (3.48) is uniformly integrable on $[0,T]$. By inequality (3.46),

$$0 \le f_m(t,u_m(t),\dots,u_m^{(n-1)}(t)) \\ \le \sum_{j=0}^{n-2}\omega_j(1) + h\Big(t, n + \sum_{j=0}^{n-1}|u_m^{(j)}(t)|\Big) + \sum_{j=0}^{n-2}\omega_j(|u_m^{(j)}(t)|)$$

for a.e. $t \in [0,T]$ and all $m \geq m_0$. From the inequality (see (3.51))

$$0 < h\Big(t, n + \sum_{j=0}^{n-1} |u_m^{(j)}(t)|\Big) \leq h\Big(t, n + r\sum_{j=0}^{n-1} T^j\Big)$$

and from $h(t, n + r\sum_{j=0}^{n-1} T^j) \in L_1[0,T]$ we see that the sequence (3.48) is uniformly integrable on $[0,T]$ if the sequences

$$\{\omega_j(|u_m^{(j)}|)\}_{m\geq m_0}, \quad 0 \leq j \leq n-2, \tag{3.52}$$

have this property. We will distinguish three cases, namely $p = n-1$, $p = 0$ and $1 \leq p \leq n-2$.

Case (i). Suppose $p = n-1$. Then Lemma 3.3 and the second inequality in (3.50) give

$$u_m^{(j)}(t) \geq \frac{\alpha}{T^{n-1}} t^{n-j-1} \quad \text{for} \ \ t \in [0,T], \ \ 0 \leq j \leq n-2, \, m \geq m_0. \tag{3.53}$$

Hence

$$\omega_j(|u_m^{(j)}(t)|) \leq \omega_j(\frac{\alpha}{T^{n-1}} t^{n-j-1})$$

and since

$$\int_0^1 \omega_j(s^{n-j-1})\, \mathrm{d}s < \infty \quad \text{for} \ \ 0 \leq j \leq n-2$$

by assumption (3.4), the sequences in (3.52) are uniformly integrable on $[0,T]$ by Criterion A.4.

Case (ii). Suppose p=0. Let $\xi_{i,m}$ denote the unique zero of $u_m^{(i)}, 1 \leq i \leq n-1$, in $(0,T)$. Then, by Lemma 3.4 and inequality (3.50),

$$0 < \xi_{n-1,m} < \xi_{n-2,m} < \cdots < \xi_{2,m} < \xi_{1,m} = T \tag{3.54}$$

and

$$u_m^{(j)}(t) \begin{cases} \geq \dfrac{\alpha}{T^{n-1}} t^{n-j-1} & \text{for} \ \ 0 \leq t \leq \xi_{j+1,m} \\ \geq \dfrac{\alpha}{T^{j+1}}(\xi_{j,m} - t) & \text{for} \ \ \xi_{j+1,m} \leq t \leq \xi_{j,m} \\ \leq \dfrac{\alpha}{T^{j+1}}(\xi_{j,m} - t) & \text{for} \ \ \xi_{j,m} \leq t \leq T \end{cases} \tag{3.55}$$

for $0 \le j \le n-2$, $m \ge m_0$. Hence for these j and m we have

$$|u_m^{(j)}(t)| \ge \begin{cases} c_j\, t^{n-j-1} & \text{for } 0 \le t \le \xi_{j+1,m} \\ c_j\, |\xi_{j,m} - t| & \text{for } \xi_{j+1,m} \le t \le T \end{cases} \tag{3.56}$$

where

$$c_j = \alpha \, \min\{T^{1-n}, T^{-1-j}\}. \tag{3.57}$$

Since

$$\int_0^1 \omega_j(s^{n-j-1})\, \mathrm{d}s < \infty \quad \text{for } 0 \le j \le n-2$$

by assumption (3.4), Criterion A.4 guarantees that the sequences in (3.52) are uniformly integrable on $[0,T]$.

Case (iii). Suppose $1 \le p \le n-2$. Then, by Lemma 3.5 and inequality (3.50), $u_m^{(i)}$ has a unique zero $\xi_{i,m}$ in $(0,T)$ for $p+1 \le i \le n-1$,

$$0 < \xi_{n-1,m} < \xi_{n-2,m} < \cdots < \xi_{p+1,m} < \xi_{p,m} = T,$$

$$u_m^{(j)}(t) \ge \frac{\alpha}{T^{n-1}}\, t^{n-j-1} \quad \text{for } t \in [0,T], \quad 0 \le j \le p-1, \quad m \ge m_0$$

and inequality (3.55) holds for $p \le j \le n-2$ and $m \ge m_0$. Now applying arguments from Case (i) for $0 \le j \le p-1$ and from Case (ii) for $p \le j \le n-2$, we can verify that the sequences in (3.52) are uniformly integrable on $[0,T]$.

Summarizing, we have proved that the sequences in (3.48) are uniformly integrable on $[0,T]$. □

Main result

Theorem 3.9. *Assume that assumptions* (3.3) *and* (3.4) *hold. Then there exists a solution* $u \in AC^{n-1}[0,T]$ *of problem* (3.1), (3.2) *such that*

$$u^{(j)} > 0 \quad \textit{on} \quad (0,T] \quad \textit{if} \quad p \ge 1 \quad \textit{and} \quad 0 \le j \le p-1 \tag{3.58}$$

and

$$u^{(p)} > 0 \quad \textit{on} \quad (0,T). \tag{3.59}$$

Proof. By Lemma 3.8, for each $m \in \mathbb{N}$, $m \geq m_0 \geq \frac{1}{K}$, there exists a solution $u_m \in AC^{n-1}[0,T]$ of problem (3.47), (3.2) satisfying inequality (3.50), which means that $\{u_m\}_{m \geq m_0}$ is bounded in $C^{n-1}[0,T]$ and the sequence (3.48) is uniformly integrable on $[0,T]$, which further implies that $\{u_m^{(n-1)})\}_{m \geq m_0}$ is equicontinuous on $[0,T]$. Thus, by the Arzelà-Ascoli theorem, we can assume without loss of generality that $\{u_m\}_{m \geq m_0}$ is convergent in $C^{n-1}[0,T]$ to a function $u \in C^{n-1}[0,T]$.

We now prove that the function $u^{(j)}$ has an at most finite number of zeros on $[0,T]$ for $0 \leq j \leq n-2$. Then $u \in AC^{n-1}[0,T]$ and u is a solution of problem (3.1), (3.2) by Theorem 1.7 since the function f in (3.1) has no singularity in its last space variable. Let $p = n-1$. Then (3.53) is true and letting $m \to \infty$ in (3.53) we obtain

$$u^{(j)}(t) \geq \frac{\alpha}{T^{n-1}} t^{n-j-1}, \quad t \in [0,T], \quad 0 \leq j \leq n-2. \tag{3.60}$$

From this inequality and from condition (3.2) we see that 0 is the unique zero of $u^{(j)}$ for $0 \leq j \leq n-2$. Let $p = 0$. Then (3.56) holds for $0 \leq j \leq n-2$ and $m \geq m_0$ where c_j is given in (3.57) and $\xi_{i,m}$ denotes the unique zero of $u_m^{(i)}$ in $(0,T)$ $(0 \leq i \leq n-1)$. The localization of $\xi_{i,m}$ is given in (3.54). Passing if necessary to subsequences, we can assume that $\{\xi_{i,m}\}_{m \geq m_0}$ is convergent; let $\lim_{m \to \infty} \xi_{i,n} = \xi_i$, $0 \leq i \leq n-1$. Letting $m \to \infty$ in inequality (3.56) yields

$$|u^{(j)}(t)| \geq \begin{cases} c_j \, t^{n-j-1} & \text{for } \ 0 \leq t \leq \xi_{j+1}, \\ c_j \, |\xi_j - t| & \text{for } \ \xi_{j+1} \leq t \leq T, \end{cases} \quad 0 \leq j \leq n-2. \tag{3.61}$$

This and condition (3.2) show that $u^{(j)}$ has at most two zeros in $[0,T]$ for $0 \leq j \leq n-2$. Finally, let $1 \leq p \leq n-2$. In this case we can show that the inequality in (3.60) holds for $t \in [0,T]$ and $0 \leq j \leq p-1$ and that in (3.61) for $t \in [0,T]$ and $p \leq j \leq n-2$. Therefore $u^{(j)}$ has at most two zeros in $[0,T]$ for $0 \leq j \leq n-2$. Summarizing, we have proved that in all the above cases $u^{(j)}$ has at most two zeros in $[0,T]$ for $0 \leq j \leq n-2$.

Finally, it follows from Lemma 3.2 that $u^{(p)} > 0$ on $(0,T)$ and if $p > 0$ then from the inequalities in (3.60) for $t \in [0,T]$ and $0 \leq j \leq p-1$ we conclude that $u^{(j)} > 0$ on $(0,T]$ for these j. □

EXAMPLE. Let γ, δ, $\beta_i \in (0,1)$, $0 < \alpha_j < \frac{1}{n-j-1}$ and let $a_j \in L_\infty[0,T]$ and $b_i \in L_1[0,T]$ be nonnegative for $0 \leq j \leq n-2, 0 \leq i \leq n-1$. Then, by

Theorem 3.9, the differential equation

$$-u^{(n)} = \frac{e^{-u}}{t^{\gamma}\,(T-t)^{\delta}} + \sum_{j=0}^{n-2} \frac{a_j(t)}{|u^{(j)}|^{\alpha_j}} + \sum_{i=0}^{n-1} b_i(t)\,|u^{(i)}|^{\beta_i}$$

has a solution $u \in AC^{n-1}[0,T]$ satisfying the boundary conditions (3.2) and inequalities (3.58), (3.59).

Bibliographical notes

Theorem 3.9 was adapted from Agarwal, O'Regan, Rachůnková and Staněk [16].

Singular (n,p) problems were considered by Agarwal and O'Regan in [9], [10] and Agarwal, O'Regan and Lakshmikantham [15]. In [9] and [10] the existence of two positive solutions in the set $C^{n-1}[0,1] \cap C^n(0,1)$ was proved for the differential equation

$$u^{(n)} + \varphi(t) f(t,u) = 0,$$

where $\varphi \in C^0(0,1) \cap L_1[0,1]$ and $f \in C^0([0,1] \times (0,\infty))$ are positive. The paper [15] dealt with the differential equation

$$u^{(n)} + \varphi(t)\, f(t, u, \ldots, u^{p-1}) = 0,$$

where $\varphi \in C^0(0,1) \cap L_1[0,1]$ and $f \in C^0([0,T] \times (0,\infty)^p)$ are positive. By a combination of regularization and sequential techniques with a nonlinear alternative of Leray-Schauder type, the authors proved the existence of a solution $u \in C^{n-1}[0,1] \cap C^n(0,1)$ with $u^{(j)} > 0$ on $(0,T]$ for $0 \le j \le p-1$.

Chapter 4

Conjugate problem

Let p be a positive integer, $1 \le p \le n-1$. Consider the $(p, n-p)$ conjugate problem

$$(-1)^p u^{(n)} = f(t, u, \dots, u^{(n-1)}), \tag{4.1}$$

$$u^{(i)}(0) = 0, \quad 0 \le i \le n-p-1, \quad u^{(j)}(T) = 0, \quad 0 \le j \le p-1, \tag{4.2}$$

where $n \ge 3$, $f \in Car([0,T] \times \mathcal{D})$, $\mathcal{D} \subset \mathbb{R}^n$ and f may be singular at the value 0 of any of its space variables. Replacing t by $T-t$ if necessary, we may assume that $p-1 \le n-p-1$, that is

$$p \in \{1, \dots, \frac{n}{2}\} \text{ for } n \text{ even} \quad \text{and} \quad p \in \{1, \dots, \frac{n-1}{2}\} \text{ for } n \text{ odd}. \tag{4.3}$$

We observe that the larger p is chosen, the more complicated structure of the set of all singular points of any solution to problem (4.1), (4.2) and its derivatives is obtained. This fact will be shown in Lemmas 4.1 and 4.2. We note that if f is positive then all solutions of problem (4.1), (4.2) have singular points of type I at $t = 0$ and $t = T$ and also singular points of type II. Problem (4.1), (4.2) with $p = 1$ is the $(n, 0)$ problem which was considered in Chapter 3 devoted to the (n, p) problem. We assume that $n \ge 3$ since problem (4.1), (4.2) for $n = 2$ is the Dirichlet problem discussed in Chapter 7.

We will use the following assumptions:

$$\begin{cases} f \in Car([0,T] \times \mathcal{D}) \text{ where } \mathcal{D} = (0,\infty) \times (\mathbb{R} \setminus \{0\})^{n-1} \text{ and} \\ \text{there exists } c > 0 \text{ such that} \\ \quad c \le f(t, x_0, \dots, x_{n-1}) \\ \text{for a.e. } t \in [0,T] \text{ and all } (x_0, \dots, x_{n-1}) \in \mathcal{D}, \end{cases} \tag{4.4}$$

$$\begin{cases} h \in Car([0,T] \times [0,\infty)) \ \text{ is positive and nondecreasing in its} \\ \text{second variable and} \\ \displaystyle \limsup_{z\to\infty} \frac{1}{z} \int_0^T h(t,z)\,\mathrm{d}t < \frac{1}{K}, \quad K = \begin{cases} \dfrac{T^n-1}{T-1} & \text{if } \ T \neq 1, \\ n & \text{if } \ T = 1, \end{cases} \end{cases} \tag{4.5}$$

$$\begin{cases} \omega_j : (0,\infty) \to (0,\infty) \ \text{ is nonincreasing and} \\ \displaystyle \int_0^1 \omega_j(s^{n-j})\,\mathrm{d}s < \infty \ \text{ for } \ 0 \le j \le n-1, \end{cases} \tag{4.6}$$

$$\begin{cases} \displaystyle f(t,x_0,\dots,x_{n-1}) \le h\Big(t, \sum_{j=0}^{n-1} |x_j|\Big) + \sum_{j=0}^{n-1} \omega_j(|x_j|) \\ \text{for a.e. } t \in [0,T] \ \text{ and all } \ (x_0,\dots,x_{n-1}) \in \mathcal{D}, \\ \text{where } \ h \ \text{ and } \ \omega_j \ \text{ satisfy (4.5) and (4.6)} \end{cases} \tag{4.7}$$

Localization analysis of zeros to solutions

Let f satisfy assumption (4.4), that is, f may be singular at the value 0 of any of its space variables and $f \ge c > 0$ on $[0,T] \times \mathcal{D}$. Then all singular points of any solution of problem (4.1), (4.2) and its derivatives coincide with zeros of this solution and its derivatives. The localization analysis of zeros of solutions to problem (4.1), (4.2) and their derivatives up to order $n-1$ can be studied by localization analysis of zeros of solutions to the differential inequality

$$(-1)^p\, u^{(n)}(t) \ge c > 0 \tag{4.8}$$

satisfying the boundary conditions (4.2). Define

$$\mathcal{B} = \{u \in AC^{n-1}[0,T] : u \text{ satisfies (4.2) and (4.8) holds for a.e. } t \in [0,T]\}.$$

Lemma 4.1. *Let* $u \in \mathcal{B}$ *and let* $p = 1$. *Then* $u > 0$ *on* $(0,T)$ *and* $u^{(j)}$ *has precisely one zero on* $(0,T)$, $1 \le j \le n-1$.

Proof. The assertion follows immediately from Lemmas 3.2 and 3.4. $\square$

Lemma 4.2. *Let* $u \in \mathcal{B}$, $p \geq 2$ *and let* (4.3) *hold. Then*

(i) $u > 0$ *on* $(0,T)$,

(ii) $u^{(k)}$ *has precisely* k *zeros in* $(0,T)$ *for* $k = 1,\dots,p-1$,

(iii) $u^{(k)}$ *has precisely* p *zeros in* $(0,T)$ *for* $k = p,\dots,n-p$,

(iv) $u^{(n-k)}$ *has precisely* k *zeros in* $(0,T)$ *for* $k = 1,\dots,p-1$.

Proof. The proof is divided into three steps.

Step 1. *Lower bounds for zeros.*

By (4.2) we see that u' has at least one zero $t_1^{(1)}$ in $(0,T)$. Hence $u'(0) = u'(t_1^{(1)}) = u'(T) = 0$, which implies that u'' has at least two zeros $t_1^{(2)}, t_2^{(2)}$ in $(0,T)$, $t_1^{(2)} < t_2^{(2)}$, and consequently (if $p \geq 3$)

$$u''(0) = u''(t_1^{(2)}) = u''(t_2^{(2)}) = u''(T) = 0.$$

By induction we conclude that $u^{(k)}$, $k = 3,\dots,p-1$, has at least k zeros $t_1^{(k)},\dots,t_k^{(k)}$ in $(0,T)$, $0 < t_1^{(k)} < \dots < t_k^{(k)} < T$ and, by (4.2) and (4.3),

$$u^{(k)}(0) = u^{(k)}(t_1^{(k)}) = \dots = u^{(k)}(t_k^{(k)}) = u^{(k)}(T) = 0, \quad k = 3,\dots,p-1.$$

Therefore $u^{(p)}$ has at least p zeros in $(0,T)$. Now we will distinguish two cases: (a) $p < \frac{n}{2}$ and (b) $p = \frac{n}{2}$.

Case (a). Let $p < \frac{n}{2}$. Then $p \leq n-p-1$ and, by (4.2),

$$u^{(j)}(0) = 0, \quad j = p,\dots,n-p-1.$$

Therefore $u^{(k)}$ has at least p zeros in $(0,T)$ for $k = p+1,\dots,n-p$.

Case (b). Let $p = \frac{n}{2}$ (clearly n is even in this case). Then $p = n-p$ and $u^{(n-p)}$ has at least p zeros in $(0,T)$.

We have shown that in both cases $u^{(n-p)}$ has at least p zeros in $(0,T)$. Since for $u^{(n-k)}$, $k = 1,\dots,p-1$, we cannot use (4.2) any more, we deduce that $u^{(n-k)}$ has at least k zeros in $(0,T)$ for $k = 1,\dots,p-1$. In particular $u^{(n-1)}$ has at least one zero in $(0,T)$.

Step 2. *Exact number of zeros.*

By inequality (4.8), $u^{(n-1)}$ is strictly monotonous on $[0,T]$ and hence it has precisely one zero in $(0,T)$. Therefore, by *Step* 1, $u^{(n-k)}$ has precisely

k zeros in $(0,T)$ for $2 \le k \le p-1$ and $u^{(k)}$ has precisely p zeros in $(0,T)$ for $p \le k \le n-p$. Similarly, $u^{(k)}$ has precisely k zeros in $(0,T)$ for $1 \le k \le p-1$ and u has no zero in $(0,T)$. We have proved that the statements (ii)–(iv) are true.

Step 3. *Positivity of* u.

Denote by $t_1^{(k)}$ the first zero of $u^{(k)}$ in $(0,T)$, $1 \le k \le n-1$. Inequality (4.8) implies that $(-1)^p\, u^{(n-1)} < 0$ on $[0, t_1^{(n-1)})$ and hence $(-1)^p\, u^{(n-2)} > 0$ on $[0, t_1^{(n-2)})$. Therefore $(-1)^{p+j}\, u^{(n-j)} > 0$ on $[0, t_1^{(n-j)})$ for $j = 3, \ldots, p$. In particular, we have $u^{(n-p)} > 0$ on $[0, t_1^{(n-p)})$, wherefore, by virtue of (4.2), we obtain $u^{(k)} > 0$ on $(0, t_1^{(k)})$, $1 \le k \le n-p-1$, and consequently $u > 0$ on $(0,T)$. $\square$

Our next result provides estimates from below of the absolute value of functions $u \in \mathcal{B}$ and their derivatives up to order $n-1$ on the interval $[0,T]$. These estimates are necessary for applying Theorem 1.7 to problem (4.1), (4.2) with f satisfying assumption (4.4).

Lemma 4.3. *Let* $u \in \mathcal{B}$ *and let* (4.3) *hold. Then for each* $i \in \{1, \ldots, n-1\}$ *there are* $p_i + 1$ *disjoint intervals* (a_k, a_{k+1}), $0 \le k \le p_i$, $p_i \le (n-1)\,p$ *such that*

$$\bigcup_{k=0}^{p_i} [a_k, a_{k+1}] = [0,T] \tag{4.9}$$

and for each $k \in \{0, \ldots, p_i\}$ *one of the inequalities*

$$|u^{(n-i)}(t)| \ge \frac{c}{i!}(t-a_k)^i \qquad \textit{for} \quad t \in [a_k, a_{k+1}] \tag{4.10}$$

or

$$|u^{(n-i)}(t)| \ge \frac{c}{i!}(a_{k+1}-t)^i \quad \textit{for} \quad t \in [a_k, a_{k+1}] \tag{4.11}$$

is satisfied.

Proof. Let $t_i^{(j)}$ be zeros of $u^{(j)}$ in $(0,T)$, $1 \le j \le n-1$, described in Lemmas 4.1 and 4.2. Integrating inequality (4.8) yields

$$\begin{cases} (-1)^{p+1}\, u^{(n-1)}(t) \ge c\,(t_1^{(n-1)} - t) & \text{for} \quad t \in [0, t_1^{(n-1)}], \\ \quad (-1)^{p}\, u^{(n-1)}(t) \ge c\,(t - t_1^{(n-1)}) & \text{for} \quad t \in [t_1^{(n-1)}, T]. \end{cases} \tag{4.12}$$

Now, integrating the first inequality in (4.12) from $t \in [0, t_1^{(n-2)})$ to $t_1^{(n-2)}$ gives

$$(-1)^p u^{(n-2)}(t) \geq \frac{c}{2}\left[(t_1^{(n-1)}-t)^2-(t_1^{(n-1)}-t_1^{(n-2)})^2\right] \geq \frac{c}{2!}\,(t_1^{(n-2)}-t)^2.$$

Hence, we get by such procedure that

$$\begin{cases} (-1)^p\, u^{(n-2)}(t) \geq \dfrac{c}{2!}\,(t_1^{(n-2)}-t)^2 & \mathit{for}\ t \in [0, t_1^{(n-2)}], \\ (-1)^{p+1}\, u^{(n-2)}(t) \geq \dfrac{c}{2!}\,(t-t_1^{(n-2)})^2 & \mathit{for}\ t \in [t_1^{(n-2)}, t_1^{(n-1)}], \\ (-1)^{p+1}\, u^{(n-2)}(t) \geq \dfrac{c}{2!}\,(t_2^{(n-2)}-t)^2 & \mathit{for}\ t \in [t_1^{(n-1)}, t_2^{(n-2)}], \\ (-1)^p\, u^{(n-2)}(t) \geq \dfrac{c}{2!}\,(t-t_2^{(n-2)})^2 & \mathit{for}\ t \in [t_2^{(n-2)}, T]. \end{cases} \tag{4.13}$$

Let us choose $i \in \{1, \ldots, n-1\}$ and take all different zeros of functions $u^{(n-1)}, \ldots, u^{(n-i)}$, which are in $(0, T)$. By Lemmas 4.1 and 4.2, there is a finite number $p_i \leq (n-1)\,p$ of these zeros. Let us put them in the natural order and denote by $a_1, \ldots, a_{p_i}$. Set $a_0 = 0$, $a_{p_i+1} = T$. Thus we get p_i+1 disjoint intervals (a_k, a_{k+1}), $0 \leq k \leq p_i$, satisfying (4.9).

If $i = 1$, then for $a_1 = t_1^{(n-1)}$ and $a_2 = T$ we get by (4.12) that

$$|u^{(n-1)}(t)| \geq c\,(a_1 - t) \quad \text{for} \quad t \in [a_0, a_1]$$

and

$$|u^{(n-1)}(t)| \geq c\,(t - a_1) \quad \text{for} \quad t \in [a_1, a_2].$$

If $i = 2$, we put $t_1^{(n-2)} = a_1$, $t_1^{(n-1)} = a_2$, $t_2^{(n-2)} = a_3$, $T = a_4$, and then inequality (4.13) gives (4.10) or (4.11).

If $i > 2$ and we integrate the inequalities in (4.13) $(i-2)$-times, we get that on each $[a_k, a_{k+1}]$, $k \in \{0, \ldots, p_i\}$ either (4.10) or (4.11) has to be fulfilled. □

Existence result

In order to prove the main result (Theorem 4.7) we will need the following three lemmas.

Lemma 4.4. *Let conditions (4.3) and (4.6) hold. Then there exist constants* $A_i > 0$, $0 \le i \le n-1$, *such that for each* $u \in \mathcal{B}$ *the estimates*

$$\int_0^T \omega_i(|u^{(i)}(t)|)\,\mathrm{d}t \le A_i, \quad 0 \le i \le n-1, \tag{4.14}$$

are satisfied.

Proof. Let $u \in \mathcal{B}$ and let $i \in \{0, \ldots, n-1\}$. By Lemma 4.3 there exist $p_i + 1$ disjoint intervals (a_k, a_{k+1}), $0 \le k \le p_i$, $p_i \le (n-1)\,p$, such that (4.9) and either (4.10) or (4.11) are satisfied. Since ω_i is nonincreasing, inequalities (4.10) and (4.11) give

$$\int_0^T \omega_i(|u^{(i)}(t)|)\,\mathrm{d}t = \sum_{k=0}^{p_i} \int_{a_k}^{a_{k+1}} \omega_i(|u^{(i)}(t)|)\,\mathrm{d}t$$

$$< \sum_{k=0}^{p_i} \Big[\int_{a_k}^{a_{k+1}} \omega_i\Big(\frac{c}{(n-i)!}\,(t-a_k)^{n-i}\Big)\,\mathrm{d}t$$

$$+ \int_{a_k}^{a_{k+1}} \omega_i\Big(\frac{c}{(n-i)!}\,(a_{k+1}-t)^{n-i}\Big)\,\mathrm{d}t\Big].$$

If we put $c_i = (\frac{c}{(n-i)!})^{1/(n-i)}$, we have

$$\int_0^T \omega_i(|u^{(i)}(t)|)\,\mathrm{d}t < \frac{2p_i}{c_i} \int_0^{c_i T} \omega_i(s^{n-i})\,\mathrm{d}s < \frac{n(n-1)}{c_i} \int_0^{c_i T} \omega_i(s^{n-i})\,\mathrm{d}s.$$

Hence inequality (4.14) holds with

$$A_i = \frac{n(n-1)}{c_i} \int_0^{c_i T} \omega_i(s^{n-i})\,\mathrm{d}s$$

and, by assumption (4.6), $A_i < \infty$ for $0 \le i \le n-1$. $\square$

Lemma 4.5. *Let conditions (4.3) and (4.6) hold and let* $\{u_m\} \subset \mathcal{B}$. *Then for* $0 \le i \le n-1$ *the sequence* $\{\omega_i(|u_m^{(i)}(t)|)\}$ *is uniformly integrable on* $[0, T]$.

Proof. Let $i \in \{0, \ldots, n-1\}$. Then, by Lemma 4.3, there exist $p_{m,i} + 1$ disjoint intervals $(a_{m,k}, a_{m,k+1})$, $0 \le k \le p_{m,i}$, $p_{m,i} \le (n-1)\,p$, such that

$$\bigcup_{k=0}^{p_{m,i}} [a_{m,k}, a_{m,k+1}] = [0, T],$$

and for each $k \in \{0, \ldots, p_{m,i}\}$ and $m \in \mathbb{N}$ one of the inequalities

$$|u_m^{(i)}(t)| \geq \frac{c}{(n-i)!}(t - a_{m,k})^{n-i} \quad \text{for} \quad t \in [a_{m,k}, a_{m,k+1}]$$

or

$$|u_m^{(i)}(t)| \geq \frac{c}{(n-i)!}(a_{m,k+1} - t)^{n-i} \quad \text{for} \quad t \in [a_{m,k}, a_{m,k+1}]$$

is satisfied. Now the uniform integrability of $\{\omega_i(|u_m^{(i)}(t)|)\}$ on $[0,T]$ follows from Criterion A.3. $\square$

Lemma 4.6. *Let conditions (4.3), (4.5) and (4.6) hold. Then there exists a positive constant $S \geq n$ such that for each $u \in \mathcal{B}$ satisfying*

$$(-1)^p\, u^{(n)}(t) \leq h\Big(t, n + \sum_{j=0}^{n-1} |u^{(j)}(t)|\Big) + \sum_{j=0}^{n-1} [\omega_j(|u^{(j)}(t)|) + \omega_j(1)] \tag{4.15}$$

for a.e. $t \in [0,T]$, the estimate

$$\|u\|_{C^{n-1}} < S \tag{4.16}$$

holds.

Proof. Let $u \in \mathcal{B}$. By Lemmas 4.1 and 4.2 and by condition (4.2) we find $t_j \in (0,T)$ such that $u^{(j)}(t_j) = 0$ for $0 \leq j \leq n-2$. Put

$$\max\{|u^{(n-1)}(t)| : 0 \leq t \leq T\} = \rho.$$

Then $-\rho \leq u^{(n-1)}(t) \leq \rho$ for $t \in [0,T]$. Integrate this inequality from t_{n-2} to $t \in (t_{n-2}, T]$ and from $t \in [0, t_{n-2})$ to t_{n-2}. We get $-\rho\, T \leq u^{(n-2)}(t) \leq \rho\, T$ on $[0,T]$. Similarly, using $u^{(j)}(t_j) = 0$ for $0 \leq j \leq n-2$ and repeating the integration, we obtain step by step

$$|u^{(j)}(t)| \leq \rho\, T^{n-j-1}, \quad t \in [0,T], \quad 0 \leq j \leq n-3.$$

Hence

$$\|u\|_{C^{n-1}} \leq \rho\, K, \tag{4.17}$$

where K is taken from condition (4.5). Now, integrating inequality (4.15) over $[0, t_{n-1}]$ and $[t_{n-1}, T]$ and using the fact that $t_{n-1} \in (0,T)$ is the unique zero of $u^{(n-1)}$ by Lemmas 4.1 and 4.2 (and therefore $(-1)^p u^{(n-1)} < 0$ on $[0, t_{n-1})$ and $(-1)^p u^{(n-1)} > 0$ on $(t_{n-1}, T]$ due to (4.8)) we get

$$\begin{aligned} 0 &< (-1)^{p+1} u^{(n-1)}(t) \\ &\leq \int_t^{t_{n-1}} h\Big(s, n + \sum_{j=0}^{n-1} |u^{(j)}(s)|\Big)\, \mathrm{d}s + \sum_{j=0}^{n-1} \int_t^{t_{n-1}} [\omega_j(|u^{(j)}(s)|) + \omega_j(1)]\, \mathrm{d}s \end{aligned}$$

for $t \in [0, t_{n-1}]$ and

$$\begin{aligned} 0 &< (-1)^{p} u^{(n-1)}(t) \\ &\leq \int_{t_{n-1}}^{t} h\Big(s, n + \sum_{j=0}^{n-1} |u^{(j)}(s)|\Big)\, \mathrm{d}s + \sum_{j=0}^{n-1} \int_{t_{n-1}}^{t} [\omega_j(|u^{(j)}(s)|) + \omega_j(1)]\, \mathrm{d}s \end{aligned}$$

for $t \in [t_{n-1}, T]$. Hence, by (4.5) and (4.17),

$$|u^{(n-1)}(t)| \leq \int_0^T h(t, n + \rho K)\, \mathrm{d}t + \sum_{j=0}^{n-1} \Big[\int_0^T \omega_j(|u^{(j)}(t)|)\, \mathrm{d}t + T\,\omega_j(1) \Big]$$

for $t \in [0, T]$. Further, by Lemma 4.4, we can find positive constants A_j, $0 \leq j \leq n-1$, independent of u and satisfying inequality (4.14). Therefore, if we put

$$A = \sum_{j=0}^{n-1} [A_j + T\,\omega_j(1)],$$

we have

$$\rho \leq \int_0^T h(t, n + \rho K)\, \mathrm{d}t + A. \tag{4.18}$$

Since, by condition (4.5), $\limsup_{z\to\infty} \frac{1}{z} \int_0^T h(t,z)\, \mathrm{d}t < \frac{1}{K}$, there exists a positive constant $S \geq n$ such that

$$\int_0^T h(t, n + Kz)\, \mathrm{d}t + A < z$$

whenever $z \geq S$. This and (4.18) give $\rho < S$, which shows that inequality (4.16) is true. $\square$

Theorem 4.7. *Let conditions* (4.3)–(4.7) *hold. Then problem* (4.1), (4.2) *has a solution* $u \in AC^{n-1}[0,T]$ *and* $u > 0$ *on* $(0,T)$.

Proof. *Step* 1. *Construction of auxiliary regular problems.*

We construct auxiliary regular problems. Let S be the constant from Lemma 4.6 satisfying inequality (4.16). Set

$$\sigma_0(x) = \begin{cases} |x| & \text{for } |x| \le S, \\ S & \text{for } |x| > S, \end{cases} \qquad \sigma(x) = \begin{cases} x & \text{for } |x| \le S, \\ \dfrac{Sx}{|x|} & \text{for } |x| > S. \end{cases}$$

Choose $m \in \mathbb{N}$ and first define an auxiliary function $h_m \in Car([0,T] \times \mathbb{R}^{n-1})$ by the following recurrent formulas:

$$h_{m,0}(t,x_0,x_1,\dots,x_{n-1}) = \begin{cases} f(t,x_0,x_1,\dots,x_{n-1}) & \text{if } x_0 \ge \frac{1}{m}, \\ f(t,\frac{1}{m},x_1,\dots,x_{n-1}) & \text{if } x_0 < \frac{1}{m}, \end{cases}$$

$$h_{m,i}(t,x_0,\dots,x_i,\dots,x_{n-1}),$$

$$= \begin{cases} h_{m,i-1}(t,x_0,\dots,x_i,\dots,x_{n-1}) & \text{if } |x_i| \ge \frac{1}{m}, \\ \frac{m}{2}\Big[h_{m,i-1}(t,x_0,\dots,x_{i-1},\frac{1}{m},x_{i+1},\dots,x_{n-1})(x_i+\frac{1}{m}) & \\ \quad -h_{m,i-1}(t,x_0,\dots,x_{i-1},-\frac{1}{m},x_{i+1},\dots,x_{n-1})(x_i-\frac{1}{m})\Big] & \\ & \text{if } |x_i| < \frac{1}{m}, \end{cases}$$

for $1 \le i \le n-1$ and

$$h_m(t,x_0,\dots,x_{n-1}) = h_{m,n-1}(t,x_0,\dots,x_{n-1}).$$

Finally, for a.e. $t \in [0,T]$ and all $(x_0,\dots,x_{n-1}) \in \mathbb{R}^n$ put

$$f_m(t,x_0,x_1,\dots,x_{n-1}) = h_m(t,\sigma_0(x_0),\sigma(x_1),\dots,\sigma(x_{n-1})). \tag{4.19}$$

Then $f_m \in Car([0,T] \times \mathbb{R}^n)$ for $m \in \mathbb{N}$ and, by (4.4) and (4.19),

$$c \le f_m(t,x_0,\dots,x_{n-1}) \le g_m(t) \tag{4.20}$$

for a.e. $t \in [0,T]$ and all $(x_0,\dots,x_{n-1}) \in \mathbb{R}^n$, where $g_m \in L_1[0,T]$. Further, for $(x_0,\dots,x_{n-1}) \in \mathbb{R}^n$ and $m \in \mathbb{N}$ we have

$$\max\{\sigma_0(x_0), \tfrac{1}{m}\} \le |x_0| + 1,$$

$$\omega_0(\max\{\sigma_0(x_0), \tfrac{1}{m}\}) < \omega_0(|x_0|) + \omega_0(S) < \omega_0(|x_0|) + \omega_0(1)$$

and similarly

$$\max\{\sigma(x_i), \tfrac{1}{m}\} \le |x_i| + 1,$$

$$\omega_i(\max\{\sigma(x_i), \tfrac{1}{m}\}) < \omega_i(|x_i|) + \omega_i(1), \quad 1 \le i \le n-1.$$

Therefore, by assumption (4.7), for each $m \in \mathbb{N}$ we have

$$f_m(t, x_0, \dots, x_{n-1}) \le h\Big(t, n + \sum_{j=0}^{n-1} |x_j|\Big) + \sum_{j=0}^{n-1} [\omega_j(|x_j|) + \omega_j(1)] \tag{4.21}$$

for a.e. $t \in [0,T]$ and all $(x_0,\dots,x_{n-1}) \in \mathbb{R}^n$.

Consider the regular differential equation

$$(-1)^p\, u^{(n)} = f_m(t, x_0, \dots, x_{n-1}). \tag{4.22}$$

Since the homogeneous problem $(-1)^p\, u^{(n)} = 0$, (4.2) has only the trivial solution and f_m satisfies inequality (4.20), the Fredholm type existence theorem guarantees that for each $m \in \mathbb{N}$ there exists a solution $u_m \in AC^{n-1}[0,T]$ of problem (4.22), (4.2). Then it follows from inequalities (4.20) and (4.21) that for each $m \in \mathbb{N}$, $u_m \in \mathcal{B}$ and inequality (4.15) holds with $u = u_m$. Hence Lemma 4.6 shows that

$$\|u_m\|_{C^{n-1}} < S, \quad m \in \mathbb{N}, \tag{4.23}$$

and, by Lemma 4.3, for each $i \in \{1,\dots,n-1\}$ there exist $p_{m,i}+1$ disjoint intervals $(a_{m,k}, a_{m,k+1})$, $0 \le k \le p_{m,i}$, $p_{m,i} \le (n-1)p$ such that

$$\bigcup_{k=0}^{p_{m,i}} [a_{m,k}, a_{m,k+1}] = [0,T],$$

and for each $k \in \{0,\dots,p_{m,i}\}$ and $m \in \mathbb{N}$ one of the inequalities

$$|u_m^{(n-i)}(t)| \ge \frac{c}{i!}(t - a_{m,k})^i \quad \text{for } t \in [a_{m,k}, a_{m,k+1}]$$

or

$$|u_m^{(n-i)}(t)| \geq \frac{c}{i!}(a_{m,k+1} - t)^i \quad \text{for} \quad t \in [a_{m,k}, a_{m,k+1}]$$

is satisfied.

Step 2. Uniform integrability.

Consider the sequence

$$\{f_m(t, u_m(t), \dots, u_m^{(n-1)}(t))\} \subset L_1[0,T]. \tag{4.24}$$

Inequalities (4.20) and (4.21) show that

$$0 < f_m(t, u_m(t), \dots, u_m^{(n-1)}(t))$$
$$\leq h\Big(t, n + \sum_{j=0}^{n-1} |u_m^{(j)}(t)|\Big) + \sum_{j=0}^{n-1} [\omega_j(|u_m^{(j)}(t)|) + \omega_j(1)]$$

for $m \in \mathbb{N}$ and a.e. $t \in [0,T]$. Since $h \in Car([0,T] \times [0,\infty))$ and u_m satisfies (4.23), there exists $h^* \in L_1[0,T]$ such that

$$h\Big(t, n + \sum_{j=0}^{n-1} |u_m^{(j)}(t)|\Big) \leq h^*(t) \quad \text{for a.e.} \quad t \in [0,T] \quad \text{and all} \quad m \in \mathbb{N}.$$

Hence, in order to prove that (4.24) is uniformly integrable on $[0,T]$ it suffices to show that the sequences

$$\{\omega_j(|u_m^{(j)}(t)|)\}, \quad j = 0, \dots, n-1,$$

are uniformly integrable on $[0,T]$. This fact follows from Lemma 4.5 since $\{u_m\} \subset \mathcal{B}$. We have proved that (4.24) is uniformly integrable on $[0,T]$.

Step 3. Existence of a solution of problem (4.1), (4.2).

Consider the sequence $\{u_m\}$ where u_m is a solution of problem (4.22), (4.2). We know that (4.23) holds and since (4.24) is uniformly integrable on $[0,T]$, the sequence $\{u_m^{(n-1)}\}$ is equicontinuous on $[0,T]$. Hence, by the Arzelà-Ascoli theorem, there exist $u \in C^{n-1}[0,T]$ and a subsequence $\{u_{l_m}\} \subset \{u_m\}$ such that

$$\lim_{m\to\infty} \|u_{l_m} - u\|_{C^{n-1}} = 0.$$

Letting $m \to \infty$ and working with subsequences if necessary, we get

$$\lim_{m\to\infty} p_{l_m,i} = p_i, \quad p_i \le (n-1)\,p, \quad 1 \le i \le n-1,$$

and

$$\lim_{m\to\infty} a_{l_m,k} = a_k, \quad 0 \le k \le p_i,$$

where $0 = a_0 \le a_1 \le \cdots \le a_{p_i} \le T$. In addition, (4.9) and either (4.10) or (4.11) hold. Hence $u^{(i)}$, $0 \le i \le n-1$, has a finite number of zeros. Therefore, by Theorem 1.7, $u \in AC^{n-1}[0,T]$ and u is a solution of problem (4.1), (4.2). From assumption (4.4) and Lemmas 4.1 and 4.2 we get $u > 0$ on $(0,T)$. □

EXAMPLE. Let p be a positive integer, $1 \le p \le n-1$. Consider the differential equation

$$(-1)^p\, u^{(n)} = \frac{1}{u^{\alpha_0}} + u^{\beta_0} + \sum_{j=1}^{n-1} \left(\frac{a_j(t)}{|u^{(j)}|^{\alpha_j}} + b_j(t)\, |u^{(j)}|^{\beta_j} \right) \tag{4.25}$$

where the functions $a_j \in L_\infty[0,T]$, $b_j \in L_1[0,T]$ are nonnegative, $\alpha_j \in (0, \frac{1}{n-j})$ and $\beta_j \in (0,1)$ for $0 \le j \le n-1$. Applying Theorem 4.7, problem (4.25), (4.2) has a solution $u \in AC^{n-1}[0,T]$ and $u > 0$ on $(0,T)$.

Bibliographical notes

Theorem 4.7 was adapted from Rachůnková and Staněk [160], [162]. Singular $(p, n-p)$ conjugate problems were discussed by Agarwal and O'Regan in [6], [10] and by Eloe and Henderson in [80] (here with $p=1$) and [81] for differential equations of the type

$$(-1)^{n-p}\, u^{(n)} = f(t,u),$$

where $f \in C^0((0,1) \times (0,\infty))$ is positive and f may be singular at $u = 0$. Here positive solutions on $(0,1)$ belong to the class $C^{n-1}[0,T] \cap C^n(0,1)$. The paper [10] discussed also the existence of two positive solutions. Existence results in [10], [80] and [81] are proved by fixed point theorems on cones, whereas those in [6] by a combination of a sequential technique and a nonlinear alternative of Leray-Schauder type.

Chapter 5

Sturm-Liouville problem

We are now concerned with the Sturm-Liouville problem for the differential equation

$$-u^{(n)} = f(t, u, \dots, u^{(n-1)}) \tag{5.1}$$

with the boundary conditions

$$\begin{cases} u^{(j)}(0) = 0, \quad 0 \le j \le n-3, \\ \alpha\, u^{(n-2)}(0) - \beta\, u^{(n-1)}(0) = 0, \\ \gamma\, u^{(n-2)}(T) + \delta\, u^{(n-1)}(T) = 0, \end{cases} \tag{5.2}$$

where $n \ge 3$, $\alpha, \gamma > 0$, $\beta, \delta \ge 0$. Here

$$f \in Car([0,T] \times \mathcal{D}) \ \text{ and } \ \mathcal{D} = (0,\infty)^{n-1} \times (\mathbb{R} \setminus \{0\}).$$

Notice that the function f may be singular at the value 0 of any of its space variables. If f is positive, the solutions of problem (5.1), (5.2) have singular points of type I at the end points of the interval $[0,T]$ and also singular points of type II.

We will impose the following conditions on the function f in (5.1):

$$\begin{cases} f \in Car([0,T] \times \mathcal{D}) \quad \text{where } \mathcal{D} = (0,\infty)^{n-1} \times (\mathbb{R} \setminus \{0\}) \\ \text{and there exist positive constants } a \text{ and } r \text{ such that} \\ \qquad a\, t^r \le f(t, x_0, \dots, x_{n-1}) \\ \text{for a.e. } t \in [0,T] \text{ and each } (x_0, \dots, x_{n-1}) \in \mathcal{D}, \end{cases} \tag{5.3}$$

$$\begin{cases} h \in Car([0,T] \times [0,\infty)) \quad \text{is positive and nondecreasing} \\ \text{in the second variable and} \\ \qquad \limsup_{v \to \infty} \dfrac{1}{v} \displaystyle\int_0^T h(t, Vv)\, \mathrm{d}t < 1 \\ \text{where } V = n\Big(\dfrac{\beta}{\alpha} + T\Big) \max\Big\{ \dfrac{T^{\,n-j-2}}{(n-j-2)!} : 0 \le j \le n-2 \Big\}, \end{cases} \tag{5.4}$$

$$\begin{cases} f(t,x_0,\dots,x_{n-1}) \le h\Big(t,\displaystyle\sum_{j=0}^{n-1}|x_j|\Big) + \displaystyle\sum_{j=0}^{n-1}\omega_j(|x_j|) \\ \text{for a.e. } t\in[0,T] \text{ and each } (x_0,\dots,x_{n-1})\in\mathcal{D}, \\ \text{where } \ \omega_j:(0,\infty)\to(0,\infty) \text{ are nonincreasing, } 0\le j\le n-1, \\ \text{and} \\ \displaystyle\int_0^1 \omega_{n-1}(t^{r+1})\,\mathrm{d}t<\infty, \int_0^1 \omega_j(t^{n-j-1})\,\mathrm{d}t<\infty,\ 0\le j\le n-2, \end{cases} \tag{5.5}$$

$$\begin{cases} f(t,x_0,\dots,x_{n-1}) \le h\Big(t,\displaystyle\sum_{j=0}^{n-1}|x_j|\Big) + \displaystyle\sum_{\substack{j=0\\ j\neq n-2}}^{n-1}\omega_j(|x_j|) \\ \qquad\qquad\qquad\qquad\qquad\qquad +q(t)\,\omega_{n-2}(|x_{n-2}|) \\ \text{for a.e. } t\in[0,T] \text{ and each } \ (x_0,\dots,x_{n-1})\in\mathcal{D}, \\ \text{where } \ q\in L_1[0,T] \text{ is nonnegative, } \omega_j:(0,\infty)\to(0,\infty) \\ \text{are nonincreasing, } \ 0\le j\le n-1, \ \text{ and} \\ \displaystyle\int_0^1 \omega_{n-1}(t^{r+1})\mathrm{d}t<\infty, \int_0^1 \omega_j(t^{n-j-2})\mathrm{d}t<\infty,\ 0\le j\le n-3. \end{cases} \tag{5.6}$$

Green function and a priori estimates

We denote by $G(t,s)$ the Green function of the problem

$$-u''=0, \tag{5.7}$$

$$\alpha\,u(0)-\beta\,u'(0)=0, \quad \gamma\,u(T)+\delta\,u'(T)=0, \tag{5.8}$$

where $\alpha,\gamma>0$ and $\beta,\delta\ge 0$. Then (see e.g. Agarwal [1])

$$G(t,s)=\begin{cases} \dfrac{1}{d}\,(\beta+\alpha\,s)\,(\delta+\gamma\,(T-t)) & \text{for } \ 0\le s\le t\le T \\ \dfrac{1}{d}\,(\beta+\alpha\,t)\,(\delta+\gamma\,(T-s)) & \text{for } \ 0\le t<s\le T, \end{cases} \tag{5.9}$$

where $d = \alpha\gamma T + \alpha\delta + \beta\gamma > 0$. We will discuss two cases, namely $\min\{\beta,\delta\} = 0$, that is, at least one of the constants β and δ equals zero, and $\min\{\beta,\delta\} > 0$, that is, both the constants β and δ are positive.

Let us choose positive constants a and r and define a set

$$\mathcal{A}(r,a) = \{u \in AC^{n-1}[0,T] : \ u \text{ fulfils } (5.2) \text{ and } (5.10)\}$$

where

$$-u^{(n)}(t) \ge a\,t^r \quad \text{for a.e. } t \in [0,T]. \tag{5.10}$$

Lemma 5.1. *Let* $\min\{\beta,\delta\} = 0$. *Let* $u \in \mathcal{A}(r,a)$ *and set*

$$A = \frac{a}{(r+1)(r+2)}\left(\frac{T}{2}\right)^{r+1}. \tag{5.11}$$

Then $u^{(n-1)}$ *is decreasing on* $[0,T]$,

$$u^{(n-1)}(t) \begin{cases} \ge \dfrac{a}{r+1}(\xi - t)^{r+1} & \text{if } t \in [0,\xi], \\ < -\dfrac{a}{r+1}(t-\xi)^{r+1} & \text{if } t \in (\xi, T] \end{cases} \tag{5.12}$$

where $\xi \in (0,T)$ *is the unique zero of* $u^{(n-1)}$,

$$u^{(n-2)}(t) \ge \begin{cases} A\,t & \text{if } t \in \left[0, \frac{T}{2}\right], \\ A\,(T-t) & \text{if } t \in \left(\frac{T}{2}, T\right] \end{cases} \tag{5.13}$$

and

$$u^{(j)}(t) \ge \frac{A}{4\,(n-j-1)!}\,t^{\,n-j-1} \quad \text{for } t \in [0,T],\ 0 \le j \le n-3. \tag{5.14}$$

Proof. From (5.9), (5.10) and the equality

$$u^{(n-2)}(t) = -\int_0^T G(t,s)\,u^{(n)}(s)\,\mathrm{d}s, \quad t \in [0,T],$$

it follows that

$$\begin{cases} u^{(n-2)}(0) = -\dfrac{\beta}{d}\displaystyle\int_0^T (\delta + \gamma\,(T-s))\,u^{(n)}(s)\,\mathrm{d}s \\ \qquad\qquad\ \ge \dfrac{a\,\beta\,\gamma}{d}\displaystyle\int_0^T (T-s)\,s^r\,\mathrm{d}s \ge 0, \end{cases} \tag{5.15}$$

$$\begin{cases} u^{(n-2)}(T) = -\dfrac{\delta}{d}\displaystyle\int_0^T (\beta + \alpha\, s)\, u^{(n)}(s)\, \mathrm{d}s \\ \qquad\qquad \geq \dfrac{a\,\alpha\,\delta}{d}\displaystyle\int_0^T s^{r+1}\, \mathrm{d}s \geq 0, \end{cases} \tag{5.16}$$

$$u^{(n-1)}(0) = -\int_0^T \frac{\partial G(t,s)}{\partial t}\Big|_{t=0} u^{(n)}(s)\, \mathrm{d}s$$

$$= -\frac{\alpha}{d}\int_0^T (\delta + \gamma\,(T-s))\, u^{(n)}(s)\, \mathrm{d}s \geq \frac{a\,\alpha\,\gamma}{d}\int_0^T (T-s)\, s^r\, \mathrm{d}s > 0$$

and

$$u^{(n-1)}(T) = -\int_0^T \frac{\partial G(t,s)}{\partial t}\Big|_{t=T} u^{(n)}(s)\, \mathrm{d}s$$

$$= \frac{\gamma}{d}\int_0^T (\beta + \alpha\, s)\, u^{(n)}(s)\, \mathrm{d}s \leq -\frac{a\,\alpha\,\gamma}{d}\int_0^T s^{r+1}\, \mathrm{d}s < 0.$$

Since $u^{(n-1)}$ is decreasing on $[0,T]$ by inequality (5.10) and

$$u^{(n-1)}(0) > 0, \quad u^{(n-1)}(T) < 0,$$

we see that $u^{(n-1)}$ has a unique zero $\xi \in (0,T)$. Then

$$-u^{(n-1)}(t) = \int_t^\xi u^{(n)}(s)\, \mathrm{d}s \leq -a\int_t^\xi s^r\, \mathrm{d}s = -\frac{a}{r+1}\,(\xi^{r+1} - t^{r+1})$$

for $t \in [0,\xi]$. Hence,

$$u^{(n-1)}(t) \geq \frac{a}{r+1}(\xi - t)^{r+1}, \quad t \in [0,\xi]$$

because of $\xi^{r+1} - t^{r+1} \geq (\xi - t)^{r+1}$ for $t \in [0,\xi]$. Similarly, using the inequality $t^{r+1} - \xi^{r+1} > (t-\xi)^{r+1}$, we get

$$u^{(n-1)}(t) = \int_\xi^t u^{(n)}(s)\, \mathrm{d}s \leq -a\int_\xi^t s^r\, \mathrm{d}s$$

$$= -\frac{a}{r+1}(t^{r+1} - \xi^{r+1}) < -\frac{a}{r+1}(t-\xi)^{r+1} \quad \text{for } t \in (\xi, T].$$

We have proved that inequality (5.12) holds.

We now verify inequality (5.13). From the first equalities in (5.15) and (5.16) and from the assumption $\min\{\beta,\delta\}=0$ it follows that

$$\min\{u^{(n-2)}(0), u^{(n-2)}(T)\}=0.$$

Moreover, by inequality (5.10), $u^{(n-2)}$ is concave on $[0,T]$ and consequently to prove (5.13) it suffices to show that

$$u^{(n-2)}\left(\frac{T}{2}\right)\geq A\,\frac{T}{2}. \tag{5.17}$$

Due to inequality (5.12) we have

$$\begin{aligned} u^{(n-2)}(t) &= u^{(n-2)}(0)+\int_0^t u^{(n-1)}(s)\,\mathrm{d}s \geq \frac{a}{r+1}\int_0^t (\xi-s)^{r+1}\,\mathrm{d}s \\ &= \frac{a}{(r+1)(r+2)}\left(\xi^{r+2}-(\xi-t)^{r+2}\right)\geq \frac{a}{(r+1)(r+2)}\,t^{r+2} \end{aligned}$$

for $t\in[0,\xi]$, since $\xi^{r+2}-(\xi-t)^{r+2}\geq t^{r+2}$ holds in such a case. Similarly, by (5.12), we obtain

$$\begin{aligned} u^{(n-2)}(t) &= u^{(n-2)}(T)-\int_t^T u^{(n-1)}(s)\,\mathrm{d}s > \frac{a}{r+1}\int_t^T (s-\xi)^{r+1}\,\mathrm{d}s \\ &= \frac{a}{(r+1)(r+2)}\left((T-\xi)^{r+2}-(t-\xi)^{r+2}\right) \\ &\geq \frac{a}{(r+1)(r+2)}\,(T-t)^{r+2} \end{aligned}$$

for $t\in(\xi,T]$, since $(T-\xi)^{r+2}-(t-\xi)^{r+2}\geq(T-t)^{r+2}$ holds in such a case. Summarizing, we have

$$u^{(n-2)}(t)\geq \frac{a}{(r+1)(r+2)}\,t^{r+2} \qquad \text{if } t\in[0,\xi] \tag{5.18}$$

and

$$u^{(n-2)}(t)\geq \frac{a}{(r+1)(r+2)}\,(T-t)^{r+2} \quad \text{if } t\in(\xi,T]. \tag{5.19}$$

We know that $\max\{u^{(n-2)}(t) : t \in [0,T]\} = u^{(n-2)}(\xi)$. Consequently if $\xi \geq \frac{T}{2}$, then (5.11) and (5.18) yield (5.17) and if $\xi < \frac{T}{2}$ then (5.17) follows from (5.11) and (5.19).

It remains to prove inequality (5.14). Using (5.13) and $u^{(n-3)}(0) = 0$, we obtain

$$u^{(n-3)}(t) = \int_0^t u^{(n-2)}(s)\,\mathrm{d}s \geq A \int_0^t s\,\mathrm{d}s = \frac{A}{2}\,t^2 \quad \text{for} \quad t \in [0, \tfrac{T}{2}].$$

In particular, $u^{(n-3)}(\frac{T}{2}) \geq \frac{A}{2}(\frac{T}{2})^2$. Since $u^{(n-3)}$ is increasing and $(\frac{t}{2})^2 \leq (\frac{T}{2})^2$, we conclude that the inequality $u^{(n-3)}(\frac{T}{2}) \leq u^{(n-3)}(t)$ holds on $[\frac{T}{2}, T]$. Thus,

$$u^{(n-3)}(t) \geq A\,\frac{t^2}{4 \cdot 2!} \quad \text{for} \quad t \in [\tfrac{T}{2}, T].$$

Consequently,

$$u^{(n-3)}(t) \geq A\,\frac{t^2}{4 \cdot 2!} \quad \text{for} \quad t \in [0, T].$$

Now, using the equalities

$$u^{(j)}(t) = \int_0^t u^{(j+1)}(s)\,\mathrm{d}s \quad \text{for} \quad t \in [0,T] \quad \text{and} \quad 0 \leq j \leq n-4$$

we can verify that inequalities (5.14) are satisfied. □

Lemma 5.2. *Let* $\min\{\beta, \delta\} > 0$. *Let* $u \in \mathcal{A}(r,a)$ *and set*

$$B = \frac{a}{d}\,\min\left\{\beta\,\gamma \int_0^T (T-s)\,s^r\,\mathrm{d}s,\ \alpha\,\delta \int_0^T s^{r+1}\,\mathrm{d}s\right\} > 0. \tag{5.20}$$

Then $u^{(n-1)}$ *is decreasing on* $[0,T]$, $u^{(n-1)}$ *satisfies inequality* (5.12) *where* $\xi \in (0,T)$ *is its unique zero,*

$$u^{(n-2)}(t) \geq B \quad \textit{for} \quad t \in [0,T] \tag{5.21}$$

and

$$u^{(j)}(t) \geq \frac{B}{(n-j-2)!}\,t^{n-j-2} \quad \textit{for} \quad t \in [0,T], \quad 0 \leq j \leq n-3. \tag{5.22}$$

Proof. The properties of $u^{(n-1)}$ follow immediately from Lemma 5.1 and its proof. Next, by relations (5.15) and (5.16),

$$u^{(n-2)}(0) \geq \frac{a\,\beta\,\gamma}{d} \int_0^T (T-s)\,s^r\,\mathrm{d}s \geq B,$$

and

$$u^{(n-2)}(T) \geq \frac{a\,\alpha\,\delta}{d} \int_0^T s^{r+1}\,\mathrm{d}s \geq B.$$

Since $u^{(n-2)}$ is concave on $[0,T]$, these inequalities show that inequality (5.21) is true. Now (5.21) and the equalities $u^{(j)}(0) = 0, \quad 0 \leq j \leq n-3$ imply that inequality (5.22) holds. $\square$

Lemma 5.3. *Let* $\min\{\beta,\delta\} = 0$ *and let* h *and* $\omega_j (0 \leq j \leq n-1)$ *have the properties given in conditions* (5.4) *and* (5.5). *Then there exists a positive constant* S_0 *such that for each* $u \in \mathcal{A}(r,a)$ *satisfying*

$$-u^{(n)}(t) \leq h\Big(t, n + \sum_{j=0}^{n-1} |u^{(j)}(t)|\Big) + \sum_{j=0}^{n-1} \big[\omega_j(|u^{(j)}(t)|) + \omega_j(1)\big] \tag{5.23}$$

for a.e. $t \in [0,T]$, *the estimates*

$$\|u^{(j)}\|_\infty < S_0 \quad \textit{for } 0 \leq j \leq n-1 \tag{5.24}$$

are valid.

Proof. Let $u \in \mathcal{A}(r,a)$ satisfy inequality (5.23) for a.e. $t \in [0,T]$. By Lemma 5.1, $u^{(n-1)}$ has a unique zero $\xi \in (0,T)$ and u satisfies inequalities (5.12)–(5.14) with A given in (5.11). From

$$u^{(n-2)}(0) = \frac{\beta}{\alpha}\,u^{(n-1)}(0) \geq 0$$

it follows that

$$|u^{(n-2)}(t)| \leq \frac{\beta}{\alpha}\,u^{(n-1)}(0) + \int_0^t |u^{(n-1)}(s)|\,\mathrm{d}s \leq \left(\frac{\beta}{\alpha} + T\right) \|u^{(n-1)}\|_\infty$$

for $t \in [0,T]$. Thus

$$\|u^{(n-2)}\|_\infty \leq \left(\frac{\beta}{\alpha} + T\right) \|u^{(n-1)}\|_\infty \tag{5.25}$$

and then the equalities

$$u^{(j)}(t) = \frac{1}{(n-j-3)!}\int_0^t (t-s)^{n-j-3}\, u^{(n-2)}(s)\,\mathrm{d}s,\ \ t\in[0,T],\ \ 0\le j\le n-3,$$

give

$$\|u^{(j)}\|_\infty \le \frac{T^{\,n-j-2}}{(n-j-2)!}\|u^{(n-2)}\|_\infty \le \frac{T^{\,n-j-2}}{(n-j-2)!}\Big(\frac{\beta}{\alpha}+T\Big)\,\|u^{(n-1)}\|_\infty,$$

that is

$$\|u^{(j)}\|_\infty \le \frac{V}{n}\|u^{(n-1)}\|_\infty, \quad 0\le j\le n-3, \tag{5.26}$$

where V is given in condition (5.4). Now inequality (5.23) yields

$$\begin{aligned} |u^{(n-1)}(t)| &= \Big|\int_\xi^t u^{(n)}(s)\,\mathrm{d}s\Big| \\ &\le \int_0^T \Big[h\Big(s, n+\sum_{j=0}^{n-1}|u^{(j)}(s)|\Big) + \sum_{j=0}^{n-1}[\omega_j(|u^{(j)}(s)|)+\omega_j(1)]\Big]\,\mathrm{d}s \\ &\le \int_0^T \Big[h(s, n+V\|u^{(n-1)}\|_\infty) + \sum_{j=0}^{n-1}[\omega_j(|u^{(j)}(s)|)+\omega_j(1)]\Big]\,\mathrm{d}s, \end{aligned}$$

for all $t\in[0,T]$, i.e.

$$\begin{cases} |u^{(n-1)}(t)| \le \displaystyle\int_0^T \Big[h(s, n+V\|u^{(n-1)}\|_\infty) \\ \qquad\qquad\qquad + \displaystyle\sum_{j=0}^{n-1}[\omega_j(|u^{(j)}(s)|)+\omega_j(1)]\Big]\,\mathrm{d}s \quad \text{for } \ t\in[0,T]. \end{cases} \tag{5.27}$$

Set

$$K = \sqrt[r+1]{\frac{a}{r+1}} \quad \text{and} \quad r_j = \sqrt[n-j-1]{\frac{A}{4(n-j-1)!}}, \quad 0\le j\le n-3.$$

Since (see inequalities (5.12)–(5.14))

$$\begin{cases} \displaystyle\int_0^T \omega_{n-1}(|u^{(n-1)}(t)|)\,dt \\ \displaystyle\le\int_0^{\xi} \omega_{n-1}\Big(\frac{a}{r+1}(\xi-t)^{r+1}\Big)\,\mathrm{d}t+\int_{\xi}^{T} \omega_{n-1}\Big(\frac{a}{r+1}(t-\xi)^{r+1}\Big)\,\mathrm{d}t \\ \displaystyle=\frac{1}{K}\Big[\int_0^{K\xi} \omega_{n-1}(t^{r+1})\,\mathrm{d}t+\int_0^{K(T-\xi)} \omega_{n-1}(t^{r+1})\,\mathrm{d}t\Big] \\ \displaystyle\le\frac{2}{K}\int_0^{KT} \omega_{n-1}(t^{r+1})\,\mathrm{d}t, \end{cases} \tag{5.28}$$

$$\begin{aligned} \int_0^T \omega_{n-2}(|u^{(n-2)}(t)|)\,\mathrm{d}t &\le \int_0^{T/2} \omega_{n-2}(At)\,\mathrm{d}t+\int_{T/2}^{T} \omega_{n-2}(A(T-t))\,\mathrm{d}t \\ &= \frac{2}{A}\int_0^{(AT)/2} \omega_{n-2}(t)\,\mathrm{d}t \end{aligned}$$

and (for $0 \le j \le n-3$)

$$\begin{aligned} \int_0^T \omega_j(|u^{(j)}(t)|)\,\mathrm{d}t &\le \int_0^T \omega_j\Big(\frac{A}{4(n-j-1)!}t^{n-j-1}\Big)\,\mathrm{d}t \\ &= \frac{1}{r_j}\int_0^{r_jT} \omega_j(t^{n-j-1})\,\mathrm{d}t, \end{aligned}$$

we deduce from inequality (5.27) that

$$\|u^{(n-1)}\|_\infty \le \int_0^T h(s, n+V\|u^{(n-1)}\|_\infty)\,\mathrm{d}s+\Lambda \tag{5.29}$$

where

$$\begin{aligned} \Lambda = \sum_{j=0}^{n-3}\frac{1}{r_j}\int_0^{r_jT} \omega_j(t^{n-j-1})\,\mathrm{d}t+\frac{2}{A}\int_0^{(AT)/2} \omega_{n-2}(t)\,\mathrm{d}t \\ +\frac{2}{K}\int_0^{KT} \omega_{n-1}(t^{r+1})\,\mathrm{d}t+T\sum_{j=0}^{n-1}\omega_j(1)<\infty. \end{aligned}$$

According to our assumption (see condition (5.4)) we have

$$\limsup_{v\to\infty} \frac{1}{v}\int_0^T h(t,Vv)\,\mathrm{d}t < 1,$$

and therefore there exists a positive constant S_* such that

$$\int_0^T h(t,n+Vv)\,\mathrm{d}t + \Lambda < v$$

whenever $v \ge S_*$. This and (5.29) show that $\|u^{(n-1)}\|_\infty < S_*$. Now using inequalities (5.25) and (5.26) we see that inequality (5.24) holds with $S_0 = S_* \max\{1, \frac{V}{n}\}$. □

Lemma 5.4. *Let* $\min\{\beta,\delta\} > 0$ *and let* h, q *and* ω_j $(0 \le j \le n-1)$ *have the properties given in conditions* (5.4) *and* (5.6). *Then there exists a positive constant* S_1 *such that*

$$\|u^{(j)}\|_\infty < S_1, \quad 0 \le j \le n-1 \tag{5.30}$$

for each $u \in \mathcal{A}(r,a)$ *satisfying the inequality*

$$\begin{cases} -u^{(n)}(t) \le h\Big(t, n + \sum_{j=0}^{n-1} |u^{(j)}(t)|\Big) + \sum_{\substack{j=0\\ j\ne n-2}}^{n-1} [\omega_j(|u^{(j)}(t)|) + \omega_j(1)] \\ \qquad\qquad + q(t)[\omega_{n-2}(|u^{(n-2)}(t)|) + \omega_{n-2}(1)] \text{ for a.e. } t \in [0,T]. \end{cases} \tag{5.31}$$

Proof. Let $u \in \mathcal{A}(r,a)$ satisfy (5.31) for a.e. $t \in [0,T]$. By Lemma 5.2, inequalities (5.12), (5.21) and (5.22) are true provided $\xi \in (0,T)$ is the unique zero of $u^{(n-1)}$ and B is given by (5.20). Since $u^{(n-2)}(0) = \frac{\beta}{\alpha} u^{(n-1)}(0)$ the same reasoning as in the proof of Lemma 5.3 shows that inequalities (5.25) and (5.26) hold if V is defined by (5.4). From inequalities (5.21) and (5.22) we obtain

$$\omega_{n-2}(|u^{(n-2)}(t)|) \le \omega_{n-2}(B), \quad t \in [0,T]$$

and

$$\begin{aligned}\int_0^T \omega_j(|u^{(j)}(t)|)\,\mathrm{d}t &\le \int_0^T \omega_j\Big(\frac{B}{(n-j-2)!} t^{n-j-2}\Big)\,\mathrm{d}t \\ &= \frac{1}{m_j}\int_0^{m_j T} \omega_j(t^{n-j-2})\,\mathrm{d}t\end{aligned}$$

for $0 \le j \le n-3$, where $m_j = \sqrt[n-j-2]{\frac{B}{(n-j-2)!}}$. Then (see (5.26), (5.28) and (5.31))

$$\begin{aligned}
|u^{(n-1)}(t)| &= \Big| \int_\xi^t u^{(n)}(s)\,\mathrm{d}s \Big| \\
&\le \int_0^T \Big[h\Big(s, n + \sum_{j=0}^{n-1} |u^{(j)}(s)|\Big) + \sum_{\substack{j=0\\ j\ne n-2}}^{n-1} [\omega_j(|u^{(j)}(s)|) + \omega_j(1)] \\
&\qquad + q(s)\,[\omega_{n-2}(|u^{(n-2)}(s)|) + \omega_{n-2}(1)] \Big]\,\mathrm{d}s \\
&\le \int_0^T h(s, n + V\|u^{(n-1)}\|_\infty)\,\mathrm{d}s + \Lambda_1 \quad \text{for} \quad t \in [0,T]
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_1 = &\sum_{j=0}^{n-3} \frac{1}{m_j} \int_0^{m_jT} \omega_j(t^{n-j-2})\,\mathrm{d}t + \|q\|_1[\omega_{n-2}(B) + \omega_{n-2}(1)] \\
&+ \frac{2}{K} \int_0^{KT} \omega_{n-1}(t^{r+1})\,\mathrm{d}t + T \sum_{\substack{j=0\\ j\ne n-2}}^{n-1} \omega_j(1) < \infty.
\end{aligned}$$

Hence

$$\|u^{(n-1)}\|_\infty \le \int_0^T h(s, n + V\|u^{(n-1)}\|_\infty)\,\mathrm{d}s + \Lambda_1$$

and using the same procedure as in the proof of Lemma 5.3, we conclude from the assumption $\limsup_{v\to\infty} \frac{1}{v}\int_0^T h(s,Vv)\,\mathrm{d}s < 1$ that inequality (5.30) is true with a positive constant S_1. □

Auxiliary regular problems

For each $m \in \mathbb{N}$ and any positive constant L define $\varrho_{L,m}, \tau_L \in C^0(\mathbb{R})$ and $f_{L,m} \in Car([0,T] \times \mathbb{R}^n)$ by the formulas

$$\varrho_{L,m}(v) = \begin{cases} \frac{1}{m} & \text{if } |v| < \frac{1}{m}, \\ |v| & \text{if } \frac{1}{m} \le |v| \le L+1, \\ L+1 & \text{if } |v| > L+1,, \end{cases} \quad \tau_L(v) = \begin{cases} v & \text{if } |v| \le L+1, \\ \dfrac{(L+1)v}{|v|} & \text{if } |v| > L+1, \end{cases}$$

and

$$f_{L,m}(t,x_0,\ldots,x_{n-2},x_{n-1})$$
$$=\begin{cases} f(t,\varrho_{L,m}(x_0),\ldots,\varrho_{L,m}(x_{n-2}),\tau_L(x_{n-1})) & \text{if } |x_{n-1}|\ge \frac{1}{m} \\ \dfrac{m}{2}\Big[f_{L,m}(t,x_0,\ldots,x_{n-2},\frac{1}{m})(x_{n-1}+\frac{1}{m}) & \\ \quad -f_{L,m}(t,x_0,\ldots,x_{n-2},-\frac{1}{m}))(x_{n-1}-\frac{1}{m})\Big] & \text{if } |x_{n-1}|<\frac{1}{m}. \end{cases}$$

Then for a.e. $t\in[0,T]$ and all $(x_0,\ldots,x_{n-1})\in\mathbb{R}^n$,

$$\begin{cases} a\,t^r \le f_{L,m}(t,x_0,\ldots,x_{n-1}) \\ \qquad \le h\Big(t,n+\displaystyle\sum_{j=0}^{n-1}|x_j|\Big)+\sum_{j=0}^{n-1}[\omega_j(|x_j|)+\omega_j(1)] \end{cases} \tag{5.32}$$

provided conditions (5.3)–(5.5) hold, and

$$\begin{cases} a\,t^r \le f_{L,m}(t,x_0,\ldots,x_{n-1}) \le h\Big(t,n+\displaystyle\sum_{j=0}^{n-1}|x_j|\Big) \\ \qquad +\displaystyle\sum_{\substack{j=0\\ j\neq n-2}}^{n-1}[\omega_j(|x_j|)+\omega_j(1)]+q(t)[\omega_{n-2}(|x_j|)+\omega_{n-2}(1)] \end{cases} \tag{5.33}$$

provided conditions (5.3), (5.4) and (5.6) hold.

Consider an auxiliary family of regular differential equations

$$-u^{(n)}=f_{L,m}(t,u,\ldots,u^{(n-1)}) \tag{5.34}$$

depending on $L>0$ and $m\in\mathbb{N}$.

Lemma 5.5. *Let* $\min\{\beta,\delta\}=0$ *and let conditions* (5.3)–(5.5) *hold. Let* S_0 *be the positive constant from Lemma* 5.3. *Then for each* $m\in\mathbb{N}$, *problem* (5.34), (5.2) *with* $L=S_0$ *has a solution* $u_m\in\mathcal{A}(r,a)$ *and*

$$\|u_m^{(j)}\|_\infty < S_0 \quad \textit{for} \quad 0\le j\le n-1. \tag{5.35}$$

In addition, the sequence

$$\{f_{S_0,m}(t, u_m(t), \ldots, u_m^{(n-1)}(t))\} \tag{5.36}$$

is uniformly integrable on $[0,T]$.

Proof. Put $g_m(t) = \sup\{f_{S_0,m}(t, x_0, \ldots, x_{n-1}) : (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n\}$. Then

$$\begin{aligned} g_m(t) \;\; &= \;\; \sup\Big\{f(t, x_0, \ldots, x_{n-1}) : \tfrac{1}{m} \le x_j \le S_0 + 1 \\ &\qquad\qquad \text{for } \; 0 \le j \le n-2 \;\text{ and }\; \tfrac{1}{m} \le |x_{n-1}| \le S_0 + 1\Big\}. \end{aligned}$$

Since $f \in Car([0,T] \times \mathcal{D})$, we have $g_m \in L_1[0,T]$. As the homogeneous problem $-u^{(n)} = 0$, (5.2) has only the trivial solution, the Fredholm type existence theorem guarantees the existence of a solution u_m of problem (5.34), (5.2) with $L = S_0$ for all $m \in \mathbb{N}$. Besides, inequality (5.32) with $L = S_0$ yields

$$a\, t^r \le -u_m^{(n)}(t) \le h\Big(t, n + \sum_{j=0}^{n-1} |u_m^{(j)}(t)|\Big) + \sum_{j=0}^{n-1} [\omega_j(|u_m^{(j)}(t)|) + \omega_j(1)]$$

for a.e. $t \in [0,T]$. Consequently $u_m \in \mathcal{A}(r,a)$ and inequality (5.35) is true by Lemmas 5.1 and 5.3. Moreover (for $m \in \mathbb{N}$),

$$u_m^{(n-1)}(t) \begin{cases} \ge \dfrac{a}{r+1}(\xi_m - t)^{r+1} & \text{for } \; t \in [0, \xi_m], \\ < -\dfrac{a}{r+1}(t - \xi_m)^{r+1} & \text{for } \; t \in (\xi_m, T], \end{cases} \tag{5.37}$$

where $\xi_m \in (0,T)$ is the unique zero of $u_m^{(n-1)}$,

$$u_m^{(n-2)}(t) \ge \begin{cases} A\, t & \text{for } \; t \in [0, \frac{T}{2}], \\ A\,(T - t) & \text{for } \; t \in (\frac{T}{2}, T] \end{cases} \tag{5.38}$$

and

$$u_m^{(j)}(t) \ge \frac{A}{4(n-j-1)!}\, t^{n-j-1} \quad \text{for } \; t \in [0,T], \quad 0 \le j \le n-3, \tag{5.39}$$

where A is defined in formula (5.11). Since

$$\begin{aligned} &0 \le f_{S_0,m}(t, u_m(t), \ldots, u_m^{(n-1)}(t)) \\ &\quad \le h(t, n(1+S_0)) + \sum_{j=0}^{n-1} [\omega_j(|u_m^{(j)}(t)|) + \omega_j(1)] \end{aligned}$$

for a.e. $t \in [0,T]$ and each $m \in \mathbb{N}$ and $h(t, n(1+S_0)) \in L_1[0,T]$ by (5.4), to prove the uniform integrability of the sequence (5.36) it suffices to show that the sequences

$$\{\omega_j(|u_m^{(j)}(t)|)\}, \quad 0 \le j \le n-1$$

are uniformly integrable on $[0,T]$. Let $0 \le j \le n-3$. Then

$$\omega_j(|u_m^{(j)}(t)|) \le \omega_j\Big(\frac{A}{4(n-j-1)!} t^{n-j-1}\Big), \quad t \in [0,T],\ m \in \mathbb{N},$$

and it follows from the properties of ω_j that $\omega_j(\frac{A}{4(n-j-1)!} t^{n-j-1}) \in L_1[0,T]$. Hence $\{\omega_j(|u_m^{(j)}(t)|)\}$ is uniformly integrable on $[0,T]$. Analogously, (5.38) gives $\omega_{n-2}(|u_m^{(n-2)}(t)|) \le \omega_{n-2}(\varphi(t))$ for $t \in [0,T]$ and $m \in \mathbb{N}$, where

$$\varphi(t) = \begin{cases} At & \text{for } t \in [0, \frac{T}{2}], \\ A(T-t) & \text{for } t \in (\frac{T}{2}, T]. \end{cases}$$

Since $\omega_{n-2}(\varphi(t)) \in L_1[0,T]$, it follows that the sequence $\{\omega_{n-2}(|u_m^{(n-2)}(t)|)\}$ is uniformly integrable on $[0,T]$. Furthermore, the uniform integrability of $\{\omega_{n-1}(|u_m^{(n-1)}(t)|)\}$ follows from Criterion A.4. We have proved that the sequence (5.36) is uniformly integrable on $[0,T]$. $\square$

Lemma 5.6. *Let* $\min\{\beta,\delta\} > 0$ *and let conditions* (5.3), (5.4) *and* (5.6) *hold. Let* S_1 *be the positive constant from Lemma* 5.4. *Then for each* $m \in \mathbb{N}$, *problem* (5.34), (5.2) *with* $L = S_1$ *has a solution* $u_m \in \mathcal{A}(r,a)$ *and*

$$\|u_m^{(j)}\|_\infty < S_1 \quad \text{for } 0 \le j \le n-1. \tag{5.40}$$

In addition, the sequence

$$\{f_{S_1,m}(t, u_m(t), \dots, u_m^{(n-1)}(t))\} \tag{5.41}$$

is uniformly integrable on $[0,T]$.

Proof. Essentially the same reasoning as in the first part of the proof of Lemma 5.5 shows that for each $m \in \mathbb{N}$ there exists a solution u_m of problem (5.34), (5.2) with $L = S_1$. The fact that $u_m \in \mathcal{A}(r,a)$ and u_m satisfies inequality (5.40) follows from Lemmas 5.2 and 5.4. It remains to verify that the sequence (5.41) is uniformly integrable on $[0,T]$. Notice that, by Lemmas 5.2 and 5.4, $u_m^{(n-1)}$ satisfies inequality (5.37) where $\xi_m \in (0,T)$ is its unique zero and

$$u_m^{(n-2)}(t) \geq B \quad \text{for} \quad t \in [0,T], \tag{5.42}$$

$$u_m^{(j)}(t) \geq \frac{B}{(n-j-2)!}\, t^{n-j-2} \quad \text{for} \quad t \in [0,T], \quad 0 \leq j \leq n-3, \tag{5.43}$$

where B is given in formula (5.20). Hence

$$\omega_{n-2}(u_m^{(n-2)}(t)) \leq \omega_{n-2}(B), \quad t \in [0,T], \quad m \in \mathbb{N} \tag{5.44}$$

and

$$\begin{cases} \omega_j(|u_m^{(j)}(t)|) \leq \omega_j\left(\dfrac{B}{(n-j-2)!}\, t^{n-j-2}\right), \\ \qquad\qquad \text{for} \quad t \in (0,T), \quad m \in \mathbb{N}, \ 0 \leq j \leq n-3. \end{cases} \tag{5.45}$$

By conditions (5.4) and (5.6) we know that the functions $h(t, n(1+S_1))$, $q(t)$ and $\omega_j(\frac{B}{(n-j-2)!}t^{n-j-2})$ belong to the set $L_1[0,T]$ for $0 \leq j \leq n-3$ and that the sequence $\{\omega_{n-1}(|u_m^{(n-1)}(t)|)\}$ is uniformly integrable on $[0,T]$, which was shown in the proof of Lemma 5.5. Hence the uniform integrability of the sequence (5.41) follows from (5.44), (5.45) and from the following inequality (see (5.33))

$$\begin{aligned} 0 \leq f_{S_1,m}(t, u_m(t), \ldots, u_m^{(n-1)}(t)) &\leq h(t, n(1+S_1)) \\ &\sum_{\substack{j=0 \\ j \neq n-2}}^{n-1} [\omega_j(|u_m^{(j)}(t)|) + \omega_j(1)] + q(t)\,[\omega_{n-2}(|u_m^{(n-2)}(t)|) + \omega_{n-2}(1)] \end{aligned}$$

for a.e. $t \in [0,T]$ and all $m \in \mathbb{N}$. □

Existence results

Theorem 5.7. *Let conditions* (5.3) − (5.5) *hold and let* $\min\{\beta, \delta\} = 0$. *Then problem* (5.1), (5.2) *has a solution* $u \in AC^{n-1}[0,T]$ *such that*

$$u^{(n-2)} > 0 \ \textit{on} \ (0,T) \ \textit{and} \ u^{(j)} > 0 \ \textit{on} \ (0,T] \ \textit{for} \ 0 \leq j \leq n-3. \tag{5.46}$$

Proof. By Lemma 5.5, for each $m \in \mathbb{N}$, there is a solution $u_m \in \mathcal{A}(r,a)$ of problem (5.34), (5.2) with $L = S_0$. Lemmas 5.1, 5.3 and 5.5 show that u_m satisfies inequalities (5.35) and (5.37)–(5.39) where $A > 0$ is given in (5.11) and the sequence (5.36) is uniformly integrable on $[0,T]$. Hence $\{u_m\}$ is bounded in $C^{n-1}[0,T]$ and $\{u_m^{(n-1)}\}$ is equicontinuous on $[0,T]$. Without loss of generality we can assume that $\{u_m\}$ is convergent in $C^{n-1}[0,T]$ and $\{\xi_m\}$ is convergent in $\mathbb{R}$ where $\xi_m \in (0,T)$ denotes the unique zero of $u_m^{(n-1)}$. Let $\lim_{m\to\infty} u_m = u$, $\lim_{m\to\infty} \xi_m = \xi$. Then

$$u^{(n-1)}(t) \begin{cases} \geq \dfrac{a}{r+1}(\xi - t)^{r+1} & \text{for } t \in [0,\xi] \\ \leq -\dfrac{a}{r+1}(\xi - t)^{r+1} & \text{for } t \in (\xi, T], \end{cases} \tag{5.47}$$

$$u^{(n-2)}(t) \geq \begin{cases} A\,t & \text{for } t \in [0,\frac{T}{2}] \\ A\,(T-t) & \text{for } t \in (\frac{T}{2}, T], \end{cases} \tag{5.48}$$

$$u^{(j)}(t) \geq \frac{A}{4(n-j-1)!} t^{n-j-1}, \quad t \in [0,T],\ 0 \leq j \leq n-3. \tag{5.49}$$

Hence $u^{(j)}$ has at most two zeros on $[0,T]$ for $0 \leq j \leq n-1$. Applying Theorem 1.7, we obtain that $u \in AC^{n-1}[0,T]$, u is a solution of problem (5.1), (5.2) and (see (5.48) and (5.49)) $u^{(n-2)} > 0$ on $(0,T)$, $u^{(j)} > 0$ on $(0,T]$ for $0 \leq j \leq n-3$. $\square$

Theorem 5.8. *Assume* (5.3), (5.4), (5.6) *and let* $\min\{\beta,\delta\} > 0$. *Then there exists a solution* $u \in AC^{n-1}[0,T]$ *of problem* (5.1), (5.2) *such that*

$$u^{(n-2)} > 0 \quad \text{on } [0,T] \text{ and } u^{(j)} > 0 \text{ on } (0,T] \text{ for } 0 \leq j \leq n-3. \tag{5.50}$$

Proof. Lemma 5.6 guarantees that for each $m \in \mathbb{N}$ there exists a solution $u_m \in \mathcal{A}(r,a)$ of problem (5.34), (5.2) with $L = S_1$. By Lemmas 5.2, 5.4 and 5.6, u_m satisfies inequalities (5.37), (5.40), (5.42) and (5.43) where $B > 0$ is defined in formula (5.20) and the sequence (5.41) is uniformly integrable on $[0,T]$. Without loss of generality we can assume that $\{u_m\}$ and $\{\xi_m\}$ are convergent in $C^{n-1}[0,T]$ and $\mathbb{R}$, respectively. Here $\xi_m \in (0,T)$ is the unique zero of $u_m^{(n-1)}$. Let $\lim_{m\to\infty} u_m = u$, $\lim_{m\to\infty} \xi_m = \xi$. Then inequalities (5.47) and

$$u^{(n-2)}(t) \geq B, \quad t \in [0,T], \tag{5.51}$$

$$u^{(j)}(t) \geq \frac{B}{(n-j-2)!} t^{n-j-2}, \quad t \in [0,T],\ 0 \leq j \leq n-3, \tag{5.52}$$

are true. Hence $u^{(j)}$ has at most one zero in $[0,T]$ for $0 \leq j \leq n-1$. Thus, by Theorem 1.7, $u \in AC^{n-1}[0,T]$ is a solution of problem (5.1), (5.2) and from (5.51) and (5.52) we see that $u^{(n-2)} > 0$ on $[0,T]$ and $u^{(j)} > 0$ on $(0,T]$ for $0 \leq j \leq n-3$. □

EXAMPLE. Consider the differential equation

$$\begin{cases} -u^{(n)} = \sin\left(\dfrac{t}{T}\right)^r + \displaystyle\sum_{j=0}^{n-2}\left(\frac{a_j(t)}{(u^{(j)})^{\alpha_j}} + b_j(t)\,(u^{(j)})^{\gamma_j}\right) \\ \qquad\qquad + \dfrac{a_{n-1}(t)}{|u^{(n-1)}|^{\alpha_{n-1}}} + b_{n-1}(t)\,|u^{(n-1)}|^{\gamma_{n-1}} \end{cases} \tag{5.53}$$

with the boundary conditions (5.2) where $\min\{\beta,\delta\} = 0$. Theorem 5.7 guarantees this problem has a solution $u \in AC^{n-1}[0,T]$ satisfying inequality (5.46) provided $r \in (0,\infty)$, $\alpha_j \in (0, \frac{1}{n-j-1})$ for $0 \leq j \leq n-2$, $\alpha_{n-1} \in (0, \frac{1}{r+1})$, $\gamma_i \in (0,1)$, and $a_i \in L_\infty[0,T]$, $b_i \in L_1[0,T]$ are nonnegative for $0 \leq i \leq n-1$.

Now consider problem (5.53), (5.2) where $\min\{\beta,\delta\} > 0$. If $r \in (0,\infty)$, $\alpha_j \in (0, \frac{1}{n-j-2})$ for $0 \leq j \leq n-3$, $\alpha_{n-2} \in (0,\infty), \alpha_{n-1} \in (0, \frac{1}{r+1})$, $\gamma_i \in (0,1)$, $b_i \in L_1[0,T]$ is nonnegative for $0 \leq i \leq n-1$ and finally $a_{n-2} \in L_1[0,T]$, $a_{n-1}, a_k \in L_\infty[0,T]$ are nonnegative for $0 \leq k \leq n-3$ then, by Theorem 5.8, problem (5.53),(5.2) has a solution satisfying inequality (5.50).

Bibliographical notes

Theorems 5.7 and 5.8 were adapted from Rachůnková and Staněk [159]. The singular Sturm-Liouville problem for the equation

$$u^{(n)} + f(t, u, \ldots, u^{(n-2)}) = 0$$

is considered in Agarwal and Wong [26] where $f \in C^0((0,1) \times (0,\infty)^{n-1})$ is positive. Here the existence of a solution $u \in C^{n-1}[0,1] \cap C^n(0,1)$ positive on $(0,1)$ is proved by a fixed point theorem for mappings that are decreasing with respect to a cone in a Banach space.

Chapter 6

Lidstone problem

Let $\mathbb{R}_- = (-\infty, 0)$, $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. We will consider the singular Lidstone problem

$$(-1)^n u^{(2n)} = f(t, u, \dots, u^{(2n-1)}), \tag{6.1}$$

$$u^{(2j)}(0) = u^{(2j)}(T) = 0, \quad 0 \le j \le n-1 \tag{6.2}$$

where $n \ge 1$ and $f \in Car([0,T] \times \mathcal{D})$ with

$$\mathcal{D} = \begin{cases} \underbrace{\mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 \times \cdots \times \mathbb{R}_+ \times \mathbb{R}_0}_{4k-2} & \text{if } n = 2k-1, \\ \underbrace{\mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 \times \cdots \times \mathbb{R}_- \times \mathbb{R}_0}_{4k} & \text{if } n = 2k \end{cases}$$

(for $n = 1$ and 2, we have $\mathcal{D} = \mathbb{R}_+ \times \mathbb{R}_0$ and $\mathcal{D} = \mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0$, respectively). If $n = 1$, problem (6.1), (6.2) reduces to the Dirichlet problem. The function f may be singular at the value 0 of its space variables. If f is positive on $[0,T] \times \mathcal{D}$, the solutions of problem (6.1), (6.2) have singular points of type I at $t = 0$ and $t = T$ and also singular points of type II.

Green functions

Let $j \in \mathbb{N}$. In our studies we will essentially use the Green functions $G_j(t,s)$ of the problems

$$u^{(2j)}(t) = 0, \quad u^{(2i)}(0) = u^{(2i)}(T) = 0, \quad 0 \le i \le j-1.$$

Then

$$G_1(t,s) = \begin{cases} \dfrac{s}{T}(t-T) & \text{for } 0 \le s \le t \le T, \\ \dfrac{t}{T}(s-T) & \text{for } 0 \le t < s \le T. \end{cases} \tag{6.3}$$

If $j > 1$ we have

$$G_j(t,s) = \underbrace{\int_0^T \cdots \int_0^T}_{(j-1)\text{ times}} G_1(t,s_{j-1})\, G_1(s_{j-1},s_{j-2}) \ldots G_1(s_1,s)\, \mathrm{d}s_1 \ldots \mathrm{d}s_{j-1}$$

for $(t,s) \in [0,T] \times [0,T]$. Therefore the Green function $G_j(t,s)$ can be expressed as

$$G_j(t,s) = \int_0^T G_1(t,\tau)\, G_{j-1}(\tau,s)\, \mathrm{d}\tau \tag{6.4}$$

for $(t,s) \in [0,T] \times [0,T]$ and $j > 1$ (see Agarwal [1], Agarwal and Wong [25], Wong and Agarwal [199]). Since $G_1(t,s) < 0$ for $(t,s) \in (0,T) \times (0,T)$, we conclude from (6.4) that

$$(-1)^j\, G_j(t,s) > 0 \quad \text{for} \quad (t,s) \in (0,T) \times (0,T). \tag{6.5}$$

The next lemma gives inequalities for the Green function $G_j(t,s)$.

Lemma 6.1. *For $(t,s) \in [0,T] \times [0,T]$ and $j \in \mathbb{N}$, the inequality*

$$|G_j(t,s)| \geq \frac{T^{2j-5}}{30^{j-1}}\, st(T-t)(T-s) \tag{6.6}$$

holds.

Proof. The validity of inequality (6.6) will be proved by induction. Since

$$|G_1(t,s)| = \begin{cases} \dfrac{s}{T}\,(T-t) \geq \dfrac{s\,t\,(T-t)\,(T-s)}{T^3} & \text{for } 0 \leq s \leq t \leq T, \\[2ex] \dfrac{t}{T}\,(T-s) \geq \dfrac{s\,t\,(T-t)\,(T-s)}{T^3} & \text{for } 0 \leq t < s \leq T, \end{cases} \tag{6.7}$$

estimate (6.6) is true for $j=1$. Assume now that (6.6) holds for $j = i \geq 1$. Then relations (6.4)–(6.7) give

$$\begin{aligned} |G_{i+1}(t,s)| &= \int_0^T |G_1(t,\tau)|\, |G_i(\tau,s)|\, \mathrm{d}\tau \\ &\geq \frac{T^{2\,i-8}}{30^{i-1}}\, s\,t\,(T-t)\,(T-s) \int_0^T \tau^2\,(T-\tau)^2\, \mathrm{d}\tau \\ &= \frac{T^{2\,i-3}}{30^{i}}\, s\,t\,(T-t)\,(T-s) \end{aligned}$$

for $(t,s)\in[0,T]\times[0,T]$ and therefore (6.6) is valid for $j=i+1$. $\square$

In the proof of Theorem 6.3 we will need the following result.

Lemma 6.2. *Let* $\xi\in(0,T)$. *Then*

$$\left|\int_\xi^t s\,(T-s)\,\mathrm{d}s\right| \ge \frac{T}{6}\,(t-\xi)^2 \quad \textit{for} \ \ t\in[0,T]. \tag{6.8}$$

Proof. It suffices to prove inequality (6.8) only for $t\in[\xi,T]$. Then

$$2\,T\,t+4\,T\,\xi-2\,(t^2+t\,\xi+\xi^2)=2\,t\,(T-t)+2\,\xi\,(T-t)+2\,\xi\,(T-\xi)>0$$

and therefore

$$\begin{aligned}\int_\xi^t s(T-s)\,\mathrm{d}s &= \frac{1}{6}\,[3\,T\,(t^2-\xi^2)-2\,(t^3-\xi^3)]\\ &= \frac{t-\xi}{6}\,[T\,(t-\xi)+2\,T\,t+4\,T\,\xi-2\,(t^2+t\,\xi+\xi^2)]\ge\frac{T}{6}\,(t-\xi)^2.\end{aligned}$$ $\square$

Main result

The next result provides sufficient conditions for the existence of a solution of the singular Lidstone problem.

Theorem 6.3. *Let* $f\in Car([0,T]\times\mathcal{D})$ *and let there exist* $a\in(0,\infty)$ *such that*

$$\begin{cases} a\le f(t,x_0,\dots,x_{2n-1}) \\ \qquad \textit{for a.e. } t\in[0,T] \textit{ and each } (x_0,\dots,x_{2n-1})\in\mathcal{D}. \end{cases} \tag{6.9}$$

Let

$$\begin{cases} f(t,x_0,\dots,x_{2n-1})\le h\Big(t,\displaystyle\sum_{j=0}^{2n-1}|x_j|\Big)+\sum_{j=0}^{2n-1}\omega_j(|x_j|) \\ \qquad \textit{for a.e. } t\in[0,T] \textit{ and each } (x_0,\dots,x_{2n-1})\in\mathcal{D}, \end{cases} \tag{6.10}$$

where $h\in Car([0,T]\times[0,\infty))$ *is positive and nondecreasing in the second variable,* $\omega_j:\mathbb{R}_+\to\mathbb{R}_+$ *is nonincreasing,* $0\le j\le 2n-1$,

$$\limsup_{v\to\infty} \frac{1}{v}\int_0^T h(t,Kv)\,\mathrm{d}t < 1 \quad \textit{with} \quad K = \begin{cases} 2n & \textit{if } \ T=1, \\ \dfrac{T^{2n}-1}{T-1} & \textit{if } \ T\neq 1, \end{cases} \tag{6.11}$$

and

$$\int_0^1 \omega_{2n-1}(s)\,\mathrm{d}s < \infty, \qquad \int_0^1 \omega_{2j}(s)\,\mathrm{d}s < \infty \quad \textit{for} \ \ 0\le j\le n-1, \tag{6.12}$$

$$\int_0^1 \omega_{2j+1}(s^2)\,\mathrm{d}s < \infty \quad \textit{for} \ \ 0\le j\le n-2. \tag{6.13}$$

Then problem (6.1), (6.2) *has a solution* $u\in AC^{2n-1}[0,T]$ *and*

$$(-1)^j\, u^{(2j)}(t) > 0 \quad \textit{for} \ \ t\in(0,T) \ \ \textit{and} \ \ 0\le j\le n-1. \tag{6.14}$$

Proof. *Step* 1. *Regularization.*

For each $m\in\mathbb{N}$, define χ_m, φ_m, $\tau_m\in C^0(\mathbb{R})$, and $\mathbb{R}_m\subset\mathbb{R}$ by the formulas

$$\chi_m(v) = \begin{cases} v & \text{if } \ v\ge \frac{1}{m}, \\ \frac{1}{m} & \text{if } \ v<\frac{1}{m}, \end{cases} \qquad \varphi_m(v) = \begin{cases} -\frac{1}{m} & \text{if } \ v>-\frac{1}{m}, \\ v & \text{if } \ v\le -\frac{1}{m}, \end{cases}$$

$$\tau_m = \begin{cases} \chi_m & \text{if } \ n=2k-1, \\ \varphi_m & \text{if } \ n=2k, \end{cases} \qquad \mathbb{R}_m = \mathbb{R}\setminus(-\tfrac{1}{m},\tfrac{1}{m}).$$

Choose $m\in\mathbb{N}$ and put

$$\begin{aligned} &f_{m,0}(t,x_0,x_1,x_2,x_3,\dots,x_{2n-2},x_{2n-1}) \\ &\quad = f(t,\chi_m(x_0),x_1,\varphi_m(x_2),x_3,\dots,\tau_m(x_{2n-2}),x_{2n-1}) \end{aligned}$$

for $(t,x_0,x_1,x_2,x_3,\dots,x_{2n-2},x_{2n-1})\in[0,T]\times\mathbb{R}\times\mathbb{R}_m\times\mathbb{R}\times\mathbb{R}_m\times\cdots\times\mathbb{R}\times\mathbb{R}_m$.

Define $f_m \in Car([0,T]\times\mathbb{R}^{2n})$ by the formula

$$f_m(t,x_0,x_1,x_2,x_3,\dots,x_{2n-2},x_{2n-1})$$
$$=\begin{cases}
\frac{m}{2}\Big[f_{m,0}(t,x_0,\frac{1}{m},x_2,x_3,\dots,x_{2n-2},x_{2n-1})(x_1+\frac{1}{m}) \\
\qquad -f_{m,0}(t,x_0,-\frac{1}{m},x_2,x_3,\dots,x_{2n-2},x_{2n-1})(x_1-\frac{1}{m})\Big] \\
\qquad \text{for } \ (t,x_0,x_1,x_2,x_3,\dots,x_{2n-2},x_{2n-1}) \\
\qquad\quad \in[0,T]\times\mathbb{R}\times[-\frac{1}{m},\frac{1}{m}]\times\mathbb{R}\times\mathbb{R}_m\times\dots\times\mathbb{R}\times\mathbb{R}_m, \\
\frac{m}{2}\Big[f_{m,0}(t,x_0,x_1,x_2,\frac{1}{m},\dots,x_{2n-2},x_{2n-1})(x_3+\frac{1}{m}) \\
\qquad -f_{m,0}(t,x_0,x_1,x_2,-\frac{1}{m},\dots,x_{2n-2},x_{2n-1})(x_3-\frac{1}{m})\Big] \\
\qquad \text{for } \ (t,x_0,x_1,x_2,x_3,\dots,x_{2n-2},x_{2n-1}) \\
\qquad\quad \in[0,T]\times\mathbb{R}^3\times[-\frac{1}{m},\frac{1}{m}]\times\dots\times\mathbb{R}\times\mathbb{R}_m, \\
\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\frac{m}{2}\Big[f_{m,0}(t,x_0,x_1,x_2,\dots,x_{2n-2},\frac{1}{m})(x_{2n-1}+\frac{1}{m}) \\
\qquad -f_{m,0}(t,x_0,x_1,x_2,\dots,x_{2n-2},-\frac{1}{m})(x_{2n-1}-\frac{1}{m})\Big] \\
\qquad \text{for } \ (t,x_0,x_1,x_2,\dots,x_{2n-2},x_{2n-1})\in[0,T]\times\mathbb{R}^{2n-1}\times[-\frac{1}{m},\frac{1}{m}].
\end{cases}$$

Then inequalities (6.9) and (6.10) imply that

$$\begin{cases}
a\le f_m(t,x_0,\dots,x_{2n-1}) \\
\quad \le \Big(t,2n+\sum_{j=0}^{2n-1}|x_j|\Big)+\sum_{j=0}^{2n-1}[\omega_j(|x_j|)+\omega_j(1)]
\end{cases} \tag{6.15}$$

for a.e. $t\in[0,T]$ and all $(x_0,\dots,x_{2n-1})\in\mathbb{R}_0^{2n}$.

Consider the sequence of approximate regular differential equations

$$(-1)^n\, u^{(2n)} = f_m(t,u,\dots,u^{(2n-1)}). \tag{6.16}$$

Step 2. Solvability of problem (6.16), (6.2).

We first give a priori bounds for solutions of problem (6.16), (6.2). To this end let $u_m \in AC^{2n-1}[0,T]$ be a solution of problem (6.16), (6.2). By inequality (6.15) we have

$$(-1)^n\, u_m^{(2n)}(t) \ge a > 0 \quad \text{ for a.e. } t \in [0,T]. \tag{6.17}$$

Furthermore, by the definitions of the Green functions $G_i(t,s)$, $i = 1, 2, \ldots, n$, the equality

$$(-1)^j\, u_m^{(2j)}(t) = (-1)^{n-j} \int_0^T G_{n-j}(t,s)\, (-1)^n\, u_m^{(2n)}(s)\, \mathrm{d}s \tag{6.18}$$

holds for $t \in [0,T]$ and $0 \le j \le n-1$. From relations (6.5) and (6.17) we see that

$$(-1)^j\, u_m^{(2j)}(t) > 0 \quad \text{for } \; t \in [0,T], \;\; 0 \le j \le n-1. \tag{6.19}$$

Hence $(-1)^j\, u_m^{(2j+1)}$ is decreasing on $[0,T]$ for $0 \le j \le n-1$. Therefore and due to boundary conditions (6.2) we conclude that $u_m^{(2j+1)}(\xi_{j,m}) = 0$ holds for a unique $\xi_{j,m} \in (0,T)$. Moreover, from relations (6.6), (6.17) and (6.18) it follows that

$$\begin{aligned} |u_m^{(2j)}(t)| &\ge a\, \frac{T^{2(n-j)-5}}{30^{n-j-1}}\, t\,(T-t) \int_0^T s\,(T-s)\, \mathrm{d}s \\ &= a\, \frac{T^{2(n-j)-2}}{6 \cdot 30^{n-j-1}}\, t\,(T-t) \quad \text{for } \; t \in [0,T], \; 0 \le j \le n-1. \end{aligned}$$

In particular,

$$|u_m^{(2j)}(t)| \ge a\, \frac{T^{2(n-j)-2}}{6 \cdot 30^{n-j-1}}\, t\,(T-t) \quad \text{for } \; t \in [0,T], \; 0 \le j \le n-1. \tag{6.20}$$

Since

$$u_m^{(2j+1)}(t) = \int_{\xi_{j,m}}^t u_m^{(2j+2)}(s)\, \mathrm{d}s \quad \text{and} \quad \left| \int_{\xi_{j,m}}^t s\,(T-s)\, \mathrm{d}s \right| \ge \frac{T}{6}\,(t - \xi_{j,m})^2$$

by Lemma 6.2, we obtain

$$
\begin{cases} |u_m^{(2j+1)}(t)| \geq a\,\dfrac{T^{2(n-j)-3}}{36\cdot 30^{n-j-2}}(t-\xi_{j,m})^2 \\[2ex] \qquad\qquad\qquad \text{for} \quad t\in[0,T] \text{ and } 0\leq j\leq n-2 \end{cases} \tag{6.21}
$$

and

$$
|u_m^{(2n-1)}(t)| \geq a\,|t-\xi_{n-1,m}| \quad \text{for} \quad t\in[0,T]. \tag{6.22}
$$

By inequality (6.17) we have $|u_m^{(2n)}(t)| \geq a > 0$ for a.e. $t\in[0,T]$. Put

$$
A = a\,\min\{1, A_1, A_2\},
$$

where

$$
A_1 = \min\left\{\frac{T^{2(n-j)-3}}{36\cdot 30^{n-j-2}} : 0\leq j\leq n-2\right\}
$$

and

$$
A_2 = \min\left\{\frac{T^{2(n-j)-2}}{6\cdot 30^{n-j-1}} : 0\leq j\leq n-1\right\}.
$$

Then inequalities (6.20)–(6.22) give

$$
\begin{cases} |u_m^{(2n-1)}(t)| & \geq A\,|t-\xi_{n-1,m}|, \\ |u_m^{(2j+1)}(t)| & \geq A\,(t-\xi_{j,m})^2 \quad \text{for} \quad 0\leq j\leq n-2, \\ |u_m^{(2j)}(t)| & \geq A\,t\,(T-t) \quad\ \ \text{for} \quad 0\leq j\leq n-1, \end{cases} \tag{6.23}
$$

for $t\in[0,T]$. Hence

$$
\begin{aligned}
\int_0^T \omega_{2n-1}(|u_m^{(2n-1)}(s)|)\,\mathrm{d}s &\leq \int_0^T \omega_{2n-1}(A\,|s-\xi_{n-1,m}|)\,\mathrm{d}s \\
&= \frac{1}{A}\int_0^{A\xi_{n-1,m}} \omega_{2n-1}(s)\,\mathrm{d}s + \frac{1}{A}\int_0^{A\,(T-\xi_{n-1,m})} \omega_{2n-1}(s)\,\mathrm{d}s \\
&< \frac{2}{A}\int_0^{A\,T} \omega_{2n-1}(s)\,\mathrm{d}s,
\end{aligned}
$$

$$\int_0^T \omega_{2j+1}(|u_m^{(2j+1)}(s)|)\,\mathrm{d}s \le \int_0^T \omega_{2j+1}(A(s-\xi_{j,m})^2)\,\mathrm{d}s$$
$$= \frac{1}{\sqrt{A}} \int_{-\sqrt{A}\,\xi_{j,m}}^{\sqrt{A}\,(T-\xi_{j,m})} \omega_{2j+1}(s^2)\,\mathrm{d}s < \frac{2}{\sqrt{A}} \int_0^{\sqrt{A}\,T} \omega_{2j+1}(s^2)\,\mathrm{d}s$$

and using the inequality

$$t\,(T-t) \ge \begin{cases} \dfrac{T\,t}{2} & \text{for } \ 0 \le t \le \frac{T}{2}, \\[2ex] \dfrac{T\,(T-t)}{2} & \text{for } \ \frac{T}{2} \le t \le T, \end{cases}$$

we compute

$$\int_0^T \omega_{2j}(|u_m^{(2j)}(s)|)\,\mathrm{d}s \le \int_0^T \omega_{2j}(A\,s\,(T-s))\,\mathrm{d}s$$
$$\le \int_0^{T/2} \omega_{2j}\Big(\frac{ATs}{2}\Big)\,\mathrm{d}s + \int_{T/2}^T \omega_{2j}\Big(\frac{AT(T-s)}{2}\Big)\,\mathrm{d}s = \frac{4}{A\,T} \int_0^{A\,T^2/2} \omega_{2j}(s)\,\mathrm{d}s.$$

So, we can summarize the above considerations as follows:

$$\int_0^T \omega_{2n-1}(|u_m^{(2n-1)}(s)|)\,\mathrm{d}s < \frac{2}{A} \int_0^{A\,T} \omega_{2n-1}(s)\,\mathrm{d}s, \tag{6.24}$$

$$\begin{cases} \displaystyle\int_0^T \omega_{2j+1}(|u_m^{(2j+1)}(s)|)\,\mathrm{d}s < \frac{2}{\sqrt{A}} \int_0^{\sqrt{A}\,T} \omega_{2j+1}(s^2)\,\mathrm{d}s, \\ \hfill j = 0, 1, \dots, n-2, \end{cases} \tag{6.25}$$

$$\begin{cases} \displaystyle\int_0^T \omega_{2j}(|u_m^{(2j)}(s)|)\,\mathrm{d}s \le \frac{4}{A\,T} \int_0^{A\,T^2/2} \omega_{2j}(s)\,\mathrm{d}s, \\ \hfill j = 0, 1, \dots, n-1, \end{cases} \tag{6.26}$$

From inequalities (6.24)–(6.26) and from (6.15) we obtain

$$|u_m^{(2n-1)}(t)| = \Big| \int_{\xi_{n-1,m}}^t f_m(s, u_m(s), \dots, u_m^{(2n-1)}(s))\,\mathrm{d}s \Big|$$
$$\le \int_0^T |f_m(s, u_m(s), \dots, u_m^{(2n-1)}(s))|\,\mathrm{d}s$$

$$\leq \int_0^T h\Big(s, 2n + \sum_{j=0}^{2n-1} |u_m^{(j)}(s)|\Big)\,\mathrm{d}s + \sum_{j=0}^{2n-1} \int_0^T \omega_j(|u_m^{(j)}(s)|)\,\mathrm{d}s$$

$$< \int_0^T h\Big(s, 2n + \sum_{j=0}^{2n-1} |u_m^{(j)}(s)|\Big)\,\mathrm{d}s + \Lambda$$

for $t \in [0, T]$, where

$$\Lambda = \frac{2}{A}\int_0^{AT} \omega_{2n-1}(s)\,\mathrm{d}s + \frac{2}{\sqrt{A}}\sum_{j=0}^{n-2}\int_0^{\sqrt{A}\,T} \omega_{2j+1}(s^2)\,\mathrm{d}s$$

$$+ \frac{4}{AT}\sum_{j=0}^{n-1}\int_0^{AT^2/2} \omega_{2j}(s)\,\mathrm{d}s + \sum_{j=0}^{2n-1}\omega_j(1)$$

In particular,

$$|u_m^{(2n-1)}(t)| < \int_0^T h\Big(s, 2n + \sum_{j=0}^{2n-1} |u_m^{(j)}(s)|\Big)\,\mathrm{d}s + \Lambda \quad \text{for} \quad t \in [0, T]. \tag{6.27}$$

Notice that $\Lambda < \infty$ due to conditions (6.12) and (6.13). Since

$$\|u_m^{(j)}\|_\infty \leq T^{2n-j-1}\|u_m^{(n-1)}\|_\infty, \quad 0 \leq j \leq 2n-2, \quad m \in \mathbb{N}, \tag{6.28}$$

which follows immediately from $u_m^{(2j+1)}(\xi_{j,m}){=}0$ and $u_m^{(2j)}(0){=}0$ $(0{\leq}j{\leq}n{-}1)$, inequality (6.27) shows that

$$\begin{cases} \|u_m^{(2n-1)}\|_\infty & < \displaystyle\int_0^T h\Big(s, 2n + \sum_{j=0}^{2n-1} \|u_m^{(j)}\|_\infty\Big)\,\mathrm{d}s + \Lambda \\ & \leq \displaystyle\int_0^T h(s, 2n + K\,\|u_m^{(2n-1)}\|_\infty)\,\mathrm{d}s + \Lambda \end{cases} \tag{6.29}$$

where K is given in (6.11). By condition (6.11),

$$\limsup_{v\to\infty} \frac{1}{v}\Big(\int_0^T h(s, 2n + Kv)\,\mathrm{d}s + \Lambda\Big) < 1$$

and therefore there exists a positive constant S such that

$$\int_0^T h(s, 2n + Kv)\,ds + \Lambda < v$$

whenever $v \geq S$. Now (6.29) shows that

$$\|u_m^{(2n-1)}\|_\infty < S, \quad m \in \mathbb{N}, \tag{6.30}$$

and then, by inequality (6.28),

$$\|u_m^{(j)}\|_\infty < T^{2n-j-1}S, \quad 0 \leq j \leq 2n-2, \quad m \in \mathbb{N}. \tag{6.31}$$

We have proved that there exists a positive constant S such that any solution u_m of problem (6.16), (6.2) satisfies inequalities (6.30) and (6.31), that is, $\|u_m\|_{C^{2n-1}} \leq KS$. Set

$$\gamma(x) = \begin{cases} 1 & \text{if } |x| \leq KS, \\ 2 - \frac{|x|}{KS} & \text{if } KS < |x| \leq 2KS, \\ 0 & \text{if } |x| > 2KS \end{cases}$$

and let $\widetilde{f}_m \in Car([0,T] \times \mathbb{R}^{2n})$ be given by

$$\widetilde{f}_m(t, x_0, \ldots, x_{2n-1}) = \gamma\Big(\sum_{j=0}^{2n-1} |x_j|\Big)\,[f_m(t, x_0, \ldots, x_{2n-1}) - a] + a.$$

Clearly, inequality (6.15) is satisfied with $\widetilde{f}_m$ instead of f_m. Hence applying the above procedure we obtain that $\|\widetilde{u}_m\|_{C^{2n-1}} \leq KS$ for any solution $\widetilde{u}_m$ of the differential equations

$$(-1)^n u^{(2n)} = \widetilde{f}_m(t, u, \ldots, u^{(2n-1)})$$

satisfying the boundary conditions (6.2). Therefore Corollary C.6 (with $\varphi(t)=a$ and with $2n$ instead of n) guarantees that problem (6.16), (6.2) has a solution $u_m \in AC^{2n-1}[0,T]$ and $\|u_m\|_{C^{2n-1}} \leq KS$.

Step 3. *Limit processes.*

By *Step* 2 we know that for each $m \in \mathbb{N}$ there exists a solution u_m of problem (6.16), (6.2) satisfying inequalities (6.23), (6.30) and (6.31). We now show that the sequence $\{f_m(u_m(t), \ldots, u_m^{(2n-1)}(t))\}$ is uniformly integrable on $[0,T]$. From inequalities (6.15) and (6.23) it follows that

$$\begin{aligned}
a &\le f_m(u_m(t),\dots,u_m^{(2n-1)}(t))\\
&\le h\Big(t, 2n+\sum_{j=0}^{2n-1}|u_m^{(j)}(t)|\Big)+\sum_{j=0}^{2n-1}[\omega_j(|u_m^{(j)}(t)|)+\omega_j(1)]\\
&\le h(t,2n+KS)+\sum_{j=0}^{2n-1}\omega_j(1)+\sum_{j=0}^{n-1}\omega_{2j}(At(T-t))\\
&\quad+\sum_{j=0}^{n-2}\omega_{2j+1}(A(t-\xi_{j,m})^2)+\omega_{2n-1}(A|t-\xi_{n-1,m}|)
\end{aligned}$$

for a.e. $t\in[0,T]$ where $\xi_{j,m}$ is the unique zero of $u_m^{(2j+1)}$ $(0\le j\le n-1$, $m\in\mathbb{N})$. We have $h(t,2n+KS)\in L_1[0,T]$ and also $\omega_{2j}(At(T-t))\in L_1[0,T]$ by (6.12). Hence, to prove that $\{f_m(u_m(t),\dots,u_m^{(2n-1)}(t))\}$ is uniformly integrable on $[0,T]$, it suffices to show that the sequences

$$\{\omega_{2j+1}(A(t-\xi_{j,m})^2)\},\quad \{\omega_{2n-1}(A|t-\xi_{n-1,m}|)\},\quad 0\le j\le n-2,$$

are uniformly integrable on $[0,T]$. Due to conditions (6.12) and (6.13) this fact follows from Criterion A.4 (with $b{=}A, r{=}2$ for $\{\omega_{2j+1}(A(t-\xi_{j,m})^2)\}$ and $b{=}A, r{=}1$ for $\{\omega_{2n-1}(A|t-\xi_{n-1,m}|)\}$). The uniform integrability of $\{f_m(u_m(t),\dots,u_m^{(2n-1)}(t))\}$ yields that $\{u_m^{(2n-1)}\}$ is equicontinuous on $[0,T]$ and consequently, by the Arzelà-Ascoli theorem and the Bolzano-Weierstrass Theorem, we can assume without loss of generality that $\{u_m\}$ is convergent in $C^{2n-1}[0,T]$ and $\{\xi_{j,m}\}$ is convergent in $\mathbb{R}$ for $0\le j\le n-1$. Let $\lim_{m\to\infty}u_m=u$ and $\lim_{m\to\infty}\xi_{j,m}=\xi_j$ $(0\le j\le n-1)$. Then $u\in C^{2n-1}[0,T]$ satisfies the boundary conditions (6.2) and letting $m\to\infty$ in inequality (6.23) we get

$$|u^{(2n-1)}(t)|\ge A|t-\xi_{n-1}|,\quad |u^{(2j+1)}(t)|\ge A(t-\xi_j)^2,\quad |u^{(2i)}(t)|\ge At(T-t)$$

for $t\in[0,T]$, $0\le j\le n-2$ and $0\le i\le n-1$. Hence $u^{(j)}$ has at most two zeros in $[0,T]$ for $0\le j\le 2n-1$ and moreover, due to inequality (6.19), u satisfies inequality (6.14). Therefore, by Theorem 1.7, u is a solution of problem (6.1), (6.2) and $u\in AC^{2n-1}[0,T]$. □

Example. Consider problem (6.1), (6.2) with

$$f(t,x_0,\dots,x_{2n-1})=p(t)+\sum_{k=0}^{2n-1}\Big(\frac{a_k(t)}{|x_k|^{\alpha_k}}+b_k(t)|x_k|^{\beta_k}\Big)$$

on $[0,T] \times \mathcal{D}$ where $a_k \in L_\infty[0,T]$, $p,\, b_k \in L_1[0,T]$ are nonnegative for $0 \le k \le 2n-1$ and $p(t) \ge a > 0$ for a.e. $t \in [0,T]$. If $\alpha_{2n-1}, \alpha_{2j} \in (0,1)$ for $0 \le j \le n-1$, $\alpha_{2j+1} \in (0, \frac{1}{2})$ for $0 \le j \le n-2$ and $\beta_k \in (0,1)$ for $0 \le k \le 2n-1$ then, by Theorem 6.3, the problem has a solution $u \in AC^{2n-1}[0,T]$ satisfying inequality (6.14).

Bibliographical notes

Theorem 6.3 was adapted from Agarwal, O'Regan, Rachůnková and Staněk [16]. The singular Lidstone problem for the differential equation

$$(-1)^n\, u^{(2n)} = f(t,u)$$

is considered in Zhao [206]. Here $f \in C^0((0,1)\times(0,\infty))$ is nonnegative and f may be singular at $u=0$, $t=0$ and/or $t=1$. The existence of positive solutions in the sets $C^{2n-2}[0,1] \cap C^{2n}(0,1)$ and $C^{2n-1}[0,1] \cap C^{2n}(0,1)$ is proved by a combination of the method of lower and upper functions with the Schauder fixed point theorem. Other singular Lidstone problem for the differential equation

$$(-1)^n\, u^{(2n)} = f(t, u, -u'', \dots, (-1)^j\, u^{(2j)}, \dots, (-1)^{n-1}\, u^{(2n-2)})$$

may be found in Wei [198], where $f \in C((0,1)\times(0,\infty)^n)$ is nonnegative and $f(t, x_0, \dots, x_{n-1})$ may be singular at $x_j = 0$, $j = 0, 1, \dots, n-1$, $t=0$ and/or $t=1$. Sufficient and necessary conditions for the existence of positive solutions in the sets $C^{2n-2}[0,1] \cap C^{2n}(0,1)$ or $C^{2n-1}[0,1] \cap C^{2n}(0,1)$ are given. The results are proved by a combination of the method of lower and upper functions with a maximal principle.

Part II

Second order singular problems with $\phi-$Laplacian

Many nonlinear evolution partial differential equations, which act as models for combusting or other processes, have solutions which develop strong singularities in a finite time, see the references in the books by Bebernes and Eberly [35], Samarskii, Galaktionov, Kurdyumov and Mikhailov [175] and in the survey paper by Levine [124]. The prototype of such problems is the semilinear parabolic equation from combustion theory

$$u_t = u_{xx} + f(u).$$

Important examples of f include $f(u) = \exp(u)$ and $f(u) = u^\beta, \quad \beta > 1.$ In many physical systems, the diffusion term is not linear but depends on the function u, for example

$$u_t = (u^\sigma\, u_x)_x + u^\beta, \quad \sigma > 0.$$

This equation has a porous-medium type diffusion term, and arises as a model for the temperature profile of a fusion reactor plasma with one source term (see Zmitrenko, Kurdyumov, Mikhailov and Samarskii [207] and for further references see the works Samarskii, Galaktionov, Kurdyumov and Mikhailov [175] or Le Roux and Wilhelmsson [123]). Another possibility is that the diffusion term depends on its gradient. It occurs in the equation

$$u_t = (|u_x|^\sigma\, u_x)_x + \exp(u)$$

which arises from studies of turbulent diffusion or the flow of a non-Newtonian liquid. This equation is invariant under the respective Lie groups of transformations (see e.g. Budd, Collins and Galaktionov [48]). Searching for solutions which are invariant under these fransformations leads to the following ordinary differential equation for u with a quasilinear differential operator:

$$(|u'|^{p-2}\, u')' - c\, t\, u' + \exp(u) - 1 = 0,$$

where c is a positive constant and $p = \sigma + 2$. Let us put

$$\phi_p(y) = |y|^{p-2} y \quad \text{for} \quad y \in \mathbb{R}.$$

If $p > 1$, then the quasilinear operator

$$u \mapsto (\phi_p(u'))'$$

is called the (one-dimensional) p – Laplacian.

Further, motivated by various significant applications to non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity and theory of capillary surfaces (see Atkinson and Bouillet [29], Esteban and Vazquez [82], Phan-Thien [151]), several authors have proposed the study of radially symmetric solutions of the p – Laplace equation

$$\operatorname{div}\left(|\nabla v|^{p-2}\,\nabla v\right) = h(|x|, v).$$

Here ∇ is the gradient, $p > 1$ and $|x|$ is the Euclidean norm in $\mathbb{R}^n$ of $x = (x_1, \ldots, x_n)$, $n > 1$. Radially symmetric solutions of this partially differential equation (i.e., solutions that depend only on the variable $r = |x|$) satisfy the ordinary differential equation

$$r^{1-n}\,(r^{n-1}\,|v'|^{p-2}\,v')' = h(r, v), \quad {}' = \frac{\mathrm{d}}{\mathrm{d}r}.$$

If $p = n$, the change of variables $t = \ln r$ transforms it into the equation

$$(|u'|^{p-2}\,u')' = e^{n\,t}\,h(e^{\,t}, u), \quad {}' = \frac{\mathrm{d}}{\mathrm{d}t}$$

and for $p \neq n$, the change of variables $t = r^{(p-n)/(p-1)}$ yields the equation

$$(|u'|^{p-2}\,u')' = \left|\frac{p-1}{p-n}\right|^p t^{\frac{p-n}{p\,(1-n)}}\,h(t^{\frac{p-1}{p-n}}, u), \quad {}' = \frac{\mathrm{d}}{\mathrm{d}t}.$$

Both these equations have (one-dimensional) p – Laplacian ϕ_p.

This operator was also discussed for systems of second order differential equations in Lu, O'Regan and Agarwal [130], Manásevich and Mawhin [131], [132], Mawhin [137], Mawhin and Ureña [139], Nowakowski and Orpel [145], Zhang [203]. Further modifications can be found in X.L. Fan and X. Fan [85], Fan, Wu and Wang [86], where the $p(t)$ – Laplacian $u \to (|u'|^{p(t)-2}u)'$ was investigated and in Dambrosio [63] who worked with the $(p_1, \ldots, p_n)$ – Laplacian. The above operators have been sometimes replaced by their abstract and more general version of the form

$$u \mapsto (\phi(u'))'$$

called the ϕ – Laplacian, where $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism. This leads to clearer exposition and better understanding of the methods that are employed to derive existence results. See also Manásevich, Mawhin [132], where $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ is a strictly monotone homeomorphism.

Most of existence results for problems with ϕ –Laplacian (or with some of its special versions) is proved under the assumption that the problems are regular. See e.g. Dambrosio [63], X.L. Fan and X. Fan [85], Fan, Wu and Wang [86], Liu [125], Lu, O'Regan and Agarwal [130], Manásevich and Mawhin [131], [132], Mawhin [138], [137], Mawhin and Ureña [139], O'Regan [147], Rachůnková and Tvrdý [169], Zhang [203] who consider two-point boundary conditions (Dirichlet, Neumann, mixed and periodic). Further we refer to the papers Agarwal, O'Regan and Staněk [20] or Nowakowski and Orpel [145] where some nonlocal boundary conditions can be found. Recently some papers dealing with singular problems with ϕ –Laplacian have been published. We can refer to Agarwal, Lü and O'Regan [3], Jiang [109], [110], Wang and Gao [197] for the Dirichlet problem, to Jebelean and Mawhin [107], [108], Liu [126], Polášek and Rachůnková [153], Rachůnková and Tvrdý [170] for the periodic problem, to Agarwal, O'Regan, Staněk [18], [20] for the mixed or nonlocal problems and to Rachůnková, Staněk and Tvrdý [163] for other references and results.

Chapter 7

Dirichlet problem

Assume that

ϕ is an increasing odd homeomorphism with $\phi(\mathbb{R}) = \mathbb{R}$.

In this chapter we consider the singular Dirichlet problem with ϕ – Laplacian of the form

$$(\phi(u'))' + f(t, u, u') = 0, \quad u(0) = u(T) = 0, \tag{7.1}$$

and its special cases, in particular, the problem of the form

$$u'' + f(t, u, u') = 0, \quad u(0) = u(T) = 0, \tag{7.2}$$

where $\phi(y) \equiv y$. We will investigate problems (7.1) and (7.2) on the set $[0,T] \times \mathcal{A}$. In general, the function f depends on the time variable $t \in [0,T]$ and on two space variables x and y, where $(x, y) \in \mathcal{A}$ and $\mathcal{A}$ is a closed subset of $\mathbb{R}^2$. We assume that problems (7.1) and (7.2) are singular, which means, by Chapter 1, that f does not satisfy the Carathéodory conditions on $[0,T] \times \mathcal{A}$. In what follows, the types of singularities of f will be exactly specified for each problem under consideration.

In accordance with Chapter 1 we define:

Definition 7.1. A function $u\colon [0,T] \to \mathbb{R}$ with $\phi(u') \in AC[0,T]$ is a *solution of problem* (7.1) if u satisfies

$$(\phi(u'(t)))' + f(t, u(t), u'(t)) = 0 \quad \text{a.e. on } [0,T]$$

and fulfils the boundary conditions $u(0) = u(T) = 0$. If $\mathcal{A} \neq \mathbb{R}^2$, we impose on u in addition the condition $(u(t), u'(t)) \in \mathcal{A}$ for $t \in [0,T]$.

A function $u \in C[0,T]$ is a *w-solution of problem* (7.1) if there exists a finite number of singular points $t_\nu \in [0,T]$, $\nu = 1, \ldots, r$, such that if we denote $J = [0,T] \setminus \{t_\nu\}_{\nu=1}^{r}$, then $\phi(u') \in AC_{loc}(J)$, u satisfies

$$(\phi(u'(t)))' + f(t, u(t), u'(t)) = 0 \quad \text{a.e. on } [0,T]$$

and fulfils the boundary conditions $u(0) = u(T) = 0$. If $\mathcal{A} \neq \mathbb{R}^2$, then $(u(t), u'(t)) \in \mathcal{A}$ for $t \in J$.

Note that the condition $\phi(u') \in AC[0,T]$ implies $u \in C^1[0,T]$ and the condition $\phi(u') \in AC_{loc}(J)$ implies $u \in C^1(J)$. If f is supposed to be continuous on $(0,T) \times \mathbb{R}^2$ and can have only time singularities at $t = 0$ and $t = T$, then any solution (any w-solution) u of problem (7.1) moreover satisfies $\phi(u') \in C^1(0,T)$. If we have a w-solution u which is not a solution, then we do not know the behaviour of u' near singular points t_ν. But we often need to know this behaviour. For example, if a singular ordinary differential equation arises from a partial differential equation with some symmetry properties we need u' to be defined on the whole interval $[0,T]$. Therefore we will focus our main attention on solutions and on such w-solutions that have bounded first derivatives on J.

REMARK 7.2. We see that the Dirichlet conditions in (7.1) can be written in the form $u \in \mathcal{B}$, where

$$\mathcal{B} = \{x \in C[0,T] : x(0) = x(T) = 0\}$$

is a closed subset of $C[0,T]$. Hence, we can carry out the investigation of problem (7.1) in the spirit of the existence principles presented in Chapter 1:

- the singular problem (7.1) is approximated by a sequence of solvable regular problems,
- a sequence $\{u_n\}$ of approximate solutions is generated,
- a convergence of a suitable subsequence $\{u_{k_n}\}$ is investigated,
- the type of this convergence determines the properties of its limit u and, among other, determines whether u is a w-solution or a solution of the original singular problem.

There are more possibilities how to construct an approximate sequence of regular problems. Their choice depends on the type of singularities of the nonlinearity f in (7.1) (time, space), on the type of singular points corresponding to a solution or a w-solution of problem (7.1) (type I, type II), on the type of results desired (existence of a solution, a positive solution, a w-solution, uniqueness), and so on. A common idea is that approximate functions f_n have no singularities, $f_n \neq f$ on neighbourhoods U_n of singular points of f, $f_n = f$ elsewhere, and $\lim_{n\to\infty} \operatorname{meas}(U_n) = 0$. Having such a sequence of $\{f_n\}$ we study regular problems

$$(\phi(u'))' + f_n(t,u,u') = 0, \quad u(0) = A_n, \quad u(T) = B_n, \quad n \in \mathbb{N},$$

where $A_n, B_n \in \mathbb{R}$, $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = 0$. In some proofs, one simply puts $A_n = B_n = 0$ for $n \in \mathbb{N}$. Solvability of these regular problems can be investigated by means of various methods which have been developed for regular Dirichlet problems (fixed point theorems, topological degree arguments – Cronin [59], Mawhin [135], the critical point theory – Drábek [77], the topological transversality method – Granas, Guenther and Lee [100], variational methods – Ambrosetti [27], Došlý and Řehák [76], Mawhin and Willem [140], lower and upper functions – De Coster and Habets [60], [61], [62], Kiguradze and Shekhter [118], Vasiliev and Klokov [194], Ważewski method – Srzednicki [180], Diblík [73], etc.). Using these methods we generate a sequence of approximate solutions $\{u_n\}$. The crucial information which enables us to realize the limit process concerns a priori estimates of the approximate solutions u_n. In the next section we present some existence results and a priori estimates of solutions of regular problems which will be used in the study of solvability of the singular problem (7.1).

7.1 Regular Dirichlet problem

In this section we will study an auxiliary regular problem of the form

$$(\phi(u'))' + g(t,u,u') = 0, \quad u(0) = A, \; u(T) = B, \tag{7.3}$$

where $g \in Car([0,T] \times \mathbb{R}^2)$, $A, B \in \mathbb{R}$.

Definition 7.3. A function $u\colon [0,T] \to \mathbb{R}$ with $\phi(u') \in AC[0,T]$ is a *solution of problem* (7.3) if u satisfies

$$(\phi(u'(t)))' + g(t,u(t),u'(t)) = 0 \quad \text{for a.e. } t \in [0,T]$$

and fulfils the boundary conditions $u(0) = A$, $u(T) = B$.

The simplest case when g has a Lebesgue integrable majorant, is described in the next theorem.

Theorem 7.4. *Assume that there is a function* $h \in L_1[0,T]$ *such that*

$$|g(t,x,y)| \le h(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } x, y \in \mathbb{R}. \tag{7.4}$$

Then problem (7.3) *has a solution.*

Proof. *Step* 1. *Solution of an auxiliary problem.*

Consider the auxiliary problem

$$(\phi(u'))' = b(t), \quad u(0) = A, \quad u(T) = B, \tag{7.5}$$

where $b \in L_1[0,T]$. It can be checked by direct computation that u is a solution of problem (7.5) if and only if $u \in C^1[0,T]$ satisfies the conditions

$$u(t) = A + \int_0^t \phi^{-1}\Big(\phi(u'(0)) + \int_0^s b(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s$$

and

$$\int_0^T \phi^{-1}\Big(\phi(u'(0)) + \int_0^s b(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s = B - A.$$

Step 2. *Definition of functional* γ.

For each $\ell \in C[0,T]$ define

$$\psi_\ell : \mathbb{R} \to \mathbb{R}, \quad \psi_\ell(x) = \int_0^T \phi^{-1}\big(x + \ell(s)\big)\,\mathrm{d}s.$$

Due to the assumption that ϕ is an increasing homeomorphism with $\phi(\mathbb{R}) = \mathbb{R}$, the function ψ_ℓ is continuous, increasing, and $\psi_\ell(\mathbb{R}) = \mathbb{R}$. Thus the equation $\psi_\ell(x) = B - A$ has exactly one root $x = \gamma(\ell) \in \mathbb{R}$. Therefore we can define the functional

$$\gamma : C[0,T] \to \mathbb{R}, \quad \psi_\ell(\gamma(\ell)) = B - A.$$

Step 3. *The functional* γ *maps bounded sets to bounded sets.*

Assume that $\mathcal{M} \subset C[0,T]$ and $c \in (0,\infty)$ are such that $\|\ell\|_\infty \le c$ for each $\ell \in \mathcal{M}$. Further assume that there exists a sequence $\{\ell_n\} \subset \mathcal{M}$ such that

$$\lim_{n\to\infty} \gamma(\ell_n) = \infty \quad \text{or} \quad \lim_{n\to\infty} \gamma(\ell_n) = -\infty.$$

Let the former possibility occur. Then

$$B - A = \lim_{n\to\infty} \psi_{\ell_n}(\gamma(\ell_n)) \geq \lim_{n\to\infty} T\phi^{-1}(\gamma(\ell_n) - c) = \infty,$$

a contradiction. The latter possibility can be argued similarly. Thus $\gamma(\mathcal{M})$ is bounded.

Step 4. *Functional* γ *is continuous.*

Consider a sequence $\{\ell_n\} \subset C[0,T]$ and assume that

$$\lim_{n\to\infty} \ell_n = \ell_0 \quad \text{in } C[0,T].$$

By Step 3, the sequence $\{\gamma(\ell_n)\} \subset \mathbb{R}$ is bounded and hence we can choose a subsequence such that $\lim_{n\to\infty} \gamma(\ell_{k_n}) = x_0 \in \mathbb{R}$. We get

$$B - A = \psi_{\ell_{k_n}}(\gamma(\ell_{k_n})) = \int_0^T \phi^{-1}\big(\gamma(\ell_{k_n}) + \ell_{k_n}(t)\big)\,\mathrm{d}t,$$

which, for $n \to \infty$, yields

$$B - A = \int_0^T \phi^{-1}\big(x_0 + \ell_0(t)\big)\,\mathrm{d}t.$$

Thus, according to Step 2, we have $x_0 = \gamma(\ell_0)$. It follows that any convergent subsequence of $\{\gamma(\ell_n)\}$ has the same limit $\gamma(\ell_0)$. Since $\{\gamma(\ell_n\}$ is bounded, we get $\gamma(\ell_0) = \lim_{n\to\infty} \gamma(\ell_n)$.

Step 5. *Definition of operator* $\mathcal{F}$.

Define operators $\mathcal{N} : C^1[0,T] \to C[0,T]$ and $\mathcal{F} : C^1[0,T] \to C^1[0,T]$ by

$$(\mathcal{N}(u))(t) = -\int_0^t g(s, u(s), u'(s))\,\mathrm{d}s$$

and

$$(\mathcal{F}(u))(t) = A + \int_0^t \phi^{-1}\Big(\gamma(\mathcal{N}(u)) + (\mathcal{N}(u))(s)\Big)\,\mathrm{d}s.$$

Step 1 and *Step* 2 yield that u is a solution of problem (7.3) if and only if $u \in C^1[0,T]$ satisfies

$$u(t) = A + \int_0^t \phi^{-1}\big(\phi(u'(0)) + (\mathcal{N}(u))(s)\big)\,\mathrm{d}s, \quad \phi(u'(0)) = \gamma(\mathcal{N}(u)).$$

Therefore the operator equation $u = \mathcal{F}(u)$ is equivalent to problem (7.3). Thus it suffices to prove that the operator $\mathcal{F}$ has a fixed point.

Step 6. Fixed point of operator $\mathcal{F}$.

Since the operators γ and $\mathcal{N}$ are continuous, it follows that $\mathcal{F}$ is continuous. Choose an arbitrary sequence $\{u_n\} \subset C^1[0,T]$ and denote $v_n = \mathcal{F}(u_n)$ for $n \in \mathbb{N}$. Then

$$v_n'(t) = \phi^{-1}\big(\gamma(\mathcal{N}(u_n)) + (\mathcal{N}(u_n))(t)\big), \quad t \in [0,T], \quad n \in \mathbb{N}.$$

By condition (7.4) there is a $c_1 \in (0,\infty)$ such that $\|(\mathcal{N}(u_n)\|_\infty \le c_1$. This implies that the sequences $\{v_n\}$ and $\{v_n'\}$ are bounded on $[0,T]$. Consequently, the sequence $\{v_n\}$ is equicontinuous on $[0,T]$. Moreover, for $t_1, t_2 \in [0,T]$ we have

$$|\phi(v_n'(t_1)) - \phi(v_n'(t_2))| = |(\mathcal{N}(u_n))(t_1) - (\mathcal{N}(u_n))(t_2)| \le \left| \int_{t_1}^{t_2} h(s)\,\mathrm{d}s \right|.$$

Thus the sequence $\{\phi(v_n')\}$ is bounded and equicontinuous on $[0,T]$. Making use of the Arzelà-Ascoli theorem we can find subsequences $\{v_{k_n}\}$ and $\{\phi(v_{k_n}')\}$ uniformly convergent on $[0,T]$. Then $\{v_{k_n}'\}$ is also uniformly convergent on $[0,T]$ and so, $\{v_{k_n}\}$ is convergent in $C^1[0,T]$. We have proved that the operator $\mathcal{F}$ is compact on $C^1[0,T]$. By the Schauder fixed point theorem, $\mathcal{F}$ has a fixed point, which is a solution of problem (7.3). $\square$

Method of a priori estimates

Using the method of a priori estimates we can get existence of solutions of problem (7.3) even for functions g which do not satisfy (7.4) with some $h \in L_1[0,T]$. To this aim the following two lemmas will be useful. Define the linear function

$$a(t) = \frac{T-t}{T}A + \frac{t}{T}B, \quad t \in [0,T]. \tag{7.6}$$

Motivated by the monographs Kiguradze [115] or Kiguradze and Shekhter [118] we will prove a priori estimates under one-sided growth conditions.

Lemma 7.5 (**A priori estimate – sublinear growth**).
Let $\alpha, \beta \in [0,1)$, $\varkappa \in (0,\infty)$. *Let* $h_1 \in L_1[0,T]$ *be nonnegative and let the function* a *be given by* (7.6). *Further assume that*

$$\lim_{y\to\infty} \frac{\phi(y)}{y} > 0. \tag{7.7}$$

Then there exists $r > 0$ *such that the estimate*

$$\|u\|_\infty + \|u'\|_\infty \le r$$

is valid for each nonnegative function $h_0 \in L_1[0,T]$ *with* $\|h_0\|_1 \le \varkappa$ *and for each function* u *satisfying*

$$\begin{cases} \phi(u') \in AC[0,T], \quad u(0) = A, \quad u(T) = B, \\ -(\phi(u'(t)))' \operatorname{sign}(u(t) - a(t)) \\ \qquad \le h_0(t) + h_1(t)(|u(t)|^\alpha + |u'(t)|^\beta) \ \ \textit{for a.e. } t \in [0,T]. \end{cases} \tag{7.8}$$

Proof. Choose an arbitrary u satisfying (7.8). Denote $\rho = \|u'\|_\infty$ and let $\rho = |u'(t_0)|$. Assume that $\rho > |\frac{B-A}{T}|$. We have $\|u\|_\infty \le \rho T + |A|$. Now, we shall consider four cases.

Case 1. Let $u'(t_0) = \rho$, $u(t_0) < a(t_0)$. This yields $t_0 \in (0,T)$ and if we put $v(t) = u(t) - a(t)$ on $[0,T]$, we have $v'(t_0) > 0$, $v(t_0) < 0$. Since $v(0) = 0$, we can find $t_1 \in [0,t_0)$ such that

$$v'(t_1) = 0, \quad v'(t) > 0 \quad \text{for } t \in (t_1, t_0).$$

This implies $u(t) - a(t) = v(t) < 0$ on $[t_1, t_0]$. Integrating the inequality in (7.8), we get

$$\int_{t_1}^{t_0} (\phi(u'(t)))' \mathrm{d}t \le \|h_0\|_1 + ((\rho T + |A|)^\alpha + \rho^\beta)\|h_1\|_1.$$

Thus

$$\begin{cases} \dfrac{\phi(\rho)}{\rho} \le \dfrac{1}{\rho}\left(\varkappa + \left|\phi\left(\frac{B-A}{T}\right)\right|\right) \\ \qquad\qquad + \left(\dfrac{(\rho T + |A|)^\alpha}{\rho} + \rho^{\beta-1}\right)\|h_1\|_1 =: F(\rho). \end{cases} \tag{7.9}$$

Since $\lim_{y\to\infty} F(y) = 0$, we deduce by assumption (7.7) that

$$\text{there exists } \rho^* > \left|\frac{B-A}{T}\right| \quad \text{such that} \quad \|u'\|_\infty \le \rho^*. \tag{7.10}$$

We see that ρ^* does not depend on the choice of u and h_0.

Case 2. Let $u'(t_0) = \rho$, $u(t_0) \ge a(t_0)$. So, for $v = u - a$ we have $v'(t_0) > 0$, $v(t_0) \ge 0$. Let $t_0 \in [0,T)$. Then there exists $t_1 \in (t_0,T)$ such that

$$v'(t_1) = 0, \quad v'(t) > 0 \quad \text{for } t \in (t_0,t_1).$$

This implies $u(t) - a(t) = v(t) > 0$ on $(t_0,t_1]$. Integrating the inequality in (7.8), we get

$$-\int_{t_0}^{t_1} (\phi(u'(t)))' \mathrm{d}t \le \|h_0\|_1 + \big((\rho T + |A|)^\alpha + \rho^\beta\big)\, \|h_1\|_1.$$

Thus relation (7.9) is valid which yields estimate (7.10). Now, let $t_0 = T$. Then there exists $t_1 \in (0,T)$ such that

$$v'(t_1) = 0, \quad v'(t) > 0 \quad \text{for } t \in (t_1,T).$$

Since $v(T) = 0$, we see that $u(t) - a(t) = v(t) < 0$ on (t_1,T). Integrating the inequality in (7.8), we get

$$\int_{t_1}^{T} (\phi(u'(t)))' \, \mathrm{d}t \le \|h_0\|_1 + \big((\rho T + |A|)^\alpha + \rho^\beta\big)\, \|h_1\|_1.$$

So, relation (7.9) and consequently estimate (7.10) are valid again.

Cases 3 and 4. Let

$$u'(t_0) = -\rho,\ u(t_0) > a(t_0) \quad \text{or} \quad u'(t_0) = -\rho,\ u(t_0) \le a(t_0).$$

Similarly, using the assumption that ϕ is odd, we can verify that estimate (7.10) is true also in this remaining two cases.

Summarizing, if we put $r = \rho^* + \rho^* T + |A|$, we get $\|u\|_\infty + \|u'\|_\infty \le r$. $\square$

REMARK 7.6. (i) If ϕ does not fulfil condition (7.7), we replace the inequality in (7.8) by

$$-(\phi(u'(t)))' \operatorname{sign}(u(t)-a(t))$$
$$\le h_0(t)+h_1(t)\left(\left|\phi\left(\frac{u(t)-A}{T}\right)\right|^{\alpha}+|\phi(u'(t))|^{\beta}\right) \qquad \text{for a.e. } t\in[0,T].$$

Then, arguing similarly to the proof of Lemma 7.5, we get

$$1\le \frac{1}{\phi(\rho)}\left(\varkappa+\left|\phi\left(\frac{B-A}{T}\right)\right|\right)+\|h_1\|_1((\phi(\rho))^{\alpha-1}+(\phi(\rho))^{\beta-1}).$$

This implies estimate (7.10) and consequently $\|u\|_\infty+\|u'\|_\infty\le r$.

(ii) If $\phi(y)=\phi_p(y)=|y|^{p-2}y$ with $p\ge 2$, then condition (7.7) is always satisfied.

Lemma 7.7 (**A priori estimate – linear growth**).
Assume that $\varkappa\in(0,\infty)$ *and that the function* a *is given by* (7.6). *Let* $h_1,h_2\in L_1[0,T]$ *be nonnegative and let*

$$\lim_{y\to\infty}\frac{\phi(y)}{y}>T\|h_1\|_1+\|h_2\|_1. \tag{7.11}$$

Then there exists $r>0$ *such that the estimate*

$$\|u\|_\infty+\|u'\|_\infty\le r$$

is valid for each nonnegative function $h_0\in L_1[0,T]$ *with* $\|h_0\|_1\le\varkappa$ *and for each function* u *satisfying*

$$\begin{cases}\phi(u')\in AC[0,T], \quad u(0)=A, \quad u(T)=B,\\ -(\phi(u'(t)))'\operatorname{sign}(u(t)-a(t))\\ \qquad \le h_0(t)+h_1(t)\,|u(t)|+h_2(t)\,|u'(t)| \quad \textit{for a.e. } t\in[0,T].\end{cases} \tag{7.12}$$

Proof. Choose an arbitrary function u satisfying condition (7.12). Denote $\rho=\|u'\|_\infty$ and let $\rho=|u'(t_0)|$. We have $\|u\|_\infty\le\rho T+|A|$. Assume that $\rho>|\frac{B-A}{T}|$. Now, we shall consider four cases as in the proof of Lemma 7.5.

Let $u'(t_0)=\rho$, $u(t_0)<a(t_0)$. We argue as in the proof of Lemma 7.5 and find $t_1\in[0,t_0)$ such that $u'(t_1)=|\frac{B-A}{T}|$ and $u(t)<a(t)$ on $[t_1,t_0]$. Integrating the inequality in (7.12), we get

$$\frac{\phi(\rho)}{\rho}\le\frac{1}{\rho}\left(\varkappa+\left|\phi\left(\frac{B-A}{T}\right)\right|+|A|\|h_1\|_1\right)+T\|h_1\|_1+\|h_2\|_1=:F_1(\rho).$$

Since $\lim_{y\to\infty}F_1(y)=T\|h_1\|_1+\|h_2\|_1$, we deduce by assumption (7.11) that estimate (7.10) holds. The remaining three cases are similar. Therefore, if we put $r=\rho^*+\rho^*T+|A|$, we get $\|u\|_\infty+\|u'\|_\infty\le r$. ☐

REMARK 7.8. (i) If condition (7.11) is not satisfied, we assume

$$T\,\|h_1\|_1+\|h_2\|_1<1$$

and replace the inequality in (7.12) by

$$-(\phi(u'(t)))'\,\mathrm{sign}(u(t)-a(t))$$
$$\le h_0(t)+h_1(t)\left|\phi\left(\frac{u(t)-A}{T}\right)\right|+h_2(t)|\phi(u'(t))|\quad\text{for a.e. }\ t\in[0,T].$$

Then, arguing similarly to the proof of Lemma 7.7 and to Remark 7.6, we get $\|u\|_\infty+\|u'\|_\infty\le r$.

(ii) We see that if $\phi(y)=\phi_p(y)=|y|^{p-2}y$ with $p>2$, then condition (7.11) is fulfilled for each $h_1,h_2\in L_1[0,T]$.

The following theorem relies on Lemma 7.5.

Theorem 7.9. *Assume that the function a is given by (7.6). Let $\alpha,\beta\in[0,1)$ and let $h\in L_1[0,T]$ be nonnegative. Further assume (7.7) and*

$$\begin{cases} g(t,x,y)\ \mathrm{sign}(x-a(t))\le h(t)(1+|x|^\alpha+|y|^\beta)\\ \qquad \textit{for a.e.}\ \ t\in[0,T]\ \textit{and all}\ \ x,y\in\mathbb{R}. \end{cases}\tag{7.13}$$

Then problem (7.3) has a solution.

Proof. Let r be the constant of Lemma 7.5 for $h_0=h_1=h$ and $\varkappa=\|h\|_1$. Put $M=\max\{|A|,|B|\}$, $\widetilde{r}=r+M$, and define

$$\chi(z)=\begin{cases}-\widetilde{r} & \text{if}\ \ z<-\widetilde{r},\\ \ \ z & \text{if}\ \ |z|\le\widetilde{r},\\ \ \ \widetilde{r} & \text{if}\ \ z>\ \widetilde{r},\end{cases}\qquad \widetilde{g}(t,x,y)=g(t,\chi(x),\chi(y))\ \text{for a.e.}\ t\in[0,T]$$

and all $x, y, z \in \mathbb{R}$. Then $\widetilde{g} \in Car([0,T] \times \mathbb{R}^2)$ and there is a function $\widetilde{h} \in L_1[0,T]$ such that $|\widetilde{g}(t,x,y)| \le \widetilde{h}(t)$ for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$. Consider the auxiliary problem

$$(\phi(u'))' + \widetilde{g}(t,u,u') = 0, \quad u(0) = A, \quad u(T) = B. \tag{7.14}$$

By Theorem 7.4, problem (7.14) has a solution u. Since $\widetilde{r} > M$, we have $\operatorname{sign}(x - a(t)) = \operatorname{sign}(\chi(x) - a(t))$ for $t \in [0,T]$, $x \in \mathbb{R}$, and

$$\begin{aligned}
-(\phi(u'(t)))' \operatorname{sign}(u(t) - a(t)) &= g(t, \chi(u(t)), \chi(u'(t))) \operatorname{sign}(\chi(u(t)) - a(t)) \\
&\le h(t)\,(1 + |\chi(u(t))|^\alpha + |\chi(u'(t))|^\beta) \\
&\le h(t)(1 + |u(t)|^\alpha + |u'(t)|^\beta) \quad \text{for a.e. } t \in [0,T].
\end{aligned}$$

Thus, by Lemma 7.5, the function u satisfies $\|u\|_\infty + \|u'\|_\infty \le r$ and hence u is also a solution of problem (7.3). □

Remark 7.10. If g satisfies inequality (7.13) with $\alpha, \beta \in [0,1)$, we will say that g has *one-sided sublinear growth in x and y*. In this case each function $g + g_0$ has also one-sided sublinear growth provided $g_0(t,x,y) \operatorname{sign}(x - a(t))$ is nonpositive on $[0,T] \times \mathbb{R}^2$.

Example. Let $A = B = 0$, $h_i \in L_1[0,T]$, $i = 0,1,2,3$, h_1, h_3 be nonnegative on $[0,T]$. For a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$ define the function

$$g(t,x,y) = h_0(t) - h_1(t)\,x^3 + h_2(t)\sqrt{|y|} - h_3(t)\,xy^4.$$

We see that g satisfies inequality (7.13) because $a(t) \equiv 0$ and we can write g in the form $g = g_0 + g_1$, where $g_1(t,x,y) = h_0(t) + h_2(t)\sqrt{|y|}$ and $g_0(t,x,y) = -h_1(t)x^3 - h_3(t)\,xy^4$. Here g_1 has a sublinear growth in x and y and $g_0(t,x,y) \operatorname{sign} x \le 0$ on $[0,T] \times \mathbb{R}^2$.

The next theorem will be applicable to problem (7.3) with $g(t,x,y)$ having *one-sided linear growth* in x and y.

Theorem 7.11. *Let the function a be given by (7.6). Let $h_0, h_1, h_2 \in L_1[0,T]$ be nonnegative and let condition (7.11) hold. Further assume*

$$\begin{cases} g(t,x,y) \operatorname{sign}(x - a(t)) \le h_0(t) + h_1(t)|x| + h_2(t)|y| \\ \qquad \text{for a.e. } t \in [0,T] \text{ and all } x, y \in \mathbb{R}. \end{cases}$$

Then problem (7.3) has a solution.

Proof. We argue as in the proof of Theorem 7.9 and use Lemma 7.7 instead of Lemma 7.5. □

EXAMPLE. Let $T=1$, $n\in\mathbb{N}$, $A=0$, $B=1$, $\phi(y)\equiv y$, $h\in L_1[0,1]$ and let $\varphi\in Car([0,1]\times\mathbb{R}^2)$ be nonnegative. Then the function

$$g(t,x,y)=h(t)+t\,x+t^2\,y-(x-t)^{2n+1}\varphi(t,x,y)$$

satisfies the conditions of Theorem 7.11 because

$$g(t,x,y)\,\mathrm{sign}(x-t)\le|h(t)|+t\,|x|+t^2\,|y|$$

for a.e. $t\in[0,1]$ and for all $x,y\in\mathbb{R}$, and

$$\lim_{y\to\infty}\frac{\phi(y)}{y}=1>\int_0^1 t\,\mathrm{d}t+\int_0^1 t^2\,\mathrm{d}t=\frac{5}{6},$$

i.e. condition (7.11) is valid.

REMARK 7.12. If ϕ does not fulfil conditions (7.7) and (7.11) in Theorems 7.9 and 7.11, respectively, we modify these theorems according to Remarks 7.6 and 7.8.

Method of lower and upper functions

It is well known that for regular second order boundary value problems *the lower and upper functions method* is a useful instrument for proofs of their solvability and for a priori estimates of their solutions. See e.g. De Coster and Habets [60], [61], [62], Kiguradze and Shekhter [118], Ladde, Lakshmikantham and Vatsala [120], Rachůnková and Tvrdý [167], [168], [169] or Vasiliev and Klokov [194]. In literature several definitions of lower and upper functions for regular boundary value problems can be found. (Note that in some papers they are called *lower and upper solutions*). Here we will use the following one.

Definition 7.13. A function $\sigma\in C[0,T]$ is called a *lower function of problem* (7.3) if there is a finite set $\Sigma\subset(0,T)$ such that $\phi(\sigma')\in AC_{loc}([0,T]\setminus\Sigma)$, $\sigma'(\tau+):=\lim_{t\to\tau+}\sigma'(t)\in\mathbb{R}$, $\sigma'(\tau-):=\lim_{t\to\tau-}\sigma'(t)\in\mathbb{R}$ for each $\tau\in\Sigma$,

$$\begin{cases}(\phi(\sigma'(t)))'+g(t,\sigma(t),\sigma'(t))\ge 0 & \text{for a.e. } t\in[0,T],\\ \sigma(0)\le A,\ \sigma(T)\le B,\ \sigma'(\tau-)<\sigma'(\tau+) & \text{for each } \tau\in\Sigma.\end{cases}\tag{7.15}$$

If the inequalities in (7.15) are reversed, then σ is called an *upper function of problem* (7.3).

We have seen that Theorems 7.9 and 7.11 can be used for problem (7.3) provided $g(t,x,y)$ satisfies sublinear or linear one-sided growth restrictions with respect to x and y. Another class of functions g is covered by the next theorem which says that if there exist lower and upper functions $\sigma_1 \le \sigma_2$ to problem (7.3), it suffices to require the inequality in (7.4) only for $x \in [\sigma_1, \sigma_2]$. This implies that $g(t,x,y)$ can grow in x arbitrarily.

Theorem 7.14. *Let σ_1 and σ_2 be a lower function and an upper function of problem* (7.3) *and let $\sigma_1(t) \le \sigma_2(t)$ for $t \in [0,T]$. Assume that there is a function $h \in L_1[0,T]$ such that*

$$|g(t,x,y)| \le h(t) \quad \textit{for a.e. } t \in [0,T] \textit{ and all } x \in [\sigma_1(t), \sigma_2(t)], y \in \mathbb{R}.$$

Then problem (7.3) *has a solution u such that*

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \quad \textit{for } t \in [0,T]. \tag{7.16}$$

Proof. *Step* 1. *Construction of an auxiliary problem.*

For a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$, $\varepsilon \in [0,1]$, define

$$\widetilde{g}(t,x,y) = \begin{cases} g(t,\sigma_1(t),y) + \omega_1\left(t, \dfrac{\sigma_1(t)-x}{\sigma_1(t)-x+1}\right) + \dfrac{\sigma_1(t)-x}{\sigma_1(t)-x+1} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } x < \sigma_1(t), \\ g(t,x,y) \qquad\qquad\qquad\qquad\qquad\quad \text{if } \sigma_1(t) \le x \le \sigma_2(t), \\ g(t,\sigma_2(t),y) - \omega_2\left(t, \dfrac{x-\sigma_2(t)}{x-\sigma_2(t)+1}\right) - \dfrac{x-\sigma_2(t)}{x-\sigma_2(t)+1} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } x > \sigma_2(t), \end{cases}$$

where

$$\omega_i(t,\varepsilon) = \sup\{|g(t,\sigma_i(t),\sigma_i'(t)) - g(t,\sigma_i(t),y)| : |y - \sigma_i'(t)| < \varepsilon\}, \quad i = 1,2.$$

We see that $\omega_i \in Car([0,T] \times [0,1])$ is nonnegative, nondecreasing in its second variable and $\omega_i(t,0) = 0$ for a.e. $t \in [0,T]$, $i = 1,2$. Further, we have

$\widetilde{g} \in Car([0,T] \times \mathbb{R}^2)$ and there exists $\widetilde{h} \in L_1[0,T]$ such that $|\widetilde{g}(t,x,y)| \le \widetilde{h}(t)$ for a.e. $t \in [0,T]$ and all $x,y \in \mathbb{R}$. Thus, by Theorem 7.4, problem (7.14) with $\widetilde{g}$ defined in this proof has a solution u.

Step 2. Solution u of the auxiliary problem lies between σ_1 and σ_2.

We will prove that estimate (7.16) holds. Denote $v(t) = u(t) - \sigma_2(t)$ for $t \in [0,T]$ and assume, on the contrary, that

$$\max\{v(t) : t \in [0,T]\} = v(t_0) > 0.$$

Since $u(0) = A, u(T) = B$ and $\sigma_2(0) \ge A, \sigma_2(T) \ge B$, we have $t_0 \in (0,T)$. Moreover, Definition 7.13 implies that $t_0 \notin \Sigma$, because $v'(\tau-) < v'(\tau+)$ for $\tau \in \Sigma$. So, we have $t_0 \in (0,T) \setminus \Sigma$ and $v'(t_0) = 0$. This guarantees the existence of $t_1 \in (t_0, T)$ such that

$$v(t) > 0 \quad \text{and} \quad |v'(t)| < \frac{v(t)}{v(t)+1} < 1$$

for $t \in [t_0, t_1]$ and $[t_0, t_1] \cap \Sigma = \emptyset$. Then

$$\begin{aligned}
&\big(\phi(u'(t))\big)' - \big(\phi(\sigma_2'(t))\big)' = -\widetilde{g}(t, u(t), u'(t)) - \big(\phi(\sigma_2'(t))\big)' \\
&\quad = -g(t, \sigma_2(t), u'(t)) + \omega_2\left(t, \frac{v(t)}{v(t)+1}\right) + \frac{v(t)}{v(t)+1} - \big(\phi(\sigma_2'(t))\big)' \\
&\quad > -g(t, \sigma_2(t), u'(t)) + \omega_2(t, |v'(t)|) - \big(\phi(\sigma_2'(t))\big)' \ge -g(t, \sigma_2(t), u'(t)) \\
&\qquad + g(t, \sigma_2(t), u'(t)) - g(t, \sigma_2(t), \sigma_2'(t)) - \big(\phi(\sigma_2'(t))\big)' \ge 0
\end{aligned}$$

for a.e. $t \in [t_0, t_1]$. Hence

$$0 < \int_{t_0}^{t} \big(\phi(u'(s))\big)' - \big(\phi(\sigma_2'(s))\big)' \mathrm{d}s = \phi(u'(t)) - \phi(\sigma_2'(t)), \; t \in (t_0, t_1].$$

Therefore $v' = u' - \sigma_2' > 0$ on $(t_0, t_1]$, which contradicts the assumption that v has its maximum value at t_0. The inequality $\sigma_1(t) \le u(t)$ can be proved similarly. Thus, u fulfils estimate (7.16) and so, u is a solution of problem (7.3). □

Example. Let $A, B \in \mathbb{R}$ and $r_1, r_2 \in \mathbb{R}$ be such that $r_1 \le \min\{0, A, B\}$ and $r_2 \ge \max\{0, A, B\}$ and

$$g(t, r_1, 0) \ge 0, \quad g(t, r_2, 0) \le 0 \quad \text{for a.e. } t \in [0, T].$$

Then the constant function $\sigma_1(t) \equiv r_1$ satisfies condition (7.15) and hence, σ_1 is a lower function of problem (7.3). Similarly $\sigma_2(t) \equiv r_2$ satisfies condition (7.15) with the reversed inequalities and so, σ_2 is an upper function of problem (7.3). Here $\Sigma = \emptyset$.

The next lemmas on a priori estimates enable us to extend the existence results of Theorems 7.9 and 7.11. The first two deal with the so called *Nagumo function* $\omega \in C[0, \infty)$ which is positive and fulfils

$$\int_0^\infty \frac{\mathrm{d}s}{\omega(s)} = \infty. \tag{7.17}$$

Similar a priori estimates for $\phi(y) \equiv y$ can be found in Kiguradze [115] or Kiguradze and Shekhter [118].

Lemma 7.15 (**A priori estimate – Nagumo condition I**).
Assume that the function a is given by (7.6). Let $r_0, \varkappa \in (0, \infty)$, let $h_0 \in L_1[0, T]$ be nonnegative and let $\omega \in C[0, \infty)$ be positive and fulfil condition (7.17). Then there exists $r > 0$ such that for each function u satisfying

$$\begin{cases} \phi(u') \in AC[0, T], \quad u(0) = A, \quad u(T) = B, \quad \|u\|_\infty \le r_0, \\ -(\phi(u'(t)))' \operatorname{sign}(u(t) - a(t)) \le \varkappa\, \omega(|\phi(u'(t))|)(h_0(t) + |u'(t)|) \end{cases} \tag{7.18}$$

the estimate $\|u'\|_\infty \le r$ is valid.

Proof. Choose an arbitrary u satisfying condition (7.18). Denote $\|u'\|_\infty = \rho$ and let $\rho = |u'(t_0)|$. Assume $\rho > |\frac{B-A}{T}|$. We will consider four cases as in the proof of Lemma 7.5.

Case 1. Let $u'(t_0) = \rho$, $u(t_0) < a(t_0)$. Then $t_0 \in (0, T)$ and since $u(0) = a(0)$, we can find $t_1 \in [0, t_0)$ such that

$$u'(t_1) = \left|\frac{B - A}{T}\right|, \quad u'(t) > \left|\frac{B - A}{T}\right| \quad \text{for } t \in (t_1, t_0).$$

This implies

$$u(t) < a(t), \quad u'(t) > 0 \quad \text{for } \ t \in [t_1, t_0]$$

and, by condition (7.18),

$$\frac{(\phi(u'(t)))'}{\omega(\phi(u'(t)))} \le \varkappa(h_0(t) + u'(t)) \quad \text{for a.e. } t \in [t_1, t_0].$$

Integration of the last inequality leads to

$$\int_{t_1}^{t_0} \frac{(\phi(u'(t)))'}{\omega(\phi(u'(t)))} \mathrm{d}t \le \varkappa(\|h_0\|_1 + 2r_0)$$

and

$$\int_0^{\phi(\rho)} \frac{\mathrm{d}s}{\omega(s)} \le \int_0^{\phi(|(B-A)/T|)} \frac{\mathrm{d}s}{\omega(s)} + \varkappa(\|h_0\|_1 + 2r_0) =: K < \infty. \tag{7.19}$$

Case 2. Let $u'(t_0) = \rho$, $u(t_0) \ge a(t_0)$. Let $t_0 \in [0, T)$. Then there exists $t_1 \in (t_0, T)$ such that

$$u'(t_1) = \left|\frac{B-A}{T}\right|, \quad u'(t) > \left|\frac{B-A}{T}\right| \quad \text{for } \ t \in (t_0, t_1).$$

This implies

$$u(t) > a(t), \quad u'(t) > 0 \quad \text{for } \ t \in (t_0, t_1]$$

and, by condition (7.18),

$$\frac{-(\phi(u'(t)))'}{\omega(\phi(u'(t)))} \le \varkappa(h_0(t) + u'(t)) \quad \text{for a.e. } t \in [t_0, t_1].$$

Integration of the last inequality leads to

$$-\int_{t_0}^{t_1} \frac{(\phi(u'(t)))'}{\omega(\phi(u'(t)))} \mathrm{d}t \le \varkappa(\|h_0\|_1 + 2r_0)$$

and we get relation (7.19).

Now, let $t_0 = T$. Then there exists $t_1 \in (0, T)$ such that

$$u'(t_1) = \left|\frac{B-A}{T}\right|, \quad u'(t) > \left|\frac{B-A}{T}\right|, \quad u(t) < a(t) \quad \text{for} \quad t \in (t_1, T).$$

We get (7.19) as in *Case* 1.

Cases 3. and 4. In the remaining two cases we prove (7.19) similarly.

By condition (7.17) there is an $r > |\frac{B-A}{T}|$ such that

$$\int_0^{\phi(r)} \frac{\mathrm{d}s}{\omega(s)} > K.$$

Thus, by virtue of relation (7.19), $\rho < r$. Hence the estimate $\|u'\|_\infty \le r$ is proved. ☐

Lemma 7.16 (**A priori estimate – Nagumo condition II**).
Let $a_1,\ a_2 \in [0, T],\ a_1 < a_2,\ y_1,\ y_2 \in \mathbb{R},\ r_0,\ \varkappa \in (0, \infty)$. *Furthermore, let* $h_0 \in L_1[0, T]$ *be nonnegative and let* $\omega \in C[0, \infty)$ *be positive and fulfil condition* (7.17). *Then there exists* $r > 0$ *such that for each function* u *satisfying*

$$\begin{cases} \phi(u') \in AC[0, T], \quad \|u\|_\infty \le r_0, \\ (\phi(u'(t)))' \operatorname{sign}(u'(t) - y_1) \\ \qquad \ge -\varkappa\, \omega(|\phi(u'(t)) - \phi(y_1)|)(h_0(t) + |u'(t) - y_1|) \\ \qquad\qquad\qquad\qquad \textit{for a.e. } t \in [0, a_2], \\ (\phi(u'(t)))' \operatorname{sign}(u'(t) - y_2) \\ \qquad \le \varkappa\, \omega(|\phi(u'(t)) - \phi(y_2)|)(h_0(t) + |u'(t) - y_2|) \\ \qquad\qquad\qquad\qquad \textit{for a.e. } t \in [a_1, T], \end{cases} \tag{7.20}$$

the estimate $\|u'\|_\infty \le r$ *is valid.*

Proof. Choose an arbitrary u satisfying condition (7.20). By the Mean Value Theorem we can find $\xi \in (a_1, a_2)$ such that $|u'(\xi)| \le \frac{2r_0}{a_2 - a_1} =: c_0$. Further we see that

$$\operatorname{sign}(\phi(u'(t)) - \phi(y_i)) = \operatorname{sign}(u'(t) - y_i), \quad i = 1, 2, \quad \text{for } t \in [0, T].$$

Put $v_i(t) = \phi(u'(t)) - \phi(y_i)$, $i = 1, 2$, for $t \in [0, T]$. Then

$$|v_i(\xi)| \le \phi(c_0) + |\phi(y_i)| =: c_i, \quad i = 1, 2.$$

Condition (7.17) implies that there exists $\rho_i \in (c_i, \infty)$, $i = 1, 2$, such that

$$\int_{c_i}^{\rho_i} \frac{\mathrm{d}s}{\omega(s)} > \varkappa(\|h_0\|_1 + 2r_0 + T|y_i|), \quad i = 1, 2. \tag{7.21}$$

Assume

$$\max\{|v_1(t)| : t \in [0, \xi]\} = |v_1(\alpha)| > \rho_1.$$

Then $\alpha < \xi$ and there exists $\beta \in (\alpha, \xi]$ such that

$$|v_1(\beta)| = c_1, \; |v_1(t)| \ge c_1 \quad \text{for} \quad t \in [\alpha, \beta].$$

By the inequality in (7.20) which holds on $[0, a_2]$, we get

$$-\frac{v_1'(t)\,\mathrm{sign}\, v_1(t)}{\omega(|v_1(t)|)} \le \varkappa\,(h_0(t) + |u'(t) - y_1|) \quad \text{for a.e.} \quad t \in [\alpha, \beta].$$

Integrating this inequality over $[\alpha, \beta]$ and using the substitution $s = |v_1'(t)|$, we arrive at

$$\int_{c_1}^{|v_1(\alpha)|} \frac{\mathrm{d}s}{\omega(s)} \le \varkappa\left(\int_\alpha^\beta h_0(t)\mathrm{d}t + \int_\alpha^\beta |u'(t) - y_1|\mathrm{d}t\right). \tag{7.22}$$

Since $|v_1(t)| = |\phi(u'(t)) - \phi(y_1)| \ge c_1$ for $t \in [\alpha, \beta]$, we see that $u'(t) - y_1$ does not change its sign on $[\alpha, \beta]$ and hence

$$\int_\alpha^\beta |u'(t) - y_1|\mathrm{d}t = \left|\int_\alpha^\beta (u'(t) - y_1)\mathrm{d}t\right| \le 2r_0 + T|y_1|.$$

So, (7.22) leads to

$$\int_{c_1}^{\rho_1} \frac{\mathrm{d}s}{\omega(s)} < \int_{c_1}^{|v_1(\alpha)|} \frac{\mathrm{d}s}{\omega(s)} \le \varkappa(\|h_0\|_1 + 2r_0 + T|y_1|),$$

which contradicts inequality (7.21). Therefore $|v_1(\alpha)| \le \rho_1$ and we have proved

$$|\phi(u'(t)) - \phi(y_1)| \le \rho_1 \quad \text{for} \quad t \in [0, \xi].$$

The estimate

$$|\phi(u'(t)) - \phi(y_2)| \le \rho_2 \text{ for } t \in [\xi, T]$$

can be proved similarly. Hence, we get $\|u'\|_\infty \le r$ if we put $r = \phi^{-1}(\rho^*)$, where $\rho^* = \max\{\rho_1, \rho_2\} + \max\{|\phi(y_1)|, |\phi(y_2)|\}$. □

If we investigate problem (7.3) with $g(t,x,y)$ having arbitrary growth in x and growth in y controlled by the Nagumo condition (7.23) , we can often use one of the following two theorems.

Theorem 7.17. *Let a be given by (7.6), let σ_1 and σ_2 be a lower function and an upper function of problem (7.3) and let $\sigma_1(t) \le \sigma_2(t)$ for $t \in [0,T]$. Assume that there exist $\varkappa \in (0,\infty)$, a nonnegative function $h_0 \in L_1[0,T]$ and a positive function $\omega \in C[0,\infty)$ fulfilling condition (7.17) and*

$$\begin{cases} g(t,x,y)\,\mathrm{sign}(x-a(t)) \le \varkappa\,\omega(|\phi(y)|)(h_0(t)+|y|) \\ \qquad \textit{for a.e. } t \in [0,T] \textit{ and all } x \in [\sigma_1(t), \sigma_2(t)],\ y \in \mathbb{R}. \end{cases} \tag{7.23}$$

Then problem (7.3) has a solution u satisfying estimate (7.16) and moreover, $\|u'\|_\infty \le r$. Here $r > 0$ is the constant independent of u and given by Lemma 7.15 for $r_0 = \max\{\|\sigma_1\|_\infty, \|\sigma_2\|_\infty\}$.

Proof. Without loss of generality we can assume

$$r > \max\{\|\sigma_1'\|_\infty, \|\sigma_2'\|_\infty\}.$$

Define

$$\chi(z) = \begin{cases} 1 & \text{if } 0 \le z \le r, \\ \dfrac{2r-z}{r} & \text{if } r < z < 2r, \\ 0 & \text{if } z \ge 2r \end{cases} \qquad \widetilde{g}(t,x,y) = \chi(|y|)g(t,x,y)$$

for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}, z \in [0,\infty)$. Then $\widetilde{g} \in Car([0,T] \times \mathbb{R}^2)$ and there is a function $\widetilde{h} \in L_1[0,T]$ such that $|\widetilde{g}(t,x,y)| \le \widetilde{h}(t)$ for a.e. $t \in [0,T]$ and all $x \in [\sigma_1(t), \sigma_2(t)]$, $y \in \mathbb{R}$. Consider problem (7.14) with $\widetilde{g}$ defined in this proof. Since σ_1 and σ_2 are also lower and upper functions to this problem, we get by Theorem 7.14 that it has a solution u satisfying estimate (7.16). Further,

$$\begin{aligned}
-(\phi(u'(t)))' \operatorname{sign}(u(t)-a(t)) &= \widetilde{g}(t,u(t),u'(t)) \operatorname{sign}(u(t)-a(t)) \\
&= \chi(|u'(t)|)\, g(t,u(t),u'(t)) \operatorname{sign}(u(t)-a(t)) \\
&\le \chi(|u'(t)|)\, \varkappa\, \omega(|\phi(u'(t))|)\,(h_0(t)+|u'(t)|) \\
&\le \varkappa\, \omega(|\phi(u'(t))|)\,(h_0(t)+|u'(t)|) \quad \text{for a.e. } \ t\in[0,T].
\end{aligned}$$

By Lemma 7.15, the function u satisfies $\|u'\|_\infty \le r$ and hence u is also a solution of problem (7.3). □

EXAMPLE. Let $k,n\in\mathbb{N}$, $A{=}B{=}1$, $c\in\mathbb{R}$, $h_1\in L_\infty[0,T]$, and let $h_2\in L_1[0,T]$ and $\varphi\in Car([0,T]\times\mathbb{R}^2)$ be nonnegative functions. For a.e. $t\in[0,T]$ and all $x,y\in\mathbb{R}$ define the function

$$g(t,x,y) = h_1(t) - x^{2n+1} + x^2(h_2(t)+cy)\phi(y) - (x-1)^{2k+1}\varphi(t,x,y).$$

We can find constant functions $\sigma_1(t)\equiv r_1<1$ and $\sigma_2(t)\equiv r_2>1$ which are respectively lower and upper functions of problem (7.3) with g defined above. Moreover, g fulfils inequality (7.23) with $\varkappa=1$,

$$\omega(s) = (1+|c|)(1+s) \quad \text{and} \quad h_0(t) = |h_1(t)| + \max\{|r_1|,r_2\}^2|h_2(t)|.$$

By Theorem 7.17 our problem has a solution u satisfying $r_1\le u(t)\le r_2$ for $t\in[0,T]$.

The second form of the Nagumo condition is condition (7.24) which is used in the next theorem.

Theorem 7.18. *Let σ_1 and σ_2 be a lower function and an upper function of problem (7.3) and let $\sigma_1(t)\le\sigma_2(t)$ for $t\in[0,T]$. Assume that there exist $a_1,a_2\in[0,T], a_1<a_2$, $y_1,y_2\in\mathbb{R}$, $\varkappa\in(0,\infty)$, a nonnegative function $h_0\in L_1[0,T]$ and a positive function $\omega\in C[0,\infty)$ fulfilling condition (7.17) and*

$$\begin{cases}
g(t,x,y)\operatorname{sign}(y-y_1) \le \varkappa\,\omega(|\phi(y)-\phi(y_1)|)(h_0(t)+|y-y_1|) \\
\qquad \text{for a.e. } t\in[0,a_2] \text{ and all } \ x\in[\sigma_1(t),\sigma_2(t)],\ y\in\mathbb{R}, \\
g(t,x,y)\operatorname{sign}(y-y_2) \ge -\varkappa\,\omega(|\phi(y)-\phi(y_2)|)(h_0(t)+|y-y_2|) \\
\qquad \text{for a.e. } t\in[a_1,T] \text{ and all } \ x\in[\sigma_1(t),\sigma_2(t)],\ y\in\mathbb{R}.
\end{cases} \tag{7.24}$$

Then problem (7.3) *has a solution* u *satisfying estimate* (7.16) *and moreover,* $\|u'\|_\infty \le r$. *Here* $r > 0$ *is the constant independent of* u *and given by Lemma* 7.16 *for* $r_0 = \max\{\|\sigma_1\|_\infty, \|\sigma_2\|_\infty\}$.

Proof. We define $\widetilde{g}$ as in the proof of Theorem 7.17 using a sufficiently large r from Lemma 7.16. Then, similarly to the proof of Theorem 7.17, we get a solution u of problem (7.14) satisfying estimate (7.16) and condition (7.20). By Lemma 7.16, the function u satisfies $\|u'\|_\infty \le r$ and hence u is also a solution of problem (7.3). □

Example. Let $k \in \mathbb{N}$ be odd, A, B, c, $r \in \mathbb{R}$, $y_1 = y_2 = 0$, a_1, $a_2 \in [0,T]$, $a_1 < a_2$, $h_1, h_2, h_3 \in L_1[0,T]$. Assume that h_1 is positive on $[0,T]$ and

$$h_2 \ge 0 \text{ a.e. on } [0,a_1], \quad h_2 = 0 \text{ a.e. on } (a_1,T],$$

$$h_3 = 0 \text{ a.e. on } [0,a_2], \quad h_3 \ge 0 \text{ a.e. on } (a_2,T].$$

Consider problem (7.3) with $\phi(y) \equiv y$ and

$$g(t,x,y) = h_1(t)(r^k - x^k) + cy^2 - h_2(t)y^3 + h_3(t)y^5$$

for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$. We can find $r_1, r_2 \in \mathbb{R}$ such that

$$r_1 \le \min\{-|r|, A, B\}, \quad r_2 \ge \max\{|r|, A, B\},$$

and

$$g(t,r_1,0) > 0, \quad g(t,r_2,0) < 0 \quad \text{for a.e. } t \in [0,T].$$

Therefore the constant function $\sigma_1(t) \equiv r_1$ satisfies condition (7.15) and hence σ_1 is a lower function of the problem. Similarly, $\sigma_2(t) \equiv r_2$ satisfies condition (7.15) with reversed inequalities and so, σ_2 is an upper function of this problem. Moreover, g fulfils both the inequalities in (7.24) with $\varkappa=1$ and

$$h_0(t) = |h_1(t)|(|r|^k + (\max\{|r_1|, r_2\})^k), \quad \omega(s) = (|c|+1)(1+s).$$

Hence, by Theorem 7.18, our problem has a solution u such that $r_1 \le u(t) \le r_2$ for $t \in [0,T]$. Note that since the growth restrictions in Theorem 7.18 are only one-sided, the function g can have not only the quadratic term $c\,y^2$ but also terms with y^3 and y^5.

7.2 Dirichlet problem with time singularities

First we will study the singular problem (7.2) under the assumption

$$f \in Car((0,T] \times \mathbb{R}^2) \quad \text{has a time singularity at} \quad t = 0, \tag{7.25}$$

i.e. there exist $x, y \in \mathbb{R}$ such that

$$\int_0^{\varepsilon} |f(t,x,y)|\mathrm{d}t = \infty \quad \text{for} \quad \varepsilon \in (0,T].$$

We want to prove the existence of a solution to (7.2) or the existence of a w-solution u to (7.2) satisfying

$$\text{there exists} \ \ r > 0 \text{ such that} \ \ |u'(t)| \le r \quad \text{for} \ \ t \in (0,T]. \tag{7.26}$$

According to Definition 7.1 and assumption (7.25), a w-solution u of problem (7.2) has a continuous derivative on $(0,T]$ but u' need not exist at the singular point t=0. However, condition (7.26) guarantees that u' must be bounded near $t = 0$. Those who are interested in the existence of a w-solution u with u' possibly unbounded near $t = 0$ can find nice results in Agarwal, Lü and O'Regan [3], Agarwal and O'Regan [4], [5], [7], [12], Kiguradze [115], [117], Kiguradze and Shekhter [118], Lomtatidze [127], Lomtatidze and Malaguti [128] or Lomtatidze and Torres [129].

If we modify theorems of Section 1.2 for the Dirichlet problem (7.2) with time singularities we can extend the results of Section 7.1 and obtain the existence of w-solutions or solutions of (7.2). To this aim we present here the version of Theorem 1.3 for $t_0 = 0$, $n = 2$ and $\mathcal{A} = \mathbb{R}^2$. Consider a sequence of regular problems

$$u'' + f_k(t,u,u') = 0, \quad u(0) = u(T) = 0, \tag{7.27}$$

where $f_k \in Car([0,T] \times \mathbb{R}^2)$, $k \in \mathbb{N}$.

Theorem 7.19. *Let assumption* (7.25) *hold. Assume*

$$\begin{cases} \text{for each } k \in \mathbb{N} \text{ and each } (x,y) \in \mathbb{R}^2, \\ f_k(t,x,y) = f(t,x,y) \ \text{a.e. on } [0,T] \setminus \triangle_k, \\ \text{where } \triangle_k = [0, \frac{1}{k}) \cap [0,T], \end{cases} \tag{7.28}$$

and

$$\begin{cases} \textit{there exists a bounded set } \Omega \subset C^1[0,T] \\ \textit{such that for each } k \in \mathbb{N} \\ \textit{the regular problem (7.27) has a solution } u_k \in \Omega. \end{cases} \tag{7.29}$$

Then

$$\begin{cases} \textit{there exist a function } u \in C[0,T] \textit{ and a subsequence} \\ \{u_{k_\ell}\} \subset \{u_k\} \textit{ such that } \lim_{\ell\to\infty} \|u_{k_\ell} - u\|_\infty = 0, \end{cases} \tag{7.30}$$

$$\lim_{\ell\to\infty} u'_{k_\ell}(t) = u'(t) \textit{ locally uniformly on } (0,T], \tag{7.31}$$

$$\begin{cases} u \in AC^1_{loc}(0,T] \textit{ and} \\ u \textit{ is a w-solution of problem (7.2) satisfying (7.26).} \end{cases} \tag{7.32}$$

Assume, moreover, that there exist $\psi \in L_1[0,T]$, $\eta > 0$, $\ell_0 \in \mathbb{N}$ *and* $\lambda \in \{-1,1\}$ *such that*

$$\begin{cases} \lambda\, f_{k_\ell}(t, u_{k_\ell}(t), u'_{k_\ell}(t)) \ge \psi(t) \\ \textit{for all } \ell \in \mathbb{N}, \ell \ge \ell_0, \textit{ and for a.e. } t \in (0,\eta]. \end{cases} \tag{7.33}$$

Then u *is a solution of problem (7.2), i.e.* $u \in AC^1[0,T]$.

If $f(t,x,y)$ in (7.2) has one-sided *sublinear growth* in x and y, we use Theorem 7.19 to modify Theorem 7.9 as follows.

Theorem 7.20. *Let assumption (7.25) hold and let* $\alpha, \beta \in [0,1)$. *Assume that there exists a nonnegative function* $h \in L_1[0,T]$ *such that*

$$\begin{cases} f(t,x,y)\,\mathrm{sign}\, x \le h(t)\,(1 + |x|^\alpha + |y|^\beta) \\ \qquad\qquad \textit{for a.e. } t \in [0,T] \textit{ and all } x,\, y \in \mathbb{R}. \end{cases}$$

Then problem (7.2) has a w-solution u *satisfying estimate (7.26).*

Proof. Choose an arbitrary $k \in \mathbb{N}$ and for $x, y \in \mathbb{R}$ define the auxiliary function

$$f_k(t,x,y) = \begin{cases} f(t,x,y) & \text{for a.e. } t \in [0,T] \setminus \Delta_k, \\ 0 & \text{for a.e. } t \in \Delta_k, \end{cases}$$

where $\Delta_k = [0,T] \cap [0,\frac{1}{k})$. We see that $f_k \in Car([0,T] \times \mathbb{R}^2)$ fulfils condition (7.28) and inequality (7.13) with $a(t) \equiv 0$ and $g = f_k$. Consider the approximate regular problem

$$u'' + f_k(t,u,u') = 0, \quad u(0) = u(T) = 0. \tag{7.34}$$

Let us put $a(t) \equiv 0$ and $\phi(y) \equiv y$. By Theorem 7.9, we deduce that problem (7.34) has a solution u_k. In this way we get a sequence $\{u_k\}$ of solutions of (7.34), $k \in \mathbb{N}$, satisfying

$$-u_k''(t)\ \mathrm{sign}\, u_k(t) \le h(t)(1 + |u_k(t)|^\alpha + |u_k'(t)|^\beta)$$

for a.e. $t \in [0,T]$ and all $k \in \mathbb{N}$. So, by Lemma 7.5, there exists $r > 0$ such that

$$\|u_k\|_\infty + \|u_k'\|_\infty \le r \quad \text{for all} \quad k \in \mathbb{N}.$$

Define the set

$$\Omega = \{x \in C^1[0,T] : \|x\|_\infty + \|x'\|_\infty \le r\}.$$

Then condition (7.29) is valid and, by Theorem 7.19, we can find a subsequence $\{u_{k_\ell}\} \subset \{u_k\}$ satisfying conditions (7.30), (7.31) and (7.32). $\square$

EXAMPLE. Let $k \in \mathbb{N}$, $\alpha \in [1,\infty)$, let $\varphi \in C(\mathbb{R}^2)$ be positive and let $h_0, h_1, h_2 \in L_1[0,T]$. Consider problem (7.2), where

$$f(t,x,y) = -\frac{x^{2k+1}\varphi(x,y)}{t^\alpha} + h_0(t) + h_1(t)x^{\frac{1}{3}} + h_2(t)|y|^{\frac{1}{2}}$$

for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$. The first term of f is singular at $t = 0$. Further, f satisfies

$$f(t,x,y)\ \mathrm{sign}\, x \le h(t)(1 + |x|^{\frac{1}{3}} + |y|^{\frac{1}{2}}) \quad \text{for a.e. } t \in [0,T] \text{ and } x, y \in \mathbb{R},$$

where $h = |h_0| + |h_1| + |h_2|$. Therefore, by Theorem 7.20, the problem has a w-solution satisfying (7.26).

If $f(t,x,y)$ in (7.2) has one-sided *linear growth* in x and y, we can decide about the existence of a w-solution by means of the following modification of Theorem 7.11.

Theorem 7.21. *Let assumption (7.25) hold. Assume that there exist non-negative functions* $h_0, h_1, h_2 \in L_1[0,T]$ *such that*

$$T\|h_1\|_1 + \|h_2\|_1 < 1,$$

$$\begin{cases} f(t,x,y)\ \mathrm{sign}\, x \le h_0(t) + h_1(t)\,|x| + h_2(t)\,|y| \\ \qquad \text{for a.e. } t \in [0,T] \text{ and all } x,\ y \in \mathbb{R}. \end{cases}$$

Then problem (7.2) has a w-solution u *satisfying estimate (7.26).*

Proof. For $k \in \mathbb{N}$ consider problem (7.34). Put $a(t) \equiv 0$ and $\phi(y) \equiv y$. Using Theorem 7.11 and Lemma 7.7 we argue as in the proof of Theorem 7.20. □

Example. Let $k \in \mathbb{N}$, $\alpha \in [1,\infty)$, $a, b \in \mathbb{R}$, $|a| + |b| < \frac{1}{2}$, let $\varphi \in C(\mathbb{R}^2)$ be positive and let $h_0 \in L_1[0,1]$. Consider problem (7.2), where $T = 1$ and

$$f(t,x,y) = -\frac{x^{2k+1}\varphi(x,y)}{t^\alpha} + h_0(t) + \frac{1}{\sqrt{t}}(a\,x + b\,y)$$

for a.e. $t \in [0,1]$ and all $x, y \in \mathbb{R}$. The first term of f is singular at $t = 0$. Further, f satisfies

$$f(t,x,y)\,\mathrm{sign}\, x \le |h_0(t)| + \frac{|a|}{\sqrt{t}}|x| + \frac{|b|}{\sqrt{t}}|y| \quad \text{for a.e. } t \in [0,1] \text{ and } x, y \in \mathbb{R}.$$

Therefore, by Theorem 7.21, the problem has a w-solution satisfying estimate (7.26).

The next theorem shows that if $f(t,x,y)$ keeps its sign for small t and x, we get a solution of problem (7.2).

Theorem 7.22. *Let all conditions of Theorem 7.20 or Theorem 7.21 be fulfilled and let* u *be a w-solution of problem (7.2) satisfying estimate (7.26). Further assume that*

$$\begin{cases} \text{there exist } \lambda \in \{-1,1\} \text{ and } \delta \in (0,T) \text{ such that} \\ \lambda\, f(t,x,y) < 0 \text{ for a.e. } t \in (0,\delta) \text{ and all } x \in (-\delta,\delta), y \in [-r,r]. \end{cases} \tag{7.35}$$

Then u *is a solution of problem (7.2).*

Proof. For $k \in \mathbb{N}$ consider problem (7.34). By the proof of Theorem 7.20 or Theorem 7.21 there exist $r > 0$ and a sequence of approximate solutions $\{u_{k_\ell}\}$ satisfying conditions (7.30), (7.31) and $\|u_{k_\ell}\|_\infty + \|u'_{k_\ell}\|_\infty \le r$ for $\ell \in \mathbb{N}$. The function u in (7.30) is a w-solution of problem (7.2) and fulfils estimate (7.26). To prove that u is a solution we will describe the behaviour of u' at the singular point $t = 0$. Since $u(0) = 0$, there exists $\eta_1 \in (0, \delta)$ such that $|u(t)| < \delta$ for $t \in (0, \eta_1)$. Then condition (7.35) gives

$$-\lambda\, u''(t) = \lambda f(t, u, u') < 0 \quad \text{for a.e. } \; t \in (0, \eta_1)$$

and hence u' is strictly monotonous on $(0, \eta_1)$. Using estimate (7.26) we see that $\lim_{t\to 0+} u'(t) \in [-r, r]$.

Let $\lim_{t\to 0+} u'(t) \neq 0$. Then

$$\begin{cases} \text{there exists } \; \eta \in (0, \eta_1) \; \text{ such that} \\ u(t) > 0 \quad \text{on } (0, \eta) \quad (\text{or } \; u(t) < 0 \text{ on } (0, \eta)). \end{cases} \tag{7.36}$$

Let $\lim_{t\to 0+} u'(t) = 0$. Since u' is strictly monotonous on $(0, \eta_1)$, we have $u'(t) \neq 0$ for $t \in (0, \eta_1)$. This implies (7.36). Moreover, conditions (7.30) and (7.36) yield $\ell_0 > 0$ such that

$$u_{k_\ell}(t) > 0 \text{ on } \; (0, \eta] \quad (\text{or } u_{k_\ell}(t) < 0 \text{ on } (0, \eta])$$

for each $\ell \in \mathbb{N}$, $\ell \ge \ell_0$. Hence, under the assumptions of Theorem 7.20 or Theorem 7.21, we have

$$\lambda_2 f_{k_\ell}(t, u_{k_\ell}(t), u'_{k_\ell}(t)) \ge \psi(t) \quad \text{for a.e. } \; t \in (0, \eta],\ \ell \ge \ell_0,$$

where $\lambda_2 = -\operatorname{sign} u_{k_\ell}(t)$. Provided the assumptions of Theorem 7.20 hold, we put $\psi(t) = -h(t)(1 + r^\alpha + r^\beta)$ and if the assumptions of Theorem 7.21 are fulfilled, we put $\psi(t) = -h_0(t) - (r+1)\,(h_1(t) + h_2(t))$. Consequently, inequality (7.33) holds and Theorem 7.19 implies $u \in AC^1[0, T]$, i.e. u is a solution of problem (7.2). □

EXAMPLE. Let $k \in \mathbb{N}$, $\alpha \in [1, \infty)$, $a, b \in \mathbb{R}$, $|a| < \frac{1}{6}$, $b < 0$ and let $\varphi \in C(\mathbb{R}^2)$ be positive. Consider problem (7.2), where $T = 1$ and

$$f(t, x, y) = -\frac{(|x| + x)^{2k+1}\varphi(x, y)}{t^\alpha} + \frac{1}{\sqrt{t}}\,(a\,x + t\,y + b)$$

for a.e. $t\in[0,1]$ and all $x,y\in\mathbb{R}$. Then f satisfies

$$f(t,x,y)\ \mathrm{sign}\,x \le \frac{|b|}{\sqrt{t}} + \frac{|a|}{\sqrt{t}}\,|x| + \sqrt{t}\,|y| \quad \text{for a.e. } t\in[0,1] \text{ and all } x,y\in\mathbb{R}.$$

Therefore, by Theorem 7.21, the problem has a w-solution satisfying estimate (7.26). We can check that there exists $\delta>0$ such that

$$f(t,x,y)<0 \quad \text{for a.e. } t\in[0,\delta] \text{ and all } x\in[-\delta,\delta], y\in[-r,r].$$

Hence, by Theorem 7.22, u is a solution of the problem.

Similarly we could modify other theorems of Section 7.1 in order to get a solution or a w-solution to problem (7.2). However, we switch our attention to the more general singular problem (7.1).

Dirichlet problem with ϕ-Laplacian

As before we assume that f fulfils condition (7.25) and we are interested in the existence of a solution to problem (7.1) or of a w-solution u to (7.1) satisfying estimate (7.26). Since problem (7.1) contains ϕ-Laplacian, we cannot now use theorems of Section 1.2 directly but we need to generalize them for problems with ϕ-Laplacian. Consider the sequence of regular problems

$$(\phi(u'))' + f_k(t,u,u') = 0, \quad u(0)=u(T)=0, \tag{7.37}$$

where $f_k\in Car([0,T]\times\mathbb{R}^2)$, $k\in\mathbb{N}$.

Theorem 7.23 (**First principle for ϕ-Laplacian and time singularities**). *Let assumptions (7.25) and (7.28) hold. Further assume that*

$$\begin{cases} \textit{there exists a bounded set } \Omega\subset C^1[0,T] \\ \textit{such that for each } k\in\mathbb{N} \\ \textit{the regular problem (7.37) has a solution } u_k\in\Omega. \end{cases} \tag{7.38}$$

Then assertions (7.30) and (7.31) are valid, $\phi(u')\in AC_{loc}(0,T]$ and u is a w-solution of problem (7.1).

If, moreover, condition (7.33) is satisfied, then u is a solution of problem (7.1), i.e. $\phi(u')\in AC[0,T]$.

Proof. *Step* 1. *Convergence of the sequence of approximate solutions.*

Condition (7.38) implies that the sequence $\{u_k\}$ is bounded and equicontinuous on $[0,T]$. By the Arzelà-Ascoli theorem assertion (7.30) is true and $u(0)=u(T)=0$. Since $\{u_k'\}$ is bounded, we get, due to assumption (7.28), that for each $\tau\in(0,T]$ there exist $k_\tau\in\mathbb{N}$ and $h_\tau\in L_1[0,T]$ such that for each $k\geq k_\tau$

$$|f_k(s,u_k(s),u_k'(s))|\leq h_\tau(s)\quad\text{for a.e. } s\in[\tau,T]. \tag{7.39}$$

Hence problem (7.37) yields for $k\geq k_\tau,\quad t_1,t_2\in[\tau,T]$

$$|\phi(u_k'(t_2))-\phi(u_k'(t_1))|\leq\left|\int_{t_1}^{t_2}h_\tau(s)\,\mathrm{d}s\right|,$$

which implies that the sequence $\{\phi(u_k')\}$ is equicontinuous on $[\tau,T]$. By virtue of the uniform continuity of ϕ^{-1} on compact intervals, the sequence $\{u_k'\}$ is also equicontinuous on $[\tau,T]$. The Arzelà-Ascoli theorem implies that for each compact subset $\mathcal{K}\subset(0,T]$ a subsequence of $\{u_k'\}$ uniformly converging to u' on $\mathcal{K}$ can be chosen. Therefore, using the diagonalization theorem, we can choose a subsequence $\{u_{k_\ell}\}$ satisfying both (7.30) and (7.31).

Step 2. *Convergence of the sequence of approximate nonlinearities.*

Let $\mathcal{V}_1$ be the set of all $t\in[0,T]$ such that $f(t,\cdot,\cdot):\mathbb{R}^2\to\mathbb{R}$ is not continuous and let $\mathcal{V}_2$ be the set of all $t\in[0,T]$ such that the equality in (7.28) is not satisfied. Then $\operatorname{meas}(\mathcal{V}_1\cup\mathcal{V}_2)=0$. Choose an arbitrary $\tau\in(0,T]\setminus(\mathcal{V}_1\cup\mathcal{V}_2)$. Then there exists $\ell_0\in\mathbb{N}$ such that for $\ell\geq\ell_0$ we have

$$f_{k_\ell}(\tau,u_{k_\ell}(\tau),u_{k_\ell}'(\tau))=f(\tau,u_{k_\ell}(\tau),u_{k_\ell}'(\tau))$$

and, by (7.30) and (7.31), the equality

$$\lim_{\ell\to\infty}f_{k_\ell}(\tau,u_{k_\ell}(\tau),u_{k_\ell}'(\tau))=f(\tau,u(\tau),u'(\tau))$$

holds. Hence,

$$\lim_{\ell\to\infty}f_{k_\ell}(t,u_{k_\ell}(t),u_{k_\ell}'(t))=f(t,u(t),u'(t))\text{ for a.e. } t\in[0,T]. \tag{7.40}$$

Step 3. *The function* u *is a w-solution of problem* (7.1).

Choose an arbitrary $\tau \in (0,T]$ and integrate the equality

$$(\phi(u'_{k_\ell}(t)))' + f_{k_\ell}(t, u_{k_\ell}(t), u'_{k_\ell}(t)) = 0 \quad \text{for a.e. } t \in [0,T].$$

We get

$$\phi(u'_{k_\ell}(T)) - \phi(u'_{k_\ell}(\tau)) + \int_\tau^T f_{k_\ell}(s, u_{k_\ell}(s), u'_{k_\ell}(s))\,\mathrm{d}s = 0.$$

Applying conditions (7.39), (7.40) and the Lebesgue dominated convergence theorem on $[\tau, T]$, we can deduce (having in mind that τ is arbitrary) that the limit u solves the equation

$$\phi(u'(T)) - \phi(u'(t)) + \int_t^T f(s, u(s), u'(s))\,\mathrm{d}s = 0 \quad \text{for } t \in (0,T]. \tag{7.41}$$

This immediately yields that $\phi(u') \in AC_{loc}(0,T]$ and u is a w-solution of problem (7.1).

Step 4. *The function* u *is a solution of problem* (7.1).

Assume, moreover, that condition (7.33) holds. Due to assumption (7.38) there is a $c \in (0,\infty)$ such that for each $\ell \in \mathbb{N}$

$$\left| \int_0^\eta f_{k_\ell}(s, u_{k_\ell}(s), u'_{k_\ell}(s))\,\mathrm{d}s \right| = |\phi(u'_{k_\ell}(0)) - \phi(u'_{k_\ell}(\eta))| \le c.$$

So, by the Fatou lemma, using also condition (7.33) and equality (7.40), we deduce that $f(t,u(t),u'(t)) \in L_1[0,\eta]$. Further, by virtue of assumption (7.38) and assertions (7.30), (7.31), the functions u and u' are bounded on $[\eta, T]$. Hence, assumption (7.25) implies $f(t,u(t),u'(t)) \in L_1[\eta,T]$, which together with the above arguments yields $f(t,u(t),u'(t)) \in L_1[0,T]$. Therefore due to equality (7.41) we have that $\phi(u') \in AC[0,T]$, i.e. u is a solution of problem (7.1). □

Now, using Theorem 7.23, we will extend Theorem 7.17 which is based on the existence of lower and upper functions to problem (7.1). Note that lower and upper functions to problem (7.1) are understood in the sense of Definition 7.13.

Theorem 7.24. *Assume that (7.25) holds. Let σ_1 and σ_2 be a lower function and an upper function of problem (7.1) and let $\sigma_1(t) \le \sigma_2(t)$ for $t \in [0,T]$. Assume that there exist a nonnegative function $h \in L_1[0,T]$ and a positive function $\omega \in C[0,\infty)$ fulfilling condition (7.17), further assume that*

$$\text{there exists } \ b>0 \quad \text{such that } \ \omega(s) \ge b \quad \text{for } \ s \in [0,\infty) \tag{7.42}$$

and

$$\begin{cases} f(t,x,y)\,\operatorname{sign} x \le \omega(|\phi(y)|)(h(t)+|y|) \\ \qquad \text{for a.e. } t \in [0,T] \text{ and all } \ x \in [\sigma_1(t),\sigma_2(t)],\ y \in \mathbb{R}. \end{cases} \tag{7.43}$$

Then problem (7.1) has a w-solution u satisfying estimate (7.16) and $\|u'\|_\infty < \infty$.

If, moreover, condition (7.35) with $r \ge \|u'\|_\infty$ holds, then u is a solution of problem (7.1).

Proof. *Step* 1. Choose an arbitrary $k \in \mathbb{N}$ and denote $\Delta_k = [0,T] \cap [0,\frac{1}{k})$, $\Delta_{k1} = \{t \in \Delta_k : \sigma_1(t) = \sigma_2(t)\}$, $\Delta_{k2} = \{t \in \Delta_k : \sigma_1(t) < \sigma_2(t)\}$. Define a function g_k by

$$g_k(t,x) = \begin{cases} (\phi(\sigma_2'(t)))' & \text{if } \ x > \sigma_2(t), \\ \dfrac{(x-\sigma_1(t))(\phi(\sigma_2'(t)))' + (\sigma_2(t)-x)\,(\phi(\sigma_1'(t)))'}{\sigma_2(t)-\sigma_1(t)} & \\ & \text{if } \ \sigma_1(t) \le x \le \sigma_2(t), \\ (\phi(\sigma_1'(t)))' & \text{if } \ x < \sigma_1(t) \end{cases}$$

for a.e. $t \in \Delta_{k2}$ and all $x \in \mathbb{R}$ and a function f_k by

$$f_k(t,x,y) = \begin{cases} f(t,x,y) & \text{if } \ t \in [0,T] \setminus \Delta_k\,, \\ -(\phi(\sigma_1'(t)))' & \text{if } \ t \in \Delta_{k1}\,, \\ -g_k(t,x) & \text{if } \ t \in \Delta_{k2} \end{cases} \tag{7.44}$$

for a.e. $t \in [0,T]$ and all $x,y \in \mathbb{R}$. Then $f_k \in Car([0,T]\times\mathbb{R}^2)$ and condition (7.28) is valid. Consider problem (7.37) with f_k defined in this proof. Then σ_1 and σ_2 are also lower and upper functions to this problem. Moreover, due to inequalities (7.42), (7.43) and formula (7.44), f_k satisfies inequality (7.23) with $g(t,x,y) = f_k(t,x,y)$, $a(t) \equiv 0$, $\varkappa = (\frac{1}{b}+1)$ and

$$h_0(t) = h(t) + |(\phi(\sigma_1'(t)))'| + |(\phi(\sigma_2'(t)))'|.$$

Hence, for each $k \in \mathbb{N}$, Theorem 7.17 gives a solution u_k of problem (7.37). Moreover, each solution u_k satisfies estimate (7.16) and $\|u_k'\|_\infty \le r$, where $r > 0$ is the constant given by Lemma 7.15 for $r_0 = \max\{\|\sigma_1\|_\infty, \|\sigma_2\|_\infty\}$ and for $A = B = 0$.

Step 2. Define a set

$$\Omega = \{x \in C^1[0,T] : \sigma_1 \le x \le \sigma_2 \ \text{ on } \ [0,T],\ \|x'\|_\infty \le r\}.$$

Then condition (7.38) is valid and, by Theorem 7.23, we can find a subsequence $\{u_{k_\ell}\} \subset \{u_k\}$ such that assertions (7.30) and (7.31) hold and the function $u \in C[0,T]$ with $\phi(u') \in AC_{loc}(0,T]$ is a w-solution of problem (7.1). Since $\{u_{k_\ell}\} \subset \Omega$, we see that u fulfils estimate (7.16) and $\|u'\|_\infty \le r$.

Step 3. Let condition (7.35) hold. Similarly to the proof of Theorem 7.22 we can show that there exist $\eta > 0$ and $\ell_0 > 0$ such that either $u_{k_\ell}(t) > 0$ on $(0,\eta]$ for each $\ell \in \mathbb{N}$, $\ell \ge \ell_0$ or $u_{k_\ell}(t) < 0$ on $(0,\eta]$ for each $\ell \in \mathbb{N}$, $\ell \ge \ell_0$. Denote

$$\omega_0 = \max\{\omega(s) : s \in [0,\phi(r)]\}$$

and

$$\psi(t) = -|(\phi(\sigma_1'(t)))'| - |(\phi(\sigma_2'(t)))'| - \omega_0\,[h(t)+r] \quad \text{for a.e. } \ t \in [0,T].$$

Since

$$-f_{k_\ell}(t, u_{k_\ell}(t), u_{k_\ell}'(t)) \operatorname{sign} u_{k_\ell}(t) \ge \psi(t) \quad \text{for a.e. } \ t \in [0,\eta] \text{ and all } \ell \ge \ell_0,$$

we see that f_{k_ℓ} fulfils condition (7.33) with $\lambda = -\operatorname{sign} u_{k_\ell}(t)$. Therefore Theorem 7.23 implies $u \in AC^1[0,T]$, i.e. u is a solution of problem (7.1).
$\square$

Example. Let $k,n \in \mathbb{N}$, $c \in \mathbb{R}$, $\alpha \in [1,\infty)$, $\varepsilon \in (0,\infty)$, $\varphi \in C(\mathbb{R}^2)$ and $\psi \in C(\mathbb{R})$. Further, assume that φ is nonnegative and $\psi(x) = 0$ if $x \le 0$

and $\psi(x) < 0$ if $x > 0$. Consider problem (7.1) where

$$f(t,x,y) = (t-\varepsilon)^{2n+1} - x^{2n+1} + c\,x^2\,y\,\phi(y) - x^{2k+1}\,\varphi(x,y) + \frac{1}{t^\alpha}\,\psi(x)$$

for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$. The last term of f is singular at $t=0$. We can find constant functions $\sigma_1(t) \equiv r_1 < 0$ and $\sigma_2(t) \equiv r_2 > 0$ which are lower and upper functions of the problem. Moreover, f satisfies inequalities (7.35) and (7.43). Indeed, we can choose $\delta > 0$ sufficiently small and put $\lambda = 1$, $r = \max\{|r_1|, r_2\}$, $\omega(s) = (|c|\,r^2 + 1)\,(1+s)$, $h(t) = |t-\varepsilon|^{2n+1}$. By Theorem 7.24, our problem has a solution u such that $r_1 \le u(t) \le r_2$ for $t \in [0,T]$.

We continue with a generalization of Theorem 1.4 to problem (7.1).

Theorem 7.25 (**Second principle for ϕ – Laplacian and time singularities**). *Let the assumptions of Theorem 7.23 be satisfied with (7.33) replaced by the assumption that there exist $\psi \in L_1[0,T]$, $\eta > 0$, $\gamma \in \mathbb{R}$, $\ell_0 \in \mathbb{N}$ and $\lambda \in \{-1, 1\}$ such that*

$$\begin{cases} \lambda\, f_{k_\ell}(t, u_{k_\ell}(t), u'_{k_\ell}(t))\ \mathrm{sign}(u'_{k_\ell}(t) - \gamma) \ge \psi(t) \\ \qquad \textit{for all } \ \ell \in \mathbb{N}, \ell \ge \ell_0 \ \textit{and for a.e. } t \in (0,\eta]. \end{cases} \tag{7.45}$$

Then the assertions of Theorem 7.23 remain valid.

Proof. By Theorem 7.23 there exist a sequence $\{u_{k_\ell}\}$ and a function u such that assertions (7.30) and (7.31) hold and u is a w-solution of problem (7.1) with $\phi(u') \in AC_{loc}(0,T]$. Arguing as in *Step* 4 of the proof of Theorem 7.23 we see that to show $\phi(u') \in AC[0,T]$ it suffices to prove that $f(t, u(t), u'(t)) \in L_1[0,\eta]$. Put $\mathcal{M} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3 \cup \mathcal{V}_4$, where

$$\mathcal{V}_1 = \{t \in [0,\eta] : f(t,\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R} \text{ is not continuous}\},$$

$$\mathcal{V}_2 = \{t \in [0,\eta] : t \text{ is an isolated zero of } \ u' - \gamma\},$$

$$\mathcal{V}_3 = \{t \in [0,\eta] : (\phi(u'(t)))' + f(t,u(t),u'(t)) = 0 \text{ is not fulfilled}\},$$

$$\mathcal{V}_4 = \{t \in [0,\eta] : \text{the equality in condition (7.28) is not fulfilled}\}.$$

Then $\mathrm{meas}\,(\mathcal{M}) = 0$. Choose an arbitrary $s \in (0,T] \setminus \mathcal{M}$.

a) Let $u'(s) \neq \gamma$. Assume for example $\operatorname{sign}(u'(s)-\gamma)=1$. Then there exists $\ell_0 \in \mathbb{N}$ such that for each $\ell \geq \ell_0$ we have $\operatorname{sign}(u'_{k_\ell}(s)-\gamma)=1$ and so, due to properties (7.28), (7.30), (7.31) and since $s \notin \mathcal{V}_1 \cup \mathcal{V}_4$, we get

$$\begin{cases} \lim_{\ell\to\infty} f_{k_\ell}(s, u_{k_\ell}(s), u'_{k_\ell}(s)) \operatorname{sign}(u'_{k_\ell}(s)-\gamma) \\ = f(s,u(s),u'(s)) \operatorname{sign}(u'(s)-\gamma). \end{cases} \tag{7.46}$$

If $\operatorname{sign}(u'(s)-\gamma)=-1$, we get equality (7.46) in the same way.

b) Let s be an accumulation point of the set $\mathcal{V}_2$ of isolated zeros of $u'-\gamma$. Then there exists a sequence $\{s_m\} \subset (0,T]$ such that $u'(s_m)=\gamma$ and $\lim_{m\to\infty} s_m = s$. Since u' is continuous on $(0,T]$, we get $u'(s)=\gamma$. Therefore $\phi(u'(s_m)) = \phi(u'(s)) = \phi(\gamma)$,

$$\lim_{m\to\infty} \frac{\phi(u'(s_m)) - \phi(u'(s))}{s_m - s} = 0,$$

and, by virtue of $s \notin \mathcal{V}_3$, we get $0 = (\phi(u'(s)))' = -f(s,u(s),u'(s))$. Since $s \notin \mathcal{V}_1 \cup \mathcal{V}_4$, we have by properties (7.28), (7.30) and (7.31)

$$\begin{aligned} &\lim_{\ell\to\infty} f_{k_\ell}(s, u_{k_\ell}(s), u'_{k_\ell}(s)) \operatorname{sign}(u'_{k_\ell}(s)-\gamma) \\ &\qquad = f(s,u(s),u'(s)) \lim_{\ell\to\infty} \operatorname{sign}(u'_{k_\ell}(s)-\gamma) = 0. \end{aligned}$$

So, we have proved that equality (7.46) is valid for a.e. $s \in [0,\eta]$.

Further, by assumption (7.38), there exist $c>0$ and $\ell_0 \in \mathbb{N}$ such that for $\ell \geq \ell_0$

$$\begin{aligned} &\int_0^\eta \lambda\, f_{k_\ell}(s, u_{k_\ell}(s), u'_{k_\ell}(s)) \operatorname{sign}(u'_{k_\ell}(s)-\gamma)\,\mathrm{d}s \leq \int_0^\eta |\phi(u'_{k_\ell}(s)) - \phi(\gamma)|'\,\mathrm{d}s \\ &\qquad \leq |\phi(u'_{k_\ell}(0)) - \phi(\gamma)| + |\phi(u'_{k_\ell}(\eta)) - \phi(\gamma)| \leq c, \end{aligned}$$

and hence, due to assumption (7.45), we can use the Fatou lemma to deduce that $\lambda\, f(t,u(t),u'(t)) \operatorname{sign}(u'(t)-\gamma) \in L_1[0,\eta]$ and, consequently, $f(t,u(t),u'(t)) \in L_1[0,\eta]$. □

Now, we are ready to extend Theorem 7.18 with the second form of Nagumo condition to problem (7.1).

Theorem 7.26. *Assume that* (7.25) *holds. Let* σ_1 *and* σ_2 *be a lower function and an upper function of problem* (7.1) *and let* $\sigma_1(t) \le \sigma_2(t)$ *for* $t \in [0, T]$. *Assume that there exist* $a_1, a_2 \in [0, T]$, $a_1 < a_2$, $y_1, y_2 \in \mathbb{R}$, *a non-negative function* $h \in L_1[0, T]$ *and a positive function* $\omega \in C[0, \infty)$ *fulfilling conditions* (7.17), (7.42) *and*

$$\begin{cases} f(t,x,y)\,\operatorname{sign}(y-y_1) \le \omega(|\phi(y)-\phi(y_1)|)(h(t)+|y-y_1|) \\ \quad \textit{for a.e. } t \in [0,a_2] \textit{ and all } x \in [\sigma_1(t),\sigma_2(t)],\ y \in \mathbb{R}, \\ f(t,x,y)\,\operatorname{sign}(y-y_2) \ge -\omega(|\phi(y)-\phi(y_2)|)(h(t)+|y-y_2|) \\ \quad \textit{for a.e. } t \in [a_1,T] \textit{ and all } x \in [\sigma_1(t),\sigma_2(t)],\ y \in \mathbb{R}. \end{cases} \tag{7.47}$$

Then problem (7.1) *has a solution* u *satisfying estimate* (7.16).

Proof. Choose an arbitrary $k \in \mathbb{N}$ and consider problem (7.37) with f_k defined in the proof of Theorem 7.24. Let us put $g(t,x,y) = f_k(t,x,y)$, $a(t) \equiv 0$, $\varkappa = (\frac{1}{b}+1)$ and

$$h_0(t) = h(t) + |(\phi(\sigma_1'(t)))'| + |(\phi(\sigma_2'(t)))'|.$$

Here $b > 0$ is given by (7.42). Using Theorem 7.18 and Lemma 7.16 and arguing similarly to the proof of Theorem 7.24 we show that conditions (7.28) and (7.38) are valid. So, by Theorem 7.25, we get a w-solution u of problem (7.1). By Theorem 7.18, u also satisfies estimates (7.16) and (7.26), where $r > 0$ is the constant found by Lemma 7.16 for $r_0 = \max\{\|\sigma_1\|_\infty, \|\sigma_2\|_\infty\}$. Moreover, the first inequality in (7.47) gives

$$-f_{k_\ell}(t, u_{k_\ell}(t), u_{k_\ell}'(t))\,\operatorname{sign}(u_{k_\ell}'(t) - y_1) \ge \psi(t) \quad \text{for a.e. } t \in [0, a_2],$$

where

$$\psi(t) = -\omega_0(h(t)+r+|y_1|) - |(\phi(\sigma_1'(t)))'| - |(\phi(\sigma_2'(t)))'|,$$

$$\omega_0 = \max\{\omega(s) : s \in [0, \phi(r) + |\phi(y_1)|]\}.$$

So, using Theorem 7.25 with $\lambda = -1$, $\eta = a_2$ and $\gamma = y_1$, we get that u is a solution of problem (7.1). $\square$

EXAMPLE. Assume that $n \in \mathbb{N}$, $c, d \in \mathbb{R}$, $\alpha \in [1, \infty)$, $\varepsilon \in (0, \infty)$. Choose $a_1 \in (0, \frac{T}{2})$, $a_2 = \frac{T}{2}$, $h_1, h_2, h_3 \in L_1[0, T]$, where $h_2(t) \ge \varepsilon$ a.e. on $[0, T]$.

Let h_3 be nonnegative a.e. on $[0,T]$ and vanish a.e. on $[0,\frac{T}{2}]$. Consider problem (7.1) where $\phi(y) \equiv y$ and

$$f(t,x,y) = -t^{-\alpha}\,y + h_1(t)\,y + c\,(y^2+1) - h_2(t)\,(x^{2n-1}-d) + h_3(t)\,y^3$$

for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$. The first term is singular at $t=0$. Let $y_1 = y_2 = 0$. We can find constant functions $\sigma_1(t) \equiv r_1 < 0$ and $\sigma_2(t) \equiv r_2 > 0$ which are lower and upper functions of the problem. Moreover, f satisfies the conditions of Theorem 7.26. We see it if we put $\omega(s) = (|c|+1)\,(s+1)$, $K = (|r_1|+r_2)^{2n-1} + |d|$ and $h(t) = a_1^{-\alpha} + |h_1(t)| + K\,h_2(t) + 1$.

7.3 Dirichlet problem with space singularities

Many papers studying problem (7.1) or (7.2) with a space singularity at $x{=}0$ concern the case that the nonlinearity f is positive. Such problems are referred to as *positone* ones in literature, see Agarwal and O'Regan [11], [12] or Staněk [183]. The positivity of f implies that each solution of (7.2) is concave and hence positive on $(0,T)$, and if, moreover, f has a space singularity at $x=0$ but not at y, then each solution has only two singular points 0, T which are of type I. This makes the study of such problems easier than of those having sign-changing f or space singularities at y because the latter problems can generate solutions with singular points of type II. First we will study the singular problem (7.2) with a positive nonlinearity f satisfying

$$\begin{cases} f \in Car([0,T]\times\mathcal{D}), \text{ where } \mathcal{D} = (0,\infty)\times\mathbb{R}, \\ f \text{ has a space singularity at } x=0, \end{cases} \tag{7.48}$$

i.e. $\limsup_{x\to 0+} |f(t,x,y)| = \infty$ for a.e. $t \in [0,T]$ and some $y \in \mathbb{R}$. In this case we can use theorems of Section 1.3 and extend the existence results of Section 7.1. To this aim we present here the version of Theorem 1.6 for $c_0 = 0$, $n = 2$ and $\mathcal{A} = [0,\infty)\times\mathbb{R}$. We will consider the sequence of regular problems

$$u'' + f_k(t,u,u') = 0, \quad u(0) = u(T) = 0, \tag{7.49}$$

where $f_k \in Car([0,T]\times\mathbb{R}^2)$.

Theorem 7.27. *Assume that (7.48) holds and that*

$$\begin{cases} f_k(t,x,y) = f(t,x,y) \text{ for a.e. } t \in [0,T], \text{ for each } k > \frac{2}{T} \\ \text{and for each } (x,y) \in [0,\infty) \times \mathbb{R}, \ x \ge \frac{1}{k}, \ |y| \ge \frac{1}{k}, \end{cases} \tag{7.50}$$

$$\begin{cases} \text{there exists a bounded set } \ \Omega \subset C^1[0,T] \\ \text{such that for each } \ k > \frac{2}{T} \\ \text{the regular problem } \ (7.49) \text{ has a solution } \ u_k \in \Omega \\ \text{and } \ u_k(t) \ge 0 \text{ for } t \in [0,T]. \end{cases} \tag{7.51}$$

Then there exist $u \in C[0,T]$ *and a subsequence* $\{u_{k_\ell}\} \subset \{u_k\}$ *such that*

$$\lim_{\ell \to \infty} u_{k_\ell}(t) = u(t) \quad \text{uniformly on } [0,T].$$

If, moreover, the set of singular points $\mathcal{S} = \{s \in [0,T] : u(s) = 0\}$ *is finite, then*

$$\lim_{\ell \to \infty} u'_{k_\ell}(t) = u'(t) \quad \text{locally uniformly on } \ [0,T] \setminus \mathcal{S}.$$

If, in addition,

$$\begin{cases} \text{on each interval } [a,b] \subset [0,T] \setminus \mathcal{S} \\ \text{the sequence } \ \{f_{k_\ell}(t, u_{k_\ell}(t), u'_{k_\ell}(t)\} \text{ is uniformly integrable,} \end{cases} \tag{7.52}$$

then $u \in AC^1_{loc}([0,T] \setminus \mathcal{S})$ *and* u *is a w-solution of problem (7.2). Finally, if there exists a function* $\psi \in L_1[0,T]$ *such that*

$$f_{k_\ell}(t, u_{k_\ell}(t), u'_{k_\ell}(t)) \ge \psi(t) \text{ for a.e. } \ t \in [0,T] \text{ and all } \ \ell \in \mathbb{N}, \tag{7.53}$$

then $u \in AC^1[0,T]$ *and* u *is a solution of problem (7.2).*

The following lemma will be useful in the subsequent proofs.

Lemma 7.28. *Let* $\varepsilon > 0$. *Then there exists* $\eta > 0$ *such that for each function* $u \in AC^1[0,T]$ *satisfying*

$$u(0) = u(T) = 0, \quad -u''(t) \ge \varepsilon \quad \text{for a.e. } t \in [0,T]$$

the estimate

$$u(t) \geq \begin{cases} \eta\, t & \textit{for} \;\; t \in [0, \frac{T}{2}], \\ \eta\,(T-t) & \textit{for} \;\; t \in [\frac{T}{2}, T]. \end{cases} \tag{7.54}$$

is valid.

Proof. Let $G(t,s)$ be the Green function of the problem $-v''(t) = 0$, $v(0) = v(T) = 0$, i.e.

$$G(t,s) = \begin{cases} \dfrac{t\,(T-s)}{T} & \text{for } 0 \leq t \leq s \leq T, \\ \dfrac{s\,(T-t)}{T} & \text{for } 0 \leq s \leq t \leq T. \end{cases}$$

Let u be an arbitrary function fulfilling $-u''(t) \geq \varepsilon$ for a.e. $t \in [0,T]$ and $u(0) = u(T) = 0$. Then we have

$$u(t) = -\int_0^T G(t,s)\, u''(s) \mathrm{d}s \geq \varepsilon \int_0^T G(t,s) \mathrm{d}s$$

$$= \frac{1}{2}\, \varepsilon\, t\,(T-t) \geq \begin{cases} \eta\, t & \text{for} \;\; t \in [0, \frac{T}{2}], \\ \eta\,(T-t) & \text{for} \;\; t \in [\frac{T}{2}, T] \end{cases}$$

if we choose $\eta \leq \varepsilon \frac{T}{4}$. □

If $f(t,x,y)$ in (7.2) has one-sided sublinear growth in x and y, we use Theorem 7.27 to modify Theorem 7.9 as follows.

Theorem 7.29. *Let* (7.48) *hold and let* $\varepsilon,\, \gamma,\, \delta \in (0,\infty)$, $\alpha,\, \beta \in [0,1)$. *Assume that there exist a nonnegative function* $g_0 \in L_1[0,T]$ *and a function* $\psi \in C(0,\infty)$ *positive and nonincreasing on* $(0,\infty)$ *satisfying*

$$\int_0^T (t^{\gamma} + t^{\delta})\, \psi(t) \mathrm{d}t < \infty,$$

$$\begin{cases} \varepsilon \leq f(t,x,y) \leq t^{\gamma}\, T-t)^{\delta}\, \psi(x) + g_0(t)\,(1 + x^{\alpha} + |y|^{\beta}) \\ \qquad \textit{for a.e.} \;\; t \in [0,T] \;\textit{and all}\;\; x \in (0,\infty), \;\; y \in \mathbb{R}. \end{cases}$$

Then problem (7.2) *has a solution positive on* $(0,T)$.

Proof. *Step* 1. *Construction of approximate regular problems.*

Choose an arbitrary $k \in \mathbb{N}$ and for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$ define the auxiliary function

$$f_k(t,x,y) = \begin{cases} f(t,x,y) & \text{if } |x| \ge \frac{1}{k}\,, \\ f(t,\frac{1}{k},y) & \text{if } |x| < \frac{1}{k}\,. \end{cases}$$

We see that $f_k \in Car([0,T] \times \mathbb{R}^2)$ fulfils condition (7.50) and

$$\begin{aligned} \varepsilon \le f_k(t,x,y) &\le t^\gamma (T-t)^\delta \psi(\tfrac{1}{k}) + g_0(t)\left(1 + \left(\tfrac{1}{k}\right)^\alpha + |x|^\alpha + |y|^\beta\right) \\ &\le h(t)\,(1 + |x|^\alpha + |y|^\beta) \end{aligned}$$

for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$, where $h(t) = t^\gamma (T-t)^\delta\, \psi(\frac{1}{k}) + 2g_0(t)$. Consider the approximate regular problem

$$u'' + f_k(t,u,u') = 0, \quad u(0) = u(T) = 0. \tag{7.55}$$

Put $a(t) \equiv 0$ and $\phi(y) \equiv y$. Then, by Theorem 7.9, problem (7.55) has a solution u_k.

Step 2. *Convergence of the sequence* $\{u_k\}$ *of approximate solutions.*

Lemma 7.28 yields $\eta \in (0,1)$ such that

$$u_k(t) \ge \begin{cases} \eta\, t & \text{for } t \in [0, \frac{T}{2}]\,, \\ \eta\,(T-t) & \text{for } t \in [\frac{T}{2}, T]\,. \end{cases} \tag{7.56}$$

Clearly $u_k > 0$ on $(0,T)$. Further, the inequality $t^\gamma\,(T-t)^\delta\,\psi(u_k(t)) \le \widetilde{\psi}(t)$ holds for a.e. $t \in [0,T]$, where

$$\widetilde{\psi}(t) = \begin{cases} t^\gamma (T-t)^\delta \psi(\eta\, t) & \text{if } t \in [0, \frac{T}{2}], \\ t^\gamma (T-t)^\delta \psi(\eta\,(T-t)) & \text{if } t \in [\frac{T}{2}, T]. \end{cases}$$

Since $\psi(\frac{1}{k}) \le \psi(x)$ if $x \in (0, \frac{1}{k}]$, we have

$$f_k(t,x,y) \le t^\gamma (T-t)^\delta \psi(x) + g_0(t)\left(2 + x^\alpha + |y|^\beta\right)$$

for a.e. $t\in[0,T]$ and all $x\in(0,\infty)$, $y\in\mathbb{R}$. Therefore

$$-u_k''(t)\le\widetilde{\psi}(t)+g_0(t)\left(2+u_k(t)^\alpha+|u_k'(t)|^\beta\right)\ \text{ for a.e. }\ t\in[0,T].$$

We can find $\varkappa_0\in(0,\infty)$ such that

$$\int_0^T\widetilde{\psi}(t)\mathrm{d}t\le\varkappa_0\quad\text{for all}\ \ k\in\mathbb{N}.$$

Thus $\|\widetilde{\psi}+g_0\|_1\le\varkappa_0+\|g_0\|_1$. Consider the sequence $\{u_k\}$ of solutions of problems (7.55), $k\in\mathbb{N}$. The functions u_k, $k\in\mathbb{N}$, satisfy condition (7.8) for $\phi(y)\equiv y$, $a(t)\equiv 0$, $h_0=\widetilde{\psi}+g_0$, with $\varkappa=\varkappa_0+\|g_0\|_1$ and $h_1=g_0$. By Lemma 7.5 there exists $r>0$ such that

$$\|u_k\|_\infty+\|u_k'\|_\infty\le r\quad\text{for}\ \ k\in\mathbb{N}.$$

Define a set $\Omega=\{x\in C^1[0,T]:\|x\|_\infty+\|x'\|_\infty\le r\}$. Then condition (7.51) is valid and, by Theorem 7.27, we can find a function $u\in C[0,T]$ and a subsequence $\{u_{k_\ell}\}\subset\{u_k\}$ such that

$$\lim_{\ell\to\infty}u_{k_\ell}(t)=u(t)\quad\text{uniformly on}\ \ [0,T].$$

Step 3. *The function u is a solution of problem* (7.2).

By estimate (7.56), u satisfies estimate (7.54), and $u\in C[0,T]$ is positive on $(0,T)$. By virtue of assumption (7.48) we know that f has only a singularity at $x=0$. The set $\mathcal{S}$ of singular points is finite because it consists of two points 0 and T. Hence, Theorem 7.27 yields

$$\lim_{\ell\to\infty}u_{k_\ell}'(t)=u'(t)\quad\text{locally uniformly on}\ \ (0,T).$$

Let us choose an arbitrary interval $[a,b]\subset(0,T)$. Then there exists $\ell_0\in\mathbb{N}$ such that for each $\ell\ge\ell_0$ the inequality $u_{k_\ell}\ge\frac{1}{\ell_0}$ is valid on $[a,b]$ and

$$f_{k_\ell}(t,u_{k_\ell}(t),u_{k_\ell}'(t))\le t^\gamma(T-t)^\delta\psi(\frac{1}{\ell_0})+g_0(t)(2+r^\alpha+r^\beta)=:\varphi(t)$$

for a.e. $t\in[a,b]$. Using Criterion A.1 and the fact that $\varphi\in L_1[a,b]$, we get that the sequence $\{f_{k_\ell}(t,u_{k_\ell}(t),u_{k_\ell}'(t))\}$ is uniformly integrable on $[a,b]$. This yields that condition (7.52) holds and consequently, $u\in AC^1_{loc}(0,T)$ is a w-solution of problem (7.2). Moreover, condition (7.53) is also satisfied because the inequality $0 \le f_{k_\ell}(t, u_{k_\ell}(t), u'_{k_\ell}(t))$ holds for a.e. $t \in [0,T]$ and for all $\ell \in \mathbb{N}$. Due to Theorem 7.27, u is a solution of problem (7.2). $\square$

EXAMPLE. Let $h_1, h_2 \in L_1[0,T]$ be nonnegative. For a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$ define a function

$$f(t,x,y) = 1 + \frac{t^{\frac{3}{2}}(T-t)^{\frac{3}{2}}}{x^2} + h_1(t)\sqrt{x} + h_2(t)\sqrt{|y|}.$$

The second term of f has a space singularity at $x = 0$. Further, f satisfies the conditions of Theorem 7.29 with $\varepsilon = 1$, $\alpha = \beta = \frac{1}{2}$, $\gamma = \delta = \frac{3}{2}$, $\psi(x) = x^{-2}$ and $g_0 = 1 + h_1 + h_2$. Therefore, by Theorem 7.29, the problem

$$u'' + 1 + \frac{t^{\frac{3}{2}}(T-t)^{\frac{3}{2}}}{u^2} + h_1(t)\sqrt{u} + h_2(t)\sqrt{|u'|} = 0, \quad u(0) = u(T) = 0$$

has a solution positive on $(0,T)$.

Now, we will present conditions ensuring solvability of problems with space singularities in the variables x and y and with singular points both of type I and of type II. The main difficulty in the study of singular points of type II is the fact that their location in $[0,T]$ is not known. This is why there are only few papers concerning solvability of such problems in mathematical literature and no results about w-solutions are known.

Consider problem (7.2) under the assumption that f satisfies

$$\begin{cases} f \in Car([0,T] \times \mathcal{D}), \text{ where } \mathcal{D} = (0,\infty) \times (\mathbb{R} \setminus \{0\}), \\ f \text{ has space singularities at } x = 0 \text{ and } y = 0, \end{cases} \tag{7.57}$$

i.e.

$$\limsup_{x \to 0+} |f(t,x,y)| = \infty \quad \text{for a.e. } t \in [0,T] \text{ and some } y \in \mathbb{R} \setminus \{0\},$$

$$\limsup_{y \to 0} |f(t,x,y)| = \infty \quad \text{for a.e. } t \in [0,T] \text{ and some } x \in (0,\infty).$$

Conditions for solvability of problem (7.2) provided $f(t,x,y)$ is positive and has one-sided linear growth in x and y are formulated in the next theorem which extends Theorem 7.11.

Theorem 7.30. *Let* (7.57) *hold and let* $\varepsilon, \gamma, \delta \in (0, \infty)$. *Assume that there are nonnegative functions* $g, h_1, h_2 \in L_1[0, T]$ *and functions* $\psi_1, \psi_2 \in C(0, \infty)$ *positive and nonincreasing on* $(0, \infty)$ *satisfying*

$$T\,\|h_1\|_1 + \|h_2\|_1 < 1,$$

$$\int_0^T (t^\gamma + t^\delta)\psi_1(t)\,\mathrm{d}t < \infty, \qquad \int_0^T \psi_2(t)\,\mathrm{d}t < \infty,$$

$$\begin{cases} \varepsilon \le f(t,x,y) \le t^\gamma (T-t)^\delta\, \psi_1(x) + \psi_2(|y|) + g(t) + h_1(t)x + h_2(t)|y| \\ \qquad \textit{for a.e. } t \in [0,T] \ \textit{ and all } \ x \in (0,\infty),\, y \in (\mathbb{R}\setminus\{0\}). \end{cases}$$

Then problem (7.2) *has a solution positive on* $(0, T)$.

Proof. Due to condition (7.57), f has also a space singularity at its last variable y and hence we cannot use Theorem 7.27, where condition (7.48) is involved. We will use some arguments from the proof of Theorem 1.6.

Step 1. *Construction of approximate regular problems.*

Choose an arbitrary $k \in \mathbb{N}$ and for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$ define the auxiliary functions

$$\widetilde{f}_k(t,x,y) = \begin{cases} f(t,|x|,y) & \text{if } |x| \ge \frac{1}{k}, \\ f(t,\frac{1}{k},y) & \text{if } |x| < \frac{1}{k} \end{cases}$$

and

$$f_k(t,x,y) = \begin{cases} \widetilde{f}_k(t,x,y) & \text{if } \ |y| \ge \frac{1}{k}\,, \\ \frac{k}{2}\,(\widetilde{f}_k(t,x,\frac{1}{k})\,(y+\frac{1}{k}) - \widetilde{f}_k(t,x,-\frac{1}{k})(y-\frac{1}{k})) \\ \qquad\qquad \text{if } \ |y| < \frac{1}{k}\,. \end{cases}$$

We see that $f_k \in Car([0,T]\times\mathbb{R}^2)$ fulfils

$$\begin{cases} f_k(t,x,y) = f(t,x,y) \\ \qquad \text{for a.e. } \ t \in [0,T] \text{ and all } \ x \in [\frac{1}{k},\infty),\ |y| \in [\frac{1}{k},\infty). \end{cases} \tag{7.58}$$

Further,

$$\begin{cases} \varepsilon \le f_k(t,x,y) \\ \quad \le t^\gamma (T-t)^\delta \psi_1(\frac{1}{k}) + \psi_2(\frac{1}{k}) + g(t) + h_1(t)(|x| + \frac{1}{k}) + h_2(t)(|y| + \frac{1}{k}) \end{cases}$$

for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$. Put $a(t) \equiv 0$, $\phi(y) \equiv y$ and

$$h_0(t) = t^\gamma (T-t)^\delta \psi_1(\tfrac{1}{k}) + \psi_2(\tfrac{1}{k}) + g(t) + h_1(t) + h_2(t).$$

Then, by Theorem 7.11, problem (7.55) with f_k defined in this proof has a solution u_k.

Step 2. Convergence of the sequence $\{u_k\}$ *of approximate solutions.*

Lemma 7.28 gives $\eta \in (0,1)$ such that u_k satisfies estimate (7.56). Clearly $u_k > 0$ on $(0,T)$ and u_k has a unique maximum point $t_k \in (0,T)$. Integrating the inequality $\varepsilon \le -u_k''(t)$ we get

$$\begin{cases} \varepsilon(t_k - t) \le u_k'(t) = |u_k'(t)| & \text{for} \quad t \in [0, t_k], \\ \varepsilon(t - t_k) \le -u_k'(t) = |u_k'(t)| & \text{for} \quad t \in [t_k, T]. \end{cases} \tag{7.59}$$

Denote

$$\widetilde{\psi}_1(t) = \begin{cases} t^\gamma (T-t)^\delta\, \psi_1(\eta t) & \text{if } t \in [0, \frac{T}{2}], \\ t^\gamma (T-t)^\delta\, \psi_1(\eta (T-t)) & \text{if } t \in [\frac{T}{2}, T] \end{cases}$$

and

$$\widetilde{\psi}_{2k}(t) = \begin{cases} \psi_2(\varepsilon(t_k - t)) & \text{if } t \in [0, t_k], \\ \psi_2(\varepsilon(t - t_k)) & \text{if } t \in [t_k, T]. \end{cases}$$

Then

$$t^\gamma (T-t)^\delta \psi_1(u_k(t)) \le \widetilde{\psi}_1(t), \quad \psi_2(|u_k'(t)|) \le \widetilde{\psi}_{2k}(t) \quad \text{for a.e. } t \in [0,T].$$

Since $\psi_1(\frac{1}{k}) \le \psi_1(x)$ if $x \in (0, \frac{1}{k}]$ and $\psi_2(\frac{1}{k}) \le \psi_2(|y|)$ if $|y| \le \frac{1}{k}$, we have

$$f_k(t,x,y) \le t^\gamma (T-t)^\delta \psi_1(x) + \psi_2(|y|) + g(t) + h_1(t)(x+1) + h_2(t)(|y|+1)$$

for a.e. $t \in [0,T]$ and all $x \in (0,\infty)$, $y \in \mathbb{R}$. Therefore

$$-u_k''(t) \le \widetilde{\psi}_1(t) + \widetilde{\psi}_{2k}(t) + g(t) + h_1(t)\,(u_k(t)+1) + h_2(t)(|u_k'(t)|+1)$$

for a.e. $t \in [0,T]$. Without loss of generality we may assume that $\varepsilon \le 1$ and we can find $\varkappa_1, \varkappa_2 \in (0,\infty)$ such that

$$\int_0^T \widetilde{\psi}_1(t)\,\mathrm{d}t \le \varkappa_1, \quad \int_0^T \widetilde{\psi}_{2k}(t)\,\mathrm{d}t \le \varkappa_2, \quad \text{for all } k \in \mathbb{N}.$$

Thus $\|\widetilde{\psi}_1 + \widetilde{\psi}_{2k} + g\|_1 \le \varkappa_1 + \varkappa_2 + \|g\|_1 =: \varkappa$. Consider the sequence $\{u_k\}$ of solutions of problems (7.55), $k \in \mathbb{N}$. The functions u_k, $k \in \mathbb{N}$, satisfy condition (7.12) for $a(t) \equiv 0$, $\phi(y) \equiv y$ and $h_0 = \widetilde{\psi}_1 + \widetilde{\psi}_{2k} + g + h_1 + h_2$. By Lemma 7.7 there exists $r \in (\eta, \infty)$ such that $\|u_k\|_\infty + \|u_k'\|_\infty \le r$ for $k \in \mathbb{N}$. By the Arzelà-Ascoli theorem we can find a function $u \in C[0,T]$ and a subsequence $\{u_{k_\ell}\} \subset \{u_k\}$ such that

$$\lim_{\ell\to\infty} u_{k_\ell}(t) = u(t) \quad \text{uniformly on } [0,T].$$

So, we have $u(0) = u(T) = 0$ and u satisfies estimate (7.54). By estimate (7.56), $u_k(\frac{T}{2}) \ge \eta\,\frac{T}{2}$ for $k \in \mathbb{N}$. Since the inequality $\|u_k'\|_\infty \le r$ holds for $k \in \mathbb{N}$, we have $\eta\frac{T}{2r} \le t_k \le T - \eta\frac{T}{2r}$ for $k \in \mathbb{N}$. Therefore we can choose the above subsequence so that $\lim_{\ell\to\infty} t_{k_\ell} = t_u \in (0,T)$.

Step 3. *Convergence of the sequence* $\{f_k\}$ *of approximate nonlinearities.*

Let us choose an arbitrary interval $[a,b] \subset (0,T)\setminus\{t_u\}$. By virtue of estimates (7.56) and (7.59) there exists $\ell_0 \in \mathbb{N}$ such that for each $\ell \ge \ell_0$

$$u_{k_\ell}(t) \ge \frac{1}{\ell_0}, \quad |u_{k_\ell}'(t)| \ge \frac{1}{\ell_0} \quad \text{for a.e. } t \in [a,b] \tag{7.60}$$

and

$$\begin{cases} f_{k_\ell}(t, u_{k_\ell}(t), u_{k_\ell}'(t)) \\ \quad \le t^\gamma (T-t)^\delta \psi_1(\frac{1}{\ell_0}) + \psi_2(\frac{1}{\ell_0}) + g(t) + h_1(t)\, r + h_2(t)\, r =: \varphi(t) \\ \text{for a.e. } t \in [a,b]. \end{cases} \tag{7.61}$$

Since $\varphi \in L_1[a,b]$, the sequence $\{u_{k_\ell}'\}$ is equicontinuous on $[a,b]$. Having in mind that $[a,b]$ is arbitrary and using the Arzelà-Ascoli theorem and the diagonalization theorem, we can choose the subsequence $\{u_{k_\ell}\}$ in such a way that

$$\lim_{\ell\to\infty} u_{k_\ell}'(t) = u'(t) \quad \text{locally uniformly on } (0,T)\setminus\{t_u\}.$$

By estimate (7.59), $u'(t) \ne 0$ for $t \in (0,T)\setminus\{t_u\}$. Denote $\mathcal{S} = \{0, t_u, T\}$ and $\mathcal{U} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{S}$, where

$$\mathcal{V}_1 = \{t \in [0,T] : f(t,\cdot,\cdot) : \mathcal{D} \to \mathbb{R} \text{ is not continuous}\},$$

$$\mathcal{V}_2 = \{t \in [0,T] : \text{the equality in condition (7.58) is not fulfilled}\}.$$

Choose an arbitrary $t \in [0,T] \setminus \mathcal{U}$. Then there exists $\ell_0 \in \mathbb{N}$ such that for each $\ell \geq \ell_0$ estimates (7.60) hold. Since $t \notin \mathcal{V}_1 \cup \mathcal{V}_2$, we have equality $f_{k_\ell}(t, u_{k_\ell}(t), u'_{k_\ell}(t)) = f(t, u(t), u'(t))$ and consequently,

$$\lim_{\ell \to \infty} f_{k_\ell}(t, u_{k_\ell}(t), u'_{k_\ell}(t)) = f(t, u(t), u'(t)). \tag{7.62}$$

Since $\operatorname{meas}(\mathcal{U}) = 0$, equality (7.62) holds for a.e. $t \in [0,T]$.

Step 4. *The function* u *is a solution of problem* (7.2).

First, we shall prove that u is a w-solution of (7.2). Choose an arbitrary interval $[a,b] \subset (0,T) \setminus \{t_u\}$. Since condition (7.61) holds for each $\ell \geq \ell_0$, we get by equality (7.62) and the Lebesgue dominated convergence theorem on $[a,b]$ that $f(t, u(t), u'(t)) \in L_1[a,b]$ and if we pass to the limit in the sequence

$$u'_{k_\ell}(t) - u'_{k_\ell}(a) + \int_a^t f_{k_\ell}(s, u_{k_\ell}(s), u'_{k_\ell}(s))\mathrm{d}s, \quad t \in [a,b],$$

we get

$$u'(t) - u'(a) + \int_a^t f(s, u(s), u'(s))\mathrm{d}s, \ t \in [a,b].$$

Having in mind that $[a,b] \subset (0,T)\setminus\{t_u\}$ is an arbitrary interval, we conclude that u is a w-solution of problem (7.2).

Finally, we shall show that u is a solution of (7.2). For each $\ell \geq \ell_0$ we have

$$\int_0^T f_{k_\ell}(t, u_{k_\ell}(t), u'_{k_\ell}(t)) = u'_{k_\ell}(0) - u'_{k_\ell}(T) \leq 2\,r$$

and

$$f_{k_\ell}(t, u_{k_\ell}(t), u'_{k_\ell}(t)) \geq \varepsilon \quad \text{for a.e. } t \in [0,T].$$

Hence, by (7.62) and the Fatou lemma, we have $f(t, u(t), u'(t)) \in L_1[0,T]$. Consequently, $u \in AC^1[0,T]$, i.e. u is a solution of problem (7.2). □

REMARK 7.31. Notice the fact that the point t_u in the proof of Theorem 7.30 is a singular point of type II, because we do not know its position in $(0,T)$.

EXAMPLE. Let $c \in (0,\infty)$. For a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R} \setminus \{0\}$ define a function

$$f(t,x,y) = \sqrt{T-t}\left(1+\frac{t^2}{x}\right) + \frac{c}{\sqrt{|y|}} + \frac{1}{6\sqrt{t\,T}}\left(\frac{x}{T}+|y|\right).$$

The first term has a space singularity at $x = 0$ and the second at $y = 0$. We can see that f satisfies the conditions of Theorem 7.30 if we put

$$\gamma = 2,\ \delta = \frac{1}{2},\ \psi_1(x) = \frac{1}{x},\ \psi_2(|y|) = \frac{c}{\sqrt{|y|}},\ g(t) = \sqrt{T-t},$$

$$h_1(t) = \frac{1}{6T\sqrt{tT}}, \quad h_2(t) = \frac{1}{6\sqrt{tT}}$$

and choose $\varepsilon > 0$ sufficiently small.

7.4 Dirichlet problem with mixed singularities

In this section we will study problems having the so called *mixed singularities*, i.e. both time and space ones. Moreover, in some theorems we omit the assumption that the nonlinearity f in the differential equation is positive. In literature we can find results about the solvability of singular Dirichlet problems with sign-changing nonlinearities which mostly concern w-solutions. Here we will prove the existence of solutions to problem (7.1) provided f has *mixed singularities*. We assume that $\mathcal{A}_1$, $\mathcal{A}_2$ are closed intervals containing 0 and

$$\begin{cases} f \in Car((0,T)\times\mathcal{D}),\ \text{where } \mathcal{D} = (\mathcal{A}_1\setminus\{0\})\times(\mathcal{A}_2\setminus\{0\}), \\ f \text{ has time singularities at } t=0 \text{ and at } t=T \\ \ \text{and space singularities at } x=0 \text{ and at } y=0, \end{cases} \tag{7.63}$$

i.e. there exists $(x,y)\in\mathcal{D}$ such that

$$\int_0^\varepsilon |f(t,x,y)|\,\mathrm{d}t = \infty \ \text{ and } \int_{T-\varepsilon}^T |f(t,x,y)|\mathrm{d}t = \infty \text{ for each } \varepsilon \in (0,\tfrac{T}{2}),$$

$$\limsup_{x\to 0} |f(t,x,y)| = \infty \quad \text{for a.e. } \ t\in[0,T] \text{ and some } \ y\in\mathcal{A}_2\setminus\{0\},$$

$$\limsup_{y\to 0} |f(t,x,y)| = \infty \quad \text{for a.e. } \ t\in[0,T] \ \text{ and some } \ x\in\mathcal{A}_1\setminus\{0\}.$$

Since problem (7.1) contains ϕ –Laplacian and has mixed singularities, we cannot use theorems of Sections 1.2 and 1.3. Hence we will prove their new generalized version. In order to do it we will consider the sequence of regular problems

$$(\phi(u'))' + f_k(t, u, u') = 0, \quad u(0) = a_k, \quad u(T) = b_k, \tag{7.64}$$

where $f_k \in Car([0,T] \times \mathbb{R}^2)$, $a_k, b_k \in \mathbb{R}$, $k \in \mathbb{N}$.

Theorem 7.32 (**Principle for ϕ –Laplacian and mixed singularities**). *Let (7.63) hold, let $\varepsilon_k > 0$, $\eta_k > 0$ for $k \in \mathbb{N}$ and let*

$$\lim_{k\to\infty} \varepsilon_k = 0, \qquad \lim_{k\to\infty} \eta_k = 0.$$

Assume that

$$\begin{cases} f_k(t,x,y) = f(t,x,y) & \text{for a.e. } t \in [\frac{1}{k}, T - \frac{1}{k}], \text{ for each } k > \frac{2}{T} \\ \quad \text{and for each } (x,y) \in A_1 \times \mathcal{A}_2, \ |x| \ge \varepsilon_k, \ |y| \ge \eta_k, \end{cases} \tag{7.65}$$

$$\begin{cases} \text{there exists a bounded set } \Omega \subset C^1[0,T] \text{ such that} \\ \text{for each } k > \frac{2}{T} \\ \text{the regular problem (7.64) has a solution } u_k \in \Omega \\ \text{and } (u_k(t), u_k'(t)) \in \mathcal{A}_1 \times \mathcal{A}_2 \text{ for } t \in [0,T]. \end{cases} \tag{7.66}$$

Then there exist $u \in C[0,T]$ and a subsequence $\{u_{k_\ell}\} \subset \{u_k\}$ such that

$$\lim_{\ell\to\infty} u_{k_\ell}(t) = u(t) \quad \text{uniformly on } [0,T].$$

Further assume that there is a finite set $\mathcal{S} = \{s_1, \ldots, s_\nu\} \subset (0,T)$ such that

$$\begin{cases} \text{the sequence } \{\phi(u_k')\} \text{ is equicontinuous} \\ \text{on each interval } [a,b] \subset (0,T) \setminus \mathcal{S}. \end{cases} \tag{7.67}$$

Then $u \in C^1((0,T) \setminus \mathcal{S})$ and

$$\lim_{\ell\to\infty} u_{k_\ell}'(t) = u'(t) \quad \text{locally uniformly on } (0,T) \setminus \mathcal{S}.$$

Assume in addition $\lim_{k\to\infty} a_k = 0$, $\lim_{k\to\infty} b_k = 0$ *and let the set* $\mathcal{S}$ *have the form*

$$\mathcal{S} = \{s \in (0,T) : u(s) = 0 \quad or \quad u'(s) = 0 \quad or \quad u'(s) \ does\ not\ exist\}. \tag{7.68}$$

Then $\phi(u') \in AC_{loc}((0,T) \setminus \mathcal{S})$ *and* u *is a w-solution of problem* (7.1).

Denote $s_0 = 0$ *and* $s_{\nu+1} = T$. *Moreover, let there be* $\eta \in (0, \frac{T}{2})$, λ_0, μ_0, $\lambda_1, \mu_1, \dots, \lambda_{\nu+1}, \mu_{\nu+1} \in \{-1, 1\}$, $\ell_0 \in \mathbb{N}$ *and* $\psi \in L_1[0,T]$ *such that*

$$\begin{cases} \lambda_i\, f_{k_\ell}(t, u_{k_\ell}(t), u'_{k_\ell}(t)) \operatorname{sign} u'_{k_\ell}(t) \ge \psi(t) \\ \qquad \textit{for a.e.} \quad t \in (s_i - \eta, s_i) \cap (0,T) \\ \qquad \textit{and for all} \quad i \in \{0, \dots \nu+1\}, \ \ell \ge \ell_0, \end{cases} \tag{7.69}$$

$$\begin{cases} \mu_i\, f_{k_\ell}(t, u_{k_\ell}(t), u'_{k_\ell}(t)) \operatorname{sign} u'_{k_\ell}(t) \ge \psi(t) \\ \qquad \textit{for a.e.} \quad t \in (s_i, s_i + \eta) \cap (0,T) \\ \qquad \textit{and for all} \quad i \in \{0, \dots \nu+1\}, \ \ell \ge \ell_0. \end{cases} \tag{7.70}$$

Then $\phi(u') \in AC[0,T]$ *and* u *is a solution of problem* (7.1). *Moreover,* $(u(t), u'(t)) \in \mathcal{A}_1 \times \mathcal{A}_2$ *holds for* $t \in [0,T]$.

Proof. *Step* 1. *Convergence of the sequence* $\{u_{k_\ell}\}$.

Assume that conditions (7.63), (7.65) and (7.66) hold. By (7.66) there exists $r > 0$ such that the sequence $\{u_k\}$ of solutions to problem (7.64) satisfies

$$\|u_k\|_{C^1} \le r \quad \text{for} \quad k > \frac{2}{T}.$$

Hence, the sequence $\{u_k\}$ is bounded and equicontinuous on $[0,T]$. Due to the Arzelà-Ascoli theorem this yields the existence of a function $u \in C[0,T]$ and a subsequence $\{u_{k_\ell}\} \subset \{u_k\}$ such that $\lim_{\ell\to\infty} u_{k_\ell}(t) = u(t)$ uniformly on $[0,T]$.

Step 2. *Convergence of the sequence* $\{u'_{k_\ell}\}$.

Assume in addition to Step 1 that condition (7.67) holds and choose an arbitrary interval $[a,b] \subset (0,T) \setminus \mathcal{S}$. Then $\{\phi(u'_k)\}$ and consequently $\{u'_k\}$ is equicontinuous on $[a,b]$. Since $\{u'_k\}$ is also bounded on $[a,b]$, we can use the Arzelà-Ascoli theorem and choose a subsequence $\{u_{k_\ell}\}$ such that it uniformly converges on $[0,T]$ and $\lim_{\ell\to\infty} u'_{k_\ell}(t) = u'(t)$ uniformly on $[a,b]$. Using the diagonalization theorem we deduce that we can choose the uniformly converging on $[0,T]$ subsequence $\{u_{k_\ell}\}$ so that

$$\lim_{\ell\to\infty} u'_{k_\ell}(t) = u'(t) \quad \text{locally uniformly on } (0,T)\setminus\mathcal{S}.$$

Therefore $u\in C^1((0,T)\setminus\mathcal{S})$.

Step 3. *Convergence of the approximate nonlinearities* $\{f_{k_\ell}\}$.

Assume in addition to Step 2 that $\lim_{k\to\infty} a_k = 0$, $\lim_{k\to\infty} b_k = 0$ and that condition (7.68) holds. Then $u(0) = u(T) = 0$. Define $\mathcal{U} = \mathcal{V}_1\cup\mathcal{V}_2\cup\mathcal{S}$, where

$$\mathcal{V}_1 = \{t\in(0,T) : f(t,\cdot,\cdot) : \mathcal{D}\to\mathbb{R} \quad \text{is not continuous}\},$$

$$\mathcal{V}_2 = \{t\in(0,T) : \text{ the equality in condition } (7.65) \text{ is not fulfilled}\}.$$

Choose an arbitrary $t\in(0,T)\setminus\mathcal{U}$. Then there exists $\ell_0\in\mathbb{N}$ such that for all $\ell\geq\ell_0$ we have $t\in[\frac{1}{k_\ell}, T-\frac{1}{k_\ell}]$ and

$$|u_{k_\ell}(t)|\geq\varepsilon_{k_\ell},\ |u'_{k_\ell}(t)|\geq\eta_{k_\ell} \ \text{ and } \ f_{k_\ell}(t,u_{k_\ell}(t),u'_{k_\ell}(t)) = f(t,u_{k_\ell}(t),u'_{k_\ell}(t)).$$

Since t is an arbitrary element in $(0,T)\setminus\mathcal{U}$ and meas $(\mathcal{U})=0$, we get

$$\lim_{\ell\to\infty} f_{k_\ell}(t,u_{k_\ell}(t),u'_{k_\ell}(t)) = f(t,u(t),u'(t)) \quad \text{a.e. on } [0,T]. \tag{7.71}$$

Step 4. *The function* u *is a w-solution.*

Now, choose an arbitrary interval $[a,b]\subset(0,T)\setminus\mathcal{S}$. Then there exist $\ell^*\in\mathbb{N}$, $\varepsilon^*>0$ and $\eta^*>0$ such that for all $\ell\geq\ell^*$

$$|f_{k_\ell}(t,u_{k_\ell}(t),u'_{k_\ell}(t))|\leq h(t) \quad \text{for a.e. } t\in[a,b]$$

where

$$h(t) = \sup\{|f(t,x,y)| : \varepsilon^*\leq|x|\leq r,\ \eta^*\leq|y|\leq r\}\in L_1[a,b].$$

Therefore we can apply the Lebesgue dominated convergence theorem and get $f(t,u(t),u'(t))\in L_1[a,b]$ and

$$\lim_{\ell\to\infty}\int_a^b f_{k_\ell}(s,u_{k_\ell}(s),u'_{k_\ell}(s))\,\mathrm{d}s = \int_a^b f(s,u(s),u'(s))\,\mathrm{d}s.$$

Integrating the equality

$$(\phi(u'_{k_\ell}(t)))' + f_{k_\ell}(t, u_{k_\ell}(t), u'_{k_\ell}(t)) = 0 \quad \text{for a.e.} \quad t \in [0,T] \tag{7.72}$$

we get

$$\phi(u'_{k_\ell}(t)) - \phi(u'_{k_\ell}(a)) + \int_a^t f_{k_\ell}(s, u_{k_\ell}(s), u'_{k_\ell}(s))\,\mathrm{d}s = 0 \quad \text{for} \quad t \in [a,b],$$

which for $\ell \to \infty$ leads to

$$\phi(u'(t)) - \phi(u'(a)) + \int_a^t f(s, u(s), u'(s))\,\mathrm{d}s = 0 \quad \text{for} \quad t \in [a,b].$$

Since $[a,b]$ can be an arbitrary interval in $(0,T) \setminus \mathcal{S}$, we deduce that $\phi(u') \in AC_{loc}((0,T) \setminus \mathcal{S})$ and u is a w-solution of problem (7.1).

Step 5. *The function u is a solution.*

Assume in addition to Step 3 that there exist $\eta \in (0, \frac{T}{2})$, $\lambda_0, \ldots, \lambda_{\nu+1}$, $\mu_0, \ldots, \mu_{\nu+1} \in \{-1, 1\}$, $\ell_0 \in \mathbb{N}$ and $\psi \in L_1[0,T]$ such that conditions (7.69) and (7.70) are valid. Since u is a w-solution of problem (7.1), it remains to prove that $\phi(u') \in AC[0,T]$. By *Step* 3, $f(t, u(t), u'(t)) \in L_1[a,b]$ for each $[a,b] \subset (0,T) \setminus \mathcal{S}$. So, it suffices to prove $f(t, u(t), u'(t)) \in L_1[c_i, d_i]$ for $i = 0, \ldots, \nu+1$, where $(c_i, d_i) = (s_i - \eta, s_i + \eta) \cap (0,T)$. Choose an arbitrary $i \in \{0, \ldots, \nu+1\}$ and $t \in (c_i, d_i) \setminus \mathcal{S}$. Then $u'(t) \neq 0$. If we use equality (7.71) and the fact that $\{u'_{k_\ell}\}$ locally uniformly converges to u' on $(0,T) \setminus \mathcal{S}$, we obtain

$$\lim_{\ell \to \infty} f_{k_\ell}(t, u_{k_\ell}(t), u'_{k_\ell}(t)) \operatorname{sign} u'_{k_\ell}(t) = f(t, u(t), u'(t)) \operatorname{sign} u'(t)$$

for a.e. $t \in [c_i, d_i]$. If we multiply equality (7.72) by $\operatorname{sign} u'_{k_\ell}(t)$ and then integrate over $[c_i, d_i]$, we get for $\ell \geq \ell_0$

$$\left| \int_{c_i}^{d_i} f_{k_\ell}(s, u_{k_\ell}(s), u'_{k_\ell}(s)) \operatorname{sign} u'_{k_\ell}(s)\,\mathrm{d}s \right|$$
$$\leq \phi(|u'_{k_\ell}(d_i)|) + \phi(|u'_{k_\ell}(c_i)|) \leq 2\,\phi(r).$$

Therefore the Fatou lemma yields $f(t, u(t), u'(t)) \in L_1[c_i, d_i]$, by conditions (7.69) and (7.70). Hence $f(t, u(t), u'(t)) \in L_1[0,T]$ and $\phi(u') \in AC[0,T]$. $\square$

REMARK 7.33. (i) Theorem 7.32 guarantees the existence of a solution u which can change its sign.

(ii) According to Step 4 of the proof of Theorem 7.32 we can claim that Theorem 7.32 remains valid if we replace (7.70) with

$$\begin{cases} f_{k_\ell}(t, u_{k_\ell}(t), u'_{k_\ell}(t)) \geq \psi(t) \\ \qquad \text{for a.e. } \ t \in (s_i - \eta, s_i + \eta) \cap (0, T) \\ \qquad \text{and all } \ i \in \{0, \dots \nu + 1\}, \ \ell \geq \ell_0. \end{cases} \tag{7.73}$$

(iii) If f has no singularity at $y = 0$, then we can put $\eta_k = 0$ for $k \in \mathbb{N}$ in Theorem 7.32. Moreover, due to Step 3 of the proof of Theorem 7.32, the set $\mathcal{S}$ in (7.68) consists only of the zeros of u. This will be accounted for in the next theorem where we will assume

$$\begin{cases} f \in Car((0,T) \times \mathcal{D}) \text{ can change its sign, } \ D = (0, \infty) \times \mathbb{R}, \\ \qquad \text{and } \ f \text{ has mixed singularities at } \ t = 0, \ t = T, \ x = 0. \end{cases} \tag{7.74}$$

Theorem 7.34. *Let (7.74) hold. Let σ_1 and σ_2 be a lower function and an upper function of problem (7.1) and let*

$$0 < \sigma_1(t) \leq \sigma_2(t) \quad \textit{for} \ \ t \in (0, T).$$

Assume that there exist $a_1, a_2 \in [0, T]$, $a_1 < a_2$, a nonnegative function $h \in L_1[0, T]$ and a positive function $\omega \in C[0, \infty)$ fulfilling conditions (7.17), (7.42) and

$$\begin{cases} f(t, x, y) \operatorname{sign} y \leq \omega(|\phi(y)|)(h(t) + |y|) \\ \qquad \textit{for a.e. } \ t \in [0, a_2] \textit{ and all } \ x \in [\sigma_1(t), \sigma_2(t)], \ y \in \mathbb{R}, \\ f(t, x, y) \operatorname{sign} y \geq -\omega(|\phi(y)|)(h(t) + |y|) \\ \qquad \textit{for a.e. } \ t \in [a_1, T] \textit{ and all } \ x \in [\sigma_1(t), \sigma_2(t)], \ y \in \mathbb{R}. \end{cases} \tag{7.75}$$

Then problem (7.1) has a solution u satisfying estimate (7.16).

Proof. Choose an arbitrary $k \in \mathbb{N}$ such that $k > \frac{2}{T}$, and denote

$$\Delta_k = [0, \tfrac{1}{k}) \cup (T - \tfrac{1}{k}, T]$$

and

$$\Delta_{k1} = \{t \in \Delta_k : \sigma_1(t) = \sigma_2(t)\}, \quad \Delta_{k2} = \{t \in \Delta_k : \sigma_1(t) < \sigma_2(t)\}.$$

Further, define

$$\alpha(t,x) = \begin{cases} \sigma_1(t) & \text{if } \ x < \sigma_1(t), \\ x & \text{if } \ \sigma_1(t) \le x \end{cases}$$

for $t \in [0,T]$ and $x \in \mathbb{R}$,

$$g_k(t,x) = \begin{cases} (\phi(\sigma_2'(t)))' & \text{if } x > \sigma_2(t), \\ \dfrac{(x - \sigma_1(t))\,(\phi(\sigma_2'(t)))' + (\sigma_2(t) - x)\,(\phi(\sigma_1'(t)))'}{\sigma_2(t) - \sigma_1(t)} & \\ & \text{if } \sigma_1(t) \le x \le \sigma_2(t), \\ (\phi(\sigma_1'(t)))' & \text{if } x < \sigma_1(t) \end{cases}$$

for a.e. $t \in \Delta_{k2}$ and $x \in \mathbb{R}$ and

$$f_k(t,x,y) = \begin{cases} f(t,\alpha(t,x),y) & \text{if } \ t \in [0,T] \setminus \Delta_k, \\ -(\phi(\sigma_1'(t)))' & \text{if } \ t \in \Delta_{k1}, \\ -g_k(t,x) & \text{if } \ t \in \Delta_{k2} \end{cases}$$

for a.e. $t \in [0,T]$ and $x, y \in \mathbb{R}$.

Then $f_k \in Car([0,T] \times \mathbb{R}^2)$ and f_k satisfies inequalities (7.24) where $g(t,x,y) = f_k(t,x,y)$, $y_1 = y_2 = 0$, $\varkappa = \frac{1}{b} + 1$ with b given by (7.42) and $h_0(t) = h(t) + |(\phi(\sigma_1'(t)))'| + |(\phi(\sigma_2'(t)))'|$. Consider problem (7.37) with f_k defined in this proof. We see that σ_1 and σ_2 are also lower and upper functions to problem (7.37). Hence, for each $k \in \mathbb{N}$, Theorem 7.18 gives a solution u_k of problem (7.37). Moreover, each solution u_k satisfies estimate (7.16) and $\|u_k'\|_\infty \le r$, where $r > 0$ is the constant found in Lemma 7.16 for $r_0 = \max\{\|\sigma_1\|_\infty, \|\sigma_2\|_\infty\}$. Define

$$\Omega = \{x \in C^1[0,T] : \sigma_1 \le x \le \sigma_2 \ \text{ on } \ [0,T],\ \|x'\|_\infty \le r\}.$$

Let us put $\mathcal{A}_1 = [0,\infty)$, $\mathcal{A}_2 = \mathbb{R}$, $\varepsilon_k = \max\{\sigma_1(\frac{1}{k}), \sigma_1(T - \frac{1}{k})\}$ and, according to Remark 7.33 (iii), we have $\eta_k = 0$ for $k \in \mathbb{N}$. Then conditions (7.65) and (7.66) are valid and, by Theorem 7.32, we can find a subsequence $\{u_{k_\ell}\} \subset \{u_k\}$ uniformly converging on $[0,T]$ to a function $u \in C[0,T]$.

Choose $[a,b] \subset (0,T)$. Then there exists $k_0 \in \mathbb{N}$ such that for $k \ge k_0$ we have $[a,b] \subset [\frac{1}{k}, T - \frac{1}{k}]$ and

$$|f_k(t, u_k(t), u_k'(t))| \le h(t) \quad \text{for a.e. } t \in [a,b],$$

where

$$h(t) = \sup\{|f(t,x,y)| : r_1 \le x \le \sigma_2(t),\ |y| \le r\}$$

and $r_1 = \min\{\sigma_1(t) : t \in [a,b]\} > 0$. Since $h \in L_1[a,b]$, we see that the sequence $\{\phi(u_k')\}$ is equicontinuous on $[a,b]$. Further, $a_k = 0$, $b_k = 0$, $k \in \mathbb{N}$. According to Remark 7.33 (iii), the set $\mathcal{S} \subset (0,T)$ consists only of the zeros of u. Since u is positive on $(0,T)$, $\mathcal{S}$ is empty and we see that conditions (7.67) and (7.68) hold. Hence, by Theorem 7.32, u is a w-solution of problem (7.1).

Denote $\omega_0 = \max\{\omega(s) : s \in [0, \phi(r)]\}$ and

$$\psi(t) = -|(\phi(\sigma_1'(t)))'| - |(\phi(\sigma_2'(t)))'| - \omega_0[h(t) + r].$$

The first inequality in (7.75) implies that

$$-f_{k_\ell}(t, u_{k_\ell}(t), u_{k_\ell}'(t)) \operatorname{sign} u_{k_\ell}'(t) \ge \psi(t) \quad \text{for a.e. } t \in [0, a_2] \text{ and all } \ell \ge \ell_0,$$

and similarly the second inequality in (7.75) gives

$$f_{k_\ell}(t, u_{k_\ell}(t), u_{k_\ell}'(t)) \operatorname{sign} u_{k_\ell}'(t) \ge \psi(t) \quad \text{for a.e. } t \in [a_1, T] \text{ and all } \ell \ge \ell_0.$$

So, if we put $\nu = 0, \mu_0 = -1, s_0 = 0$ and $\lambda_1 = 1, s_1 = T, \eta = \min\{a_2, T - a_1\}$, we get inequalities (7.69) and (7.70). Therefore, by Theorem 7.32, u is a solution of problem (7.1). $\square$

EXAMPLE. Let $\alpha, \beta \in [1, \infty)$, $a \in \mathbb{R}$, $b \in (0, \frac{1}{\sqrt{2}})$, $c \in (0, \infty)$, $d \in (0, \frac{1}{b} - 2b)$. Consider problem (7.1) where $\phi(y) \equiv y$ and

$$f(t,x,y) = \left((T-t)^{-\beta} - t^{-\alpha} + a\right)(x - b\,t\,(T-t))\, y + c\,y^2 - d + \frac{t(T-t)}{x}$$

for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$. The first term of f has time singularities at $t = 0$, $t = T$ and the last term of f has a space singularity at $x = 0$. Let us put $\sigma_1(t) = b\,t\,(T-t)$, $\sigma_2(t) \equiv r_2 \ge \frac{T^2}{4}\,(\frac{1}{d} + b)$, $\omega(s) = (c+1)\,(s+1)$, $a_1 = \frac{T}{3}$, $a_2 = \frac{T}{2}$. If we choose a sufficiently large positive constant K and put $h(t) \equiv K$, we can check that all conditions of Theorem 7.34 are fulfilled. Therefore our problem has a solution u satisfying (7.16).

The next theorem deals with problem (7.1) provided f has singularities in all its variables.

Theorem 7.35. *Let* $\nu \in (0, \frac{T}{2})$, $\varepsilon \in (0, \frac{\phi(\nu)}{\nu})$, $c_1, c_2 \in (\nu, \infty)$, *and let assumption* (7.63) *hold with* $\mathcal{A}_1 = [0, \infty)$, $\mathcal{A}_2 = [-c_1, c_2]$. *Denote*

$$\sigma(t) = \min\{c_2\, t, c_1\,(T-t)\} \quad \textit{for} \quad t \in [0,T]$$

and assume that

$$\begin{cases} f(t, \sigma(t), \sigma'(t)) = 0 \quad \textit{for a.e.} \quad t \in [0,T], \\ 0 \le f(t,x,y) \\ \qquad \textit{for a.e.} \quad t \in [0,T] \textit{ and all } x \in (0, \sigma(t)],\ y \in [-c_1, c_2], \\ \varepsilon \le f(t,x,y) \\ \qquad \textit{for a.e.} \quad t \in [0,T] \textit{ and all } x \in (0, \sigma(t)],\ y \in [-\nu, \nu]. \end{cases} \tag{7.76}$$

Then problem (7.1) *has a solution* u *satisfying*

$$0 < u(t) \le \sigma(t), \quad -c_1 \le u'(t) \le c_2 \quad \textit{for} \quad t \in (0,T). \tag{7.77}$$

Proof. *Step* 1. *Existence of approximate solutions.*

Choose $k \in \mathbb{N}$, $k > \frac{2}{T}$ and put $\varepsilon_k = \min\{\sigma(\frac{1}{k}), \sigma(T - \frac{1}{k})\}$. For $x, y \in \mathbb{R}$ define

$$\alpha_k(x) = \begin{cases} x & \text{if } \varepsilon_k \le x, \\ \varepsilon_k & \text{if } x < \varepsilon_k, \end{cases} \qquad \beta(y) = \begin{cases} c_2 & \text{if } y > c_2, \\ y & \text{if } -c_1 \le y \le c_2, \\ -c_1 & \text{if } y < -c_1, \end{cases}$$

and

$$\gamma(y) = \begin{cases} \varepsilon & \text{if } |y| \le \nu, \\ 0 & \text{if } y \le -c_1 \text{ or } y \ge c_2, \\ \varepsilon\, \frac{c_2 - y}{c_2 - \nu} & \text{if } \nu < y < c_2, \\ \varepsilon\, \frac{c_1 + y}{c_1 - \nu} & \text{if } -c_1 < y < -\nu. \end{cases}$$

Further, for a.e. $t\in[0,T]$ and all $x,y\in\mathbb{R}$ define auxiliary functions

$$\widetilde{f}_k(t,x,y)=\begin{cases}\gamma(y) & \text{if } \ t\in[0,\frac{1}{k})\cup(T-\frac{1}{k},T],\\[2mm] f(t,\alpha_k(x),\beta(y)) & \text{if } \ t\in[\frac{1}{k},T-\frac{1}{k}],\end{cases}$$

and

$$f_k(t,x,y)=\begin{cases}\widetilde{f}_k(t,x,y) & \text{if } \ |y|\geq\frac{1}{k},\\[2mm] \frac{k}{2}\left(\widetilde{f}_k(t,x,\frac{1}{k})(y+\frac{1}{k})-\widetilde{f}_k(t,x,-\frac{1}{k})(y-\frac{1}{k})\right) & \\ & \text{if } \ |y|<\frac{1}{k}.\end{cases}\tag{7.78}$$

Then $f_k\in Car([0,T]\times\mathbb{R}^2)$ and we can find a function $m_k\in L_1[0,T]$ such that

$$|f_k(t,x,y)|\leq m_k(t)\quad\text{for a.e. }\ t\in[0,T]\text{ and all }\ x\in[0,\sigma(t)],\ y\in\mathbb{R}.$$

Moreover, f_k satisfies condition (7.65) with $\varepsilon_k=\min\{\sigma(\frac{1}{k}),\sigma(T-\frac{1}{k})\}$, $\eta_k=\frac{1}{k}$. Due to (7.76) we have

$$f_k(t,\sigma(t),\sigma'(t))=0,\quad f_k(t,0,0)\geq 0\quad\text{for a.e. }t\in[0,T],$$

and $\sigma_1\equiv 0$ and σ are respectively a lower and an upper function of problem (7.55) with f_k defined in this proof. Hence, by Theorem 7.14, this problem has a solution u_k and

$$0\leq u_k(t)\leq\sigma(t)\quad\text{for }\ t\in[0,T].$$

Step 2. A priori estimates of approximate solutions.

Since $f_k(t,x,y)\geq 0$ for a.e. $t\in[0,T]$ and all $x,y\in\mathbb{R}$, we have

$$(\phi(u_k'(t)))'\leq 0\quad\text{for a.e. }\ t\in[0,T].$$

This yields that $\phi(u_k')$ and u_k' are nonincreasing functions on $[0,T]$. Moreover,

$$-c_1\leq u_k'(t)\leq c_2\quad\text{for }\ t\in[0,T],\tag{7.79}$$

because $u_k(0) = \sigma(0) = u_k(T) = \sigma(T) = 0$ and $\sigma'(0) = c_2$, $\sigma'(T) = -c_1$. Let $t_k \in (0,T)$ be a point of maximum of u_k. Then $u_k'(t_k) = 0$ and

$$\begin{cases} u_k'(t) \geq 0 & \text{for } t \in [0, t_k], \\ u_k'(t) \leq 0 & \text{for } t \in [t_k, T]. \end{cases}$$

(i) Let $t_k - \nu \geq 0$. Then there exists $a_k \in [0, t_k)$ such that $u_k'(t) \leq \nu$ for $t \in [a_k, t_k]$. Assuming $a_k \leq t_k - \nu$ and integrating the last inequality in assumption (7.76), we get

$$\varepsilon\,(t_k - t) \leq \phi(u_k'(t)) \quad \text{for} \quad t \in [t_k - \nu, t_k]. \tag{7.80}$$

If $a_k > t_k - \nu$ and $u_k'(t) > \nu$ for $t \in [0, a_k)$, then similarly

$$\varepsilon\,(t_k - t) \leq \phi(u_k'(t)) \quad \text{for} \quad t \in [a_k, t_k].$$

Since $\phi(u_k'(t)) > \phi(\nu) > \varepsilon\,\nu > \varepsilon\,(t_k - t)$ for $t \in [t_k - \nu, a_k]$, we get estimate (7.80) again. Integration of (7.80) over $[t_k - \nu, t_k]$ yields the estimate

$$u_k(t_k) \geq \int_0^\nu \phi^{-1}(\varepsilon s)\,\mathrm{d}s = \nu_0 > 0. \tag{7.81}$$

(ii) Let $t_k - \nu < 0$. Then $t_k + \nu \leq T$ and there exists $b_k \in (t_k, T]$ such that $-u_k'(t) \leq \nu$ for $t \in [t_k, b_k]$. Assuming $b_k \geq t_k + \nu$ and integrating the last inequality in assumption (7.76) we obtain

$$\varepsilon\,(t - t_k) \leq -\phi(u_k'(t)) \quad \text{for} \quad t \in [t_k, t_k + \nu]. \tag{7.82}$$

If $b_k < t_k + \nu$ and $u_k'(t) < -\nu$ for $t \in (b_k, T]$, then similarly

$$\varepsilon\,(t - t_k) \leq -\phi(u_k'(t)) \quad \text{for} \quad t \in [t_k, b_k].$$

Since $-\phi(u_k'(t)) > \phi(\nu) > \varepsilon\nu > \varepsilon(t - t_k)$ for $t \in [b_k, t_k + \nu]$, we get inequality (7.82) again. Integration of (7.82) over $[t_k, t_k + \nu]$ yields estimate (7.81). Using this estimate and the fact the u_k' is nonincreasing on $[0, T]$ we conclude that

$$\alpha_k(t) \leq u_k(t) \leq \sigma(t) \quad \text{for} \quad t \in [0, T],$$

where

$$\alpha_k(t) = \begin{cases} \dfrac{\nu_0}{T}\, t & \text{for } \ t \in [0, t_k], \\ \dfrac{\nu_0}{T}\,(T - t) & \text{for } \ t \in (t_k, T]. \end{cases}$$

Step 3. *Convergence of the sequence of approximate solutions.*

Consider the sequence of solutions $\{u_k\}$, $k > \frac{2}{T}$. Define

$$\Omega = \{x \in C^1[0, T] : \ 0 \le x \le \sigma(t), \ -c_1 \le x' \le c_2 \text{ on } \ [0, T]\}.$$

Then condition (7.66) is valid and by Theorem 7.32 we can choose a subsequence $\{u_{k_\ell}\} \subset \{u_k\}$ which is uniformly converging on $[0, T]$ to a function $u \in C[0, T]$. By estimates (7.79) and (7.81) we get $0 < \frac{\nu_0}{c_2} \le t_k$ and $t_k \le T - \frac{\nu_0}{c_1} < T$ for $k \in \mathbb{N}$. So, we can choose a subsequence $\{u_{k_\ell}\}$ in such a way that $\lim_{\ell \to \infty} t_{k_\ell} = t_u \in (0, T)$ and

$$\alpha_u(t) \le u(t) \le \sigma(t) \quad \text{for } \ t \in [0, T], \tag{7.83}$$

where

$$\alpha_u(t) = \begin{cases} \dfrac{\nu_0}{T}\, t & \text{for } \ t \in [0, t_u], \\ \dfrac{\nu_0}{T}\,(T - t) & \text{for } \ t \in (t_u, T]. \end{cases}$$

Put $\mathcal{S} = \{t_u\}$ and choose $[a, b] \subset (0, t_u)$. Then there exists $k_0 \in \mathbb{N}$ such that for $k \ge k_0$ we have $|t_k - t_u| \le \frac{1}{2}(t_u - b)$, $[a, b] \subset (\frac{1}{k}, t_k)$,

$$u_k(t) \ge \frac{\nu_0\, a}{T} =: m_0, \quad \phi(u_k'(t)) \ge \frac{\varepsilon}{2}(t_u - b) =: m_1 \text{ on } \ [a, b].$$

Thus for a.e. $t \in [a, b]$

$$|f_k(t, u_k(t), u_k'(t))| \le h(t) \in L_1[a, b],$$

where $h(t) = \sup\{|f(t, x, y)| : m_0 \le x \le \sigma(t), \phi^{-1}(m_1) \le y \le c_2\}$. If we choose $[a, b] \subset (t_u, T)$, we argue similarly and obtain also a Lebesgue integrable majorant for f_k, $k \ge k_0$, on $[a, b]$. So, we have proved that condition (7.67) holds. By Theorem 7.32, we get $u \in C^1((0, T) \setminus \mathcal{S})$ and $\lim_{\ell \to \infty} u_{k_\ell}'(t) = u'(t)$ locally uniformly on $(0, T) \setminus \mathcal{S}$.

Step 4. *The function* u *is a solution.*

Since u_k' is nonincreasing on $[0,T]$ for $k \geq k_0$, u' is nonincreasing on $(0,t_u)$ and on (t_u,T). Therefore

$$0 \leq u'(t) \leq c_2 \quad \text{for} \quad t \in [0,t_u), \qquad -c_1 \leq u'(t) \leq 0 \quad \text{for} \quad t \in (t_u,T] \tag{7.84}$$

and the limits $\lim_{t\to t_u-} u'(t)$ and $\lim_{t\to t_u+} u'(t)$ exist.

(i) Let $\lim_{t\to t_u-} u'(t) = 0$. Assume that there exists $t^* \in (0,t_u)$ such that $u'(t^*) = 0$. Then $u'(t) = 0$ for $t \in [t^*,t_u]$ and, by the last inequality in assumption (7.76),

$$0 < \phi^{-1}(\varepsilon(t_u - t)) \leq u'(t) \quad \text{for} \quad t \in [t^*,t_u),$$

a contradiction. Similarly for $\lim_{t\to t_u+} u'(t) = 0$.

(ii) Let $\lim_{t\to t_u-} u'(t) > 0$. Since u' is nonincreasing, we have $u'(t) > 0$ for $t \in [0,t_u)$. Similarly for $\lim_{t\to t_u+} u'(t) < 0$. Hence we have shown that t_u is the unique point in $[0,T]$ where either $u'(t_u) = 0$ or $u'(t_u)$ does not exist. By estimate (7.83) u is positive in $(0,T)$. This implies that $\mathcal{S}$ satisfies condition (7.68). Having in mind that $a_k = b_k = 0$, $k \in \mathbb{N}$, we get by Theorem 7.32 that $\phi(u') \in AC_{loc}((0,T)\setminus\mathcal{S})$ and u is a w-solution of problem (7.1). Finally, by assumption (7.76) and definition (7.78), we have

$$f_{k_\ell}(t, u_{k_\ell}(t), u'_{k_\ell}(t)) \geq 0 \quad \text{for a.e. } t \in [0,T], \ \ell \in \mathbb{N}.$$

Hence condition (7.73) holds. According to Theorem 7.32 and Remark 7.33, u is a solution of problem (7.1). Estimates (7.83) and (7.84) yield the required estimate (7.77). $\square$

Example. Let α_1, α_2, β_1, $\beta_2 \in (0,\infty)$, and let functions $h_i \in L_{loc}(0,T)$, $i = 1,2,3,4$, be nonnegative. For a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$ define

$$f(t,x,y) = (1-y^2)\left(\frac{1}{2t} + h_1(t)\,x^{\alpha_1} + h_2(t)\,|y|^{\alpha_2} + h_3(t)\,\frac{1}{x^{\beta_1}} + h_4(t)\frac{1}{|y|^{\beta_2}}\right).$$

We can check that f satisfies the conditions of Theorem 7.35.

Bibliographical notes

Modified versions of Theorems 7.20 and 7.21 were published in Kiguradze and Shekhter [118]. Theorem 7.22 is new. Theorems 7.23 and 7.25 are adapted from Polášek and Rachůnková [153]. The existence of w-solutions under the assumptions of Theorems 7.24 and 7.26 was proved for $\phi(y) \equiv y$ in Kiguradze and Shekhter [118]. Theorems 7.29 and 7.30 are taken from Rachůnková and Stryja [164]. Theorems 7.32, 7.34 and 7.35 were proved by Rachůnková and Stryja in [165] and [166], respectively.

The singular Dirichlet problem has been studied almost 30 years and hundreds of papers have been written till now. From monographs investigating singular Dirichlet problems we would like to highlight Agarwal and O'Regan [12], Kiguradze and Shekhter [118], O'Regan [148] or Rachůnková, Staněk and Tvrdý [163], where also further historical and bibliographical notes are presented.

Chapter 8

Periodic Problem

The main goal of this chapter is to present existence results for singular periodic problems of the form

$$(\phi(u'))' = f(t, u, u'), \tag{8.1}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{8.2}$$

where $0 < T < \infty$, $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing and odd homeomorphism such that $\phi(\mathbb{R}) = \mathbb{R}$ and

$$\begin{cases} f \in Car([0, T] \times ((0, \infty) \times \mathbb{R})) \\ \text{and } \ f \ \text{ has a space singularity at } \ x = 0. \end{cases} \tag{8.3}$$

In accordance with Section 1.3, this means that

$$\limsup_{x \to 0+} |f(t, x, y)| = \infty \quad \text{for a.e. } t \in [0, T] \text{ and some } y \in \mathbb{R}.$$

Physicists say that f has an *attractive singularity* at $x = 0$ if

$$\liminf_{x \to 0+} f(t, x, y) = -\infty \quad \text{for a.e. } t \in [0, T] \text{ and some } y \in \mathbb{R}$$

since near the origin the force is directed inward. Alternatively, f is said to have a *repulsive singularity* at $x = 0$ if

$$\limsup_{x \to 0+} f(t, x, y) = \ \infty \quad \text{for a.e. } t \in [0, T] \text{ and some } y \in \mathbb{R}$$

Second order nonlinear differential equations or systems with singularities appear naturally in the description of particles subject to Newtonian type forces or to forces caused by compressed gases. Their mathematical study started in the sixties by Forbat and Huaux [91], Huaux [106], Derwidué [70] and Faure [87] , who considered positive solutions of equations describing e.g. the motion of a piston in a cylinder closed at one extremity and subject to a periodic exterior force, to the restoring force of a perfect gas and to a viscosity friction. The equations they studied may be after suitable substitutions transformed to

$$u'' + c\,u' = \frac{\beta}{u} + e(t),$$

where $c \neq 0$ and $\beta \neq 0$ can be either positive or negative. Equations of this form are usually called *Forbat equations* and their *Liénard* type generalizations like

$$u'' + h(u)\,u' = g(t,u) + e(t)$$

are sometimes also referred to as the generalized Forbat equations.

In the setting of Section 1.3, problem (8.1), (8.2) is investigated on the set $[0,T]\times\mathcal{A}$, where $\mathcal{A} = [0,\infty)\times\mathbb{R}$. In contrast to the Dirichlet problem (7.1), where each solution vanishes at $t=0$ and $t=T$ and hence enters the space singularity $x=0$ of f, all known existence results for the periodic problem (8.1), (8.2) under assumption (8.3) concern positive solutions which do not touch the space singularity $x=0$ of the function f.

Definition 8.1. A function $u\colon [0,T]\to\mathbb{R}$ is called a *solution of problem* (8.1), (8.2) if $\phi(u')\in AC[0,T]$, $(u(t),u'(t))\in\mathcal{A}$ for $t\in[0,T]$,

$$(\phi(u'(t)))' = f(t,u(t),u'(t)) \qquad \text{for a.e. } t\in[0,T]$$

and condition (8.2) is satisfied. If $u>0$ on $[0,T]$, then u is called a *positive solution.*

By Definition 8.1 and assumption (8.3) and with respect to the choice $\mathcal{A} = [0,\infty)\times\mathbb{R}$ we see that each solution of problem (8.1), (8.2) must be nonnegative and can vanish just on a set of zero measure. The restriction to positive solutions causes that the general existence principles in Theorems 1.6 and 1.7 about the limit of a sequence of approximate solutions need not be employed here. On the other hand, the singular problem (8.1), (8.2) will be also investigated through regular approximate periodic problems governed by differential equations of the form

$$(\phi(u'))' = h(t,u,u'), \tag{8.4}$$

where $h \in Car([0,T] \times \mathbb{R}^2)$. As usual, by a *solution of the regular problem* (8.4), (8.2) we understand a function u such that $\phi(u') \in AC[0,T]$, (8.2) is true and

$$(\phi(u'(t)))' = h(t, u(t), u'(t)) \quad \text{for a.e. } t \in [0,T].$$

Notice that the requirement $\phi(u') \in AC[0,T]$ implies that $u \in C^1[0,T]$.

In this chapter we will extensively utilize the Leray-Schauder degree and its finite dimensional special case – the Brouwer degree. For the definitions and basic properties of these notions we refer to Appendix C. In particular, see the Leray-Schauder degree theorem, the Borsuk antipodal theorem and Remark C.4.

We will also discuss various special cases of equation (8.1) including the classical one with $\phi(y) \equiv y$ or those with f not depending on u' or with f depending on u' linearly. Let us notice that the assumption that ϕ is an odd function is only technical. We employ it just to simplify some formulas occurring in this section.

8.1 Method of lower and upper functions

Regular problems

First, we will consider problem (8.4), (8.2), where $h \in Car([0,T] \times \mathbb{R}^2)$. We bring some results which will be exploited in the investigation of the singular problem (8.1), (8.2). The lower and upper functions method combined with the topological degree argument is an important tool for proofs of solvability of regular periodic problems.

Definition 8.2. We say that a function $\sigma \in C[0,T]$ is a *lower function of problem* (8.4), (8.2) if there is an at most finite set $\Sigma \subset (0,T)$ such that $\phi(\sigma') \in AC_{loc}([0,T] \setminus \Sigma)$,

$$\sigma'(t+) := \lim_{\tau \to t+} \sigma'(\tau) \in \mathbb{R}, \quad \sigma'(t-) := \lim_{\tau \to t-} \sigma'(\tau) \in \mathbb{R} \quad \text{for each } t \in \Sigma,$$

$$(\phi(\sigma'(t)))' \geq h(t, \sigma(t), \sigma'(t)) \quad \text{for a.e. } t \in [0,T], \tag{8.5}$$

$$\sigma(0) = \sigma(T), \ \sigma'(0) \geq \sigma'(T) \ \text{ and } \ \sigma'(t+) > \sigma'(t-) \ \text{ for all } t \in \Sigma. \tag{8.6}$$

If the inequalities in (8.5) and (8.6) are reversed, σ is called an *upper function of problem* (8.4), (8.2).

REMARK 8.3. It follows immediately from Definition 8.2 that $\|\sigma_1'\|_\infty < \infty$ and $\|\sigma_2'\|_\infty < \infty$ hold for each lower function σ_1 and each upper function σ_2 of problem (8.4), (8.2).

The role of lower and upper functions is demonstrated by the following *maximum principle*:

Lemma 8.4. *Let σ_1 and σ_2 be lower and upper functions of problem* (8.4), (8.2) *and let $\sigma_1 \le \sigma_2$ on $[0,T]$. Then for each $\widetilde{f} \in Car([0,T]\times\mathbb{R}^2)$ and each $d \in [\sigma_1(0), \sigma_2(0)]$ such that*

$$\begin{cases} \widetilde{f}(t,x,y) < h(t,\sigma_1(t),\sigma_1'(t)) \text{ for a.e. } t\in[0,T], \text{ all } x\in(-\infty,\sigma_1(t)) \\ \qquad \text{and all } y\in\mathbb{R} \text{ such that } |y-\sigma_1'(t)| < \dfrac{\sigma_1(t)-x}{\sigma_1(t)-x+1}, \\ \widetilde{f}(t,x,y) > h(t,\sigma_2(t),\sigma_2'(t)) \text{ for a.e. } t\in[0,T], \text{ all } x\in(\sigma_2(t),\infty) \\ \qquad \text{and all } y\in\mathbb{R} \text{ such that } |y-\sigma_2'(t)| < \dfrac{x-\sigma_2(t)}{x-\sigma_2(t)+1}, \end{cases} \tag{8.7}$$

any solution u of the problem

$$(\phi(u'))' = \widetilde{f}(t,u,u'), \quad u(0)=u(T)=d \tag{8.8}$$

satisfies $\sigma_1 \le u \le \sigma_2$ on $[0,T]$.

Proof. Let u be a solution of the auxiliary Dirichlet problem (8.8). Denote $v = u - \sigma_1$ and assume that

$$v(t_0) = \min\{v(t) : t\in[0,T]\} < 0.$$

Since $d \in [\sigma_1(0), \sigma_2(0)]$ and thanks to property (8.6) where $\sigma = \sigma_1$, we may assume that $t_0 \in (0,T)\setminus\Sigma$, $v'(t_0) = 0$, and there is $t_1 \in (t_0, T]$ such that $(t_0,t_1]\cap\Sigma = \emptyset$ and

$$v(t) < 0 \text{ and } |v'(t)| < \frac{-v(t)}{1-v(t)} \text{ for all } t\in[t_0,t_1].$$

Using property (8.5) and the first inequality in (8.7), we obtain

$$(\phi(u'(t))-\phi(\sigma_1'(t)))' < h(t,\sigma_1(t),\sigma_1'(t)) - (\phi(\sigma_1'(t)))' \le 0 \text{ for a.e. } t \in [t_0,t_1].$$

Hence

$$0 > \int_{t_0}^{t} (\phi(u'(s)) - \phi(\sigma_1'(s)))' \, \mathrm{d}s = \phi(u'(t)) - \phi(\sigma_1'(t)) \quad \text{for a.e. } t \in (t_0,t_1],$$

which leads to a contradiction with the definition of t_0, i.e. $u \ge \sigma_1$ on $[0,T]$. Similarly we can show that $u \le \sigma_2$ on $[0,T]$. □

REMARK 8.5. Let $h \in Car([0,T]\times\mathbb{R})$ and let $\sigma_1, \sigma_2 \in C[0,T]$ be such that $\sigma_1 < \sigma_2$ on $[0,T]$. Furthermore, assume that there is $\psi \in L_1[0,T]$ such that

$$|h(t,x,y)| \le \psi(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in [\sigma_1(t),\sigma_2(t)] \times \mathbb{R}.$$

Then it is always possible to construct a function $\widetilde{f} \in Car([0,T]\times\mathbb{R}^2)$ having the following properties:

(i) $\widetilde{f}(t,x,y) = h(t,x,y)$ whenever $x \in [\sigma_1(t),\sigma_2(t)]$,

(ii) there is $\widetilde{\psi} \in L_1[0,T]$ such that $|\widetilde{f}(t,x,y)| \le \widetilde{\psi}(t)$ for a.e. $t \in [0,T]$ and all $(x,y) \in \mathbb{R}^2$.

(iii) $\widetilde{f}$ satisfies inequalities (8.7).

Indeed, let us define

$$\omega_i(t,\zeta) = \sup_{z\in\mathbb{R},\, |\sigma_i'(t)-z|\le\zeta} |h(t,\sigma_i(t),\sigma_i'(t)) - h(t,\sigma_i(t),z)|$$

for $i = 1,2$ and $(t,\zeta) \in [0,T]\times[0,1]$ and

$$\widetilde{f}(t,x,y) = \begin{cases} h(t,\sigma_1(t),y) - \omega_1\left(t, \dfrac{\sigma_1(t)-x}{\sigma_1(t)-x+1}\right) - \dfrac{\sigma_1(t)-x}{\sigma_1(t)-x+1} & \\ \qquad\qquad \text{if } x < \sigma_1(t), & \\ h(t,x,y) & \text{if } x \in [\sigma_1(t),\sigma_2(t)], \\ h(t,\sigma_2(t),y) + \omega_2\left(t, \dfrac{x-\sigma_2(t)}{x-\sigma_2(t)+1}\right) + \dfrac{x-\sigma_2(t)}{x-\sigma_2(t)+1} & \\ \qquad\qquad \text{if } x > \sigma_2(t) & \end{cases}$$

for a.e. $t \in [0,T]$ and $(x,y) \in \mathbb{R}^2$.

One can verify that the functions ω_i, $i = 1,2$, belong to the class $Car([0,T]\times[0,1])$ and map the set $[0,T]\times[0,1]$ into $[0,\infty)$. In particular, $\widetilde{f} \in Car([0,T]\times\mathbb{R}^2)$. Furthermore, it is easy to verify that $\widetilde{f}$ has properties (i) and (ii). We will show that $\widetilde{f}$ satisfies also the first inequality in (8.7). Indeed, let

$$x < \sigma_1(t) \quad \text{and} \quad |y - \sigma_1'(t)| < \frac{\sigma_1(t) - x}{\sigma_1(t) - x + 1}.$$

Then, since ω_1 is nondecreasing in the second variable, we have

$$|h(t,\sigma_1(t),\sigma_1'(t)) - h(t,\sigma_1(t),y)| \le \omega_1\left(t, \frac{\sigma_1(t) - x}{\sigma_1(t) - x + 1}\right),$$

i.e.

$$h(t,\sigma_1(t),y) \le h(t,\sigma_1(t),\sigma_1'(t)) + \omega_1\left(t, \frac{\sigma_1(t) - x}{\sigma_1(t) - x + 1}\right)$$

for a.e. $t \in [0,T]$. Consequently,

$$\begin{aligned}\widetilde{f}(t,x,y) = h(t,\sigma_1(t),y) - \omega_1\left(t, \frac{\sigma_1(t) - x}{\sigma_1(t) - x + 1}\right) - \frac{\sigma_1(t) - x}{\sigma_1(t) - x + 1} \\ < h(t,\sigma_1(t),\sigma_1'(t)) \quad \text{for a.e.} \quad t \in [0,T].\end{aligned}$$

Similarly, we can show that $\widetilde{f}$ satisfies also the second inequality in (8.7).

Now we will transform problem (8.4), (8.2) to a fixed point problem. Having in mind that the periodic conditions (8.2) can be equivalently written as

$$u(0) = u(T) = u(0) + u'(0) - u'(T),$$

we can proceed similarly to the proof of Theorem 7.4:

Let us consider the *quasilinear Dirichlet problem*

$$(\phi(x'))' = b(t) \quad \text{a.e. on } [0,T], \quad x(0) = x(T) = d \tag{8.9}$$

with $b \in L_1[0,T]$ and $d \in \mathbb{R}$. A function $x \in C^1[0,T]$ is a solution of (8.9) if and only if there is $\gamma \in \mathbb{R}$ such that

$$x(t) = d + \int_0^t \phi^{-1}\Big(\gamma + \int_0^s b(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s \quad \text{for } t \in [0,T]$$

and

$$\int_0^T \phi^{-1}\Big(\gamma + \int_0^s b(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s = 0.$$

As in the proof of Theorem 7.4, we can see that for each $\ell \in C[0,T]$ there is a uniquely determined $c := \gamma(\ell) \in \mathbb{R}$ such that

$$\int_0^T \phi^{-1}(c + \ell(s))\,\mathrm{d}s = 0.$$

The functional $\gamma : C[0,T] \to \mathbb{R}$ is continuous and maps bounded sets to bounded sets (see *Step* 3 of the proof of Theorem 7.4). Thus, we can define an operator $\mathcal{K}\colon C[0,T] \to C^1[0,T]$ by

$$(\mathcal{K}(\ell))(t) = \int_0^t \phi^{-1}\big(\gamma(\ell) + \ell(s)\big)\,\mathrm{d}s. \tag{8.10}$$

Due to the continuity of γ and of ϕ^{-1}, the operator $\mathcal{K}$ is continuous as well. Let $\mathcal{N}\colon C^1[0,T] \to C[0,T]$ and $\mathcal{F}\colon C^1[0,T] \to C^1[0,T]$ be given by

$$(\mathcal{N}(u))(t) = \int_0^t h(s,u(s),u'(s))\,\mathrm{d}s$$

and

$$(\mathcal{F}(u))(t) = u(0) + u'(0) - u'(T) + (\mathcal{K}(\mathcal{N}(u)))(t). \tag{8.11}$$

In view of the definition of $\mathcal{F}$, a function $u \in C^1[0,T]$ is a solution to problem (8.4), (8.2) if and only if it is a fixed point of $\mathcal{F}$. Furthermore, since the operators $\mathcal{K}$ and $\mathcal{N}$ are continuous, it follows that $\mathcal{F}$ is continuous. The properties of the operator $\mathcal{F}$ are summarized by the following lemma.

Lemma 8.6. *Let $\mathcal{F}\colon C^1[0,T] \to C^1[0,T]$ be defined by* (8.11). *Then $\mathcal{F}$ is completely continuous and $u \in C^1[0,T]$ is a solution to problem* (8.4), (8.2) *if and only if $\mathcal{F}(u) = u$.*

Proof. It remains to show that $\mathcal{F}$ is completely continuous. Let $\{u_n\}$ be an arbitrary sequence bounded in $C^1[0,T]$. Denote $v_n = \mathcal{F}(u_n)$ for $n \in \mathbb{N}$. Then

$$v_n'(t) = \phi^{-1}\big(\gamma(\mathcal{N}(u_n)) + (\mathcal{N}(u_n))(t)\big) \quad \text{for } t \in [0,T] \text{ and } n \in \mathbb{N}.$$

We can see that the sequences $\{v_n\}$ and $\{v_n'\}$ are bounded on $[0,T]$. In particular, the sequence $\{v_n\}$ is equicontinuous on $[0,T]$. Further, since $h \in Car([0,T] \times \mathbb{R}^2)$, there is $m \in L_1[0,T]$ such that

$$|h(t, u_n(t), u_n'(t))| \le m(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } n \in \mathbb{N}.$$

So, for $t_1, t_2 \in [0,T]$ we get

$$|\phi(v_n'(t_1)) - \phi(v_n'(t_2))| = |(\mathcal{N}(u_n))(t_1) - (\mathcal{N}(u_n))(t_2)| \le \left| \int_{t_1}^{t_2} m(s)\mathrm{d}s \right|.$$

Therefore the sequence $\{\phi(v_n')\}$ is bounded and equicontinuous on $[0,T]$. Making use of the Arzelà-Ascoli theorem we can find subsequences $\{v_{k_n}\}$ and $\{\phi(v_{k_n}')\}$ uniformly convergent on $[0,T]$. Then $\{v_{k_n}'\}$ is also uniformly convergent on $[0,T]$ and so, $\{v_{k_n}\}$ is convergent in $C^1[0,T]$. We have proved that the operator $\mathcal{F}$ maps any sequence bounded in $C^1[0,T]$ to a set relatively compact in $C^1[0,T]$. Since we already know that $\mathcal{F}$ is continuous, we can conclude that it is completely continuous in $C^1[0,T]$. $\square$

The next lemma describes the relationship between lower and upper functions and the Leray-Schauder degree. We will consider the class of auxiliary problems

$$(\phi(v'))' = \eta(v')\, h(t,v,v'), \quad v(0) = v(T),\ v'(0) = v'(T), \tag{8.12}$$

where η may be an arbitrary continuous function mapping $\mathbb{R}$ into $[0,1]$.

Lemma 8.7. *Let σ_1 and σ_2 be lower and upper functions of problem (8.4) (8.2) and let $\sigma_1 < \sigma_2$ on $[0,T]$. Furthermore, assume that there exists $r^* > 0$ such that*

$$\begin{cases} \|v'\|_\infty < r^* \text{ for each continuous } \eta\colon \mathbb{R} \to [0,1] \text{ and} \\ \text{for each solution } v \text{ of (8.12) such that } \sigma_1 \le v \le \sigma_2 \text{ on } [0,T]. \end{cases} \tag{8.13}$$

Finally, assume that $\mathcal{F}: C^1[0,T] \to C^1[0,T]$ *is defined by* (8.11) *and, for* $\rho > 0$, *denote*

$$\Omega_\rho = \big\{u \in C^1[0,T] \colon \sigma_1 < u < \sigma_2 \text{ on } [0,T] \text{ and } \|u'\|_\infty < \rho\big\}. \tag{8.14}$$

Then

$$\deg(\mathcal{I} - \mathcal{F}, \Omega_\rho) = 1 \quad \text{for each } \rho \ge r^* \text{ such that } \mathcal{F}(u) \ne u \text{ on } \partial\Omega_\rho.$$

Proof. *Step* 1. *The Leray-Schauder degree of an auxiliary operator* $\widetilde{\mathcal{F}}$.

Denote $\Omega = \Omega_{r^*}$ and assume

$$\mathcal{F}(u) \ne u \quad \text{for all } u \in \partial\Omega. \tag{8.15}$$

Furthermore, since $\sigma_1', \sigma_2' \in L_\infty[0,T]$ (see Remark 8.3), we can define

$$R^* = r^* + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty \quad \text{and} \quad \eta(y) = \begin{cases} 1 & \text{if } |y| \le R^*, \\ 2 - \dfrac{|y|}{R^*} & \text{if } R^* < |y| < 2\,R^*, \\ 0 & \text{if } |y| \ge 2\,R^*. \end{cases} \tag{8.16}$$

Then σ_1 and σ_2 are lower and upper functions for problem (8.12) and there exists a function $\psi \in L_1[0,T]$ satisfying

$$|\eta(y)\, h(t,x,y)| \le \psi(t) \text{ for a.e. } t \in [0,T] \text{ and all } (x,y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}.$$

Now, let $\widetilde{f} \in Car([0,T] \times \mathbb{R}^2)$ and $\widetilde{\psi} \in L_1[0,T]$ be such that

$$\begin{cases} \widetilde{f}(t,x,y) = \eta(y)\, h(t,x,y) \\ \qquad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}, \end{cases} \tag{8.17}$$

$$|\widetilde{f}(t,x,y)| \le \widetilde{\psi}(t) \qquad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathbb{R}^2 \tag{8.18}$$

and $\widetilde{f}$ satisfies inequalities (8.7) with $\eta(y)\, h(t,x,y)$ in place of $h(t,x,y)$. Such a function can be certainly constructed, see Remark 8.5.

Let an operator $\widetilde{\mathcal{F}}: C^1[0,T] \to C^1[0,T]$ be given by

$$\widetilde{\mathcal{F}}(u) = \alpha(u(0) + u'(0) - u'(T)) + \mathcal{K}(\widetilde{\mathcal{N}}(u)), \tag{8.19}$$

where

$$(\widetilde{\mathcal{N}}(u))(t) = \int_0^t \widetilde{f}(s, u(s), u'(s))\,\mathrm{d}s \quad \text{for } u \in C^1[0,T] \text{ and } t \in [0,T],$$

$$\alpha(x) = \begin{cases} \sigma_1(0) & \text{if } x < \sigma_1(0), \\ x & \text{if } \sigma_1(0) \le x \le \sigma_2(0), \\ \sigma_2(0) & \text{if } x > \sigma_2(0) \end{cases}$$

and $\mathcal{K}\colon C[0,T] \to C^1[0,T]$ is defined by (8.10). According to Lemma 8.6, the operator $\widetilde{\mathcal{F}}$ is completely continuous. Moreover, it follows from the definition of the operator $\widetilde{\mathcal{F}}$ that the problem

$$(\phi(u'))' = \widetilde{f}(t,u,u'), \quad u(0) = u(T) = \alpha(u(0) + u'(0) - u'(T)) \tag{8.20}$$

is equivalent to the operator equation $\widetilde{\mathcal{F}}(u) = u$. Due to relations (8.18) and (8.19) we can find $r_0 \in (0,\infty)$ such that for any $\lambda \in [0,1]$, each fixed point u of the operator $\lambda\widetilde{\mathcal{F}}$ belongs to the set

$$\mathcal{B}(r_0) = \{x \in C^1[0,T] : \|x\|_\infty + \|x'\|_\infty < r_0\}.$$

So, by the normalization property and the homotopy property from the Leray-Schauder degree theorem, where

$$\mathcal{H}(\lambda, x) = (\mathcal{I} - \lambda\widetilde{\mathcal{F}})(x) \quad \text{and} \quad \Omega = \mathcal{B}(r_0),$$

we get

$$\deg(\mathcal{I} - \widetilde{\mathcal{F}}, \mathcal{B}(r_0)) = \deg(\mathcal{I}, \mathcal{B}(r_0)) = 1. \tag{8.21}$$

Step 2. Fixed points of the operator $\widetilde{\mathcal{F}}$.

Denote

$$\mathcal{Q} = \big\{u \in \Omega\colon \sigma_1(0) < u(0) + u'(0) - u'(T) < \sigma_2(0)\big\}.$$

Obviously, $\widetilde{\mathcal{F}} = \mathcal{F}$ on $\overline{\mathcal{Q}}$ and $\sigma_1(0) < u(0) = u(0) + u'(0) - u'(T) < \sigma_2(0)$ whenever $\mathcal{F}(u) = u$ and $u \in \Omega$. In other words, we have

$$\Big(\mathcal{F}(u) = u \quad \text{and} \quad u \in \Omega\Big) \implies u \in \mathcal{Q}. \tag{8.22}$$

We shall show that the implication

$$\Big(\widetilde{\mathcal{F}}(u) = u\Big) \implies u \in \mathcal{Q} \tag{8.23}$$

is true, as well. To this end, assume that $\widetilde{\mathcal{F}}(u)=u$. Then

$$\sigma_1(0)\le u(0)=u(T)=\alpha(u(0)+u'(0)-u'(T))\le\sigma_2(0). \tag{8.24}$$

This, together with Lemma 8.4, proves that the estimate

$$\sigma_1\le u\le\sigma_2 \quad \text{on} \quad [0,T] \tag{8.25}$$

holds. Furthermore, taking into account relation (8.17), we conclude that

$$\widetilde{f}(t,u(t),u'(t))=\eta(u'(t))\,h(t,u(t),u'(t)) \qquad \text{for a.e. } t\in[0,T]. \tag{8.26}$$

We already know that $u(0)=u(T)$. We shall show that u satisfies the second condition from (8.2), i.e. that $u'(0)=u'(T)$ holds. By virtue of (8.19), this is true whenever

$$\sigma_1(0)\le u(0)+u'(0)-u'(T)\le\sigma_2(0). \tag{8.27}$$

If the inequality $u(0)+u'(0)-u'(T)>\sigma_2(0)$ were valid then, in accordance with property (8.6) of lower functions, with inequality (8.24) and with the definition of α, we would obtain

$$u(0)=u(T)=\sigma_2(0)=\sigma_2(T) \quad \text{and} \quad u'(0)>u'(T).$$

However, this together with the already justified estimate (8.25) can hold only if $\sigma_2'(0)\ge u'(0)>u'(T)\ge\sigma_2'(T)$, which contradicts property (8.6) of lower functions. Therefore, $u(0)+u'(0)-u'(T)\le\sigma_2(0)$. Similarly we could prove that $u(0)+u'(0)-u'(T)\ge\sigma_1(0)$ is true as well. Consequently, relation (8.27) and hence also the equality $u'(0)=u'(T)$ hold. To summarize, if $\widetilde{\mathcal{F}}(u)=u$, then u solves problem (8.20), satisfies the periodicity condition (8.2) and relation (8.26). Therefore, it is a solution to problem (8.12). Furthermore, having in mind that (8.25) holds and by virtue of relations (8.13) and (8.16), we conclude that

$$\|u'\|_\infty<r^*\le R^*. \tag{8.28}$$

Therefore $\eta(u'(t))\equiv 1$ on $[0,T]$ and u is a solution to problem (8.4), (8.2) (cf. (8.16)). In other words, $\mathcal{F}(u)=u$ and $u\in\Omega$ due to relations (8.15), (8.25) and (8.28). Now, recalling that $\sigma_1(0)<u(0)+u'(0)-u'(T)<\sigma_2(0)$ holds whenever $\mathcal{F}(u)=u$ and $u\in\Omega$, we conclude that $u\in\mathcal{Q}$. This completes the proof of implication (8.23).

Step 3. *The Leray-Schauder degree of the operator* $\mathcal{F}$.

Having in mind implication (8.22) and applying the excision property of the Leray-Schauder degree we get

$$\deg(\mathcal{I}-\mathcal{F},\Omega)=\deg(\mathcal{I}-\mathcal{F},\mathcal{Q}).$$

The equality $\widetilde{\mathcal{F}}=\mathcal{F}$ on $\overline{\mathcal{Q}}$ implies that $\deg(\mathcal{I}-\mathcal{F},\mathcal{Q})=\deg(\mathcal{I}-\widetilde{\mathcal{F}},\mathcal{Q})$. On the other hand, implication (8.23) gives

$$\deg(\mathcal{I}-\widetilde{\mathcal{F}},\mathcal{Q})=\deg(\mathcal{I}-\widetilde{\mathcal{F}},\mathcal{B}(R_0)).$$

Therefore, by (8.21),

$$\deg(\mathcal{I}-\mathcal{F},\Omega)=\deg(\mathcal{I}-\mathcal{F},\mathcal{Q})=\deg(\mathcal{I}-\widetilde{\mathcal{F}},\mathcal{B}(R_0))=1.$$

Finally, notice that due to assumption (8.13) the implication

$$\Big(\mathcal{F}(u)=u \;\text{ and }\; \sigma_1<u<\sigma_2 \text{ on } [0,T]\Big) \implies u\in\Omega$$

is valid. So, we have proved that

$$\deg(\mathcal{I}-\mathcal{F},\Omega_\rho)=\deg(\mathcal{I}-\mathcal{F},\Omega)=1$$

for each $\rho\ge r^*$ such that $\mathcal{F}(u)\ne u$ on $\partial\Omega_\rho$. $\square$

Lemma 8.7 offers a possibility to get existence results for problems having a pair of lower and upper functions σ_1 and σ_2 satisfying

$$\sigma_1\le\sigma_2 \quad \text{on } [0,T]. \tag{8.29}$$

In such a case we say that σ_1 and σ_2 are *well-ordered* and the existence of a constant r^* with property (8.13) is usually ensured by conditions of Nagumo type. A suitable version of such conditions is provided by the next lemma.

Lemma 8.8. *Let* $\alpha,\,\beta\in C[0,T]$ *be such that* $\alpha\le\beta$ *on* $[0,T]$ *and assume that*

$$\begin{cases}\psi\in L_1[0,T] \textit{ is nonnegative, } \;\varepsilon_1,\,\varepsilon_2\in\{-1,1\},\\ \omega\in C(\mathbb{R}) \textit{ is positive and } \displaystyle\int_{-\infty}^0\frac{\mathrm{d}t}{\omega(t)}=\int_0^\infty\frac{\mathrm{d}t}{\omega(t)}=\infty.\end{cases} \tag{8.30}$$

Then there is an $r^* > 0$ *such that*

$$\|u'\|_\infty < r^* \tag{8.31}$$

holds for each function $u \in C^1[0,T]$ *fulfilling the periodicity conditions* (8.2) *and, in addition, possessing the following properties:* $\phi(u') \in AC[0,T]$,

$$\alpha \le u \le \beta \quad \textit{on} \quad [0,T]$$

and

$$\begin{cases} \varepsilon_1\,(\phi(u'(t)))' \le (\psi(t) + u'(t))\,\omega(\phi(u'(t))) & \textit{if} \;\; u'(t) > 0, \\ \varepsilon_2\,(\phi(u'(t)))' \le (\psi(t) - u'(t))\,\omega(\phi(u'(t))) & \textit{if} \;\; u'(t) < 0 \\ & \textit{for a.e.} \;\; t \in [0,T]. \end{cases} \tag{8.32}$$

Proof. Denote

$$\begin{aligned} \mathcal{Q} = \{u \in C^1[0,T] \colon \phi(u') \in AC[0,T],\, u(0) = u(T),\; u'(0) = u'(T), \\ \alpha \le u \le \beta \text{ on } [0,T]\} \end{aligned}$$

and

$$\mathcal{N}_u = \{t \in [0,T] \colon u'(t) = 0\} \quad \text{for } u \in \mathcal{Q}\,.$$

Let a function $u \in \mathcal{Q}$ fulfilling inequalities (8.32) a.e. on $[0,T]$ be given. We want to show that then the a priori estimate (8.31) holds with r^* independent of the choice of $u \in \mathcal{Q}$. Without any loss of generality we may assume that $\|u'\|_\infty > 0$. Let $t_u \in [0,T]$ be such that $|u'(t_u)| = \|u'\|_\infty$. Since $u(0) = u(T)$, we have $\mathcal{N}_u \ne \emptyset$.

(i) First, let $u'(t_u) > 0$ and $\varepsilon_1 = 1$. We may assume that $t_u \in (0,T]$. Moreover, let $\mathcal{N}_u \cap [0,t_u) \ne \emptyset$. Then there is $t_1 \in \mathcal{N}_u \cap [0,t_u)$ such that $u'(t) > 0$ on $(t_1,t_u]$. Hence, in view of estimates (8.32), we have

$$(\phi(u'(t)))' \le (\psi(t) + u(t))\,\omega(u'(t)) \text{ for a.e. } t \in [t_1,t_u].$$

Consequently,

$$\begin{aligned} \int_0^{\phi(\|u'\|_\infty)} \frac{\mathrm{d}t}{\omega(t)} &= \int_{t_1}^{t_u} \frac{(\phi(u'(t)))'}{\omega(u'(t))}\,\mathrm{d}t \le \int_{t_1}^{t_u} (\psi(t) + u(t))\,\mathrm{d}t \\ &\le \|\psi\|_1 + 2\|u\|_\infty \le \|\psi\|_1 + 2\,(\|\alpha\|_\infty + \|\beta\|_\infty), \end{aligned}$$

i.e.

$$\int_0^{\phi(\|u'\|_\infty)} \frac{\mathrm{d}t}{\omega(t)} \le \|\psi\|_1 + 2\left(\|\alpha\|_\infty + \|\beta\|_\infty\right). \tag{8.33}$$

On the other hand, if $\mathcal{N}_u \cap [0, t_u) = \emptyset$, then $u' > 0$ on $[0, t_u]$. Therefore, $u'(T) = u'(0) > 0$ and there is $t_2 \in \mathcal{N}_u$ such that $u' > 0$ on $(t_2, T]$. Using estimates (8.32), we get

$$\int_0^{\phi(u'(0))} \frac{\mathrm{d}t}{\omega(t)} = \int_{\phi(u'(t_2))}^{\phi(u'(T))} \frac{\mathrm{d}t}{\omega(t)} = \int_{t_2}^{T} \frac{(\phi(u'(t)))'}{\omega(u'(t))}\,\mathrm{d}t$$
$$\le \int_{t_2}^{T} (\psi(t) + u(t))\,\mathrm{d}t \le \|\psi\|_1 + 2\left(\|\alpha\|_\infty + \|\beta\|_\infty\right)$$

and

$$\int_{\phi(u'(0))}^{\phi(u'(t_u))} \frac{\mathrm{d}t}{\omega(t)} = \int_0^{t_u} \frac{(\phi(u'(t)))'}{\omega(u'(t))}\,\mathrm{d}t$$
$$\le \int_0^{t_u} (\psi(t) + u(t))\,\mathrm{d}t \le \|\psi\|_1 + 2\left(\|\alpha\|_\infty + \|\beta\|_\infty\right).$$

Thus,

$$\int_0^{\phi(\|u'\|_\infty)} \frac{\mathrm{d}t}{\omega(t)} = \int_0^{\phi(u'(0))} \frac{\mathrm{d}t}{\omega(t)} + \int_{\phi(u'(0))}^{\phi(u'(t_u))} \frac{\mathrm{d}t}{\omega(t)}$$
$$\le 2\left(\|\psi\|_1 + 2\left(\|\alpha\|_\infty + \|\beta\|_\infty\right)\right),$$

i.e.

$$\int_0^{\phi(\|u'\|_\infty)} \frac{\mathrm{d}t}{\omega(t)} \le 2\left(\|\psi\|_1 + 2\left(\|\alpha\|_\infty + \|\beta\|_\infty\right)\right). \tag{8.34}$$

(ii) Now, let $u'(t_u) > 0$ and $\varepsilon_1 = -1$. Since $u(0) = u(T)$, we may assume that $t_u \in [0, T)$. Moreover, let $\mathcal{N}_u \cap (t_u, T] \ne \emptyset$. Then there is $t_3 \in \mathcal{N}_u \cap (t_u, T]$ such that $u' > 0$ on $[t_u, t_3)$. Using estimates (8.32) we obtain

$$(\phi(u'(t)))' \ge -(\psi(t) + u(t))\,\omega(u'(t)) \text{ for a.e. } t \in [t_u, t_3].$$

Therefore,

$$\int_0^{\phi(\|u'\|_\infty)} \frac{\mathrm{d}t}{\omega(t)} = -\int_{t_u}^{t_3} \frac{(\phi(u'(t)))'}{\omega(u'(t))}\,\mathrm{d}t \le \int_{t_u}^{t_3} (\psi(t)+u(t))\,\mathrm{d}t$$
$$\le \|\psi\|_1 + 2\,(\|\alpha\|_\infty + \|\beta\|_\infty),$$

i.e. (8.33) holds also in this case.

If $\mathcal{N}_u \cap (t_u,T] = \emptyset$, then $u' > 0$ on $[t_u,T]$. Furthermore, $u'(0) = u'(T) > 0$ and there is $t_4 \in \mathcal{N}_u$ such that $u' > 0$ on $[0,t_4)$. Using estimates (8.32), we obtain

$$(\phi(u'(t)))' \ge -(\psi(t)+u(t))\,\omega(u'(t)) \text{ for a.e. } t \in [0,t_4] \cup [t_u,T].$$

Hence

$$\int_0^{\phi(\|u'\|_\infty)} \frac{\mathrm{d}t}{\omega(t)} = \int_0^{\phi(u'(0))} \frac{\mathrm{d}t}{\omega(t)} + \int_{\phi(u'(0))}^{\phi(u'(t_u))} \frac{\mathrm{d}t}{\omega(t)}$$
$$= -\int_0^{t_4} \frac{(\phi(u'(t)))'}{\omega(u'(t))}\,\mathrm{d}t - \int_{t_u}^{T} \frac{(\phi(u'(t)))'}{\omega(u'(t))}\,\mathrm{d}t$$
$$\le 2\left(\|\psi\|_1 + 2\,(\|\alpha\|_\infty + \|\beta\|_\infty)\right),$$

i.e. (8.34) is again true.

To summarize, inequality (8.34) is true whenever $u'(t_u) > 0$. Analogously we can prove that

$$\int_{-\phi(\|u'\|_\infty)}^{0} \frac{\mathrm{d}t}{\omega(t)} \le 2\left(\|\psi\|_1 + 2\,(\|\alpha\|_\infty + \|\beta\|_\infty)\right) \tag{8.35}$$

holds provided $u'(t_u) < 0$.

On the other hand, conditions (8.30) imply that we can choose $r^* > 0$ such that

$$\min\left\{\int_{-\phi(r^*)}^{0} \frac{\mathrm{d}t}{\omega(t)}, \int_0^{\phi(r^*)} \frac{\mathrm{d}t}{\omega(t)}\right\} > 2\left(\|\psi\|_1 + 2\,(\|\alpha\|_\infty + \|\beta\|_\infty)\right).$$

However, this may hold simultaneously with inequalities (8.34) and (8.35) only if estimate (8.31) is true for all $u \in \mathcal{Q}$ fulfilling (8.32). $\square$

In the case that the given problem possesses only lower and upper functions σ_1 and σ_2 which are not well-ordered , i.e. if

$$\sigma_1(\tau) > \sigma_2(\tau) \quad \text{for some} \quad \tau \in [0,T], \tag{8.36}$$

the following a priori estimate is available.

Lemma 8.9. *Let* $\psi \in L_1[0,T]$, $r^* = \phi^{-1}(\|\psi\|_1)$ *and* $\varepsilon \in \{-1, 1\}$. *Then the estimate* $\|u'\|_\infty \le r^*$ *holds for each* $u \in C^1[0,T]$ *fulfilling the periodicity conditions* (8.2) *and such that* $\phi(u') \in AC[0,T]$ *and*

$$\varepsilon\,(\phi(u'(t)))' \ge \psi(t) \quad \text{for a.e. } t \in [0,T].$$

Analogously, $\|u'\|_\infty < r^*$ *holds for each* $u \in C^1[0,T]$ *fulfilling the periodicity conditions* (8.2) *and such that* $\phi(u') \in AC[0,T]$ *and*

$$\varepsilon\,(\phi(u'(t)))' > \psi(t) \quad \text{for a.e. } t \in [0,T].$$

Proof. Let $u \in C^1[0,T]$ fulfil $\phi(u') \in AC[0,T]$, the periodicity conditions (8.2) and let

$$(\phi(u'(t)))' > \psi(t) \qquad \text{for a.e. } t \in [0,T].$$

Put $v = \phi(u')$. Then $v \in AC[0,T]$, $v(0) = v(T)$, $v' > \psi$ a.e. on $[0,T]$ and there is a $t_v \in (0,T)$ such that $v(t_v) = 0$. We have

$$-\|\psi\|_1 \le -\int_{t_v}^{t} |\psi(s)|\,\mathrm{d}s < v(t) \quad \text{for} \quad t \in (t_v, T] \tag{8.37}$$

and

$$-\|\psi\|_1 \le -\int_{t}^{t_v} |\psi(s)|\,\mathrm{d}s < -v(t) \quad \text{for} \quad t \in [0, t_v).$$

In particular, since $v(0) = v(T)$,

$$-\|\psi\|_1 \le -\int_{t_v}^{T} |\psi(s)|\,\mathrm{d}s < v(T) = v(0) < \int_0^{t_v} |\psi(s)|\,\mathrm{d}s \le \|\psi\|_L. \tag{8.38}$$

Furthermore, if $t \in [0, t_v]$, then using (8.37) and (8.38) we obtain

$$v(t) \geq v(0) - \int_0^t |\psi(s)| \, \mathrm{d}s > - \int_{t_v}^T |\psi(s)| \, \mathrm{d}s - \int_0^t |\psi(s)| \, \mathrm{d}s \geq -\|\psi\|_1.$$

Similarly, for $t \in [t_v, T]$ we get

$$v(t) < v(T) + \int_t^T |\psi(s)| \, \mathrm{d}s < \int_0^{t_v} |\psi(s)| \, \mathrm{d}s + \int_t^T |\psi(s)| \, \mathrm{d}s \leq \|\psi\|_1.$$

Summarizing, we can see that the estimates $\|v\|_\infty = \|\phi(u')\|_\infty < \|\psi\|_1$ and $\|u'\|_\infty < \phi^{-1}(\|\psi\|_1)$ are satisfied.

In the cases $(\phi(v'(t)))' < \psi(t)$ or $\varepsilon\,(\phi(v'(t)))' \geq \psi(t)$ the proof follows a similar argument. $\square$

The next assertion provides an existence principle which covers also the case (8.36):

Theorem 8.10. *Let σ_1 and σ_2 be lower and upper functions of problem (8.4), (8.2) and let assumption (8.36) hold. Furthermore, let there be $m \in L_1[0,T]$ and $\varepsilon \in \{-1, 1\}$ such that*

$$\varepsilon\, h(t, x, y) > m(t) \quad \textit{for a.e.} \quad t \in [0,T] \quad \textit{and all} \quad x, y \in \mathbb{R}$$

and let $\psi = -(|m| + 2)$.

Then problem (8.4), (8.2) has a solution u satisfying

$$\|u'\|_\infty < \phi^{-1}(\|\psi\|_1) \tag{8.39}$$

and

$$\begin{cases} \min\{\sigma_1(\tau_u), \sigma_2(\tau_u)\} \leq u(\tau_u) \leq \max\{\sigma_1(\tau_u), \sigma_2(\tau_u)\} \\ \hfill \textit{for some } \tau_u \in [0,T]. \end{cases} \tag{8.40}$$

Proof. Let $\varepsilon = 1$.

Step 1. *Auxiliary problem and operator representation.*

Put $r^* = \phi^{-1}(\|\psi\|_1)$. By Lemma 8.9 we have

$$\begin{cases} \|u'\|_\infty < r^* \text{ for each } u \in C^1[0,T] \text{ fulfilling (8.2) and such that} \\ \quad \phi(u') \in AC[0,T] \text{ and } (\phi(u'(t))' > \psi(t) \text{ for a.e. } t \in [0,T]. \end{cases} \tag{8.41}$$

Furthermore, put $c^* = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + T\,r^*$ and define for a.e. $t \in [0,T]$ and all $(x,y) \in \mathbb{R}^2$

$$\widetilde{f}(t,x,y) = \begin{cases} -(|m(t)|+1) & \text{if } x \le -(c^*+1), \\ h(t,x,y) + (x+c^*)\,(|m(t)|+1+h(t,x,y)) & \\ & \text{if } -(c^*+1) < x < -c^*, \\ h(t,x,y) & \text{if } -c^* \le x \le c^*, \\ h(t,x,y) + (x-c^*)\,|m(t)| & \text{if } c^* < x < c^*+1, \\ h(t,x,y) + |m(t)| & \text{if } x \ge c^*+1. \end{cases}$$

Let us consider the auxiliary problem

$$(\phi(u'))' = \widetilde{f}(t,u,u'), \quad u(0) = u(T), \quad u'(0) = u'(T). \tag{8.42}$$

We have

$$\begin{cases} \widetilde{f}(t,x,y) < 0 & \text{if } x \le -(c^*+1), \\ \widetilde{f}(t,x,y) > 0 & \text{if } x \ge c^*+1, \\ \widetilde{f}(t,x,y) = h(t,x,y) & \text{if } x \in [-c^*,c^*] \\ & \text{for a.e. } t \in [0,T] \text{ and all } x,y \in \mathbb{R} \end{cases} \tag{8.43}$$

and

$$\widetilde{f}(t,x,y) > \psi(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } x,y \in \mathbb{R}. \tag{8.44}$$

Furthermore, σ_1 and σ_2 are lower and upper functions of (8.42) and, moreover, $\sigma_3(t) \equiv -c^* - 2$ and $\sigma_4(t) \equiv c^* + 2$ form another pair of lower and upper functions for (8.42). We have

$$\sigma_3 < \min\{\sigma_1,\sigma_2\} \le \max\{\sigma_1,\sigma_2\} < \sigma_4 \text{ on } [0,T].$$

Denote

$$\Omega_0 = \{u \in C^1[0,T] : \sigma_3 < u < \sigma_4 \text{ on } [0,T],\ \|u'\|_\infty < r^*\},$$
$$\Omega_1 = \{u \in \Omega_0 : \sigma_3 < u < \sigma_2 \text{ on } [0,T]\},$$
$$\Omega_2 = \{u \in \Omega_0 : \sigma_1 < u < \sigma_4 \text{ on } [0,T]\}$$

and

$$\Omega = \Omega_0 \setminus \overline{\Omega_1 \cup \Omega_2}.$$

Let $\mathcal{F}$ be given by (8.11). Clearly, Ω is the set of all $u \in \Omega_0$ for which the relations $\|u'\|_\infty < r^*$ and

$$u(t_u) < \sigma_1(t_u) \quad \text{and} \quad u(s_u) > \sigma_2(s_u) \quad \text{for some } t_u, s_u \in [0,T] \tag{8.45}$$

are satisfied. Furthermore, $\Omega_1 \cap \Omega_2 = \emptyset$ and $\partial\Omega = \partial\Omega_0 \cup \partial\Omega_1 \cup \partial\Omega_2$.

By Lemma 8.6, problem (8.42) is equivalent to the operator equation $\widetilde{\mathcal{F}}(u) = u$ in $C^1[0,T]$, where

$$(\widetilde{\mathcal{N}}(u))(t) = \int_0^t \widetilde{f}(s, u(s), u'(s))\,\mathrm{d}s$$

and

$$\widetilde{\mathcal{F}}(u) = u(0) + u'(0) - u'(T) + \mathcal{K}(\widetilde{\mathcal{N}}(u))$$

and $\mathcal{K}\colon C[0,T] \to C^1[0,T]$ is given by (8.10). Let $\mathcal{F}$ be given by (8.11). Clearly, $\widetilde{\mathcal{F}}(u) = \mathcal{F}(u)$ for $u \in C^1[0,T]$ such that $\|u\|_\infty \le c^*$.

Step 2. *First a priori estimate.*

We will prove the implication

$$\left(\widetilde{\mathcal{F}}(u) = u \ \text{ and } \ u \in \overline{\Omega}_0\right) \implies u \in \Omega_0. \tag{8.46}$$

To this aim, first notice that by (8.41) and (8.44) the implication

$$\left(\widetilde{\mathcal{F}}(u) = u\right) \implies \|u'\|_\infty < r^* \tag{8.47}$$

holds. Now, assume that $\widetilde{\mathcal{F}}(u) = u$ and $u \in \partial\Omega_0$. Taking into account (8.47), we can see that this can happen only if

$$u(\alpha) = \max_{t \in [0,T]} u(t) = c^* + 2 \quad \text{or} \quad u(\alpha) = \min_{t \in [0,T]} u(t) = -(c^* + 2) \tag{8.48}$$

for some $\alpha \in [0,T)$. In the former case, we have $u'(\alpha) = 0$ and $u(t) > c^*+1$ on $[\alpha, \beta]$ for some $\beta \in (\alpha, T]$. Due to (8.43), we have also

$$(\phi(u'(t)))' = \widetilde{f}(t, u(t), u'(t)) > 0 \quad \text{for a.e. } t \in [\alpha, \beta],$$

i.e. $u'(t) > 0$ on $(\alpha, \beta]$, a contradiction. Similarly we can prove that the latter case in (8.48) is impossible. This shows that u satisfies the estimate

$$\|u\|_\infty < c^* + 2, \tag{8.49}$$

wherefrom, with respect to (8.47), implication (8.46) follows.

Step 3. *Second a priori estimate.*

Next we will prove that the implication

$$\left(\widetilde{\mathcal{F}}(u) = u \;\text{ and }\; u \in \overline{\Omega}\right) \implies \|u\|_\infty < c^* \tag{8.50}$$

is true. Indeed, let $\widetilde{\mathcal{F}}(u) = u$ and $u \in \partial\,\Omega$. By (8.47) we have $\|u'\|_\infty < r^*$ and (8.49). Consequently, either $u \in \partial\,\Omega_1$ or $u \in \partial\,\Omega_2$. This means that there is a $\tau_u \in [0, T]$ such that either $u(\tau_u) = \sigma_1(\tau_u)$ or $u(\tau_u) = \sigma_2(\tau_u)$. In both these cases we have $|u(\tau_u)| \le \|\sigma_1\|_\infty + \|\sigma_2\|_\infty$. Consequently,

$$|u(t)| \le |u(\tau_u)| + \int_{\tau_u}^{t} |u'(s)|\,\mathrm{d}s < \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + T\,r^* = c^*.$$

This completes the proof of estimate (8.50).

Step 4. *Existence of a solution to problem* (8.4), (8.2).

(i) Let $\widetilde{\mathcal{F}}(u) = u$ and $u \in \partial\Omega$. By (8.50), we have $\mathcal{F}(u) = \widetilde{\mathcal{F}}(u) = u$ and u is a solution to problem (8.4), (8.2).

(ii) Let $\widetilde{\mathcal{F}}(u) \neq u$ on $\partial\Omega$. Then using successively Lemma 8.7 for three well-ordered couples: $\{\sigma_3, \sigma_4\}$, $\{\sigma_3, \sigma_2\}$ and $\{\sigma_1, \sigma_4\}$ of lower and upper functions for problem (8.4), (8.2), we get

$$\deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_0) = \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_1) = \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_2) = 1.$$

Since by (8.36) we have $\Omega_1 \cap \Omega_2 = \emptyset$, the additivity property of the degree yields that the equalities

$$\deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega) = \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_0) - \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_1) - \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_2) = -1$$

hold. So $\widetilde{\mathcal{F}}$ has a fixed point u in Ω. Moreover, by *Step* 3, we have $\|u\|_\infty < c^*$ and hence

$$\widetilde{f}(t, u(t), u'(t)) = h(t, u(t), u'(t))$$

holds for a.e. $t \in [0, T]$. This means that u is a solution to (8.4), (8.2).

We can proceed analogously when $\varepsilon = -1$. $\square$

Singular problems

Now we are going to consider problem (8.1), (8.2) where f satisfies condition (8.3). We will present sufficient conditions in terms of lower and upper functions for the existence of positive solutions to the singular problem (8.1), (8.2). Lower and upper functions σ_1 and σ_2 are defined similarly to those for the regular problem (8.4), (8.2) (see Definition 8.2). However, since problem (8.1), (8.2) is investigated on $[0,T]\times\mathcal{A}$ where $\mathcal{A}=[0,\infty)\times\mathbb{R}$, only such σ_1 and σ_2 which are positive a.e. on $[0,T]$ make sense.

Definition 8.11. We say that a function $\sigma\in C[0,T]$ is a *lower function of problem* (8.1), (8.2) provided $\sigma(t)\in(0,\infty)$ for a.e. $t\in[0,T]$ and there is a finite set $\Sigma\subset(0,T)$ such that $\phi(\sigma')\in AC_{loc}([0,T]\setminus\Sigma)$ and (8.5) and (8.6) are satisfied.

If the inequalities in (8.5) and (8.6) are reversed, σ is called an *upper function of problem* (8.4), (8.2).

The first existence result concerns problem (8.1), (8.2) possessing well-ordered lower and upper functions.

Theorem 8.12. *Let there exist lower and upper functions σ_1 and σ_2 of problem* (8.1), (8.2) *such that $\sigma_2\ge\sigma_1>0$ on $[0,T]$. Furthermore, let for a.e. $t\in[0,T]$ and each $(x,y)\in[\sigma_1(t),\sigma_2(t)]\times\mathbb{R}$ the inequalities*

$$\begin{cases}\varepsilon_1\,f(t,x,y)\le(\psi(t)+y)\,\omega(\phi(y)) & \text{if } y>0,\\ \varepsilon_2\,f(t,x,y)\le(\psi(t)-y)\,\omega(\phi(y)) & \text{if } y<0\end{cases} \tag{8.51}$$

hold with $\varepsilon_1,\varepsilon_2,\omega$ and ψ satisfying (8.30).

Then problem (8.1), (8.2) *has a positive solution u such that*

$$\sigma_1\le u\le\sigma_2 \quad \text{on } [0,T]. \tag{8.52}$$

Proof. *Step* 1. *The case $\sigma_1<\sigma_2$.*

Assume that $\sigma_1<\sigma_2$ on $[0,T]$. Consider the auxiliary regular problem (8.4), (8.2) with h defined for a.e. $t\in[0,T]$ and $(x,y)\in\mathbb{R}^2$ by

$$h(t,x,y)=\begin{cases}f(t,\sigma_1(t),y) & \text{if } x<\sigma_1(t),\\ f(t,x,y) & \text{if } x\in[\sigma_1(t),\sigma_2(t)],\\ f(t,\sigma_2(t),y) & \text{if } x>\sigma_2(t).\end{cases}$$

Clearly, $h \in Car([0,T] \times \mathbb{R}^2)$ and σ_1 and σ_2 are a lower and an upper function of problem (8.4), (8.2). Choose an arbitrary continuous function $\eta\colon \mathbb{R} \to [0,1]$ and let v be an arbitrary solution of problem (8.12) fulfilling $\sigma_1 \le v \le \sigma_2$ on $[0,T]$. Since (8.51) is satisfied with h instead of f, we have for a.e. $t \in [0,T]$

$$\begin{aligned}\varepsilon_1\,(\phi(v'(t)))' &= \varepsilon_1\,\eta(v'(t))\,h(t,v(t),v'(t))\\ &\le \eta(v'(t))\,(\psi(t)+v'(t))\,\omega(\phi(v'(t)))\\ &\le (\psi(t)+v'(t))\,\omega(\phi(v'(t))) \qquad \text{if } v'(t)>0\end{aligned}$$

and

$$\varepsilon_2\,(\phi(v'(t)))' \le (\psi(t)-v'(t))\,\omega(\phi(v'(t))) \qquad \text{if } v'(t)<0.$$

Hence we can apply Lemma 8.8 to deduce that (8.13) is satisfied. Let $\mathcal{F} : C^1[0,T] \to C^1[0,T]$ and $\Omega = \Omega_{r^*}$ be defined by (8.11) and (8.14), respectively. Then there are two possibilities: either $\mathcal{F}$ has a fixed point $u \in \partial\Omega$ or $\mathcal{F}(u) \ne u$ on $\partial\Omega$.

(i) Let $\mathcal{F}(u) = u$ for some $u \in \partial\Omega$. In view of Lemma 8.6 and of the definition of h, it follows that u is a solution to (8.1), (8.2) fulfilling (8.52).

(ii) If $\mathcal{F}(u) \ne u$ on $\partial\Omega$, then by Lemma 8.7 we have $\deg(\mathcal{I}-\mathcal{F},\Omega)=1$, which implies that $\mathcal{F}$ has a fixed point $u \in \Omega$. As in (a), this fixed point is a solution to (8.1), (8.2) fulfilling (8.52).

Step 2. *The case* $\sigma_1 \le \sigma_2$.

For each $k \in \mathbb{N}$ the function $\widetilde{\sigma}_k = \sigma_2 + \frac{1}{k}$ is also an upper function of problem (8.4), (8.2) and $\sigma_1 < \widetilde{\sigma}_k$ on $[0,T]$. Hence, in the general case when the strict inequality between σ_1 and σ_2 need not hold, we can use *Step* 1 to show that for each $k \in \mathbb{N}$ there exists a solution u_k to (8.4), (8.2) such that

$$u_k(t) \in [\sigma_1(t), \sigma_2(t) + \tfrac{1}{k}] \quad \text{for } t \in [0,T] \quad \text{and} \quad \|u_k'\|_\infty < r^*,$$

where $r^* > 0$ is the constant given by Lemma 8.8 where $\alpha = \sigma_1$ and $\beta = \sigma_2 + 1$. Using the Arzelà-Ascoli theorem and the Lebesgue dominated convergence theorem for the sequences $\{u_k\}$ and $\{h(t,u_k(t),u_k'(t))\}$ we get a solution u of (8.1), (8.2) as the limit of a subsequence of $\{u_k\}$ on $C^1[0,T]$.

□

REMARK 8.13. Let functions α and β continuous on $[0,T]$ and such that $\beta \geq \alpha > 0$ on $[0,T]$ be given. We say that a function f satisfies the *Nagumo conditions* with respect to the couple α, β if there are $\varepsilon_1, \varepsilon_2 \in \{-1,1\}$ and functions ω, ψ having properties (8.30) and such that (8.51) is satisfied for a.e. $t \in [0,T]$ and all $(x,y) \in [\alpha(t), \beta(t)] \times \mathbb{R}$. Notice that the Nagumo conditions with respect to α, β are satisfied in particular if $f(t,x,y) = -h(x)y + g(t,x)$, where $h \in C[0,\infty)$ and $g \in Car([0,T] \times (0,\infty))$. Indeed, for a.e. $t \in [0,T]$ and each $(x,y) \in [\alpha(t), \beta(t)] \times \mathbb{R}$ we have

$$|f(t,x,y)| \leq |h(x)|\,|y| + |g(t,x)| \leq K\,(\psi(t) + |y|)$$

where

$$K = 1 + \max\{|h(x)| : x \in [\delta, \|\beta\|_\infty]\},\ \ \psi(t) = \sup\{|g(t,x)| : x \in [\delta, \|\beta\|_\infty]\}$$

and $\delta = \min\{\alpha(t) : t \in [0,T]\}$. (By assumption, we have $\delta > 0$.)

EXAMPLE. Theorem 8.12 provides the existence of a positive solution to problem (8.1), (8.2) also for

$$f(t,x,y) = g(t,x)\,y^{2n+1} + h(x)\,y\,\phi(y) - a\,x^{-\lambda_1} + b\,x^{\lambda_2}$$

for a.e. $t \in [0,T]$ and all $(x,y) \in (0,\infty) \times \mathbb{R}$, where $g \in Car([0,T] \times \mathbb{R})$ is nonnegative, $n \in \mathbb{N}$, $a, b, \lambda_1, \lambda_2 \in (0,\infty)$ and $h \in C[0,\infty)$.

The last result of this section concerns the case when the given problem possesses lower and upper functions, but no pair of them is well-ordered. We will restrict ourselves to the equation

$$(\phi(u'))' = g(u) + p(t,u,u'), \tag{8.53}$$

where p is a well-behaved function $(p \in Car([0,T] \times \mathbb{R}^2))$ and g has a singularity at the origin. Recall that problem (8.53), (8.2) is investigated on the set $[0,T] \times \mathcal{A}$, where $\mathcal{A} = [0,\infty) \times \mathbb{R}$.

The key assumption is that

$$\lim_{x \to 0+} \int_x^1 g(s)\,\mathrm{d}s = +\infty. \tag{8.54}$$

Clearly, condition (8.54) implies that

$$\limsup_{x \to 0+} g(x) = +\infty, \tag{8.55}$$

which means that g has a space *repulsive singularity* at the origin. Repulsive singularities having property (8.54) are called *strong singularities* and the function g is then usually said to be a strong repulsive singular force. We will refer to condition (8.54) as to the *strong repulsive singularity condition* . On the other hand, if condition (8.55) is satisfied together with

$$\lim_{x\to 0+} \int_x^1 g(s)\,\mathrm{d}s \in \mathbb{R},$$

then the singularity of f at $x = 0$ is called a *weak singularity* and g is said to be a weak repulsive singular force.

The meaning of the strong repulsive singularity condition is revealed by the following lemma.

Lemma 8.14. *Let* $p \in Car([0,T]\times\mathbb{R}^2)$ *and let* $g \in C(0,\infty)$. *Furthermore, let* g *satisfy the strong repulsive singularity condition* (8.54) *and let there be a function* $m \in L_1[0,T]$ *such that*

$$g(x) + p(t,x,y) > m(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } x > 0,\ y \in \mathbb{R}. \tag{8.56}$$

Then each lower function σ_1 *of problem* (8.53), (8.2) *is positive on the whole interval* $[0,T]$.

Proof. Let σ_1 be a lower function for (8.53), (8.2) and $\rho := \|\sigma_1'\|_\infty$. Then $\rho < \infty$ and, by virtue of the property (8.5) for $\sigma = \sigma_1$, we have

$$(\phi(\sigma_1'(t)))'\,(\sigma_1'(t)-\rho) \le g(\sigma_1(t))\,(\sigma_1'(t)-\rho) + p(t,\sigma_1(t),\sigma_1'(t))\,(\sigma_1'(t)-\rho)$$

for a.e. $t \in [0,T]$. Furthermore, due to (8.54) there is $\delta > 0$ such that

$$\lim_{x\to 0+} \int_x^{\delta'} g(s)\,\mathrm{d}s = \infty \qquad \text{for all } \delta' \in (0,\delta). \tag{8.57}$$

Let an arbitrary $\varepsilon > 0$ be given. Since in view of Definition 8.11 we have $\sigma_1 > 0$ a.e. on $[0,T]$, we can choose $t_0 \in (0,\varepsilon]$ in such a way that $\sigma_1(t_0) > 0$. Put $t^* = \sup\{t \in [t_0,T] : \sigma_1(s) > 0 \text{ on } [t_0,t]\}$. Let $\sigma_1(t^*) = 0$. Then there is a $t' \in (t_0,t^*)$ such that

$$\sigma_1(t) \in [0,\delta) \qquad \text{for all } t \in [t',t^*]. \tag{8.58}$$

Let $t_n \in (t', t^*)$ be an increasing sequence such that $\lim_{n\to\infty} t_n = t^*$. Then

$$\lim_{n\to\infty} \sigma_1(t_n) = \sigma_1(t^*) = 0 \tag{8.59}$$

and

$$\begin{aligned}
&\int_{t'}^{t_n} (\phi(\sigma_1'(t)))' \, (\sigma_1'(t) - \rho) \, \mathrm{d}t \\
&\quad \le \int_{t'}^{t_n} g(\sigma_1(t)) \, (\sigma_1'(t) - \rho) \, \mathrm{d}t + \int_{t'}^{t_n} p(t, \sigma_1(t), \sigma_1'(t)) \, (\sigma_1'(t) - \rho) \, \mathrm{d}t \\
&\quad = - \int_{\sigma_1(t_n)}^{\sigma_1(t')} g(s) \, \mathrm{d}s - \rho \int_{t'}^{t_n} (g(\sigma_1(t)) + p(t, \sigma_1(t), \sigma_1'(t)) \, \mathrm{d}t \\
&\qquad + \int_{t'}^{t_n} p(t, \sigma_1(t), \sigma_1'(t)) \, \sigma_1'(t) \, \mathrm{d}t.
\end{aligned}$$

Therefore, for each $n \in \mathbb{N}$ we have

$$\begin{aligned}
&\int_{\sigma_1(t_n)}^{\sigma_1(t')} g(s) \, \mathrm{d}s \\
&\quad \le \int_{t'}^{t_n} |(\phi(\sigma_1'(t)))'| \, |\sigma_1'(t) - \rho| \, \mathrm{d}t + \int_{t'}^{t_n} |p(t, \sigma_1(t), \sigma_1'(t)| \, |\sigma_1'(t)| \, \mathrm{d}t \\
&\qquad - \rho \int_{t'}^{t_n} (g(\sigma_1(t)) + p(t, \sigma_1(t), \sigma_1'(t)) \, \mathrm{d}t \le c,
\end{aligned}$$

where $\quad c = \rho \left(2\|\phi(\sigma_1')'\|_1 + \displaystyle\int_0^T |p(t, \sigma_1(t), \sigma_1'(t))| \, \mathrm{d}t + \|m\|_1 \right) < \infty.$

On the other hand, thanks to relations (8.57)–(8.59) we have

$$\lim_{n\to\infty} \int_{\sigma_1(t_n)}^{\sigma_1(t')} g(s) \, \mathrm{d}s = \infty,$$

a contradiction. Thus, $\sigma_1(t^*) > 0$. It follows that $t^* = T$, since otherwise we would get a contradiction with the definition of t^*. In particular, we can see that $\sigma_1(t)$ is positive on any interval $(\varepsilon, T]$, $\varepsilon > 0$, and, as we also have $\sigma_1(0) = \sigma_1(T) > 0$ in view of the periodicity condition (8.6), this completes the proof of the lemma. □

REMARK 8.15. Lemma 8.14 says, in particular, that under assumptions (8.54) and (8.56), where $m \in L_1[0,T]$, each solution $u \in C^1[0,T]$ of problem (8.53), (8.2) must be positive at each $t \in [0,T]$.

Theorem 8.16. *Let $p \in Car([0,T] \times \mathbb{R}^2)$ and $g \in C(0,\infty)$. Furthermore, let the strong repulsive singularity condition (8.54) and condition (8.56) with some $m \in L_1[0,T]$ be satisfied. Finally, let there be lower and upper functions σ_1 and σ_2 of problem (8.53), (8.2) such that relation (8.36) is true and $\sigma_2 > 0$ on $[0,T]$.*

Then problem (8.53), (8.2) possesses a positive solution u having properties (8.39) and (8.40).

Proof. Put $r^* = \phi^{-1}(\|\psi\|_1)$, where $\psi = |m| + 2$. Let us define

$$R = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty, \quad r = r^* + \|\sigma_1'\|_\infty \text{ and } B = R + r^* T. \tag{8.60}$$

Since $p \in Car([0,T] \times \mathbb{R}^2)$, there is $\widetilde{p} \in L_1[0,T]$ such that

$$|p(t,x,y)| \le \widetilde{p}(t) \text{ for a.e. } t \in [0,T] \text{ and all } (x,y) \in [0,B] \times [-r,r]. \tag{8.61}$$

By Lemma 8.14, $\sigma_1 > 0$ on $[0,T]$. Since we assume $\sigma_2 > 0$ on $[0,T]$, it follows that $\delta := \min\{\{\sigma_1(t), \sigma_2(t)\} : t \in [0,T]\} > 0$. Now, put

$$K = \|\widetilde{p}\|_1 r^* + \int_\delta^B |g(s)| \, \mathrm{d}s.$$

By (8.54) there exists $\varepsilon \in (0,\delta)$ such that $g(\varepsilon) > 0$ and

$$\int_\varepsilon^\delta g(s) \, \mathrm{d}s > K. \tag{8.62}$$

For a.e. $t \in [0,T]$ and all $(x,y) \in \mathbb{R}^2$, define

$$h(t,x,y) = \widetilde{g}(x) + p(t,x,y), \quad \text{where } \widetilde{g}(x) = \begin{cases} g(\varepsilon) & \text{if } x < \varepsilon, \\ g(x) & \text{if } x \ge \varepsilon. \end{cases}$$

Then $h \in Car([0,T] \times \mathbb{R}^2)$, σ_1 and σ_2 are a lower and an upper function of problem (8.4), (8.2) and, by assumption (8.56),

$$h(t,x,y) > m(t) \text{ for a.e. } t \in [0,T] \text{ and all } x > 0, y \in \mathbb{R}.$$

By Theorem 8.10, problem (8.4), (8.2) has a solution u satisfying estimate (8.39) and $\delta \le u(t_u) \le R$ for some $t_u \in [0,T]$. In particular, $u \le B$ for all $t \in [0,T]$. It remains to show that $u \ge \varepsilon$ on $[0,T]$. Let $t_0, t_1 \in [0,T]$ be such that

$$u(t_0) = \min\{u(t)\colon t \in [0,T]\} \quad\text{and}\quad u(t_1) = \max\{u(t)\colon t \in [0,T]\}.$$

We have $u'(t_0) = u'(t_1) = 0$ and $u(t_1) \in [\delta, B]$. Put $v(t) = \phi(u'(t))$ for $t \in [0,T]$. Then $u'(t) = \phi^{-1}(v(t))$ on $[0,T]$, $v(t_0) = v(t_1) = \phi(0)$ and

$$\int_{t_0}^{t_1} (\phi(u'(s)))'\, u'(s)\, \mathrm{d}s = \int_{t_0}^{t_1} v'(s)\, \phi^{-1}(v(s))\, \mathrm{d}s = \int_{v(t_0)}^{v(t_1)} \phi^{-1}(y)\, \mathrm{d}y = 0.$$

Thus, multiplying both sides of the equality

$$(\phi(u'(t)))' = h(t, u(t), u'(t))$$

by $u'(t)$ and integrating from t_0 to t_1, we get

$$\int_{u(t_0)}^{u(t_1)} \widetilde{g}(s)\, \mathrm{d}s \le \int_{t_0}^{t_1} |p(t, u(t), u'(t))|\, |u'(t)|\, \mathrm{d}t \le \|\widetilde{p}\|_1\, r^*.$$

Therefore

$$\begin{aligned} g(\varepsilon)\,(\varepsilon - u(t_0)) + \int_{\varepsilon}^{\delta} g(s)\, \mathrm{d}s &= \int_{u(t_0)}^{\delta} \widetilde{g}(s)\, \mathrm{d}s \\ \le \int_{u(t_0)}^{u(t_1)} \widetilde{g}(s)\, \mathrm{d}s + \int_{\delta}^{B} |g(s)|\, \mathrm{d}s &\le \|\widetilde{p}\|_1\, r^* + \int_{\delta}^{B} |g(s)|\, \mathrm{d}s = K. \end{aligned}$$

Since $g(\varepsilon) > 0$, this contradicts inequality (8.62) whenever

$$u(t_0) = \min\{u(t)\colon t \in [0,T]\} \le \varepsilon.$$

Hence, $u(t) > \varepsilon$ on $[0,T]$, which means that u is a solution to problem (8.53), (8.2). □

Example. Let

$$g(x) = a\, x^{-\lambda_1} + b\, x^{\lambda_2} \quad \text{for} \quad x \in (0, \infty),$$

where $a, b, \lambda_2 \in (0,\infty)$ and $\lambda_1 \ge 1$. Then Theorem 8.16 provides the existence of a positive solution to problem (8.53), (8.2) if $p \in Car([0,T] \times \mathbb{R}^2)$ is bounded below, i.e. there is $m \in L_1[0,T]$ such that $p(t,x,y) \ge m(t)$ for a.e. $t \in [0,T]$ and all $(x,y) \in \mathbb{R}^2$.

8.2 Attractive singular forces

This section is devoted to the singular problem (8.1), (8.2) where f has an attractive singularity at $x = 0$, which means that, in addition to (8.3), it has also the following property:

$$\liminf_{x \to 0+} f(t,x,y) = -\infty \qquad \text{for a.e. } t \in [0,T] \text{ and some } y \in \mathbb{R}.$$

Such a situation can be treated by means of lower and upper functions associated with the problem. We can decide whether the problem has constant lower and upper functions and to find them provided they exist. In general, however, it is easy neither to find lower and upper functions which need not be constant nor to prove their existence, which can make the application of theorems like Theorem 8.12 difficult. A simple possibility how to find non-constant lower or upper functions to problem (8.1), (8.2) is offered by the following lemma. In what follows we use the standard notation for *mean values of integrable functions*: for $y \in L_1[0,T]$, the symbol $\overline{y}$ stands for

$$\overline{y} := \frac{1}{T} \int_0^T y(t)\, \mathrm{d}t.$$

Lemma 8.17. (i) *Let there exist* $A > 0$ *and* $b \in L_1[0,T]$ *such that* $\overline{b} \geq 0$,

$$\begin{cases} f(t,x,y) \geq b(t) \\ \textit{for a.e. } t \in [0,T] \textit{ and all } x \in [A,B],\ |y| \leq \phi^{-1}(\|b\|_1), \end{cases} \tag{8.63}$$

where $B - A \geq 2\,T\,\phi^{-1}\big(\|b\|_1\big)$.

Then problem (8.1), (8.2) *possesses an upper function* σ_2 *such that*

$$A \leq \sigma_2 \leq B \quad \textit{on} \quad [0,T].$$

(ii) *If* A, B *and* $b \in L_1[0,T]$ *satisfy analogous conditions but with* $\overline{b} \leq 0$ *and*

$$\begin{cases} f(t,x,y) \leq b(t) \\ \textit{for a.e. } t \in [0,T] \textit{ and all } x \in [A,B],\ |y| \leq \phi^{-1}(\|b\|_1), \end{cases} \tag{8.64}$$

then problem (8.1), (8.2) *possesses a lower function* σ_1 *such that*

$$A \leq \sigma_1 \leq B \quad \textit{on} \quad [0,T].$$

Proof. (i) Assume that $\bar b \ge 0$ and relation (8.63) is true. For a given $d \in \mathbb{R}$, let x_d be a solution of the quasilinear auxiliary Dirichlet problem (8.9). Then

$$\phi(x_d'(t)) = \phi(x_d'(t_0)) + \int_{t_0}^{t} b(s)\,\mathrm{d}s \quad \text{for all } \ t, t_0 \in [0,T].$$

Since $\bar b \ge 0$, it follows that $x_d'(T) \ge x_d'(0)$. Since $x_d(0) = x_d(T)$, there is a $t_d \in (0,T)$ such that $x_d'(t_d) = 0$. Thus

$$\phi(x_d'(t)) = \int_{t_d}^{t} b(s)\,\mathrm{d}s \quad \text{for } t \in [0,T]$$

and so $\|x_d'\|_\infty \le \phi^{-1}(\|b\|_1)$ for each $d \in \mathbb{R}$ and $\|x_0\|_\infty \le T\,\phi^{-1}(\|b\|_1)$. Put $\sigma_2 = A + T\,\phi^{-1}(\|b\|_1) + x_0$. Then

$$A \le \sigma_2 \le A + 2T\,\phi^{-1}(\|b\|_1) \le B \quad \text{ on } [0,T].$$

Having in mind assumption (8.63) and the definition of x_d, we can see that σ_2 is an upper function of problem (8.1), (8.2).

(ii) If $\bar b \le 0$ and assumption (8.64) is valid, then $\sigma_1 = A + T\,\phi^{-1}(\|b\|_1) + x_0$ is a lower function of problem (8.1), (8.2) and $A \le \sigma_1 \le B$ on $[0,T]$. $\square$

Corollary 8.18. *Let there exist* $r > 0$, $A > r$ *and* $b \in L_1[0,T]$ *such that* $\bar b \ge 0$, (8.63) *with* $B - A \ge 2T\,\phi^{-1}\big(\|b\|_1\big)$ *and*

$$f(t,r,0) \le 0 \quad \textit{for a.e.} \ \ t \in [0,T]$$

hold. Furthermore, let for a.e. $t \in [0,T]$ *and each* $(x,y) \in [r,B] \times \mathbb{R}$ *inequalities* (8.51) *be true with* ε_1, ε_2, ω, ψ *satisfying* (8.30).

Then problem (8.1), (8.2) *has a positive solution* u *such that*

$$r \le u \le B \ \textit{on} \ [0,T]. \tag{8.65}$$

Proof. By Lemma 8.17, problem (8.1), (8.2) has an upper function σ_2 such that $\sigma_2 \in [A,B]$ on $[0,T]$. Furthermore, $\sigma_1 = r$ is a lower function of (8.1), (8.2) and $0 < \sigma_1 < \sigma_2$ on $[0,T]$. By Theorem 8.12, problem (8.1), (8.2) has a positive solution u satisfying (8.65). $\square$

Now, let us consider the *Liénard equation*

$$(\phi(u'))' + h(u)\,u' = g(u) + e(t), \tag{8.66}$$

where

$$h \in C[0,\infty),\;\; g \in C(0,\infty),\;\; e \in L_1[0,T] \tag{8.67}$$

and g has an attractive space singularity at $x = 0$, i.e.

$$\liminf_{x\to 0+} g(x) = -\infty. \tag{8.68}$$

The next lemma shows that problem (8.66), (8.2) possesses an upper function whenever

$$\liminf_{x\to\infty}\,[g(x) + \overline{e}] > 0. \tag{8.69}$$

Lemma 8.19. *Let conditions* (8.67) *and* (8.69) *hold. Furthermore, assume that there exists* $\alpha \in (0,\infty)$ *such that*

$$\liminf_{|y|\to\infty} \frac{|\phi(y)|}{|y|^{\alpha}} > 0. \tag{8.70}$$

Then for an arbitrary $r \in (0,\infty)$, *problem* (8.66), *(8.2) possesses an upper function* σ_2 *such that* $\sigma_2 > r$ *on* $[0,T]$.

Proof. *Step* 1. *Construction of operator* $\mathcal{F}_\lambda$.

Choose $r \in (0,\infty)$. By assumption (8.69) there is $R > r$ such that

$$g(x) + \overline{e} > 0 \quad \text{for} \quad x \ge R. \tag{8.71}$$

Take an arbitrary $c \in \mathbb{R}$ and consider the auxiliary Dirichlet problem

$$(\phi(v'))' + \lambda\, h(v+c)\, v' = \lambda\, b(t), \quad v(0) = v(T) = 0, \tag{8.72}$$

where $b(t) = g_0 + e(t)$ for a.e. $t \in [0,T]$, $g_0 = \inf\{g(x) : x \in [R,\infty)\}$ and $\lambda \in [0,1]$ is a parameter. For a given $\lambda \in [0,1]$ define an operator $\mathcal{F}_\lambda \colon C^1[0,T] \times \mathbb{R} \to C^1[0,T] \times \mathbb{R}$ by

$$\begin{aligned}\mathcal{F}_\lambda \colon (v,a) \to \Bigg(&\int_0^t \phi^{-1}\Big(a + \lambda \int_0^s [b(\tau) - h(v(\tau)+c)\, v'(\tau)]\,\mathrm{d}\tau\Big)\,\mathrm{d}s, \\ &a - \int_0^T \phi^{-1}\Big(a + \lambda \int_0^s [b(\tau) - h(v(\tau)+c)\, v'(\tau)]\,\mathrm{d}\tau\Big)\,\mathrm{d}s \Bigg).\end{aligned}$$

Taking into account that the second component of $\mathcal{F}_\lambda$ has a finite dimensional range and using an argument analogous to those applying to the proof of Lemma 8.6 (see also the proof of Theorem 7.4) we can show that the operator $\mathcal{F}_\lambda$ is completely continuous for each $\lambda \in [0,1]$. Furthermore, v is a solution of the Dirichlet problem (8.72) satisfying $\phi(v'(0)) = a$ if and only if $\mathcal{F}_\lambda(v,a) = (v,a)$.

Step 2. A priori estimates of fixed points of $\mathcal{F}_\lambda$.

Choose $\lambda \in (0,1]$ and assume that $(v,a) \in C^1[0,T] \times \mathbb{R}$ is a fixed point of the operator $\mathcal{F}_\lambda$. We have

$$(\phi(v'(t)))' + \lambda\, h(v(t)+c)\, v'(t) = \lambda\, b(t) \quad \text{for a.e. } t \in [0,T], \tag{8.73}$$

$v(0) = v(T) = 0$ and $\phi(v'(0)) = a$. Multiplying equality (8.73) by $v(t)$ and integrating over $[0,T]$, we get

$$-\int_0^T \phi(v'(t))\, v'(t)\, \mathrm{d}t = \lambda \int_0^T b(t)\, v(t)\, \mathrm{d}t. \tag{8.74}$$

Let $\alpha \in (0,\infty)$ be such that relation (8.70) holds. Then there are $k > 0$ and $y_0 > 0$ such that

$$\frac{\phi(|y|)}{|y|^\alpha} > \frac{k}{2} \quad \text{for } |y| \ge y_0.$$

Consequently, if we define $\beta(y) = \phi(y) - k\, y^\alpha$ for $y \ge 0$, then $\beta \in C[0,\infty)$ and

$$-\frac{\beta(y)}{y^\alpha} < \frac{k}{2} \quad \text{for } \; y \ge y_0. \tag{8.75}$$

Next, since ϕ is odd, we have $\phi(y)\, y \ge 0$ and $|\phi(y)| = \phi(|y|)$ for each $y \in \mathbb{R}$. In particular, $\phi(|y|)\, |y| = \phi(y)\, y$ for all $y \in \mathbb{R}$. Relation (8.74) can be now rewritten as

$$-k \|v'\|_{\alpha+1}^{\alpha+1} - \int_0^T \beta(|v'(t)|)\, |v'(t)|\, \mathrm{d}t = \lambda \int_0^T b(t)\, v(t)\, \mathrm{d}t. \tag{8.76}$$

Denote $J = \{t \in [0,T] : |v'(t)| \ge y_0\}$ and $M = \max\{\beta(y) : y \in [0,y_0]\}$ and assume that $\|v\|_\infty \geq 1$. Then relations (8.75) and (8.76) imply

$$\begin{aligned} k\|v'\|_{\alpha+1}^{\alpha+1} &\leq \|b\|_1\|v\|_\infty + My_0T - \int_J \frac{\beta(|v'(t)|)}{|v'(t)|^\alpha}\,|v'(t)|^{\alpha+1}\,\mathrm{d}t \\ &\leq \left(\|b\|_1 + My_0T\right)\|v\|_\infty + \frac{k}{2}\|v'\|_{\alpha+1}^{\alpha+1}, \end{aligned}$$

i.e.

$$\|v'\|_{\alpha+1}^{\alpha+1} \leq \frac{2}{k}\left(\|b\|_1 + My_0T\right)\|v\|_\infty.$$

Further, as the Hölder inequality yields

$$\|v\|_\infty \leq \int_0^T |v'(s)|\,\mathrm{d}s \leq T^{\frac{\alpha}{\alpha+1}}\|v'\|_{\alpha+1}, \tag{8.77}$$

we conclude that

$$\|v'\|_{\alpha+1} \leq \left(\frac{2}{k}\left(\|b\|_1 + My_0T\right)\right)^{\frac{1}{\alpha}} T^{\frac{1}{\alpha+1}}.$$

Now, using (8.77) once more, we get

$$\|v\|_\infty \leq T\left(\frac{2}{k}\left(\|b\|_1 + My_0T\right)\right)^{\frac{1}{\alpha}}.$$

Thus, including into our consideration also the case $\|v\|_\infty < 1$, we conclude that v satisfies the estimate

$$\|v\|_\infty < d := T\left(\frac{2}{k}\left(\|b\|_1 + My_0T\right)\right)^{\frac{1}{\alpha}} + 1. \tag{8.78}$$

As $v(0) = v(T)$, there is $\tau_0 \in (0,T)$ such that $v'(\tau_0) = 0$. Hence, integrating equality (8.73) we obtain

$$\phi(v'(t)) + \lambda\int_{v(\tau_0)}^{v(t)} h(x+c)\,\mathrm{d}x = \lambda\int_{\tau_0}^{t} b(s)\,\mathrm{d}s \quad \text{for } t \in [0,T],$$

wherefrom the estimate

$$|\phi(v'(t))| \leq \varkappa := \|b\|_1 + 2\,d\,\max\{|h(x)| \colon |x| \leq |c| + d\} \quad \text{for all } \ t \in [0,T]$$

follows. Consequently,

$$\|v'\|_\infty \le \phi^{-1}(\varkappa) \quad \text{and} \quad |a| = |\phi(v'(0))| \le \varkappa. \tag{8.79}$$

On the other hand, it is easy to see that $\mathcal{F}_0(v,a) = (v,a)$ if and only if $(v,a) = (0,0)$. This, together with (8.79), imply that if we choose

$$\rho > d + \phi^{-1}(\varkappa) + \varkappa,$$

we get $(v,a) \in \mathcal{B}(\rho)$, where

$$\mathcal{B}(\rho) = \{(v,a) \in C^1[0,T] \times \mathbb{R} : \|v\|_\infty + |a| < \rho\}.$$

Step 3. *Properties of the Leray-Schauder degree of* $\mathcal{F}_\lambda$.

By *Step* 2 and by the homotopy property from the Leray-Schauder degree theorem, where $\mathcal{H}(\lambda,x) = (\mathcal{I} - \mathcal{F}_\lambda)(x)$ and $\Omega = \mathcal{B}(\rho)$, we get

$$\deg(\mathcal{I} - \mathcal{F}_1, \mathcal{B}(\rho)) = \deg(\mathcal{I} - \mathcal{F}_0, \mathcal{B}(\rho)).$$

Moreover, $\mathcal{F}_0$ is an odd mapping, and hence by the Borsuk antipodal theorem we see that

$$\deg(\mathcal{I} - \mathcal{F}_0, \mathcal{B}(\rho)) \ne 0.$$

Therefore, by the existence property of the Leray-Schauder degree, we deduce that for each $c \in \mathbb{R}$ the operator $\mathcal{F}_1$ has a fixed point (v_c, a_c). It follows from the construction of the operator $\mathcal{F}_1$ that v_c is a solution of the auxiliary Dirichlet problem (8.72) with $\lambda = 1$ and $a_c = \phi(v_c'(0))$. Moreover, $\|v_c\|_\infty < d$ on $[0,T]$ holds due to (8.78).

Step 4. *Construction of an upper function* σ_2.

Put $c = R + d$ and $\sigma_2 = v_c + c$. Then $\sigma_2(0) = \sigma_2(T) = c$ and, due to (8.71), we have

$$\phi(\sigma_2'(T)) - \phi(\sigma_2'(0)) = T\,\overline{b} = T\,(g_0 + \overline{e}) \ge 0.$$

Furthermore, $\sigma_2(t) > c - d = R$ on $[0,T]$. Therefore, due to inequality (8.71),

$$\begin{aligned}(\phi(\sigma_2'(t)))' &= -h(\sigma_2(t))\,\sigma_2'(t) + g_0 + e(t)\\ &\le -h(\sigma_2(t))\,\sigma_2'(t) + g(\sigma_2(t)) + e(t) \quad \text{for a.e. } t \in [0,T].\end{aligned}$$

This shows that σ_2 is an upper function for (8.66), (8.2). $\square$

The following alternative assertion can be proved by an argument analogous to that used in the proof of the previous lemma.

Lemma 8.20. *Assume* (8.67) *and*

$$\limsup_{x\to\infty}\,[g(x)+\overline{e}]<0.$$

Then for an arbitrary $r\in(0,\infty)$, *problem* (8.66), (8.2) *possesses a lower function* σ_1 *such that* $\sigma_1>r$ *on* $[0,T]$.

A straightforward application of Theorem 8.12 and Lemma 8.19 gives the following result.

Theorem 8.21. *Assume* (8.67)–(8.70) *and let there exist* $r\in(0,\infty)$ *such that*

$$g(r)+e(t)\le 0 \qquad \textit{for a.e. } t\in[0,T].$$

Then problem (8.66), (8.2) *has a positive solution* u *such that* $u\ge r$ *on* $[0,T]$.

Proof. Let $r\in(0,\infty)$ be such that $g(r)+e(t)\le 0$ for a.e. $t\in[0,T]$. Then $\sigma_1(t)\equiv r$ is a lower function of problem (8.66), (8.2). Furthermore, due to assumption (8.70) and Lemma 8.19, problem (8.66), (8.2) has an upper function σ_2 such that $\sigma_2>r=\sigma_1>0$ on $[0,T]$. Thus, by Theorem 8.12 and Remark 8.13, problem (8.66), (8.2) has a positive solution u such that $u(t)\in[r,\sigma_2(t)]$ for each $t\in[0,T]$. $\square$

EXAMPLE. Let $g\in C(0,\infty)$ satisfy (8.68). Then we can guarantee the existence of a positive constant r for which the inequality $g(r)+e(t)\le 0$ holds a.e. on $[0,T]$ provided

$$\liminf_{x\to 0+}\,(g(x)+\|e\|_\infty)<0.$$

This occurs e.g. if $\sup\operatorname{ess}\{e(t)\colon t\in[0,T]\}<\infty$. In particular, Theorem 8.21 applies to problem (8.66), (8.2) if

$$\phi=\phi_p,\ p\in(1,\infty),\ \overline{e}>0,\ \sup\operatorname{ess}\{e(t)\colon t\in[0,T]\}<\infty$$

and $g(x) = -a\,x^{-\lambda_1} + b\,x^{\lambda_2}$, where $a, b, \lambda_1, \lambda_2 \in (0, \infty)$.

Further, notice that condition (8.70) is satisfied e.g. by

$$\phi(y) = (|y|\,y + y)\,\ln(1 + \frac{1}{|y|}) \quad \text{or} \quad \phi(y) = y\,(\exp(y^2) - 1).$$

8.3 Strong repulsive singular forces

In this section we study the singular problem (8.1), (8.2) with f having a *repulsive singularity* at $x = 0$. Recall that this means that, in addition to (8.3), the relation

$$\limsup_{x \to 0+} f(t, x, y) = \infty \quad \text{for a.e. } t \in [0, T] \text{ and some } y \in \mathbb{R}$$

is true. In general, in this case, the existence of a pair of associated lower and upper functions having the opposite order is typical. This causes that such a case is more difficult and more interesting than that of an attractive singularity.

The next assertion deals with equation (8.53) and is a direct corollary of Theorem 8.16.

Theorem 8.22. *Assume that $g \in C(0, \infty)$ and $p \in Car([0, T] \times \mathbb{R}^2)$ satisfy the strong repulsive singularity condition (8.54) and inequality (8.56) with some $m \in L_1[0, T]$. Furthermore, let there be a function $b \in L_1[0, T]$ and constants $r, A, B \in (0, \infty)$ such that*

$$\bar{b} \le 0, \;\; A > r, \;\; B - A \ge 2\,T\,\phi^{-1}\big(\|b\|_1\big),$$

$$g(r) + p(t, r, 0) \ge 0 \quad \textit{for a.e.} \;\; t \in [0, T]$$

and

$$\begin{cases} g(x) + p(t, x, y) \le b(t) \\ \quad \textit{for a.e. } t \in [0, T] \textit{ and all } x \in [A, B] \textit{ and } |y| \le \phi^{-1}(\|b\|_1). \end{cases}$$

Then problem (8.53), (8.2) has a positive solution u such that

$$u(t_u) \in [r, B] \quad \textit{for some} \;\; t_u \in [0, T].$$

Proof. By Lemma 8.17 (ii) there is a lower function σ_1 of problem (8.53), (8.2) such that $A \le \sigma_1 \le B$ on $[0,T]$. Moreover, by our assumptions, $\sigma_2(t) \equiv r$ is an upper function of problem (8.53), (8.2). Using Theorem 8.16 we complete the proof. □

In particular, Theorem 8.22 provides for the *Duffing equation* with the ϕ –Laplacian

$$(\phi(u'))' = g(u) + e(t) \tag{8.80}$$

the following immediate corollary.

Corollary 8.23. *Let* $e \in L_1[0,T]$, $\inf \operatorname{ess}\{e(t) : t \in [0,T]\} > -\infty$ *and let* $g \in C(0,\infty)$ *satisfy the strong repulsive singularity condition* (8.54). *Further, let*

$$g_* := \inf\{g(x) : x \in (0,\infty)\} > -\infty$$

and let there be $A > 0$ *such that*

$$g(x) + \overline{e} \le 0 \quad \textit{for} \quad x \in [A,B], \quad \textit{where} \quad B - A \ge 2\,T\,\phi^{-1}(\|e - \overline{e}\|_1).$$

Then problem (8.80), (8.2) *has a positive solution* u *such that* $u(t_u) \le B$ *for some* $t_u \in [0,T]$.

Proof. By the strong singularity condition (8.54) we have (8.55). Since, moreover, we assume $\inf \operatorname{ess}\{e(t) : t \in [0,T]\} > -\infty$, we can certainly find an $r \in (0,A)$ such that $g(r) + e(t) \ge 0$ for a.e. $t \in [0,T]$. The assertion then follows by Theorem 8.22 if we put $b(t) = e(t) - \overline{e}$ and $m(t) = g_* + e(t)$ a.e. on $[0,T]$. □

In the remaining part of the section we will consider the *Liénard equation*

$$(\phi_p(u'))' + h(u)\,u' = g(u) + e(t) \tag{8.81}$$

with the p-Laplacian $\phi_p(y) = |y|^{p-2}\,y$. To this aim, the following continuation type principle will be helpful.

Lemma 8.24. *Let* $p \in (1,\infty)$, $h \in C[0,\infty)$, $g \in C(0,\infty)$ *and* $e \in L_1[0,T]$. *Furthermore, assume that there exist* $r > 0$, $R > r$ *and* $R' > 0$ *such that*

(i) *the inequalities* $r < v < R$ *on* $[0,T]$ *and* $\|v'\|_\infty < R'$ *hold for each* $\lambda \in (0,1]$ *and for each positive solution* v *of the problem*

$$\begin{cases} (\phi_p(v'))' = \lambda\,(-h(v)\,v' + g(v) + e(t)), \\ v(0) = v(T), \quad v'(0) = v'(T), \end{cases} \tag{8.82}$$

(ii) $(g(x) + \overline{e} = 0) \implies r < x < R,$

(iii) $(g(r) + \overline{e})\,(g(R) + \overline{e}) < 0.$

Then problem (8.81), (8.2) *has at least one solution* u *such that* $r < u < R$ *on* $[0,T]$.

Proof. *Step* 1. *Construction of the operator* $\mathcal{F}_\lambda$.

First, notice that integrating the differential equation in (8.82) over the interval $[0,T]$ and taking into account the periodicity conditions we arrive at

$$\begin{cases} 0 = \displaystyle\int_0^T g(v(s))\,\mathrm{d}s + T\,\overline{e} \\ \qquad \text{for all solutions } u \text{ of problem (8.82).} \end{cases} \tag{8.83}$$

Let us consider the problems

$$(\phi_p(v'))' = f_\lambda(t,v)(t), \quad v(0) = v(T),\ v'(0) = v'(T), \tag{8.84}$$

where $\lambda \in [0,1]$ and

$$f_\lambda(t,v)(t) = \lambda\,\big(-h(v(t))\,v'(t) + g(v(t)) + e(t)\big) + (1-\lambda)\,w_0(v),$$

$$w_0(v) = \frac{1}{T}\left(\int_0^T g(v(s))\,\mathrm{d}s + T\,\overline{e}\right)$$

for $v \in C^1[0,T]$ and for a.e. $t \in [0,T]$. Due to (8.83), we can see that for each $\lambda \in [0,1]$ problems (8.82) and (8.84) are equivalent. Furthermore, for $\lambda = 1$ problem (8.84) reduces to problem (8.81), (8.2) (with v instead of u).

As in the proof of Theorem 7.4 (see also the introduction to Lemma 8.6 in Section 8.1), we denote by γ the functional on $C[0,T]$ which is uniquely determined by the relation

$$\int_0^T \phi^{-1}(\gamma(\ell) + \ell(s))\,\mathrm{d}s = 0. \tag{8.85}$$

Similarly, the operator $\mathcal{K}\colon C[0,T] \to C^1[0,T]$ is defined by (8.10), i.e.

$$(\mathcal{K}(\ell))(t) = \int_0^t \phi^{-1}(\gamma(\ell) + \ell(s))\,\mathrm{d}s.$$

Recall that both γ and $\mathcal{K}$ are continuous. Now, for $\lambda \in [0,1]$, we define operators $\mathcal{N}_\lambda\colon C_1[0,T] \to C[0,T]$ and $\mathcal{F}_\lambda\colon C^1[0,T] \to C^1[0,T]$ by

$$\begin{cases} \mathcal{N}_\lambda(u)(t) = \displaystyle\int_0^t f_\lambda(s,u)(s)\,\mathrm{d}s, \\ \text{and} \\ \mathcal{F}_\lambda(u)(t) = u(0) + u'(0) - u'(T) + \mathcal{K}(\mathcal{N}_\lambda(u)). \end{cases} \tag{8.86}$$

Arguing as in the proof of Lemma 8.6, we can show that for each $\lambda \in [0,1]$ the operator $\mathcal{F}_\lambda$ is completely continuous. Moreover, a function $v \in C^1[0,T]$ solves problem (8.84) if and only if it is a fixed point of $\mathcal{F}_\lambda$. In particular, $u \in C^1[0,T]$ is a solution of (8.81), (8.2) if and only if $\mathcal{F}_1(u) = u$.

Step 2. *Properties of the fixed points of* $\mathcal{F}_\lambda$.

We state that

$$\mathcal{F}_\lambda(v) \neq v \quad \text{for all } \lambda \in [0,1] \text{ and } v \in \partial\Omega, \tag{8.87}$$

where

$$\Omega = \{v \in C^1[0,T] \colon r < v < R \text{ and } |v'| < R' \text{ on } [0,T]\}.$$

Indeed, if $\lambda > 0$, then relation (8.87) follows from assumption (i), while for $\lambda = 0$ it is a corollary of the following claim.

CLAIM. $v \in C^1[0,T]$ is a fixed point of $\mathcal{F}_0$ if and only if there is $x \in (r,R)$ such that $v(t) \equiv x$ on $[0,T]$ and

$$g(x) + \overline{e} = 0. \tag{8.88}$$

Proof of CLAIM. For each $v \in C^1[0,T]$ and each $t \in [0,T]$ we have $f_0(t,v)(t) = w_0(v)$ and $(\mathcal{N}_0(v))(t) = t\,w_0(v)$. Let $c \in \mathbb{R}$. If $w_0(v) \neq 0$, then

$$\int_0^T \phi_p^{-1}(c + \mathcal{N}_0(v)(t))\,\mathrm{d}t = \int_0^T \phi_q(c + t\,w_0(v))\,\mathrm{d}t = \frac{|c + T w_0(v)|^q - |c|^q}{q\,w_0(v)},$$

where $q = \frac{p}{p-1}$. In particular,

$$\int_0^T \phi_p^{-1}\,(c+\mathcal{N}_0(v)(t))\,\mathrm{d}t = 0 \quad \text{if and only if} \quad c = -\tfrac{T}{2}\,w_0(v).$$

On the other hand, if $w_0(v) = 0$, then

$$\int_0^T \phi_p^{-1}\,(c+\mathcal{N}_0(v)(t))\,\mathrm{d}t = T\,\phi_p^{-1}(c) = 0 \;\; \text{if and only if} \;\; c = 0.$$

Since $\gamma(\mathcal{N}_0(v))$ is the only solution of equation (8.85) with $\ell = \mathcal{N}_0(v)$, we can summarize that

$$c = \gamma\,(\mathcal{N}_0(v)) = -\tfrac{T}{2}\,w_0(v) \quad \text{for} \quad v \in C^1[0,T].$$

Inserting this into the definition of $\mathcal{F}_0$, we get

$$\begin{aligned}\mathcal{F}_0(v)(t) &= v(0) + v'(0) - v'(T) + \int_0^t \phi_p^{-1}\left(w_0(v)\left(s - \tfrac{T}{2}\right)\right)\mathrm{d}s\\ &= v(0) + v'(0) - v'(T) + \tfrac{1}{q}\,\phi_q\big(w_0(v)\big)\left(\left|t - \tfrac{T}{2}\right|^q - \left(\tfrac{T}{2}\right)^q\right).\end{aligned}$$

Consequently, $v \in C^1[0,T]$ is a fixed point of $\mathcal{F}_0$ if and only if

$$v(t) = v(0) + v'(0) - v'(T) + \tfrac{1}{q}\,\phi_q(w_0(v))\left(\left|t-\tfrac{T}{2}\right|^q - \left(\tfrac{T}{2}\right)^q\right) \quad \text{for } t \in [0,T].$$

In particular, for $t = 0$, this relation reduces to $v(0) = v(0) + v'(0) - v'(T)$, which yields $v'(0) = v'(T)$. Similarly, inserting $t = T$ gives $v(T) = v(0)$. On the other hand,

$$v'(t) = \phi_q\big(w_0(v)\big)\left|t-\tfrac{T}{2}\right|^{q-1}\operatorname{sign}\left(t-\tfrac{T}{2}\right) \quad \text{for} \quad t \neq \tfrac{T}{2}$$

and

$$v'(T) - v'(0) = 2\,\phi_q\big(w_0(v)\big)\left(\tfrac{T}{2}\right)^{q-1}.$$

Thus, $v'(0) = v'(T)$ can hold if and only if $w_0(v) = 0$, which gives $v(t) \equiv v(0)$ on $[0,T]$. Denoting $x = v(0)$, we can see that $w_0(v) = 0$ if and only if $g(x)+\overline{e} = 0$. However, by assumption (ii), any $x \in \mathbb{R}$ satisfying this equation must belong to the interval (r,R). On the other hand, if $g(x) + \overline{e} = 0$ and $v \equiv x$ on $[0,T]$, then v is obviously a fixed point of $\mathcal{F}_0$. This completes the proof of the claim. $\diamondsuit$

Step 3. *Properties of the Leray-Schauder degree of* $\mathcal{F}_\lambda$.

By (8.87) and by the homotopy property of the degree we have

$$\deg(\mathcal{I}-\mathcal{F}_1,\Omega)=\deg(\mathcal{I}-\mathcal{F}_0,\Omega). \tag{8.89}$$

Denote

$$\mathbb{X}=\{v\in C^1[0,T]\colon v(t)\equiv v(0) \text{ on } [0,T]\} \quad\text{and}\quad \Omega_0=\Omega\cap\mathbb{X}.$$

Then $\Omega_0=\{v\in\mathbb{X}\colon r<v(0)<R\}$ and, by CLAIM in *Step* 2, each fixed point of $\mathcal{F}_0$ belongs to Ω_0. Consequently, the excision property of the topological degree yields

$$\deg(\mathcal{I}-\mathcal{F}_0,\Omega)=\deg(\mathcal{I}-\mathcal{F}_0,\Omega_0). \tag{8.90}$$

Step 4. *Construction and properties of the operator* $\widetilde{\mathcal{F}}_\mu$.

For $\mu\in[0,1]$ define $\widetilde{\mathcal{F}}_\mu\colon\mathbb{X}\to C^1[0,T]$ by

$$\widetilde{\mathcal{F}}_\mu(v)(t)=v(0)+\phi_q\Big(g(v(0))+\overline{e}\Big)\left[1-\mu+\tfrac{\mu}{q}\left(\left|t-\tfrac{T}{2}\right|^q-\left(\tfrac{T}{2}\right)^q\right)\right].$$

We have

$$\widetilde{\mathcal{F}}_0(v)=v(0)+\phi_q\left(g(v(0))+\overline{e}\right) \quad\text{and}\quad \widetilde{\mathcal{F}}_1(v)=\mathcal{F}_0(v) \quad\text{for}\quad v\in\mathbb{X}.$$

Similarly to $\mathcal{F}_\lambda$, the operators $\widetilde{\mathcal{F}}_\mu$, $\mu\in[0,1]$, are also completely continuous. By CLAIM in *Step* 2, $\widetilde{\mathcal{F}}_1(v)\neq v$ for all $v\in\partial\Omega_0$. Let i be the natural isometrical isomorphism $\mathbb{R}\to\mathbb{X}$, i.e.

$$i(x)(t)\equiv x \quad\text{for}\quad x\in\mathbb{R} \quad\text{and}\quad i^{-1}(v)=v(0) \quad\text{for}\quad v\in\mathbb{X}.$$

Assume that $\mu\in[0,1), x\in\mathbb{R}, v=i(x)$ and $\widetilde{\mathcal{F}}_\mu(v)=v$. Then

$$\phi_q\left(g(x)+\overline{e}\right)\left[1-\mu+\tfrac{\mu}{q}\left(\left|t-\tfrac{T}{2}\right|^q-\left(\tfrac{T}{2}\right)^q\right)\right]=0 \quad\text{for all}\quad t\in[0,T].$$

If $t=0$, this relation reduces to $g(x)+\overline{e}=0$, which is due to assumption (ii) possible only if $x\in(r,R)$. To summarize,

$$\widetilde{\mathcal{F}}_\mu(v)\neq v \quad\text{for all } v\in\partial\,\Omega_0 \quad\text{and all}\quad \mu\in[0,1].$$

Therefore, using the homotopy property of the degree and taking into account that $\dim \mathbb{X} = 1$, we conclude that

$$\deg(\mathcal{I} - \mathcal{F}_0, \Omega_0) = \deg(\mathcal{I} - \widetilde{\mathcal{F}}_1, \Omega_0) = \mathrm{d}_B(\mathcal{I} - \widetilde{\mathcal{F}}_0, \Omega_0), \tag{8.91}$$

where $\mathrm{d}_B(\mathcal{I} - \widetilde{\mathcal{F}}_0, \Omega_0)$ stands for the Brouwer degree of $I - \widetilde{\mathcal{F}}_0$ with respect to Ω_0.

Step 5. *The Brouwer degree of* $\mathcal{I} - \widetilde{\mathcal{F}}_0$.

Define $\Phi \colon \mathbb{R} \to \mathbb{R}$ by $\Phi(x) = g(x) + \overline{e}$. Then

$$(\mathcal{I} - \widetilde{\mathcal{F}}_0)(i(x)) = i(\Phi(x)) \quad \text{for each} \quad x \in \mathbb{R}.$$

In other words, $\Phi = i^{-1} \circ (\mathcal{I} - \widetilde{\mathcal{F}}_0) \circ i$. Consequently, by Remark C.4, we have

$$\mathrm{d}_B(\mathcal{I} - \widetilde{\mathcal{F}}_0, \Omega_0) = \mathrm{d}_B(\Phi, (r, R)). \tag{8.92}$$

Put

$$\Psi(x) = \Phi(r)\,\frac{R - x}{R - r} + \Phi(R)\,\frac{x - r}{R - r}.$$

Then Ψ has a unique zero $x_0 \in (r, R)$ and

$$\Psi'(x_0) = \frac{\Phi(R) - \Phi(r)}{R - r}.$$

Hence, by the definition of the Brouwer degree in $\mathbb{R}$ we have

$$\mathrm{d}_B(\Psi, (r, R)) = \operatorname{sign} \Psi'(x_0) = \operatorname{sign}\left(\Phi(R) - \Phi(r)\right).$$

By the homotopy property and thanks to our assumption (iii), we conclude that

$$\mathrm{d}_B(\Phi, (r, R)) = \mathrm{d}_B(\Psi, (r, R)) = \operatorname{sign}\left(\Phi(R) - \Phi(r)\right) \neq 0. \tag{8.93}$$

Step 6. *Fixed point of* $\mathcal{F}_1$.

To summarize, by (8.89)–(8.93) we have

$$\deg(\mathcal{I} - \mathcal{F}_1, \Omega) \neq 0,$$

which, in view of the existence property of the topological degree, shows that $\mathcal{F}_1$ has a fixed point $u \in \Omega$. By *Step* 1 this means that problem (8.81), (8.2) has a solution. ☐

Lemma 8.24 enables us to prove the following result, where we meet the symbol π_p defined for $p \in (1, \infty)$ by

$$\pi_p = \frac{2\,\pi\,(p-1)^{\frac{1}{p}}}{p\,\sin\left(\frac{\pi}{p}\right)}.$$

Clearly $\pi_2 = \pi$. Furthermore, $(\frac{\pi_p}{T})^p$ is the first eigenvalue of the *quasilinear Dirichlet problem*

$$(\phi_p(u'))' + \lambda\,\phi_p(u) = 0, \quad u(0) = u(T) = 0$$

(see Appendix D).

Theorem 8.25. *Assume that* $p \in (1, \infty)$, $h \in C[0, \infty)$, $e \in L_1[0, T]$. *Furthermore, let* $g \in C(0, \infty)$ *satisfy the strong repulsive singularity condition* (8.54) *and conditions*

$$\liminf_{x \to 0+} [g(x) + \overline{e}\,] > 0 > \limsup_{x \to \infty} [g(x) + \overline{e}\,] \tag{8.94}$$

and

$$\begin{cases} \text{there exist nonnegative constants } a,\,\gamma \text{ such that } a < \left(\frac{\pi_p}{T}\right)^p \\ \quad \text{and} \quad g(x)\,x \ge -(a\,x^p + \gamma) \quad \text{for all } x > 0. \end{cases} \tag{8.95}$$

Then problem (8.81), (8.2) *has a positive solution.*

Proof. We will verify that the assumptions of Lemma 8.24 are satisfied.

Step 1. *One-point estimate.*

First, we will show that

$$\begin{cases} \text{there are } R_0 > 0 \text{ and } R_1 > R_0 \text{ such that} \\ \qquad v(t_v) \in (R_0, R_1) \quad \text{for some } t_v \in [0, T] \\ \text{holds for each } \lambda \in (0, 1] \text{ and each positive solution } v \text{ of (8.82)}. \end{cases} \tag{8.96}$$

So, assume that $\lambda \in (0,1]$ and that v is a positive solution to the auxiliary problem (8.82). By the first inequality in assumption (8.94), there is $R_0 > 0$ such that

$$g(x) + \overline{e} > 0 \quad \text{whenever } x \in (0, R_0). \tag{8.97}$$

If $g(v(t)) + \overline{e} > 0$ were valid on $[0,T]$, we would have

$$\int_0^T (g(v(t)) + e(t))\,\mathrm{d}t = \int_0^T (g(v(t)) + \overline{e})\,\mathrm{d}t > 0,$$

which contradicts (8.83). This shows that $\max\{v(t) \colon t \in [0,T]\} > R_0$.

Similarly, by the second inequality in assumption (8.94), there is $R_1 > R_0$ such that

$$g(x) + \overline{e} < 0 \quad \text{whenever } x > R_1 \tag{8.98}$$

and $\min\{v(t) \colon t \in [0,T]\} < R_1$. This proves (8.96).

Step 2. Upper estimate of solutions to the auxiliary problem (8.82).

We claim that

$$\begin{cases} \text{there is } R > 0 \text{ such that} \\ \qquad v < R \text{ on } [0,T] \\ \text{holds for each } \lambda \in (0,1] \text{ and each positive solution } v \text{ of } (8.82). \end{cases} \tag{8.99}$$

Indeed, assume that $\lambda \in (0,1]$ and v is a positive solution to the auxiliary problem (8.82). Multiplying the differential equation in (8.82) by $v(t)$ and integrating over $[0,T]$ we get

$$-\|v'\|_p^p = \int_0^T g(v(s))\,v(s)\,\mathrm{d}s + \int_0^T e(s)\,v(s)\,\mathrm{d}s,$$

and using assumption (8.95) we arrive at the inequality

$$\|v'\|_p^p \le a\|v\|_p^p + \|e\|_1\|v\|_\infty + \gamma T. \tag{8.100}$$

Further, by (8.96) we have

$$0 < v(t) = v(t_v) + \int_{t_v}^t v'(s)\,\mathrm{d}s < R_1 + T^{\frac{1}{q}}\|v'\|_p \quad \text{for } t \in [0,T], \tag{8.101}$$

where $q = \frac{p}{p-1}$. Now put

$$y(t) = \begin{cases} v(t+t_v) - v(t_v) & \text{if } 0 \le t \le T - t_v, \\ v(t+t_v-T)) - v(t_v) & \text{if } T - t_v \le t \le T. \end{cases}$$

Since $y \in C^1[0,T]$, $y(0) = y(T) = 0$ and $\|y+v(t_v)\|_p^p = \|v\|_p^p$, we can apply the sharp Poincaré inequality (see Lemma D.2) to show that

$$\|y\|_p \le \tfrac{T}{\pi_p}\|y'\|_p = \tfrac{T}{\pi_p}\|v'\|_p.$$

Now, we can see that for arbitrary positive numbers ε and c_0 we can always find a positive constant c_2 such that $(x+c_0)^p \le (1+\varepsilon)\,x^p + c_2$ holds for each $x \ge 0$. Indeed, the inequality $(x+c_0)^p < (1+\varepsilon)\,x^p$ holds whenever $x > x_0 := c_0\,((1+\varepsilon)^{1/p}-1)^{-1}$ and the expression $|(x+c_0)^p - (1+\varepsilon)\,x^p|$ is certainly bounded on the interval $[0, x_0]$. As a result, we can state that for an arbitrary $\varepsilon > 0$ there is $c_1 > 0$ such that

$$\|v\|_p^p \le \left(\|y\|_p + v(t_v)\,T^{\frac{1}{p}}\right)^p \le (1+\varepsilon)\left(\tfrac{T}{\pi_p}\right)^p \|v'\|_p^p + c_1.$$

Inserting this into inequality (8.100), choosing $\varepsilon \in \big(0, \frac{1}{a}\,(\frac{\pi_p}{T})^p - 1\big)$ and having in mind estimate (8.101), we deduce that we can choose $c_2 > 0$ such that

$$\alpha\,\|v'\|_p^p \le T^{\frac{1}{q}}\,\|e\|_1\,\|v'\|_p + c_2$$

holds with

$$\alpha = \left(1 - a(1+\varepsilon)\left(\tfrac{T}{\pi_p}\right)^p\right) > 0.$$

However, this is possible only if there is $R_p \in (0,\infty)$ independent of λ and v and such that $\|v'\|_p < R_p$. Therefore

$$0 < v(t) < R_1 + T^{\frac{1}{q}}\,R_p + 1 \quad \text{on } [0,T]$$

for each $\lambda \in (0,1]$ and each positive solution v of (8.82), i.e., statement (8.99) is true with $R = R_1 + T^{\frac{1}{q}}\,R_p + 1$.

Step 3. *Estimate of the derivatives of solutions to problem* (8.82).

Now we show that

$$\begin{cases} \text{there is } R' > 0 \text{ such that} \\ \qquad |v'| < R' \text{ on } [0,T] \\ \text{for each } \lambda \in (0,1] \text{ and each positive solution } v \text{ of (8.82).} \end{cases} \tag{8.102}$$

Let $\lambda \in (0,1]$ and let v be a positive solution to the auxiliary problem (8.82). In particular, we have $v(0) = v(T)$ and, therefore, there is $t' \in [0,T]$ such that $v'(t') = 0$. Integrating the differential equation in (8.82) over the interval $[t', t]$ and taking into account statement (8.99), we obtain

$$\begin{cases} |v'(t)|^{p-1} \le \lambda \left(\displaystyle\int_0^R |h(x)|\,\mathrm{d}x + \|e\|_1 + \left| \int_{t'}^t |g(v(s))|\,\mathrm{d}s \right| \right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \in [0,T]. \end{cases} \tag{8.103}$$

Thanks to assumption (8.94), we can choose $b > 0$ in such a way that $\inf\{g(x) : x \in (0,R]\} \ge -b$ and, by (8.99), also $g(v(t)) \ge -b$ on $[0,T]$. Therefore, $|g(v(t))| \le g(v(t)) + 2\,b$ holds for all $t \in [0,T]$. From this inequality, using (8.83), we deduce that

$$\left| \int_{t'}^t |g(v(s))|\,\mathrm{d}s \right| \le 2\,b\,T + \|e\|_1,$$

which inserted into (8.103) yields (8.102) with

$$R' = \left(\int_0^R |h(x)|\,\mathrm{d}x + 2\,(b\,T + \|e\|_1) \right)^{\frac{1}{p-1}} > 0.$$

Step 4. *Lower estimate of solutions to problem* (8.82).

Choose $\lambda \in (0,1]$ and let v be a positive solution of problem (8.82). Put

$$H = \max\{|h(x)| \colon x \in [0,R]\} \text{ and } K = R'^2\,T\,H + \int_{R_0}^R |g(x)|\,\mathrm{d}x + R'\|e\|_1.$$

By (8.54) there is $r \in (0, R_0)$ such that

$$\int_x^{R_0} g(x)\,\mathrm{d}x > K \quad \text{for all } x \in (0,r]. \tag{8.104}$$

Let $t_1, t_2 \in [0,T]$ be such that

$$v(t_1) = \min\{v(t) : t \in [0,T]\} \text{ and } v(t_2) = \max\{v(t) : t \in [0,T]\}.$$

In view of (8.2) we have $v'(t_1) = v'(t_2) = 0$. Denote $w(t) = \phi_p(v'(t))$ for $t \in [0,T]$. Then $v'(t) = \phi_p^{-1}(w(t))$ on $[0,T]$ and $w(t_1) = w(t_2) = \phi_p(0) = 0$. Let, as before, $q = \frac{p}{p-1}$. Then $\phi_q = \phi_p^{-1}$ and we have also

$$\int_{t_1}^{t_2} (\phi_p(v'(t)))'\, v'(t)\, \mathrm{d}t = \int_{t_1}^{t_2} w'(t)\, \phi_q(w(t))\, \mathrm{d}t = \int_{w(t_1)}^{w(t_2)} \phi_q(x)\, \mathrm{d}x = 0.$$

Thus, multiplying the differential equation in (8.82) by $v'(t)$ and integrating from t_1 to t_2 yields

$$0 = -\int_{t_1}^{t_2} h(v(t))\, v'^2(t)\, \mathrm{d}t + \int_{v(t_1)}^{R_0} g(x)\, \mathrm{d}x + \int_{R_0}^{v(t_2)} g(x)\, \mathrm{d}x + \int_{t_1}^{t_2} e(t)\, v'(t)\, \mathrm{d}t.$$

It follows that

$$\int_{v(t_1)}^{R_0} g(x)\, \mathrm{d}x \le R'^2\, T\, H + \int_{R_0}^{R} |g(x)|\, \mathrm{d}x + R' \|e\|_1,$$

which is, owing to (8.104), possible only when $v(t_1) > r$.

Step 5. *Final conclusion.*

To summarize, there are r, R and R' such that assumption (i) from Lemma 8.24 is satisfied. Furthermore, since by *Step* 1 we have

$$g(x) + \overline{e} > 0 \quad \text{if } 0 < x < R_0 \quad \text{and} \quad g(x) + \overline{e} < 0 \quad \text{if } x > R_1$$

and $0 < r < R_0 < R_1 < R$, it is easy to see that also assumptions (ii) and (iii) of Lemma 8.24 are satisfied. Hence, applying Lemma 8.24, we complete the proof of the theorem. □

The following two results are consequences of Theorem 8.25 and its proof.

Corollary 8.26. *Let all assumptions of Theorem 8.25 be satisfied but with (8.95) replaced by*

$$\liminf_{x \to \infty} \frac{g(x)}{x^{p-1}} > -\left(\tfrac{\pi_p}{T}\right)^p.$$

Then problem (8.81), (8.2) has a positive solution.

Proof. Let

$$\liminf_{x\to\infty} \frac{g(x)}{x^{p-1}} > -a > -\left(\tfrac{\pi_p}{T}\right)^p.$$

Then there exists $A > 0$ such that

$$g(x)\,x \ge -a\,x^p \quad \text{for all} \quad x \ge A.$$

Furthermore, by (8.94) we have $g_* = \inf\{g(x) : x \in (0, A)\} > -\infty$. Therefore, $g(x)\,x \ge -|g_*|\,A > -\infty$ for all $x \in (0, A)$. So, we can summarize that condition (8.95) is satisfied. The proof is completed by means of Theorem 8.25. □

Corollary 8.27. *Let all assumptions of Theorem 8.25 be satisfied but with (8.95) replaced by*

$$e \in L_2[0, T] \ \textit{and} \ \ h(x) \ge h_* > 0 \ (\textit{or} \ h(x) \le -h_* < 0) \ \textit{for all} \ \ x \in [0, \infty).$$

Then problem (8.81), (8.2) has a positive solution.

Proof. Assume that the *dissipativity condition*

$$h(x) \ge h_* > 0 \quad \text{for all } x \in [0, \infty)$$

is satisfied. Then the proof is analogous to that of Theorem 8.25, just the estimate (8.99) is now obtained more easily. Indeed: let $\lambda \in (0, 1]$ and let v be a positive solution of (8.82). Let R_0, R_1 and t_v be found as in (8.96), i.e., R_0 is such that (8.97) is true, $R_1 > R_0$, $g(x) + \overline{e} < 0$ for $x \ge R_1$ and $v(t_v) \in (R_0, R_1)$. Put $w(t) = \phi(v'(t))$ for $t \in [0, T]$. Then $v'(t) = \phi^{-1}(w(t))$ on $[0, T]$, $w(0) = \phi(v'(0)) = \phi(v'(T)) = w(T)$ and

$$\int_0^T (\phi(v'(s)))'\, v'(s)\, \mathrm{d}s = \int_0^T w'(s)\, \phi^{-1}(w(s))\, \mathrm{d}s = \int_{w(0)}^{w(T)} \phi^{-1}(y)\, \mathrm{d}y = 0.$$

Thus, multiplying the differential equation in (8.82) by v' and integrating over the interval $[0, T]$, we obtain $h_* \|v'\|_2 \le \|e\|_2$ and, consequently,

$$v(t) = v(t_v) + \int_{t_v}^t v'(s)\, \mathrm{d}s < R_1 + \sqrt{T}\, \frac{\|e\|_2}{h_*} + 1 \quad \text{for all } t \in [0, T].$$

Thus, (8.99) is true with $R = R_1 + \sqrt{T}\, \frac{\|e\|_2}{h_*} + 1$. Now, we can repeat *Steps* 3–5 of the proof of Theorem 8.25. □

EXAMPLES. (i) Clearly, if $g \in C(0,\infty)$ fulfils condition (8.94) and, in addition, also $\liminf_{x\to\infty} g(x) > -\infty$, it satisfies also condition (8.95) and, hence, in such a case Theorem 8.25 ensures the existence of a positive solution to problem (8.81), (8.2). In particular, Theorem 8.25 implies that problem (8.81), (8.2) with $g(x) = \beta\, x^{-\alpha}$ on $(0,\infty)$, $\beta > 0$, $\alpha \ge 1$, $h \in C[0,\infty)$ and $e \in L^1[0,T]$ has a positive solution if $\overline{e} < 0$. Moreover, integrating both sides of the differential equation in (8.81) over $[0,T]$ and taking into account that g is positive on $(0,\infty)$, we can see that the condition $\overline{e} < 0$ is also necessary for the existence of a positive solution to (8.81), (8.2).

(ii) Let $p \in (1,\infty)$, $h \in C[0,\infty)$, $0 < a < (\frac{\pi_p}{T})^p$, $\beta > 0$ and $\alpha \ge 1$. Then, by Corollary 8.26, the problem

$$\begin{aligned}&(|u'|^{p-2}\, u')' + h(u)\, u' = -a\, u^{p-1} + \frac{\beta}{u^\alpha} + \sin u + e(t),\\ &u(0) = u(T), \quad u'(0) = u'(T)\end{aligned}$$

has a positive solution for each $e \in L_1[0,T]$.

Similarly, if in addition $p > 2$, m is the integer part of $p-2$ and

$$g(x) = -a\, x^{p-1} + \sum_{i=0}^{m} c_i\, x^i + \frac{\beta}{x^\alpha} \quad \text{for} \quad x > 0,$$

then, by Corollary 8.26, problem (8.81), (8.2) has a positive solution for arbitrary coefficients $c_i \in \mathbb{R}$, $i = 0,1,\dots,m$, and each $e \in L_1[0,T]$.

(iii) Let $p \in (1,\infty)$, $c \ne 0$, $a > 0$, $\beta > 0$, $\alpha \ge 1$. Then, by Corollary 8.27, the problem

$$(|u'|^{p-2}\, u')' + c\, u' = \frac{\beta}{u^\alpha} - a\, \exp(u) + e(t), \quad u(0) = u(T),\ u'(0) = u'(T)$$

has a solution for each $e \in L_2[0,T]$.

8.4 Weak repulsive singular forces

Here, unlike the previous section, we do not assume the strong singularity condition. We will restrict ourselves to the case that f does not depend on u', i.e., we consider the equation

$$(\phi_p(u'))' = f(t,u), \tag{8.105}$$

where $f \in Car([0,T] \times (0,\infty))$ can have a *weak repulsive singularity* at the origin, i.e.

$$\limsup_{x \to 0+} f(t,x) = \infty \qquad \text{for a.e. } t \in [0,T]$$

can hold.

The next existence principle relies on the comparison of the given problem with the related quasilinear problem fulfilling the antimaximum principle.

Theorem 8.28. *Let* $f \in Car([0,T] \times (0,\infty))$ *and* $p \in [2,\infty)$. *Further, let* $r \in (0,\infty)$, $A \in [r,\infty)$ *and* $\mu \in L_1[0,T]$, $\beta \in L_1[0,T]$ *be such that* $\mu(t) \ge 0$ *for a.e.* $t \in [0,T]$, $\overline{\mu} > 0$, $\overline{\beta} \le 0$,

$$f(t,x) \le \beta(t) \text{ for a.e. } t \in [0,T] \text{ and all } x \in [A,B] \tag{8.106}$$

and

$$f(t,x) + \mu(t)\,\phi_p(x-r) \ge 0 \text{ for a.e. } t \in [0,T] \text{ and all } x \in [r,B], \tag{8.107}$$

where

$$\begin{cases} B - A \ge \frac{T}{2}\,\phi_p^{-1}(\|m\|_1), \\ m(t) = \max\big\{ \sup\{f(t,x) : x \in [r,A]\}, \beta(t), 0 \big\} \text{ for a.e. } t \in [0,T] \end{cases} \tag{8.108}$$

and

$$\begin{cases} v \ge 0 \text{ on } [0,T] \text{ holds for each } v \in C^1[0,T] \text{ such that} \\ \qquad \phi_p(v') \in AC[0,T], \\ \qquad (\phi_p(v'(t)))' + \mu(t)\,\phi_p(v(t)) \ge 0 \quad \text{for a.e. } t \in [0,T], \\ \qquad v(0) = v(T), \quad v'(0) = v'(T). \end{cases} \tag{8.109}$$

Then problem (8.105), (8.2) *has a solution* u *such that*

$$r \le u \le B \quad \text{on } [0,T] \quad \text{and} \quad \|u'\|_\infty < \phi_p^{-1}(\|m\|_1). \tag{8.110}$$

Proof. *Part* I. First, assume that $\overline{\beta} < 0$.

Step 1. *Upper and lower functions of an auxiliary regular problem.*

Put

$$\widetilde{f}(t,x) = \begin{cases} f(t,r) - \mu(t)\,\phi_p(x-r) & \text{if } \ x \le r, \\ f(t,x) & \text{if } \ x \in [r,B], \\ f(t,B) & \text{if } \ x \ge B \end{cases} \tag{8.111}$$

and consider an auxiliary problem

$$(\phi_p(u'))' = \widetilde{f}(t,u), \quad u(0) = u(T), \quad u'(0) = u'(T). \tag{8.112}$$

We have $\widetilde{f} \in Car([0,T] \times \mathbb{R})$. Furthermore, by (8.106), (8.107) and (8.111), the inequalities

$$\widetilde{f}(t,x) \le \beta(t) \quad \text{if } \ x \in [A,\infty) \tag{8.113}$$

and

$$\widetilde{f}(t,x) + \mu(t)\,\phi_p(x-r) \ge 0 \quad \text{for all } \ x \in \mathbb{R} \tag{8.114}$$

are valid for a.e. $t \in [0,T]$. In particular, in view of (8.111) we have

$$\widetilde{f}(t,x) \ge h(t) := -\mu(t)\,\phi_p(B-r) \ \text{ for a.e. } t \in [0,T] \text{ and all } x \in \mathbb{R}, \tag{8.115}$$

with $h \in L_1[0,T]$.

By (8.114), $\sigma_2 \equiv r$ is an upper function of (8.112). Further, if $b = \beta - \overline{\beta}$, then $b \in L_1[0,T]$ and $\overline{b} = 0$ and, similarly to the proofs of Lemma 8.6 or of Theorem 7.4, we can see that there is a uniquely defined $\sigma_0 \in C^1[0,T]$ such that $\phi_p(\sigma_0') \in AC[0,T]$,

$$(\phi_p(\sigma_0'(t)))' = b(t) \quad \text{for a.e. } \ t \in [0,T] \quad \text{and} \quad \sigma_0(0) = \sigma_0(T) = 0.$$

Now, let us choose $c^* > 0$ such that $c^* + \sigma_0 \ge A$ on $[0,T]$ and define $\sigma_1 = c^* + \sigma_0$. We have $\sigma_1(0) = \sigma_1(T) = c^*$, $\phi_p(\sigma_0'(T)) - \phi_p(\sigma_0'(0)) = T\,\overline{b} = 0$ and, by (8.113),

$$(\phi_p(\sigma_1'(t)))' = b(t) = \beta(t) - \overline{\beta} > \beta(t) \ge \widetilde{f}(t,\sigma_1(t)) \text{ for a.e. } t \in [0,T].$$

Consequently, σ_1 is a lower function of (8.112). Therefore, by (8.115) and by Theorem 8.10, the regular problem (8.112) has a solution u such that $u(t_u) \ge r$ for some $t_u \in [0,T]$.

Step 2. A priori estimates of the solution u *of the regular problem.*

We shall show that

$$u(t) \geq r \quad \text{for} \quad t \in [0, T]. \tag{8.116}$$

To this aim, set $v = u - r$. By virtue of (8.114) we have

$$(\phi_p(v'(t)))' + \mu(t)\,\phi_p(v(t)) = \widetilde{f}(t, u(t)) + \mu(t)\,\phi_p(u(t) - r) \geq 0$$

for a.e. $t \in [0, T]$. By (8.109) it follows that $v(t) \geq 0$ on $[0, T]$, i.e. (8.116) is true.

Now, we show that

$$u(t) \leq B \quad \text{for} \quad t \in [0, T]. \tag{8.117}$$

Indeed, by the definition of m and by (8.111) and (8.113) we have

$$\widetilde{f}(t, x) \leq m(t) \quad \text{ for a.e. } t \in [0, T] \text{ and all } x \geq r.$$

Hence, we can use Lemma 8.9 to get the estimate

$$\|u'\|_\infty \leq \phi_p^{-1}(\|m\|_1). \tag{8.118}$$

If $u \geq A$ were valid on $[0, T]$, then taking into account the periodicity of u' and (8.113) we would get

$$0 = \int_0^T \widetilde{f}(t, u(t))\,\mathrm{d}t \leq \int_0^T \beta(t)\,\mathrm{d}t = T\,\overline{\beta} < 0,$$

a contradiction. Hence,

$$\min\{u(s) : s \in [0, T]\} < A.$$

Now, assume that

$$u^* := \max\{u(s) : s \in [0, T]\} > A$$

and extend u to be T-periodic on $\mathbb{R}$. There are s_1, s_2 and $s^* \in \mathbb{R}$ such that

$$s_1 < s^* < s_2, \quad s_2 - s_1 < T, \quad u(s_1) = u(s_2) = A \text{ and } u(s^*) = u^* > A.$$

In particular, due to (8.118),

$$2\,(u(s^*)-A)=\int_{s_1}^{s^*}u'(s)\,\mathrm{d}s+\int_{s_2}^{s^*}u'(s)\,\mathrm{d}s\le T\,\phi_p^{-1}(\|m\|_1),$$

wherefrom the estimate

$$u(t)-A\le \tfrac{T}{2}\,\phi_p^{-1}(\|m\|_1)\le B-A \text{ on } [0,T]$$

follows. Thus, (8.117) is true.

Estimates (8.116) and (8.117) mean that $r\le u\le B$ holds on $[0,T]$. In view of (8.111), we conclude that u is a solution to (8.1), (8.2).

Part II. Now, let $\overline{\beta}=0$. Put $n_0=\max\{\frac{1}{r},\frac{1}{B-A},3\}$. For an arbitrary $n\in\mathbb{N}$, define

$$\widetilde{f}_n(t,x)=\begin{cases} f(t,r) & \text{if } x\le r,\\ f(t,x) & \text{if } x\in[r,A],\\ f(t,x)-\mu(t)\,\phi_p\left(\frac{1}{n}\,\frac{x-A}{x-A+1}\right) & \text{if } x\in(A,B],\\ f(t,B)-\mu(t)\,\phi_p\left(\frac{1}{n}\,\frac{B-A}{B-A+1}\right) & \text{if } x\ge B. \end{cases} \tag{8.119}$$

If $x\in[A+\frac{1}{n},B]$, then using (8.106) we deduce that

$$\begin{aligned}\widetilde{f}_n(t,x)&= f(t,x)-\mu(t)\phi_p\left(\tfrac{1}{n}\tfrac{x-A}{x-A+1}\right)\le\beta(t)-\mu(t)\phi_p\left(\tfrac{1}{n}\tfrac{x-A}{x-A+1}\right)\\ &\le\beta(t)-\mu(t)\,\phi_p\left(\tfrac{1}{n\,(n+1)}\right)\le\beta(t)-\mu(t)\,\phi_p\left(\tfrac{1}{2n^2}\right)\end{aligned}$$

is true for a.e. $t\in[0,T]$ and all $n\in\mathbb{N}$ such that $n\ge n_0$. Similarly, if $x>B$, then

$$\widetilde{f}_n(t,x)=f(t,B)-\mu(t)\,\phi_p\left(\tfrac{1}{n}\,\tfrac{B-A}{B-A+1}\right)\le\beta(t)-\mu(t)\,\phi_p\left(\tfrac{1}{2n^2}\right).$$

Thus,

$$\begin{cases}\widetilde{f}_n(t,x)\le\beta_n(t):=\beta(t)-\mu(t)\,\phi_p\left(\frac{1}{2n^2}\right)\\ \qquad \text{for } \ x\ge A+\frac{1}{n}, \ \text{ for a.e. } \ t\in[0,T] \ \text{ and all } \ n\ge n_0.\end{cases} \tag{8.120}$$

Clearly,

$$\overline{\beta}_n < 0 \ \text{ and } \ \beta_n(t) \le \beta(t) \text{ for a.e. } t \in [0,T]. \tag{8.121}$$

Furthermore, by (8.107) and (8.119) we have

$$\widetilde{f}_n(t,x) + \mu(t)\,\phi_p\left(x - (r - \tfrac{1}{n})\right) \ge f(t,r) \ge 0 \qquad \text{if } \ x \in [r - \tfrac{1}{n}, r],$$

$$\widetilde{f}_n(t,x) + \mu(t)\,\phi_p\left(x - (r - \tfrac{1}{n})\right) \ge f(t,x) + \mu(t)\,\phi_p\,(x - r) \ge 0$$
$$\text{if } \ x \in [r, A]$$

and, taking into account that $\xi^{p-1} + \eta^{p-1} \le (\xi + \eta)^{p-1}$ holds for all $\xi,\, \eta \ge 0$ and each $p \ge 2$,

$$\begin{aligned}
&\widetilde{f}_n(t,x) + \mu(t)\,\phi_p\left(x - (r - \tfrac{1}{n})\right) \\
&\qquad = f(t,x) - \mu(t)\,\phi_p\left(\tfrac{1}{n}\,\tfrac{x-A}{x-A+1}\right) + \mu(t)\,\phi_p\left(x - r + \tfrac{1}{n}\right) \\
&\qquad \ge f(t,x) + \mu(t)\,\phi_p\,(x - r) \ge 0 \quad \text{if } \ x \in [A, B]
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{f}_n(t,x) + \mu(t)\,\phi_p\left(x - (r - \tfrac{1}{n})\right) \\
&\qquad = f(t,B) - \mu(t)\,\phi_p\left(\tfrac{1}{n}\,\tfrac{B-A}{B-A+1}\right) + \mu(t)\,\phi_p\left(x - r + \tfrac{1}{n}\right) \\
&\qquad \ge f(t,B) + \mu(t)\,\phi_p\,(B - r) \ge 0 \quad \text{if } \ x \ge B.
\end{aligned}$$

To summarize,

$$\widetilde{f}_n(t,x) + \mu(t)\,\phi_p\left(x - (r - \tfrac{1}{n})\right) \ge 0 \quad \text{for all } \ x \ge r - \tfrac{1}{n}. \tag{8.122}$$

For a.e. $t \in [0,T]$ and all $n \in \mathbb{N}$, put

$$\widetilde{m}_n(t) := \max\left\{\sup\{\widetilde{f}_n(t,x) : x \in [r - \tfrac{1}{n}, A + \tfrac{1}{n}]\}, \beta_n(t), 0\right\}.$$

In view of (8.119) and (8.121) we have

$$0 \le \widetilde{m}_n(t) \le m(t) \qquad \text{for a.e. } t \in [0,T] \ \text{ and } \ n \ge n_0.$$

This together with (8.120)–(8.122) means that, for each $n \in \mathbb{N}$ large enough, *Part* I of this proof ensures the existence of a solution u_n to the auxiliary problem

$$(\phi_p(u_n'))' = \widetilde{f}_n(t, u_n), \quad u_n(0) = u_n(T), \quad u_n'(0) = u_n'(T)$$

which satisfies the estimates

$$r - \tfrac{1}{n} \le u_n(t) \le B + \tfrac{1}{n} \quad \text{on} \quad [0,T] \quad \text{and} \quad \|u_n'\|_\infty \le \phi_p^{-1}(\|m\|_1).$$

Now, notice that

$$|\widetilde{f}_n(t,x) - h(t,x)| \le \mu(t)\,\phi_p\left(\tfrac{1}{n}\right)$$
$$\text{for a.e. } t \in [0,T], \text{ all } \ x \in \mathbb{R} \text{ and all } n \in \mathbb{N},$$

where

$$h(t,x) = \begin{cases} f(t,r) & \text{if } \ x \le r, \\ f(t,x) & \text{if } \ x \in [r,B], \\ f(t,B) & \text{if } \ x \ge B. \end{cases}$$

In particular, $h \in Car([0,T] \times \mathbb{R})$,

$$\lim_{n\to\infty} \widetilde{f}_n(t,x) = h(t,x) \quad \text{for a.e. } \ t \in [0,T] \text{ and all } \ x \in \mathbb{R}$$

and the sequence $\{\widetilde{f}_n(t,u_n(t))\}$ has a common Lebesgue integrable majorant on $[0,T]$. Thus, using the Arzelà-Ascoli theorem and the Lebesgue dominated convergence theorem for the sequences $\{u_n\}$ and $\{\widetilde{f}_n(t,u_n(t))\}$, we can show that the sequence $\{u_n\}$ contains a subsequence which converges in $C^1[0,T]$ to a solution u of the problem

$$(\phi_p(u'))' = h(t,u), \quad u(0) = u(T), \quad u'(0) = u'(T).$$

Since u satisfies estimate (8.110), u solves also problem (8.1), (8.2). $\square$

The next supplementary assertion concerning the case $p \in (1,2)$ follows immediately from the first part of the previous proof.

Theorem 8.29. *Let all assumptions of Theorem 8.28 be satisfied, with the exceptions that $1 < p < 2$ is allowed and $\overline{\beta} < 0$ is required in (8.106). Then problem (8.105), (8.2) has a solution u such that (8.110) is true.*

It is well-known that the function

$$G(t,s) = \tfrac{T}{2\pi}\sin\left(\tfrac{\pi}{T}\,|t-s|\right), \quad t,s \in [0,T],$$

is the Green function for the linear periodic problem

$$v'' + \left(\tfrac{\pi}{T}\right)^2 v = 0, \quad v(0) = v(T), \ v'(0) = v'(T)$$

and $G(t,s)$ is nonnegative on $[0,T] \times [0,T]$. Therefore, each T-periodic function $v \in AC^1[0,T]$ fulfilling the inequality

$$v''(t) + \left(\tfrac{\pi}{T}\right)^2 v(t) \geq 0 \quad \text{for a.e. } t \in [0,T]$$

must be nonnegative on $[0,T]$. More generally, for linear periodic problems the following *antimaximum principle* is valid:

Let $\mu \in L_1[0,T]$ be such that $0 \leq \mu(t) \leq \left(\tfrac{\pi}{T}\right)^2$ for a.e. $t \in [0,T]$ and $\overline{\mu} > 0$ and let $v \in AC^1[0,T]$ satisfy the periodic conditions (8.2) and

$$v''(t) + \mu(t)\, v(t) \geq 0 \quad \textit{for a.e. } t \in [0,T].$$

Then v is nonnegative on $[0,T]$.

Next, we will show that for quasilinear periodic problems an analogous assertion holds although, in general, no tools like the Green function are available.

Theorem 8.30. *Let $1 < p < \infty$ and $\mu \in L_1[0,T]$ be such that*

$$\overline{\mu} > 0 \quad \textit{and} \quad 0 \leq \mu(t) \leq \left(\tfrac{\pi_p}{T}\right)^p \quad \textit{for a.e. } t \in [0,T] \tag{8.123}$$

and let $v \in C^1[0,T]$ be such that $\phi_p(v') \in AC[0,T]$,

$$(\phi_p(v'(t)))' + \mu(t)\, \phi_p(v(t)) \geq 0 \ \textit{for a.e. } t \in [0,T] \tag{8.124}$$

and

$$v(0) = v(T), \quad v'(0) = v'(T). \tag{8.125}$$

Then $v \geq 0$ on $[0,T]$.

Proof. Let $v \in C^1[0,T]$ be such that $\phi_p(v') \in AC[0,T]$ and (8.123)–(8.125) hold. Without any loss of generality we may assume that v is not trivial.

Step 1. First, we show that

$$v^* := \max\{v(t) : t \in [0,T]\} > 0. \tag{8.126}$$

Assuming, on the contrary, that $v \le 0$ on $[0,T]$, we get by (8.124)

$$(\phi_p(v'(t)))' \ge -\mu(t)\,\phi_p(v(t)) \ge 0 \text{ for a.e. } t \in [0,T].$$

Therefore, v' is nondecreasing on $[0,T]$ and, taking into account (8.125), we deduce that $v' = 0$ on $[0,T]$. Consequently, $v(t) \equiv v(0) \le 0$ on $[0,T]$. Hence, (8.124) reduces to

$$-\mu(t)\,(-v(0))^{p-1} \ge 0 \text{ for a.e. } t \in [0,T].$$

However, as $\mu \ge 0$ a.e. on $[0,T]$ and $\overline{\mu} > 0$, this is possible if and only if $v(0) = 0$, i.e. $v \equiv 0$ on $[0,T]$, which contradicts our assumption that v does not vanish identically on $[0,T]$. Thus, (8.126) is true.

Step 2. Assume that $\min\{v(t)\colon t \in [0,T]\} < 0$. Let us extend v and μ to T-periodic functions on $\mathbb{R}$. In view of *Step* 1, there are $a,\, b \in \mathbb{R}$ such that $v > 0$ on (a,b), $v(a) = v(b) = 0$ and

$$0 < b - a < T. \tag{8.127}$$

In virtue of (8.123) and (8.124), we have

$$\begin{cases} (\phi_p(v'(t)))' + \left(\frac{\pi_p}{T}\right)^p \phi_p(v(t)) \ge (\phi_p(v'(t)))' + \mu(t)\,\phi_p(v(t)) \ge 0 \\ \hfill \text{for a.e. } t \in [a,b]. \end{cases} \tag{8.128}$$

Furthermore, put

$$a_0 = a - \tfrac{1}{2}\,(T - b + a), \qquad b_0 = a_0 + T > b$$

and

$$\sigma_2(t) = d\,\tfrac{T}{\pi_p}\,\sin_p\left(\left(\tfrac{\pi_p}{T}\right)(t - a_0)\right) \quad \text{for } \ t \in \mathbb{R}$$

with $d > 0$ such that $\sigma_2(t) > v(t) \ge 0$ on $[a,b]$. We have

$$(\phi_p(\sigma_2'(t)))' + \left(\tfrac{\pi_p}{T}\right)^p \phi_p(\sigma_2(t)) = 0 \text{ for a.e. } t \in [a,b]. \tag{8.129}$$

Thus, σ_2 is an upper function for the problem

$$(\phi_p(u'))' + \lambda\,\phi_p(u) = 0, \qquad u(a) = u(b) = 0. \tag{8.130}$$

Moreover, in view of (8.128), $\sigma_1 = v$ is a lower function for (D.3). It follows easily from Theorem 7.14 where we put $g(t,x,y) = -\left(\frac{\pi_p}{T}\right)^p \phi_p(x)$ for t, $x, y \in \mathbb{R}$, that there exists a nontrivial solution u to (8.130). This, due to (8.127), contradicts Lemma D.2. $\square$

Theorems 8.28–8.30 yield the following new existence criterion.

Theorem 8.31. *Let $f \in Car([0,T]\times(0,\infty))$ and $1<p<\infty$. Furthermore, let $r \in (0,\infty)$, $A \in [r,\infty)$ and $\beta \in L_1[0,T]$ be such that estimates (8.106) and (8.108) hold, where $\overline{\beta} < 0$ if $1<p<2$ and $\overline{\beta} \le 0$ if $2 \le p < \infty$.*

Finally, let $\mu \in L_1[0,T]$ be such that $\overline{\mu} > 0$

$$0 \le \mu(t) \le \left(\tfrac{\pi_p}{T}\right)^p \quad \textit{for a.e. } t \in [0,T]$$

and estimate (8.107) is true.

Then problem (8.105), (8.2) has a solution u such that (8.110) is true.

In particular, for the *Duffing equation* $(\phi_p(u'))' = g(u) + e(t)$ we have

Corollary 8.32. *Let $1<p<\infty$. Suppose that $f(t,x) = g(x) + e(t)$ for $x \in (0,\infty)$ and a.e. $t \in [0,T]$, where $g \in C(0,\infty)$, $e \in L_1[0,T]$,*

$$\overline{e} + \limsup_{x\to\infty} g(x) < 0 \tag{8.131}$$

and

$$\begin{cases} \textit{there exists } r > 0 \textit{ such that} \\ \qquad e(t) + g(x) + \left(\frac{\pi_p}{T}\right)^p (x-r)^{p-1} \ge 0 \\ \textit{for a.e. } t \in [0,T] \textit{ and all } x \ge r. \end{cases} \tag{8.132}$$

Then problem (8.1), (8.2) has a solution u such that $u(t) \ge r$ on $[0,T]$.

Proof. Denote $f(t,x) = g(x) + e(t)$. Due to (8.131) we can find $A \ge r$ such that

$$g(x) + \overline{e} < \tfrac{1}{2}\left(\overline{e} + \limsup_{x\to\infty} g(x)\right) < 0 \quad \text{for } x \in [A,\infty).$$

Consequently,

$$f(t,x) = g(x) + \overline{e} + e(t) - \overline{e} < \tfrac{1}{2}\left(\overline{e} + \limsup_{x\to\infty} g(x)\right) + e(t) - \overline{e}$$

for a.e. $t \in [0,T]$ and all $x \in [A,\infty)$. Therefore (8.106) holds with

$$\beta(t) := e(t) + \tfrac{1}{2}\left(\limsup_{x\to\infty} g(x) - \overline{e}\right),$$

$\overline{\beta} < 0$ and $B > A$ arbitrarily large. Furthermore, by virtue of (8.132), we have

$$f(t,x) + \left(\tfrac{\pi_p}{T}\right)^p (x-r)^{p-1} \geq 0 \quad \text{for } \ x \in [r,\infty).$$

The assertion now follows by Theorem 8.31. □

REMARK 8.33. Notice that the assertion of Corollary 8.32 remains valid also when assumption (8.131) is replaced by a slightly weaker assumption that there is an $A > r$ such that $g(x) + \overline{e} \leq 0$ for $x \geq A$.

EXAMPLE. Consider the problem

$$(\phi_p(u'))' = g(u) + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{8.133}$$

with $1 < p < \infty$, $e \in L_1[0,T]$ essentially bounded below and

$$g(x) := -k\,x^{p-1} + \frac{a}{x^\alpha} \quad \text{for } x > 0; \ a > 0, \ \alpha > 0, \ k \geq 0.$$

We will apply Corollary 8.32. To this aim we need to verify that conditions (8.131) and (8.132) are satisfied.

It is easy to see that if $k > 0$, then assumption (8.131) of Corollary 8.32 is satisfied for all $e \in L_1[0,T]$, while in the case $k = 0$ this condition holds whenever $\overline{e} < 0$.

Furthermore, denote $e_* := \inf \operatorname{ess}\{e(t) : t \in [0,T]\}$, $\mu = \left(\tfrac{\pi_p}{T}\right)^p$,

$$\begin{cases} h(x,r) := \dfrac{a}{x^\alpha} + \mu\,(x-r)^{p-1} - k\,x^{p-1} \\ \qquad\qquad\qquad\qquad\qquad \text{for } \ r > 0 \ \text{ and } \ x \geq r \ \text{ or } \ r = 0 \ \text{ and } \ x > r \end{cases}$$

and

$$\varkappa(r) := \inf\{h(x,r)\colon x \in (r,\infty)\} \quad \text{for} \quad r \geq 0.$$

Then condition (8.132) is satisfied if and only if there is $r > 0$ such that $e_* + \varkappa(r) \geq 0$. We can show that this occurs if $e_* + \varkappa(0) > 0$. Notice that

$$\begin{cases} \varkappa(0) = a \left(\dfrac{\alpha+p-1}{p-1}\right) \left(\dfrac{(p-1)\,(\mu-k)}{\alpha\, a}\right)^{\frac{\alpha}{\alpha+p-1}} & \\ & \text{if } k \in [0,\mu) \text{ and } 1 < p \leq \infty, \\ \varkappa(0) = 0 & \text{if } k = \mu \text{ and } 1 < p \leq 2. \end{cases}$$

Thus, making use of Corollary 8.32, we can summarize that problem (8.133) has a positive solution if

$$k = 0,\ 1 < p < \infty,\ \bar{e} < 0 \quad \text{and} \quad e_* > -a \left(\frac{\alpha+p-1}{p-1}\right) \left(\frac{(p-1)\,\mu}{\alpha\, a}\right)^{\frac{\alpha}{\alpha+p-1}}$$

or

$$0 < k < \mu,\ 1 < p < \infty \quad \text{and} \quad e_* > -a \left(\frac{\alpha+p-1}{p-1}\right) \left(\frac{(p-1)\,(\mu-k)}{\alpha\, a}\right)^{\frac{\alpha}{\alpha+p-1}}$$

or

$$k = \mu,\ 1 < p \leq 2 \quad \text{and} \quad e_* > 0.$$

Notice that $\lim_{x\to\infty} h(x,r) = -\infty$ if $k > \mu,\ p > 1$ and $r \geq 0$ and also if $k = \mu,\ p > 2$ and $r > 0$. We have $\varkappa(r) = -\infty$ in these cases. In particular, condition (8.132) cannot be satisfied when

$$k > \mu,\ p > 1 \quad \text{or} \quad k = \mu,\ p > 2.$$

8.5 Periodic problem with time singularities

In this section we will study the periodic problem (8.1), (8.2) under the assumption

$$f \in Car((0,T) \times \mathbb{R}^2) \text{ has time singularities at } t = 0, t = T, \tag{8.134}$$

i.e., there exist $x,\ y \in \mathbb{R}$ such that

$$\int_0^{\varepsilon} |f(t,x,y)|\mathrm{d}t = \infty \quad \text{and} \quad \int_{T-\varepsilon}^{T} |f(t,x,y)|\mathrm{d}t = \infty$$

for each sufficiently small $\varepsilon > 0$.

We will provide conditions for the existence of solutions to problem (8.1), (8.2) which can change their sign on $[0, T]$. Solutions of problem (8.1), (8.2) are understood in the sense of Definition 8.1 where $\mathcal{A} = \mathbb{R}^2$.

Theorem 8.34. *Let (8.134) hold. Assume that there exist $a_1, a_2 \in [0, T]$, $a_1 < a_2$, $\alpha, \gamma, r_1, r_2 \in \mathbb{R}$, a nonnegative function $h_0 \in L_1[0, T]$ and a positive function $\omega \in C[0, \infty)$ fulfilling condition (7.17) such that*

$$\begin{cases} r_1 + t\,\gamma \le \alpha \le r_2 + t\,\gamma \quad \text{for } t \in [0, T], \\ f(t, r_1 + t\,\gamma, \gamma) \le 0, \quad f(t, r_2 + t\gamma, \gamma) \ge 0 \ \text{ for a.e. } t \in [0, T], \end{cases} \tag{8.135}$$

$$\begin{cases} f(t, x, y)\ \operatorname{sign}(y - \gamma) \ge -\omega(|\phi(y) - \phi(\gamma)|)\,(h_0(t) + |y - \gamma|) \\ \text{for a.e. } t \in [0, a_2] \text{ and all } x \in [r_1 + t\,\gamma, r_2 + t\,\gamma], y \in \mathbb{R}, \end{cases} \tag{8.136}$$

$$\begin{cases} f(t, x, y)\ \operatorname{sign}(y - \gamma) \le \omega(|\phi(y) - \phi(\gamma)|)\,(h_0(t) + |y - \gamma|) \\ \text{for a.e. } t \in [a_1, T] \text{ and all } x \in [r_1 + t\,\gamma, r_2 + t\,\gamma], y \in \mathbb{R}. \end{cases} \tag{8.137}$$

Further assume that r is the constant given by Lemma 7.16 for $y_1 = y_2 = \gamma$, $r_0 = \max\{|r_1|, |r_2|\} + T\,|\gamma|$, $\varkappa = 1$ and that there exist $\eta \in (0, \frac{T}{2})$, $\psi_0 \in L_1[0, T]$ and a nonnegative function $h \in L_{loc}(0, T)$ satisfying (A.20), (A.24),

$$\begin{cases} f(t, x, y)\ \operatorname{sign}(y - \gamma) \ge h(t)\,|\phi(y) - \phi(\gamma)| + \psi_0(t) \\ \text{for a.e. } t \in (0, \eta) \text{ and all } x \in [r_1 + t\,\gamma, r_2 + t\,\gamma],\ y \in [-r, r], \end{cases} \tag{8.138}$$

and

$$\begin{cases} f(t, x, y)\ \operatorname{sign}(y - \gamma) \le -h(t)\,|\phi(y) - \phi(\gamma)| + \psi_0(t) \\ \text{for a.e. } t \in [T - \eta, T] \text{ and all } x \in [r_1 + t\,b, r_2 + t\,b],\ y \in [-r, r]. \end{cases} \tag{8.139}$$

Then problem (8.1), (8.2) has a solution u satisfying

$$u(0) = u(T) = \alpha, \qquad u'(0) = u'(T) = \gamma. \tag{8.140}$$

Proof. *Step 1. Approximate regular problems.*

Choose an arbitrary $k \in \mathbb{N}$, $k > \frac{2}{T}$, and for $x, y \in \mathbb{R}$ define the auxiliary function

$$f_k(t, x, y) = \begin{cases} -f(t, x, y) & \text{for a.e. } t \in [0, T] \setminus \Delta_k, \\ 0 & \text{for a.e. } t \in \Delta_k, \end{cases} \tag{8.141}$$

where $\Delta_k = [0, \frac{1}{k}) \cup (T - \frac{1}{k}, T]$. We see that $f_k \in Car([0,T] \times \mathbb{R}^2)$ fulfils the inequalities

$$f_k(t,x,y)\,\mathrm{sign}(y-\gamma) \le \omega(|\phi(y) - \phi(\gamma)|)(h_0(t) + |y-\gamma|)$$

for a.e. $t \in [0, a_2]$ and all $x \in [r_1 + t\gamma, r_2 + t\gamma], y \in \mathbb{R}$, and

$$f_k(t,x,y)\,\mathrm{sign}(y-\gamma) \ge -\omega(|\phi(y) - \phi(\gamma)|)(h_0(t) + |y-\gamma|)$$

for a.e. $t \in [a_1, T]$ and all $x \in [r_1 + t\,\gamma, r_2 + t\,\gamma]$, $y \in \mathbb{R}$. Let us put $\sigma_1(t) = r_1 + t\,b$, $\sigma_2(t) = r_2 + t\,b$ for $t \in [0,T]$. Then f_k satisfies condition (7.24) with $g = f_k$, $y_1 = y_2 = \gamma$, $\varkappa = 1$. Moreover, by assumption (8.135) and Definition 7.13, the functions σ_1 and σ_2 are respectively lower and upper functions of the regular Dirichlet problem

$$(\phi(u'))' + f_k(t,u,u') = 0, \quad u(0) = u(T) = \alpha. \tag{8.142}$$

Hence, by Theorem 7.18, problem (8.142) has a solution u_k satisfying

$$r_1 + t\gamma \le u_k(t) \le r_2 + t\gamma \quad \text{for } t \in [0,T], \quad \|u_k'\|_\infty \le r. \tag{8.143}$$

Step 2. Convergence of the sequence of approximate solutions $\{u_k\}$.

Condition (8.143) implies that the sequence $\{u_k\}$ is bounded and equicontinuous on $[0,T]$. By the Arzelà-Ascoli theorem this yields a function $u \in C[0,T]$ and a subsequence uniformly converging to u on $[0,T]$. Therefore the limit u satisfies

$$u(0) = u(T) = \alpha. \tag{8.144}$$

Choose an arbitrary interval $[a,b] \subset (0,T)$. Since the sequence $\{u_k'\}$ is also bounded, assumption (8.134) and formula (8.141) provide a function $m \in L_1[0,T]$ such that for each $k > \frac{2}{T}$

$$|f_k(t, u_k(t), u_k'(t))| \le m(t) \quad \text{for a.e. } t \in [a,b]. \tag{8.145}$$

Hence the equation in (8.142) yields

$$|\phi(u_k'(t_2)) - \phi(u_k'(t_1))| \le \left| \int_{t_1}^{t_2} m(s)\,\mathrm{d}s \right|$$

for $k > \frac{2}{T}$ $t_1, t_2 \in [a, b]$, which implies that the sequence $\{\phi(u_k')\}$ is equicontinuous on $[a, b]$. By virtue of the uniform continuity of ϕ^{-1} on compact intervals, the sequence $\{u_k'\}$ is also equicontinuous on $[a, b]$. The Arzelà-Ascoli theorem guarantees that for each compact subset $\mathcal{K} \subset (0, T)$ a subsequence of $\{u_k'\}$ uniformly converging to u' on $\mathcal{K}$ can be chosen. Therefore, using the diagonalization theorem, we can choose a subsequence $\{u_{k_\ell}\}$ satisfying

$$\begin{cases} \lim_{\ell\to\infty} u_{k_\ell}(t) = u(t) & \text{uniformly on } [0, T], \\ \lim_{\ell\to\infty} u_{k_\ell}'(t) = u'(t) & \text{locally uniformly on } (0, T). \end{cases} \tag{8.146}$$

By (8.143) the limit u fulfils

$$r_1 + t\gamma \le u(t) \le r_2 + t\gamma \quad \text{for } t \in [0, T], \quad \|u'\|_\infty \le r.$$

Step 3. *Convergence of the sequence of approximate nonlinearities* $\{f_k\}$.

Let $\mathcal{V}_1$ be the set of all $t \in [0, T]$ such that $f(t, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ is not continuous and let $\mathcal{V}_2$ be the set of all $t \in [0, T]$ such that the equality in (8.141) is not satisfied. Then $\operatorname{meas}(\mathcal{V}_1 \cup \mathcal{V}_2) = 0$. Choose an arbitrary $\xi \in (0, T) \setminus (\mathcal{V}_1 \cup \mathcal{V}_2)$. Then there exists $\ell_0 \in \mathbb{N}$ such that for $\ell \ge \ell_0$ we have

$$f_{k_\ell}(\xi, u_{k_\ell}(\xi), u_{k_\ell}'(\xi)) = -f(\xi, u_{k_\ell}(\xi), u_{k_\ell}'(\xi))$$

and, by (8.146),

$$\lim_{\ell\to\infty} f_{k_\ell}(\xi, u_{k_\ell}(\xi), u_{k_\ell}'(\xi)) = -f(\xi, u(\xi), u'(\xi)).$$

Hence,

$$\lim_{\ell\to\infty} f_{k_\ell}(t, u_{k_\ell}(t), u_{k_\ell}'(t)) = -f(t, u(t), u'(t)) \text{ for a.e. } t \in [0, T]. \tag{8.147}$$

Step 4. *The function* u *is a w-solution of problem* (8.1), (8.144).

Choose an arbitrary $t \in (0, T)$. Then there exists an interval $[a, b] \subset (0, T)$ such that $t, \frac{T}{2} \in [a, b]$. Integrate the equality

$$(\phi(u_{k_\ell}'(t)))' + f_{k_\ell}(t, u_{k_\ell}(t), u_{k_\ell}'(t)) = 0 \quad \text{for a.e. } t \in [0, T].$$

We get

$$\phi(u'_{k_\ell}(t)) - \phi(u'_{k_\ell}(\tfrac{T}{2})) + \int_{\frac{T}{2}}^{t} f_{k_\ell}(s, u_{k_\ell}(s), u'_{k_\ell}(s))\,\mathrm{d}s = 0.$$

According to conditions (8.145), (8.147) and the Lebesgue dominated convergence theorem on $[a,b]$, we can deduce that the limit u solves the equation

$$\phi(u'(t)) - \phi(u'(\tfrac{T}{2})) - \int_{\frac{T}{2}}^{t} f(s, u(s), u'(s))\,\mathrm{d}s = 0 \quad \text{for } t \in (0,T), \tag{8.148}$$

$\phi(u') \in AC_{loc}(0,T)$ and u is a w-solution of problem (8.1), (8.144).

Step 5. *The function u is a solution of problem* (8.1), (8.2).

First we prove that

$$f(t, u(t), u'(t)) \in L_1[0,\eta] \quad \text{and} \quad f(t, u(t), u'(t)) \in L_1[T-\eta, T].$$

Assumption (8.138), formula (8.141) and estimate (8.143) imply

$$-f_k(t, u_k(t), u'_k(t))\,\mathrm{sign}(u'_k(t) - \gamma) \ge -|\psi_0(t)|$$

for a.e. $t \in (0,\eta)$ and all $k > \frac{2}{T}$. By conditions (8.146) and (8.147) we have

$$\lim_{\ell\to\infty} (-f_{k_\ell}(t, u_{k_\ell}(t), u'_{k_\ell}(t))\,\mathrm{sign}(u'_{k_\ell}(t)-\gamma)) = f(t, u(t), u'(t))\,\mathrm{sign}(u'(t)-\gamma)$$

for a.e. $t \in [0,T]$ and all $k > \frac{2}{T}$. Finally, having in mind that $\mathrm{sign}(y-\gamma) = \mathrm{sign}(\phi(y) - \phi(\gamma))$ for $y \in \mathbb{R}$, we compute

$$\left| \int_0^\eta f_{k_\ell}(t, u_{k_\ell}(t), u'_{k_\ell}(t))\ \mathrm{sign}(u'_{k_\ell}(t) - \gamma)\,\mathrm{d}t \right| \le \int_0^\eta |\phi(u'_{k_\ell}(t)) - \phi(\gamma)|'\,\mathrm{d}t$$

$$\le \phi(|u'_{k_\ell}(\eta)|) + 2\phi(|\gamma|) + \phi(|u'_{k_\ell}(0)|) \le 2\,\phi(r) + 2\,\phi(|\gamma|)$$

for each $\ell \in \mathbb{N}$. Therefore, the Fatou lemma implies $f(t,u(t),u'(t)) \in L_1[0,\eta]$. The condition $f(t,u(t),u'(t)) \in L_1[T-\eta, T]$ can be proved similarly. Hence $f(t,u(t),u'(t)) \in L_1[0,T]$ and $u \in AC^1[0,T]$.

In order to prove that u fulfils condition (8.2) we put

$$g^*(t) = |\psi_0(t)|, \quad h^*(t) = 0 \quad \text{for a.e. } t \in [0,T]$$

and

$$v_k(t) = \phi(u_k'(t)) - \phi(\gamma) \quad \text{for } t \in [0, T].$$

Then, according to (8.141) and (8.142),

$$v_k'(t) = \begin{cases} f(t, u_k(t), u_k'(t)) & \text{for a.e. } t \in [0, T] \setminus \Delta_k, \\ 0 & \text{for a.e. } t \in \Delta_k. \end{cases}$$

By estimate (8.143) there exists $\beta_0 \in (0, \infty)$ such that

$$|v_k(\eta)| \le \beta_0, \quad |v_k(T - \eta)| \le \beta_0.$$

Further, due to assumption (8.138), we have

$$v_k'(t) \operatorname{sign} v_k(t) \ge h(t)|v_k(t)| - g^*(t) \quad \text{for a.e. } t \in [\tfrac{1}{k}, \eta].$$

So, we see that conditions (A.21), (A.22) and (A.23) are fulfilled and, by Criterion A.12, the sequence $\{v_k\}$ is equicontinuous at 0 from the right and $\lim_{k\to\infty} v_k(0) = 0$. Similarly, due to (8.139) we have

$$v_k'(t) \operatorname{sign} v_k(t) \le -h(t)|v_k(t)| + g^*(t) \quad \text{for a.e. } t \in [T - \eta, T - \tfrac{1}{k}].$$

Hence, conditions (A.17), (A.18) and (A.19) hold and Criterion A.11 guarantees that the sequence $\{v_k\}$ is equicontinuous at T from the left and $\lim_{k\to\infty} v_k(T) = 0$. Consequently, the sequences $\{\phi(u_k')\}$ and $\{u_k'\}$ are also equicontinuous at 0 from the right and at T from the left and

$$\lim_{k\to\infty} u_k'(0) = \gamma, \quad \lim_{k\to\infty} u_k'(T) = \gamma.$$

This yields that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $t \in (0, \delta)$ we can find $k_t \in \mathbb{N}$ such that

$$|u'(t) - \gamma| \le |u'(t) - u_{k_t}'(t)| + |u_{k_t}'(t) - u_{k_t}'(0)| + |u_{k_t}'(0) - \gamma| < 3\varepsilon.$$

So, $\lim_{t\to 0+} u'(t) = \gamma$. The relation $\lim_{t\to T-} u'(t) = \gamma$ can be proved similarly. This together with (8.144) yields that u satisfies the periodic conditions (8.2). ☐

Corollary 8.35. *Let all assumptions of Theorem 8.34 be fulfilled and let* $\alpha = 0$ *and* $\gamma \ne 0$. *Then problem* (8.1), (8.2) *has a sign-changing solution.*

EXAMPLE. Assume that $\lambda,\ \mu \in (1,\infty),\ \gamma,\ c,\ r \in \mathbb{R},\ \ n\in\mathbb{N}$ and that $\psi\in L_1[0,T]$ is positive. For a.e. $t\in[0,T]$ and all $x,\ y\in\mathbb{R}$ define the function

$$f(t,x,y)=\left(\frac{1}{t^\lambda}-\frac{1}{(T-t)^\mu}\right)(\phi(y)-\phi(\gamma))+c\,\phi(y)\,y+\psi(t)\,(x-r)^{2n-1}.$$

Then for an arbitrary $\alpha\in\mathbb{R}$ the conditions of Theorem 8.34 are satisfied. Indeed, choose $\alpha\in\mathbb{R}$ and $a_1,\ a_2\in(0,T),\ \ a_1<a_2$. Then we can find a large positive number r_2 and a negative number r_1 with a large modulus such that condition (8.135) holds. Denote

$$\psi_1(t)=\psi(t)\,\max\{|x-r|^{2n-1}\colon r_1+t\,\gamma\le x\le r_2+t\,\gamma\}\quad\text{for a.e. }\ t\in[0,T]$$

and

$$\psi_2(t)=\begin{cases}(T-t)^{-\mu} & \text{for a.e. } t\in[0,a_1),\\ (T-t)^{-\mu}+t^{-\lambda} & \text{for a.e. } t\in[a_1,a_2],\\ t^{-\lambda} & \text{for a.e. } t\in(a_2,T].\end{cases}$$

Then $\psi_1,\psi_2\in L_1[0,T]$ are positive and for a.e. $t\in[0,a_2]$ and for each $x\in[r_1+t\gamma,r_2+t\gamma],\ \ y\in\mathbb{R}$ we have

$$\begin{aligned}f(t,x,y)\,\mathrm{sign}(y-\gamma)&=f(t,x,y)\,\mathrm{sign}(\phi(y)-\phi(\gamma))\\&>-\frac{1}{(T-t)^\mu}|\phi(y)-\phi(\gamma)|-|c||\phi(y)-\phi(\gamma)||y|-|c||\phi(\gamma)||y|-\psi_1(t)\\&>-(|\phi(y)-\phi(\gamma)|+1)(|c|+1)(|\phi(\gamma)|+1)(\psi_1(t)+\psi_2(t)+|y|).\end{aligned}$$

So, if we put

$$\omega(s)=(s+1)\,(|c|+1)\,(|\phi(\gamma)|+1)\quad\text{and } h_0=\psi_1+\psi_2,$$

we get inequality (8.136). Similarly we can derive inequality (8.137).

Finally, let us assume that r is the constant given by Lemma 7.16 for $y_1=y_2=\gamma,\ \ r_0=\max\{|r_1|,r_2\}+T|\gamma|,\ \ \varkappa=1,$ and put

$$h(t)=\begin{cases}t^{-\lambda} & \text{for a.e. } t\in(0,\eta),\\ 0 & \text{for a.e. } \in[\eta,T-\eta],\\ (T-t)^{-\mu} & \text{for a.e. } t\in(T-\eta,T),\end{cases}$$

$$\psi_3(t)=|c|\phi(r)r+\psi_2(t)(\phi(r)+|\phi(\gamma)|)+\psi_1(t),$$

and

$$\psi_0(t) = \begin{cases} -\psi_3(t) & \text{for a.e. } t \in (0, \eta), \\ 0 & \text{for a.e. } \ \in [\eta, T - \eta], \\ \psi_3(t) & \text{for a.e. } t \in (T - \eta, T). \end{cases}$$

Then $\psi_0 \in L_1[0,T]$, $h \in L_{loc}(0,T)$ and h is nonnegative and satisfies conditions (A.20) and (A.24). Further, for a.e. $t \in (0,\eta)$ and for each $x \in [r_1 + t\gamma, r_2 + t\gamma]$, $y \in [-r, r]$ we obtain

$$\begin{aligned} f(t,x,y)\,\mathrm{sign}(y-\gamma) &= f(t,x,y)\,\mathrm{sign}(\phi(y) - \phi(\gamma)) \\ &> \frac{1}{t^\lambda}|\phi(y) - \phi(\gamma)| - |c|\phi(r)r - \psi_2(t)(\phi(r) + |\phi(\gamma)|) - \psi_1(t) \\ &= h(t)|\phi(y) - \phi(\gamma)| + \psi_0(t). \end{aligned}$$

Hence condition (8.138) is valid. Similarly we show that condition (8.139) holds. Therefore, by Theorem 8.34, problem (8.1), (8.2), where f is defined at the beginning of our example, has a solution u satisfying (8.140). Since α is chosen arbitrarily, problem (8.1), (8.2) has infinitely many solutions. In particular, if we choose $\alpha = 0$ and $\gamma \neq 0$, the corresponding solution of problem (8.1), (8.2) changes its sign on $[0,T]$.

Bibliographical notes

Lemma 8.8 is a modified version of Lemma 1 in Staněk [181]. Lemma 8.7 and Theorems 8.10 and 8.21 are contained in the papers [169] and [170] by Rachůnková and Tvrdý. Lemma 8.24 was stated by Jebelean and Mawhin (see [107, Lemma 3]) as a corollary of a general continuation principle by Manásevich and Mawhin from [131, Theorem 3.1]. Its proof is given here for the reader's convenience. Theorem 8.25 is a slightly modified scalar version of the results by Liu [126, Theorem 1] and Rachůnková and Tvrdý [170, Theorem3.5]. The assertion of Theorem 8.27 is due to Jebelean and Mawhin, see [107, Theorem 2] and [108, Theorem 3]. The results of Section 8.4 are taken from the paper by Cabada, Lomtatidze and Tvrdý [52]. Section 8.5 is based on the paper [153] by Polášek and Rachůnková.

Several rather general definitions of lower and upper functions are available, see e.g. De Coster and Habets [60], [61], Fabry and Habets [84], Kiguradze and Shekhter [118] or Rachůnková and Tvrdý [167]. For other possibilities of operator representation of problem (8.4), (8.2), see e.g. Cabada and Pouso [49], Manásevich and Mawhin [131], Mawhin [138] or Yan [200]).

The singular periodic problem for ordinary differential equations (when ϕ_p is the identity operator) has been studied for about 40 years and many papers have been written till now. However, the attention paid to this problem considerably increased after 1987 due to the paper [122] by Lazer and Solimini. Motivated by the model equation $u'' = a\,u^{-\alpha} + e(t)$ with $\alpha > 0$, $a \neq 0$ and e integrable on $[0, T]$, they investigated the existence of positive solutions to the Duffing equation $u'' = g(u) + e(t)$ using topological arguments and the lower and upper functions method. The restoring force g was allowed to have an attractive space singularity or a strong repulsive space singularity at origin. The results by Lazer and Solimini have been generalized or extended e.g. by Habets and Sanchez [101], Mawhin [135], del Pino, Manásevich and Montero [68], Omari and Ye [146], Zhang [202] and [204], Ge and Mawhin [95], Rachůnková and Tvrdý [168] or Rachůnková, Tvrdý and Vrkoč [172]. All of these papers, when dealing with the repulsive singularity, supposed that the strong force condition is satisfied. For the case of weak singularity, first results were delivered by Rachůnková, Tvrdý and Vrkoč in [171]. Further results were delivered later also by Bonheure and De Coster [45] and Torres [192]. For more historical details and more detailed description of some of the above results, see also Rachůnková, Staněk and Tvrdý [163].

Chapter 9

Mixed problem

Various mathematical models of phenomena from physics, chemistry and technical practice take on the form of partial differential equations subject to initial or boundary conditions. For the investigation of stationary solutions many of these models can be reduced to singular ordinary differential equations of the second order, especially when, due to symmetries in the geometry of the problem data, polar, cylindrical or spherical coordinates can be used. We can refer to the Thomas-Fermi equation occuring in problems from quantum mechanics and astrophysics in Chan and Hon [57] and the Ginzburg-Landau equation describing ferromagnetic systems and arising in superconductivity models in Rentrop [174]. Further examples are singular Sturm-Liouville eigenvalue problems in Reddien [173], problems in the theory of diffusion and reaction according to Langmuir-Hinshelwood kinetics in Bobisud [43], [44], problems from chemical reactor theory in Parter, M.L. Stein and P.R. Stein [149] and applications from mechanics, especially from the buckling theory of spherical shells in Drmota, Scheidl, Troger and Weinmüller [79].

In this chapter we will study a class of nonlinear singular boundary value problems whose importance is derived, in part, from the fact that they arise when searching for positive, *radially symmetric solutions* to the nonlinear elliptic partial differential equation

$$\Delta\, u + g(r, u) = 0 \quad \text{on } \Omega, \quad u\,|_{\Gamma} = 0,$$

where Δ is the Laplace operator, Ω is the open unit disk in $\mathbb{R}^n$ (centered at the origin), Γ is its boundary, and r is the radial distance from the origin. Radially symmetric solutions to this problem are solutions of the ordinary differential equation

$$u'' + \frac{n-1}{t}\, u' + g(t, u) = 0$$

with *mixed boundary conditions* $u'(0) = 0$, $u(1) = 0$. See e.g. Berestycki, Lions and Peletier [36] or Gidas, Ni and Nirenberg [96].

9.1 Problem with singularities in all variables

Similarly to Chapter 7 we will assume that ϕ is an increasing odd homeomorphism with $\phi(\mathbb{R}) = \mathbb{R}$ and consider now the singular *mixed problem* of the form

$$(\phi(u'))' + f(t,u,u') = 0, \quad u'(0) = u(T) = 0. \tag{9.1}$$

We will investigate problem (9.1) on the set $[0,T]\times\mathcal{A}$, where $\mathcal{A}$ is a closed subset of $\mathbb{R}^2$, and we will assume that f has singularities, i.e. f does not satisfy the Carathéodory conditions on the whole set $[0,T]\times\mathcal{A}$. Singularities of f will be specified later for each problem under consideration. Since the mixed and the Dirichlet problem are close to each other, a lot of results and comments are valid for both of them. In accordance with Chapters 1 and 7 we define:

Definition 9.1. A function $u\colon [0,T] \to \mathbb{R}$ with $\phi(u') \in AC[0,T]$ is a *solution of problem* (9.1) if u satisfies $(\phi(u'(t)))' + f(t,u(t),u'(t)) = 0$ a.e. on $[0,T]$ and fulfils the boundary conditions $u'(0) = u(T) = 0$. If $\mathcal{A} \neq \mathbb{R}^2$ we impose on u in addition the condition $(u(t),u'(t)) \in \mathcal{A}$ for $t \in [0,T]$.

A function $u \in C[0,T]$ is a *w-solution of problem* (9.1) if there exists a finite number of singular points $t_\nu \in [0,T]$, $\nu = 1,\dots,r$, such that if we denote $J = [0,T]\setminus\{t_\nu\}_{\nu=1}^{r}$, then $\phi(u') \in AC_{loc}(J)$, u satisfies

$$(\phi(u'(t)))' + f(t,u(t),u'(t)) = 0 \quad \text{for a.e. } t \in [0,T]$$

and fulfils the boundary conditions $u'(0) = u(T) = 0$. If $\mathcal{A} \neq \mathbb{R}^2$, then $(u(t),u'(t)) \in \mathcal{A}$ for $t \in J$.

First, we consider the auxiliary regular mixed problem of the form

$$(\phi(u'))' + g(t,u,u') = 0, \quad u'(0) = 0,\ u(T) = 0, \tag{9.2}$$

where $g \in Car([0,T]\times\mathbb{R}^2)$. In the previous chapters we have defined solutions of regular problems in the same way as those of singular ones. In particular, we define:

Definition 9.2. A function $u\colon [0,T]\to\mathbb{R}$ with $\phi(u') \in AC[0,T]$ is a *solution of problem* (9.2) if u satisfies $(\phi(u'(t)))' + g(t,u(t),u'(t)) = 0$ a.e. on $[0,T]$ and fulfils the boundary conditions $u'(0) = 0$, $u(T) = 0$.

All theorems of Section 7.1 can be modified to suit problem (9.2). However, we present here only one of them which is based on the existence of lower and upper functions to problem (9.2) and will be used further in the investigation of the singular mixed problem (9.1).

Definition 9.3. A function $\sigma \in C[0,T]$ is called a *lower function of problem* (9.2) if there exists a finite set $\Sigma \subset (0,T)$ such that $\phi(\sigma') \in AC_{loc}([0,T] \setminus \Sigma)$, $\sigma'(\tau+) := \lim_{t\to\tau+} \sigma'(t) \in \mathbb{R}$, $\sigma'(\tau-) := \lim_{t\to\tau-} \sigma'(t) \in \mathbb{R}$ for each $\tau \in \Sigma$,

$$\begin{cases} (\phi(\sigma'(t)))' + g(t,\sigma(t),\sigma'(t)) \geq 0 \quad \text{for a.e. } t \in [0,T], \\ \sigma'(0) \geq 0, \ \sigma(T) \leq 0, \quad \sigma'(\tau-) < \sigma'(\tau+) \ \text{ for each } \ \tau \in \Sigma. \end{cases} \tag{9.3}$$

If the inequalities in (9.3) are reversed, then σ is called an *upper function of problem* (9.2).

The next theorem can be proved similarly to Theorem 7.14 of Section 7.1.

Theorem 9.4. *Let σ_1 and σ_2 be a lower function and an upper function of problem* (9.2) *and let $\sigma_1(t) \leq \sigma_2(t)$ for $t \in [0,T]$. Assume that there is a function $h \in L_1[0,T]$ satisfying*

$$|g(t,x,y)| \leq h(t) \quad \textit{for a.e. } t \in [0,T] \ \textit{ and all } \ x \in [\sigma_1(t),\sigma_2(t)],\ y \in \mathbb{R}.$$

Then problem (9.2) *has a solution u such that*

$$\sigma_1(t) \leq u(t) \leq \sigma_2(t) \quad \textit{for} \ \ t \in [0,T]. \tag{9.4}$$

We will apply Theorem 9.4 to the singular mixed problem (9.1) under the assumption

$$\begin{cases} f \in Car((0,T) \times \mathcal{D}), \quad \mathcal{D} = (0,\infty) \times (-\infty,0), \\ f \ \text{ has time singularities at } \ t=0,\ t=T \\ \qquad \text{and space singularities at } \ x=0,\ y=0. \end{cases} \tag{9.5}$$

We are interested in the existence of a solution positive and decreasing on $[0,T)$ and so we will investigate problem (9.1) on the set $[0,T] \times \mathcal{A}$, where $\mathcal{A} = [0,\infty) \times (-\infty,0]$.

Theorem 9.5. *Let* (9.5) *hold. Assume that there exist* $c \in (\nu, \infty), \nu \in (0, T)$ *and* $\varepsilon \in (0, \frac{\phi(\nu)}{\nu})$ *such that*

$$\begin{cases} f(t, c\,(T-t), -c) = 0 \quad \text{for a.e. } t \in [0, T], \\ 0 \le f(t, x, y) \\ \quad \text{for a.e. } t \in [0, T] \text{ and all } x \in (0, c\,(T-t)],\ y \in [-c, 0), \\ \varepsilon \le f(t, x, y) \\ \quad \text{for a.e. } t \in [0, \nu] \text{ and all } x \in (0, c\,(T-t)],\ y \in [-\nu, 0). \end{cases} \tag{9.6}$$

Then problem (9.1) *has a solution* $u \in AC^1[0, T]$ *satisfying*

$$0 < u(t) \le c\,(T-t), \quad -c \le u'(t) < 0 \text{ for } t \in (0, T). \tag{9.7}$$

Proof. *Step* 1. *Approximate solutions.*

Choose $k \in \mathbb{N}$ such that $k > \frac{2}{T}$. For $t \in [\frac{1}{k}, T - \frac{1}{k}]$, $x, y \in \mathbb{R}$ put

$$\alpha_k(t, x) = \begin{cases} c\,(T-t) & \text{if } x > c\,(T-t), \\ x & \text{if } \frac{c}{k} \le x \le c\,(T-t), \\ \frac{c}{k} & \text{if } x < \frac{c}{k}, \end{cases}$$

$$\beta_k(y) = \begin{cases} -\frac{\varepsilon}{k} & \text{if } y > -\frac{\varepsilon}{k}, \\ y & \text{if } -c \le y \le -\frac{\varepsilon}{k}, \\ -c & \text{if } y < -c, \end{cases} \quad \gamma(y) = \begin{cases} \varepsilon & \text{if } y \ge -\nu, \\ \varepsilon\,\dfrac{c+y}{c-\nu} & \text{if } -c < y < -\nu, \\ 0 & \text{if } y \le -c. \end{cases}$$

For a.e. $t \in [0, T]$ and all $x, y \in \mathbb{R}$ define

$$f_k(t, x, y) = \begin{cases} \gamma(y) & \text{if } t \in [0, \frac{1}{k}), \\ f(t, \alpha_k(t, x), \beta_k(y)) & \text{if } t \in [\frac{1}{k}, T - \frac{1}{k}], \\ 0 & \text{if } t \in (T - \frac{1}{k}, T]. \end{cases}$$

Then $f_k \in Car([0, T] \times \mathbb{R}^2)$ and there is $\psi_k \in L_1[0, T]$ such that

$$|f_k(t, x, y)| \le \psi_k(t) \quad \text{for a.e. } t \in [0, T] \text{ and all } x, y \in \mathbb{R}. \tag{9.8}$$

Moreover, assumption (9.6) yields

$$f_k(t, c\,(T-t), -c) = 0 \quad \text{and} \quad f_k(t, 0, 0) \ge 0 \qquad \text{for a.e. } t \in [0, T].$$

We have arrived at the auxiliary regular problem

$$(\phi(u'))' + f_k(t, u, u') = 0, \quad u'(0) = 0, \quad u(T) = 0. \tag{9.9}$$

Put $\sigma_1(t) = 0$, $\sigma_2(t) = c\,(T-t)$ for $t \in [0, T]$. Then σ_1 is a lower function and σ_2 is an upper function of problem (9.9). Hence, by Theorem 9.4, problem (9.9) has a solution u_k and

$$0 \le u_k(t) \le c\,(T-t) \quad \text{for} \quad t \in [0, T].$$

Step 2. A priori estimates of the approximate solutions u_k.

Since $f_k(t,x,y) \ge 0$ for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$, we get that $\phi(u_k'(t))$ as well as $u_k'(t)$ are nonincreasing on $[0,T]$. Therefore $u_k'(0)=0$ implies $u_k'(t) \le 0$ on $[0,T]$. By $u_k(T)=0$ we get $u_k(T) - u_k(t) \ge c\,(T-t)$, which yields $u_k'(T) \ge -c$ and

$$-c \le u_k'(t) \le 0 \quad \text{for} \quad t \in [0, T]. \tag{9.10}$$

Due to $u_k'(0) = 0$, there is $t_k \in (0, T]$ such that

$$-\nu \le u_k'(t) \le 0 \quad \text{for} \quad t \in [0, t_k].$$

If $t_k \ge \nu$, the last inequality in assumption (9.6) implies

$$\phi(u_k'(t)) \le -\varepsilon\, t \quad \text{for} \quad t \in [0, \nu]. \tag{9.11}$$

Assume that $t_k < \nu$ and $u_k'(t) < -\nu$ for $t \in (t_k, \nu]$. Then

$$\phi(u_k'(t)) \le -\varepsilon\, t \quad \text{for } t \in [0, t_k].$$

Since $\phi(u_k'(t)) < -\phi(\nu) < -\varepsilon t$ for $t \in (t_k, \nu]$, we get inequality (9.11) again. Integrating (9.11) over $[0, \nu]$ and using the the fact that u_k' is nonincreasing on $[0, T]$ and so u_k is concave here we deduce that

$$\frac{\nu_0}{T}\,(T-t) \le u_k(t) \le c\,(T-t) \quad \text{on} \quad [0, T],$$

where $\nu_0 = \displaystyle\int_0^{\nu} \phi^{-1}(\varepsilon s)\,\mathrm{d}s > 0$.

Step 3. *Convergence of the sequences* $\{u_k\}$ *and* $\{u'_k\}$.

Consider the sequence $\{u_k\}$. Choose an arbitrary interval $[a,b] \subset (0,T)$. By virtue of estimates (9.10) and (9.11) there is $k_0 \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq k_0$,

$$\frac{c}{k_0} \leq u_k(t) \leq c(T-t), \quad -c \leq u'_k(t) \leq -\frac{\varepsilon}{k_0} \quad \text{for} \quad t \in [a,b], \tag{9.12}$$

and hence there is $\psi \in L_1[a,b]$ such that

$$|f_k(t, u_k(t), u'_k(t))| \leq \psi(t) \quad \text{for a.e. } t \in [a,b]. \tag{9.13}$$

The sequences $\{u_k\}$ and $\{u'_k\}$ are bounded on $[0,T]$ and, due to inequality (9.13), $\{u'_k\}$ is equicontinuous on $[a,b]$. Therefore, using the Arzelà-Ascoli theorem and the diagonalization theorem, we can choose $u \in C[0,T] \cap C^1(0,T)$ and a subsequence of $\{u_k\}$ (which we denote for the sake of simplicity in the same way) such that

$$\begin{cases} \lim\limits_{k\to\infty} u_k = u \quad \text{uniformly on} \quad [0,T], \\ \lim\limits_{k\to\infty} u'_k = u' \quad \text{locally uniformly on} \quad (0,T). \end{cases} \tag{9.14}$$

Consequently, we have $u(T) = 0$.

Step 4. *Convergence of the sequence of approximate nonlinearities* $\{f_k\}$.

Let $\xi \in (0,T)$ be such that $f(\xi,\cdot,\cdot)$ is continuous on $(0,\infty) \times (-\infty,0)$. By estimate (9.12) there exist an interval $[a_\xi, b_\xi] \subset (0,T)$ and $k_\xi \in \mathbb{N}$ such that $\xi \in [a_\xi, b_\xi]$ and for each $k \geq k_\xi$

$$c(T-\xi) \geq u_k(\xi) > \frac{c}{k_\xi}, \quad -c \leq u'_k(\xi) < -\frac{\varepsilon}{k_\xi}, \quad [a_\xi, b_\xi] \subset [\tfrac{1}{k}, T - \tfrac{1}{k}].$$

Therefore $f_k(\xi, u_k(\xi), u'_k(\xi)) = f(\xi, u_k(\xi), u'_k(\xi))$ and, by virtue of property (9.14), we get

$$\lim_{k\to\infty} f_k(t, u_k(t), u'_k(t)) = f(t, u(t), u'(t)) \quad \text{for a.e. } t \in [0,T]. \tag{9.15}$$

Step 5. *The function* u *is a solution.*

Choose an arbitrary $t\in(0,T)$. Then there exists an interval $[a,b]\subset(0,T)$ such that $t,\ \frac{T}{2}\in[a,b]$ and inequality (9.13) holds for all sufficiently large k with $\psi\in L_1[a,b]$. Integrating the equality in (9.9) we get

$$\phi(u_k'(\tfrac{T}{2}))-\phi(u_k'(t))=\int_{\frac{T}{2}}^{t} f_k(s,u_k(s),u_k'(s))\,\mathrm{d}s.$$

Letting $k\to\infty$ and using conditions (9.13), (9.14), (9.15) and the Lebesgue dominated convergence theorem on $[a,b]$, we get

$$\phi(u'(\tfrac{T}{2}))-\phi(u'(t))=\int_{\frac{T}{2}}^{t} f(s,u(s),u'(s))\,\mathrm{d}s \quad \text{for each} \ \ t\in(0,T).$$

Therefore $\phi(u')\in AC_{loc}(0,T)$ satisfies

$$(\phi(u'(t)))'+f(t,u(t),u'(t))=0 \qquad \text{for a.e. } t\in[0,T]. \tag{9.16}$$

Further, according to problem (9.9), we have

$$\int_0^T f_k(s,u_k(s),u_k'(s))\,\mathrm{d}s=-\phi(u_k'(T))\le\phi(c) \quad \text{for each} \ \ k>\tfrac{2}{T},$$

which together with the nonnegativity of f_k and equality (9.15) yields, by the Fatou lemma, that $f(t,u(t),u'(t))\in L_1[0,T]$. Therefore, by equality (9.16), we have $\phi(u')\in AC[0,T]$. Moreover,

$$\begin{aligned}&|\phi(u_k'(t))|\\ &\quad\le\int_0^t|f_k(s,u_k(s),u_k'(s))-f(s,u(s),u'(s))|\,\mathrm{d}s+\int_0^t|f(s,u(s)u'(s))|\,\mathrm{d}s.\end{aligned}$$

for each $k>\frac{2}{T}$ and $t\in(0,T)$. So, by (9.15), for each $\varepsilon_0>0$ there exists $\delta>0$ such that

$$|\phi(u_k'(t))|<\varepsilon_0 \quad \text{for all} \ \ t\in[0,\delta],\ k>\tfrac{2}{T}.$$

Then

$$\begin{aligned}|\phi(u'(t))|&\le|\phi(u'(t))-\phi(u_k'(t))|+|\phi(u_k'(t))|\\ &<|\phi(u'(t))-\phi(u_k'(t))|+\varepsilon_0 \quad \text{for all} \ \ t\in(0,\delta],\ k>\tfrac{2}{T}.\end{aligned}$$

Hence, by property (9.14),

$$|\phi(u'(t))| \le \lim_{k\to\infty} |\phi(u'(t)) - \phi(u_k'(t))| + \varepsilon_0 = \varepsilon_0 \quad \text{for each } t \in (0,\delta).$$

It means that $u'(0) = \lim_{t\to 0+} u'(t) = 0$. We have proved that u is a solution of problem (9.1). ☐

EXAMPLE. Let $\alpha > 0$, $\beta, \gamma, \delta \ge 0$ be arbitrary numbers. By Theorem 9.5 the problem

$$u'' + \frac{1}{t^\gamma (1-t)^\delta}\left(\frac{1}{u^\alpha} + u^\beta + 1\right)(1 + (u')^3) = 0, \quad u'(0) = u(1) = 0$$

has a solution $u \in AC^1[0,1]$ satisfying

$$0 < u(t) \le 1 - t, \quad -1 \le u'(t) < 0 \quad \text{for} \quad t \in (0,1).$$

Note that Theorem 9.5 guarantees solvability of our problem even for the nonlinearity

$$f(t,x,y) = \frac{1}{t^\gamma (1-t)^\delta}\left(\frac{1}{x^\alpha} + x^\beta + 1\right)(1 + y^3)$$

having a strong space singularity $(\alpha \ge 1)$ at $x = 0$.

9.2 Problem arising in the shallow membrane caps theory

Now we will investigate solvability of the singular differential equation

$$(t^3 u')' + t^3 \left(\frac{1}{8u^2} - \frac{a_0}{u} - b_0 t^{2\gamma-4}\right) = 0 \tag{9.17}$$

subject to the mixed boundary conditions

$$\lim_{t\to 0+} t^3 u'(t) = 0, \quad u(1) = 0, \tag{9.18}$$

where $a_0 \ge 0$, $b_0 > 0$, $\gamma > 1$, arising in the theory of shallow membrane caps , see Baxley and Robinson [34], Dickey [72], Johnson [112], Kannan and O'Regan [113]. For close problems see Agarwal and O'Regan [13], [14], Baxley [32], Goldberg [97].

Our aim is to prove existence of a positive w-solution to problem (9.17), (9.18) which is defined as follows.

Definition 9.6. A function u is a *positive w-solution of problem* (9.17), (9.18) if u satisfies the following conditions:

(i) $u \in C[0,1] \cap C^2(0,1)$,

(ii) $u(t) > 0$ for all $t \in (0,1)$,

(iii) u satisfies equation (9.17) for $t \in (0,1)$ and the boundary conditions (9.18).

Note that problem (9.17), (9.18) is singular and exhibits both the time and space singularities. We can see this by transforming equation (9.17) into the first order system by means of the substitution $x_1(t)=u(t)$, $x_2(t)=t^3\, u'(t)$, viz.

$$x_1' = f_1(t, x_1, x_2) = \frac{1}{t^3}\, x_2,$$
$$x_2' = f_2(t, x_1, x_2) = -t^3 \left(\frac{1}{8\, x_1^2} - \frac{a_0}{x_1} - b_0\, t^{2\gamma-4} \right).$$

Because of the term $\frac{1}{t^3}$ in the first equation we see that the function f_1 is not integrable in t on any right neighborhood of $t = 0$ and so f_1 has a time singularity at $t = 0$. Moreover, the function f_2 is not continuous in x_1, having a space singularity at $x_1 = 0$. In particular, since the powers of x_1 in f_2 are -2 and -1, f_2 has strong singularities at $x_1 = 0$.

The present investigation of problem (9.17), (9.18) is strongly motivated by the results given in Kannan and O'Regan [113], where the second boundary condition in (9.18) has the form $u(1) = u_1 > 0$. It turns out that in this case the solutions of problem (9.17), (9.18) are positive on $[0,1]$ and consequently, the problem has no space singularities. As a technical tool in the existence proof, the lower and upper functions method has been used in [113]. In our case, since $u_1 = 0$, we need to cope with a space singularity at $u = 0$ and therefore it is necessary to generalize the approach. To this aim we consider the following auxiliary boundary value problem

$$(p(t)u')' + p(t)\,q(t)\,f(t,u) = 0, \tag{9.19}$$

$$\lim_{t\to 0+} p(t)\,u'(t) = 0, \quad u(T) = 0, \tag{9.20}$$

where $p\colon [0,T]\to\mathbb{R}$, $q\colon (0,T]\to\mathbb{R}$ are continuous and f satisfies the Carathéodory conditions on the set $(0,T)\times\mathcal{D}$, where $\mathcal{D}\subset\mathbb{R}$.

Definition 9.7. A function $u\in C[0,T]\cap C^1(0,T]$ with $pu'\in AC[0,T]$ is called *a solution of problem* (9.19), (9.20) if it satisfies equation (9.19) for a.e. $t\in[0,T]$ and if the boundary conditions (9.20) hold.

We now define a lower function and an upper function of problem (9.19), (9.20).

Definition 9.8. A function $\sigma\in C[0,T]$ is called a *lower function of problem* (9.19), (9.20) if there is a finite set $\Sigma\subset(0,T)$ such that $\sigma'(\tau+), \sigma'(\tau-)\in\mathbb{R}$ for each $\tau\in\Sigma$ and $p\,\sigma'\in AC_{loc}((0,T)\setminus\Sigma)$. Moreover, σ has to satisfy

$$\begin{cases} (p(t)\,\sigma'(t))' + p(t)\,q(t)\,f(t,\sigma(t)) \ge 0 \quad \text{for a.e. } t\in[0,T], \\ \lim_{t\to 0+} p(t)\,\sigma'(t) \ge 0, \quad \sigma(T)\le 0, \\ \sigma'(\tau-) < \sigma'(\tau+) \quad \text{for each } \tau\in\Sigma. \end{cases} \tag{9.21}$$

If the inequalities in (9.21) are reversed, then σ is called an *upper function of problem* (9.19), (9.20).

Note that, in contrast to Definition 9.3, Definition 9.8 admits lower and upper functions having first derivatives unbounded at the endpoints $t=0$ and $t=T$.

For the subsequent analysis we make the following assumptions:

$$p\in C[0,T],\ q\in C(0,T], \quad p(t)>0,\ q(t)>0 \text{ for } t\in(0,T], \tag{9.22}$$

$$\int_0^T p(s)\,q(s)\,\mathrm{d}s<\infty, \quad \int_0^T \frac{1}{p(t)}\left(\int_0^t p(s)\,q(s)\,\mathrm{d}s\right)\mathrm{d}t<\infty, \tag{9.23}$$

$$f \text{ satisfies the } L_\infty - \text{Carathéodory conditions on } [0,T]\times\mathbb{R}, \tag{9.24}$$

i.e., $f \in Car([0,T]\times\mathbb{R})$ and for each compact set $\mathcal{K} \subset \mathbb{R}$ there is a constant $m_{\mathcal{K}} > 0$ such that

$$|f(t,x)| \le m_{\mathcal{K}} \quad \text{for a.e. } t \in [0,T] \text{ and all } x \in \mathcal{K}.$$

To prove the existence of a solution u to problem (9.19), (9.20) we use the lower and upper functions method. The related fundamental statement is given in Theorem 9.9.

Theorem 9.9. *Let σ_1 and σ_2 be a lower function and an upper function of problem (9.19), (9.20). Assume that $\sigma_1(t) \le \sigma_2(t)$ for $t \in [0,T]$. Let us also assume that conditions (9.22), (9.23) and (9.24) hold. Then problem (9.19), (9.20) has a solution u satisfying estimate (9.4). If, moreover,*

$$\lim_{t\to 0+} \frac{1}{p(t)} \int_0^t p(s)\, q(s)\, \mathrm{d}s = 0, \tag{9.25}$$

then

$$u \in C^1[0,T], \quad u'(0) = 0. \tag{9.26}$$

Proof. *Step* 1. *Existence of a solution u of an auxiliary problem.*

For a.e. $t \in [0,T]$ and all $x \in \mathbb{R}$ define

$$f^*(t,x) = \begin{cases} f(t,\sigma_2(t)) - \dfrac{x-\sigma_2(t)}{x-\sigma_2(t)+1} & \text{if } x > \sigma_2(t), \\ f(t,x) & \text{if } \sigma_1(t) \le x \le \sigma_2(t), \\ f(t,\sigma_1(t)) + \dfrac{\sigma_1(t)-x}{\sigma_1(t)-x+1} & \text{if } x < \sigma_1(t), \end{cases}$$

and consider the equation

$$(p(t)\, u')' + p(t)\, q(t)\, f^*(t,u) = 0. \tag{9.27}$$

Define an operator $\mathcal{F} : C[0,T] \to C[0,T]$ by

$$(\mathcal{F}u)(t) := \int_t^T \left(\frac{1}{p(\tau)} \int_0^\tau p(s)q(s)f^*(s,u(s))\, \mathrm{d}s \right) \mathrm{d}\tau. \tag{9.28}$$

Since condition (9.24) holds, we can find $m^* \in (0,\infty)$ such that

$$|f^*(t,x)| \le m^* \quad \text{for a.e. } t \in [0,T] \text{ and all } x \in \mathbb{R}. \tag{9.29}$$

Therefore, due to assumption (9.23), $\mathcal{F}$ is continuous and compact, and the Schauder fixed point theorem guarantees that a fixed point $u \in C[0,T]$ of $\mathcal{F}$ exists. According to (9.28) we now have

$$u(t) = \int_t^T \left(\frac{1}{p(\tau)} \int_0^\tau p(s)q(s)f^*(s,u(s))\mathrm{d}s \right) \mathrm{d}\tau \quad \text{for} \quad t \in [0,T].$$

Hence, u satisfies equation (9.27) a.e. in $[0,T]$, the boundary conditions (9.20) hold, and $pu' \in AC[0,T]$. The assumptions $p \in C[0,T]$ and $p > 0$ on $(0,T]$ result in $u \in C^1(0,T]$. This means that u is a solution of problem (9.27), (9.20).

If additionally, assumption (9.25) holds, we can use inequality (9.29) to conclude

$$\begin{aligned}
\lim_{t\to 0+} |u'(t)| &= \lim_{t\to 0+} \left| -\frac{1}{p(t)} \int_0^t p(s)\,q(s)\,f^*(s,u(s))\,\mathrm{d}s \right| \\
&\le m^* \lim_{t\to 0+} \frac{1}{p(t)} \int_0^t p(s)\,q(s)\,\mathrm{d}s = 0.
\end{aligned}$$

Finally, we set $u'(0) = \lim_{t\to 0+} u'(t) = 0$, and assertion (9.26) follows.

Step 2. The function u *solves equation* (9.19).

To this end we verify that estimate (9.4) holds. Let us set $v = u - \sigma_2(t)$ and assume that

$$\max\{v(t) \colon t \in [0,T]\} = v(t_0) > 0.$$

Since $\sigma_2(T) \ge 0$ and $u(T) = 0$, it follows that $t_0 \in [0,T)$. Moreover, Definitions 9.7 and 9.8 imply that $t_0 \notin \Sigma$, because $v'(\tau-) < v'(\tau+)$ for $\tau \in \Sigma$. Let $t_0 = 0$. We have from (9.20) and the inequality $\lim_{t\to 0+} p(t)\,\sigma_2'(t) \le 0$ (see (9.21)) that $\lim_{t\to 0+} p(t)\,v'(t) \ge 0$. Let $\lim_{t\to 0+} p(t)\,v'(t) > 0$. Then $\lim_{t\to 0+} v'(t) > 0$, which contradicts the assumption that v has its maximum value at $t_0 = 0$. Therefore, $\lim_{t\to 0+} p(t)\,v'(t) = 0$ holds. Now, let $t_0 \in (0,T) \setminus \Sigma$. Then $v'(t_0) = 0$. So, we have $t_0 \in [0,T) \setminus \Sigma$ and we can find a $\delta > 0$ such that $v(t) > 0$ on $(t_0, t_0 + \delta) \subset (0, T)$ and

$$\begin{aligned}(p(t)\, v'(t))' &= (p(t)\, u'(t))' - (p(t)\, \sigma_2'(t))' \\ &\ge -p(t)\, q(t) \left(f(t, \sigma_2(t)) - \frac{u(t) - \sigma_2(t)}{u(t) - \sigma_2(t) + 1} \right) + p(t)\, q(t)\, f(t, \sigma_2(t)) \\ &= p(t)\, q(t)\, \frac{v(t)}{v(t) + 1} > 0\end{aligned}$$

a.e. in $(t_0, t_0 + \delta)$. This yields

$$0 < \int_{t_0}^{t} p(s)\, q(s)\, \frac{v(s)}{v(s)+1}\, \mathrm{d}s \le \int_{t_0}^{t} (p(s)\, v'(s))'\, \mathrm{d}s = p(t)\, v'(t)$$

for $t \in (t_0, t_0 + \delta)$, contradicting the fact that v has its maximum at t_0. We have shown that $u(t) \le \sigma_2(t)$ for $t \in [0, T]$. The inequality $\sigma_1(t) \le u(t)$ for $t \in [0, T]$ follows analogously. The definition of f^* finally implies that u is also a solution of equation (9.19). □

Example. Let $a > 0$, $\varepsilon > 0$, $p(t) = t^a$, $q(t) = t^{\varepsilon - 1}$. Then p and q satisfy conditions (9.22), (9.23) and (9.25).

The main difficulty in applying Theorem 9.9 is to find a lower function σ_1 and an upper function σ_2 for problem (9.19), (9.20) which are well ordered, i.e., $\sigma_1(t) \le \sigma_2(t)$ for $t \in [0, T]$. If $f(\cdot, x)$ in equation (9.19) changes its sign on $[0, T]$, for instance, then lower and upper functions of problem (9.19), (9.20) have to be nonconstant and therefore their computation can be difficult. In Lemmas 9.10 and 9.11 we present two pairs of well ordered lower and upper functions for problem (9.17), (9.18), where $f(t, x) = \frac{1}{8\, x^2} - \frac{a_0}{x} - b_0\, t^{\,2\gamma - 4}$ changes its sign on $(0, 1) \times (0, \infty)$.

Lemma 9.10. *Let $\gamma \ge 2$. Then there exist constants ν_*, $c_* \in (0, \infty)$ such that for each $\nu \in (0, \nu_*]$ and $c \ge c_*$ the functions*

$$\sigma_1(t) = \nu\, (t + \nu)\, (1 - t) \quad \text{and} \quad \sigma_2(t) = c\, \sqrt{1 - t^2} \quad \text{for } t \in [0, 1], \tag{9.30}$$

are a lower and an upper function of problem (9.17), (9.18).

Proof. It follows from (9.30) that $\sigma_1'(t) = \nu(1 - 2t - \nu)$ and $\sigma_2'(t) = \frac{-ct}{\sqrt{1 - t^2}}$ for $t \in [0, 1)$. Thus,

$$\lim_{t \to 0+} t^3 \sigma_1'(t) = 0, \quad \lim_{t \to 0+} t^3 \sigma_2'(t) = 0, \quad \sigma_1(1) = \sigma_2(1) = 0. \tag{9.31}$$

By inserting σ_1 into equation (9.17) we obtain

$$\begin{aligned}(t^3\sigma_1'(t))' + t^3\left(\frac{1}{8\,\sigma_1^2(t)} - \frac{a_0}{\sigma_1(t)} - b_0 t^{2\gamma-4}\right)&\\ = t^2\left(\nu\varphi_1(t,\nu) + \frac{t}{\nu^2(1-t)^2(t+\nu)^2}\varphi_2(t,\nu)\right) &\quad \text{for} \quad t\in(0,1),\end{aligned}$$

where

$$\begin{aligned}\varphi_1(t,\nu) &= 3 - 3\,\nu - 8\,t,\\ \varphi_2(t,\nu) &= \frac{1}{8} - a_0\,\nu\,(1-t)\,(t+\nu) - b_0\,t^{2\gamma-4}\,\nu^2\,(1-t)^2\,(t+\nu)^2.\end{aligned}$$

Let us choose $\nu_0 \in (0, \frac{3}{11})$ such that

$$a_0\,\nu_0\,(1+\nu_0) + b_0\,\nu_0^2\,(1+\nu_0)^2 < \frac{1}{16}.$$

Then for all $\nu \in (0,\nu_0)$ we have $\varphi_1(t,\nu) > 0$, $\varphi_2(t,\nu) > 0$ for $t\in[0,\nu]$. Moreover, we can find $\nu_* \in (0,\nu_0)$ such that

$$\nu_*\,\varphi_1(t,\nu_*) + \frac{1}{16\,\nu_*(1+\nu_*)^2} > 0 \quad \text{for } t\in[\nu_*,1],$$

and consequently, for all $\nu\in(0,\nu_*]$, we have

$$(t^3\,\sigma_1'(t))' + t^3\left(\frac{1}{8\,\sigma_1^2(t)} - \frac{a_0}{\sigma_1(t)} - b_0 t^{2\gamma-4}\right) \geq 0 \quad \text{for } t\in[0,1). \tag{9.32}$$

By properties (9.31) and (9.32), σ_1 is a lower function of problem (9.17), (9.18).

We now insert σ_2 into equation (9.17) and obtain

$$(t^3\,\sigma_2'(t))' + t^3\left(\frac{1}{8\,\sigma_2^2(t)} - \frac{a_0}{\sigma_2(t)} - b_0\,t^{2\gamma-4}\right) \leq t^3\,\varphi_3(t,c) \quad \text{for } t\in[0,1),$$

where

$$\varphi_3(t,c) = -c(1-t^2)^{-\frac{3}{2}}\left(1 - \frac{\sqrt{1-t^2}}{8\,c^3}\right) \leq -c(1-t^2)^{-\frac{3}{2}}\left(1 - \frac{1}{8\,c^3}\right) \leq 0$$

for $t \in [0,1)$ and $c \geq \frac{1}{2}$. Hence, for all $c \in [\frac{1}{2}, \infty)$ in the definition of σ_2, cf. (9.30), we have

$$(t^3 \sigma_2'(t))' + t^3 \left(\frac{1}{8\,\sigma_2^2(t)} - \frac{a_0}{\sigma_2(t)} - b_0\, t^{2\gamma-4} \right) \leq 0 \quad \text{for } t \in [0,1). \tag{9.33}$$

Finally, we conclude from properties (9.31) and (9.33) that σ_2 is an upper function of problem (9.17), (9.18), which completes the proof. ☐

Lemma 9.11. *Assume $\gamma \in (1,2)$. Then there exist constants ν_*, $c_* \in (0,\infty)$ such that for each $\nu \in (0, \nu_*]$ and $c \geq c_*$ the functions*

$$\sigma_1(t) = \nu\, t^{2-\gamma}\,(1-t) \quad \textit{and} \quad \sigma_2(t) = c\sqrt{1-t^2} \quad \textit{for } t \in [0,1] \tag{9.34}$$

are a lower and an upper function of problem (9.17), (9.18).

Proof. We first calculate the derivatives of σ_1 and σ_2 :

$$\sigma_1'(t) = \nu\, t^{1-\gamma}\,(2-\gamma-(3-\gamma)\,t), \quad \sigma_2'(t) = \frac{-c\,t}{\sqrt{1-t^2}} \quad \text{for } t \in [0,1).$$

Clearly, σ_1 and σ_2 satisfy condition (9.31). By inserting σ_1 into equation (9.17) we obtain

$$\begin{aligned}
&(t^3\, \sigma_1'(t))' + t^3 \left(\frac{1}{8\,\sigma_1^2(t)} - \frac{a_0}{\sigma_1(t)} - b_0 t^{2\gamma-4} \right) \\
&\quad = \nu\, t^{3-\gamma}\,[(4-\gamma)\,(2-\gamma) - (5-\gamma)\,(3-\gamma)t] + \frac{t^{2\gamma-1}}{\nu^2\,(1-t)^2}\,\psi(t,\nu)
\end{aligned}$$

for $t \in (0,1)$, where $\psi(t,\nu) = \frac{1}{8} - a_0\,\nu(1-t)\,t^{2-\gamma} - b_0\nu^2\,(1-t)^2$. We now find a constant $\nu_0 > 0$ such that $\psi(t,\nu) > 0$ for $t \in [0,1]$ and $\nu \in (0,\nu_0]$. Furthermore, if $t_0 = \frac{(4-\gamma)(2-\gamma)}{(5-\gamma)(3-\gamma)}$, we have $(4-\gamma)(2-\gamma) - (5-\gamma)(3-\gamma)\,t \geq 0$ for $t \in [0, t_0]$. Further, we get

$$\lim_{\nu \to 0+} \frac{t^{2\gamma-1}}{\nu^2\,(1-t)^2}\,\psi(t,\nu) = \infty$$

uniformly on $[t_0, 1)$. Therefore, we are able to provide a constant $\nu_* \in (0, \nu_0]$ such that for any $\nu \in (0, \nu_*]$ in the definition of σ_1, see (9.34),

$$(t^3\, \sigma_1'(t))' + t^3 \left(\frac{1}{8\sigma_1^2(t)} - \frac{a_0}{\sigma_1(t)} - b_0 t^{2\gamma-4} \right) > 0 \quad \text{for } t \in (0,1)$$

holds. This means that, by condition (9.31), σ_1 is a lower function of problem (9.17), (9.18). Since σ_2 is as in Lemma 9.10, we can similarly show that it is an upper function, and the result follows. $\square$

The main results characterizing solvability of problem (9.17), (9.18) are contained in the next two theorems. We begin with considering the case $\gamma \geq 2$. This study will utilize results provided by Lemma 9.10.

Theorem 9.12. *Let $\gamma \geq 2$. Then there exists a positive w-solution u of problem* (9.17), (9.18). *Moreover, this solution satisfies*

$$u(0) > 0, \qquad \lim_{t \to 0+} u'(t) = 0. \tag{9.35}$$

Proof. *Step* 1. *Construction of auxiliary functions f_k.*

Our arguments are based on Theorem 9.9. We set

$$T = 1, \; p(t) = t^3, \; q(t) = 1, \; f(t,x) = \frac{1}{8\,x^2} - \frac{a_0}{x} - b_0\, t^{2\gamma-4}.$$

It is easily seen that p and q satisfy conditions (9.22), (9.23), and (9.25), but condition (9.24) does not hold for f. To remedy the situation, we introduce a sequence of functions f_k, $k \in \mathbb{N}$, $k > 3$. Let σ_1 and σ_2 be specified by formulas (9.30), where $\nu \leq \nu_* \leq \frac{1}{9}$ and $c \geq c_* > 1$, and for $t \in [0,1]$, $x \in \mathbb{R}$ define

$$f_k(t,x) = \begin{cases} 0 & \text{if } \; t \in [0, \frac{1}{k}), \\ f(t, \alpha(t,x)) & \text{if } \; t \in [\frac{1}{k}, 1 - \frac{1}{k}], \\ 1 & \text{if } \; t \in (1 - \frac{1}{k}, 1], \end{cases} \tag{9.36}$$

where

$$\alpha(t,x) = \begin{cases} \sigma_2(t) & \text{if } \; x > \sigma_2(t), \\ x & \text{if } \; \sigma_1(t) \leq x \leq \sigma_2(t), \\ \sigma_1(t) & \text{if } \; x < \sigma_1(t). \end{cases}$$

Note that all functions f_k satisfy condition (9.24).

Step 2. *Lower and upper functions.*

By Lemma 9.10, σ_1 is a lower function and σ_2 is an upper function of problem (9.17), (9.18). For $k \in \mathbb{N}$, $k > 3$, consider the equation

$$(t^3\, u')' + t^3\, f_k(t,u) = 0. \tag{9.37}$$

Since $k > 3$, we have

$$(t^3\, \sigma_1'(t))' = t^2\, \nu\, (3 - 3\,\nu - 8\,t) \geq 0 \qquad \text{for } t \in [0, \tfrac{1}{k}),$$

and

$$(t^3\, \sigma_1'(t))' + t^3 = t^2\, (\nu\, (3 - 3\,\nu - 8\,t) + t) > 0 \qquad \text{for } t \in (1 - \tfrac{1}{k}, 1].$$

Similarly,

$$(t^3\, \sigma_2'(t))' = -c\, t^3\, (1 - t^2)^{-\frac{3}{2}}\, (4 - 3\,t^2) \leq 0 \qquad \text{for } t \in [0, \tfrac{1}{k}),$$

and

$$(t^3\, \sigma_2'(t))' + t^3 = t^3\, (-c\, (1 - t^2)^{-\frac{3}{2}} (4 - 3\,t^2) + 1) < 0 \quad \text{for } t \in (1 - \tfrac{1}{k}, 1).$$

Therefore σ_1 and σ_2 are also lower and upper functions of problem (9.37), (9.18). With no loss of generality, we can choose $\nu \in (0, \nu_*)$ and $c \geq c_*$ in such a way that $\nu(1+\nu) < c$ holds. Then $\sigma_1 \leq \sigma_2$ on $[0,1]$ and, by Theorem 9.9, problem (9.37), (9.18) has a solution $u_k \in C^1[0,1]$ for $k > 3$ satisfying

$$\sigma_1(t) \leq u_k(t) \leq \sigma_2(t) \quad \text{for } t \in [0,1], \quad u_k'(0) = 0. \tag{9.38}$$

Step 3. *Convergence of the sequence of approximate solutions* $\{u_k\}$.

We regard the sequence $\{u_k\}$ of solutions to problem (9.37), (9.18) as a sequence of approximations to u, and first discuss the convergence properties of $\{u_k\}$. Let us choose an interval $[0,b] \subset [0,1)$. Then there exists an index $k_1 \in \mathbb{N}$ such that $[0,b] \subset [0, 1 - \frac{1}{k}]$ for $k \geq k_1$, and due to the boundary conditions (9.18) and equation (9.37) we have

$$t^3\, u_k'(t) + \int_0^t s^3\, f_k(s, u_k(s))\, \mathrm{d}s = 0 \quad \text{for } t \in [0,b] \tag{9.39}$$

for $k \geq k_1$. Let

$$r_b = \min\{\sigma_1(t) \colon t \in [0,b]\}, \quad m_b = \frac{1}{8\, r_b^2} + \frac{a_0}{r_b}.$$

It follows from the first formula in (9.30) that $r_b > 0$ and hence, (9.36) and (9.38) yield

$$|t^3 f_k(t, u_k(t))| \le m_b\, t^3 + b_0\, t^{2\gamma-1} \quad \text{for } \; t \in [0, b] \text{ and } k \ge k_1. \tag{9.40}$$

Consequently, by equality (9.39),

$$|t^3 u_k'(t)| \le \frac{m_b}{4}\, t^4 + \frac{b_0}{2\gamma}\, t^{2\gamma} \quad \text{for } \; t \in [0, b] \text{ and } k \ge k_1. \tag{9.41}$$

Due to estimates (9.38), (9.41) and the condition $\gamma \ge 2$, the sequences $\{u_k\}$ and $\{u_k'\}$ are bounded on $[0, b]$, which implies that $\{u_k\}$ is equicontinuous on $[0, b]$. Furthermore, for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $t_1,\, t_2 \in [0, b]$ and $k \ge k_1$, if $|t_1 - t_2| < \delta$ holds, then

$$|t_1^3\, u_k'(t_1) - t_2^3\, u_k'(t_2)| \le m_b \left| \int_{t_1}^{t_2} s^3 \,\mathrm{d}s \right| + b_0 \left| \int_{t_1}^{t_2} s^{2\gamma-1} \mathrm{d}s \right| < \varepsilon.$$

Hence the sequence $\{t^3 u_k'\}$ is equicontinuous on $[0, b]$ and, by inequality (9.41), it is bounded on $[0, b]$. The Arzelà-Ascoli theorem now implies that there exists a subsequence $\{u_{k_\ell}\} \subset \{u_k\}$ such that

$$\lim_{\ell\to\infty} u_{k_\ell} = u \quad \text{uniformly on } [0, b],$$

$$\lim_{\ell\to\infty} t^3\, u_{k_\ell}' = t^3\, u' \quad \text{locally uniformly on } (0, b].$$

Finally, by the diagonalization theorem, we find a subsequence (for simplicity we denote it by $\{u_k\}$) satisfying

$$\begin{cases} \lim_{k\to\infty} u_k = u \quad \text{locally uniformly on } [0, 1), \\ \lim_{k\to\infty} t^3\, u_k' = t^3\, u' \quad \text{locally uniformly on } (0, 1). \end{cases} \tag{9.42}$$

Step 4. *Properties of the function* u.

We conclude the proof by establishing the properties of the limit function u. By (9.41) and (9.42) we obtain

$$|t^3 u'(t)| \le \frac{m_b}{4}\, t^4 + \frac{b_0}{2\gamma}\, t^{2\gamma} \quad \text{for } \; t \in (0, \delta).$$

Therefore

$$\lim_{t\to 0+} t^3 u'(t) = 0 \tag{9.43}$$

and due to (9.38) and (9.42) we have

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \quad \text{for } t\in[0,1), \quad u\in C[0,1). \tag{9.44}$$

Since $\sigma_1(1) = \sigma_2(1) = 0$, we get

$$\lim_{t\to 1-} u(t) = 0. \tag{9.45}$$

Moreover, (9.36) and (9.42) imply

$$\lim_{k\to\infty} t^3 f_k(t, u_k(t)) = t^3 f(t, u(t)) \quad \text{for } t\in(0,1).$$

Consequently, due to (9.40) we can use the Lebesgue dominated convergence theorem on $[0,b]$. Having in mind that $b\in(0,1)$ is arbitrary and letting $k\to\infty$ in equality (9.39), we conclude that

$$t^3\, u'(t) + \int_0^t s^3\, f(s, u(s))\mathrm{d}s = 0 \quad \text{for } t\in(0,1). \tag{9.46}$$

Thus $u\in C^2(0,1)$ and u satisfies equation (9.17) for $t\in(0,1)$. Setting $u(1) = \lim_{t\to 1-} u(t)$, we obtain $u(1) = 0$ and $u\in C[0,1]$. These smoothness properties of u together with properties (9.43)–(9.46) guarantee that u is a positive w-solution of problem (9.17), (9.18). It remains to show that assertion (9.35) holds. The first condition in (9.35) follows from $\sigma_1(0) > 0$. The second condition results by noting that

$$\lim_{t\to 0+} |u'(t)| \le \lim_{t\to 0+} \frac{m_b}{4}\, t + \lim_{t\to 0+} \frac{b_0}{2\gamma}\, t^{2\gamma-3} = 0$$

due to (9.41) and (9.42). □

Now, we apply the results from Lemma 9.11 in order to cover the case $\gamma\in(1,2)$.

Theorem 9.13. *Let* $\gamma\in(1,2)$. *Then there exists a w-positive solution* u *of problem* (9.17), (9.18). *If* $\gamma > \frac{3}{2}$, *then assertion* (9.35) *holds and for* $\gamma = \frac{3}{2}$ *the w-solution* u *satisfies*

$$u(0) > 0, \qquad \lim_{t\to 0+} u'(t) = \frac{b_0}{3}. \tag{9.47}$$

Proof. *Step* 1. The arguments for the construction of the auxiliary sequence $\{f_k\}$ and of the upper function σ_2 are analogous to those given in *Steps* 1 and 2 of the proof of Theorem 9.12. The only difference is the definition of the lower function σ_1 which is now specified by the first formula in (9.34), with $\nu \le \nu_* \le \frac{1}{8}$. By Lemma 9.11, σ_1 is a lower function of problem (9.17), (9.18). Choose $k_0 \in \mathbb{N}$, $k_0 > \frac{4}{2-\gamma}$. For $k \ge k_0$ we have

$$(t^3\,\sigma_1'(t))' = \nu\, t^{3-\gamma}\left((4-\gamma)(2-\gamma) - (5-\gamma)(3-\gamma)\, t\right) \ge 0$$

if $t \in [0, \frac{1}{k})$ and

$$(t^3\,\sigma_1'(t))' + t^3 = \nu\, t^{3-\gamma}\left((4-\gamma)(2-\gamma) - (5-\gamma)(3-\gamma)\, t\right) + t^3 > 0$$

if $t \in (1-\frac{1}{k}, 1]$, which implies that σ_1 is also a lower function of problem (9.37), (9.18). Since σ_2 is the same as in the previous proof, it is an upper function of problem (9.37), (9.18). Now, arguing as in the proof of Theorem 9.12, we get the sequence $\{u_k\}$ of solutions to problems (9.37), (9.18), $k \in \mathbb{N}$, $k \ge k_0$. Also, $u_k \in C^1[0,1]$ and it satisfies conditions (9.38).

Step 2. Consider an interval $[0,b] \subset [0,1)$ and the sequence $\{u_k\}$, $k \in \mathbb{N}$, $k \ge k_0$. Then equality (9.39) holds. If we put

$$a_1 = \frac{a_0}{\nu(1-b)}, \quad b_1 = \frac{1}{8\nu^2(1-b)^2} + b_0,$$

we get

$$\frac{t^3}{8\,\sigma_1^2(t)} + \frac{a_0\, t^3}{\sigma_1(t)} + b_0\, t^{2\gamma-1} \le a_1\, t^{\gamma+1} + b_1\, t^{2\gamma-1} \quad \text{for } t \in [0,b]. \tag{9.48}$$

Assume that $k_1 \ge k_0$. Thus, (9.38), (9.39) and (9.48) yield

$$|t^3\, f_k(t, u_k(t))| \le a_1\, t^{\gamma+1} + b_1\, t^{2\gamma-1}, \quad |t^3\, u_k'(t)| \le \frac{a_1}{\gamma+2}\, t^{\gamma+2} + \frac{b_1}{2\gamma}\, t^{2\gamma}$$

for $t \in [0,b]$ provided that $k \ge k_1$. Hence, for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $t_1, t_2 \in [0,b]$ and $k \ge k_1$,

$$|t_1 - t_2| < \delta \Rightarrow |t_1^3\, u_k'(t_1) - t_2^3\, u_k'(t_2)| \le \left| \int_{t_1}^{t_2} (a_1\, t^{\gamma+1} + b_1\, t^{2\gamma-1})\, \mathrm{d}t \right| < \varepsilon$$

and

$$|t_1-t_2|<\delta \Rightarrow |u_k(t_1)-u_k(t_2)| \le \left|\int_{t_1}^{t_2}\left(\frac{a_1}{\gamma+2}\,t^{\gamma-1}+\frac{b_1}{2\gamma}\,t^{2\gamma-3}\right)\mathrm{d}t\right| < \varepsilon.$$

Therefore the sequences $\{u_k\}$ and $\{t^3 u_k'\}$ are bounded and equicontinuous on $[0,b]$ and condition (9.42) results due to the arguments given in the proof of Theorem 9.12.

Step 3. Properties (9.44), (9.45), (9.46) and $u\in C[0,1]\cap C^2(0,1)$ can be shown as in the proof of Theorem 9.12. Equality (9.46) leads to

$$t^3\,u'(t)=\int_0^t \frac{s^3}{u^2(s)}\left(a_0\,u(s)-\tfrac{1}{8}\right)\mathrm{d}s+\frac{b_0}{2\gamma}\,t^{2\gamma} \quad \text{for} \quad t\in(0,1). \tag{9.49}$$

Assume that $u(0)>0$. Having in mind that $\gamma>1$ and $\lim_{t\to 0+} t^3u'(t)=0$, equation (9.49) yields

$$\lim_{t\to 0+}\int_0^t s^3\left(\frac{a_0}{u(s)}-\frac{1}{8\,u^2(s)}\right)\mathrm{d}s=0.$$

Hence, by the l'Hospital rule, we have

$$\begin{aligned}\lim_{t\to 0+} u'(t) &= \lim_{t\to 0+}\frac{1}{t^3}\int_0^t s^3\left(\frac{a_0}{u(s)}-\frac{1}{8\,u^2(s)}\right)\mathrm{d}s+\lim_{t\to 0+}\frac{b_0}{2\gamma}\,t^{2\gamma-3}\\ &=\tfrac{1}{3}\lim_{t\to 0+}\frac{t}{u^2(t)}\left(a_0\,u(t)-\tfrac{1}{8}\right)+\frac{b_0}{2\gamma}\lim_{t\to 0+}t^{2\gamma-3}=\frac{b_0}{2\gamma}\lim_{t\to 0+}t^{2\gamma-3},\end{aligned}$$

i.e.

$$\lim_{t\to 0+} u'(t)=\frac{b_0}{2\gamma}\lim_{t\to 0+}t^{2\gamma-3}. \tag{9.50}$$

On the other hand, since $\sigma_1(0)=0$ and $\lim_{t\to 0+}\sigma_1'(t)=\infty$, we conclude that

$$u(0)=0 \;\Rightarrow\; \lim_{t\to 0+} u'(t)=\infty \tag{9.51}$$

by virtue of the first inequality in (9.44).

Now, assume that $\gamma\ge\frac{3}{2}$. If $u(0)=0$, then there is $\delta_0\in(0,1)$ such that

$$\int_0^t \frac{s^3}{u^2(s)}\left(a_0\,u(s)-\tfrac{1}{8}\right)\mathrm{d}s<0 \quad \text{for } t\in(0,\delta_0)$$

and consequently, by (9.49),

$$u'(t) < \frac{b_0}{2\gamma} t^{2\gamma-3} < c_0 \quad \text{for } t \in (0, \delta_0),$$

where $c_0 = \frac{b_0}{2\gamma} \delta_0^{2\gamma-3} \in (0, \infty)$. This contradicts (9.51). So we have proved that if $\gamma \geq \frac{3}{2}$, then $u(0) > 0$. Thus if $\gamma > \frac{3}{2}$, relation (9.50) gives $\lim_{t\to 0+} u'(t) = 0$ and if $\gamma = \frac{3}{2}$, we get from (9.50) that $\lim_{t\to 0+} u'(t) = \frac{b_0}{3}$. This completes the proof. □

REMARK 9.14. Consider a positive w-solution u of problem (9.17), (9.18) for $\gamma > 1$. We first recapitulate the behaviour of u' at the singular point $t = 0$.

If $\gamma \in (\frac{3}{2}, \infty)$, then, by (9.35), we know that $u'(0+) = 0$ holds.

If $\gamma = \frac{3}{2}$, then, by (9.47), the derivative satisfies $u'(0+) = \frac{b_0}{3}$.

If $\gamma \in (1, \frac{3}{2})$, then $u'(0+) = \infty$. This follows from (9.51) for $u(0) = 0$ and from (9.50) for $u(0) > 0$.

Now, let us consider the singular point $t = 1$. Since $u(1) = 0$, there exists $\xi \in (0,1)$ such that $a_0 u(t) \leq \frac{1}{16}$ for $t \in [\xi, 1]$. Let σ_2 be an upper function given by the second formula in (9.30) and let $u(t) \leq \sigma_2(t)$ for $t \in [0,1]$. Then it follows that

$$-\int_\xi^t \frac{\mathrm{d}s}{u^2(s)} \leq -\int_\xi^t \frac{\mathrm{d}s}{\sigma_2^2(s)} \leq -\frac{1}{2\,c^2}\int_\xi^t \frac{\mathrm{d}s}{1-s} = \frac{1}{2\,c^2} \ln\left(1-\frac{t}{1-\xi}\right), \quad t \in (\xi, 1).$$

Integration of equation (9.17) yields

$$\begin{aligned} t^3 u'(t) &= \xi^3 u'(\xi) + \int_\xi^t \frac{s^3}{u^2(s)} \left(a_0 u(s) - \tfrac{1}{8}\right) \mathrm{d}s + b_0 \int_\xi^t s^{2\gamma-1}\, \mathrm{d}s \\ &\leq \xi^3 u'(\xi) + \frac{\xi^3}{32\,c^2} \ln\left(1-\frac{t}{1-\xi}\right) + \frac{b_0}{2\gamma} \quad \text{for } t \in (\xi, 1), \end{aligned}$$

and therefore $\lim_{t\to 1-} t^3 u'(t) = u'(1-) = -\infty$.

Bibliographical notes

Theorem 9.5 can be found in Rachůnková [157] or in Rachůnková, Staněk and Tvrdý [163]. Theorem 9.9 was proved in Rachůnková, Koch, Pulverer and Weinmüller [158] and its similar version appeared in Agarwal and O'Regan [14]. Theorems 9.12 and 9.13 are taken from [158].

In literature we can find other papers studying singular mixed boundary value problems. Let us mention here the monographs Kiguradze and Shekhter [118], O'Regan [148], Rachůnková, Staněk and Tvrdý [163] and references contained in them.

Chapter 10

Nonlocal problems

In this chapter we discuss problems for second order differential equations with ϕ – Laplacian and with nonlinearities which may have singularities in both their space variables. Boundary conditions under discussion are generally nonlinear and nonlocal. Using regularization and sequential techniques we present general existence principles for the solvability of regular and singular nonlocal problems and show their applications.

We consider singular differential equations of the form

$$(\phi(u'))' = f(t, u, u') \tag{10.1}$$

where

$$\phi \text{ is an increasing and odd homeomorphism and } \phi(\mathbb{R}) = \mathbb{R}. \tag{10.2}$$

Here $f \in Car([0,T] \times \mathcal{D})$, $\mathcal{D} \subset \mathbb{R}^2$ is not necessarily closed, and f may have singularities in its space variables.

Let $\mathcal{A}$ denote the set of functionals $\alpha : C^1[0,T] \to \mathbb{R}$ which are

(a) continuous and

(b) bounded, that is, $\alpha(\Omega)$ is bounded for any bounded $\Omega \subset C^1[0,T]$.

For $\alpha, \beta \in \mathcal{A}$, consider the (generally nonlinear and nonlocal) boundary conditions

$$\alpha(u) = 0, \quad \beta(u) = 0, \tag{10.3}$$

where α and β satisfy the *compatibility condition* requiring that for each $\mu \in [0,1]$ there exists a solution of the problem

$$(\phi(u'))' = 0, \quad \alpha(u) - \mu\,\alpha(-u) = 0, \; \beta(u) - \mu\,\beta(-u) = 0.$$

This is true if and only if the system

$$\begin{cases} \alpha(A+Bt) - \mu\,\alpha(-A-Bt) = 0, \\ \beta(A+Bt) - \mu\,\beta(-A-Bt) = 0 \end{cases} \tag{10.4}$$

has a solution $(A,B) \in \mathbb{R}^2$ for each $\mu \in [0,1]$.

Definition 10.1. A function $u : [0,T] \to \mathbb{R}$ is said to be a *solution of problem* (10.1), (10.3) if $\phi(u') \in AC[0,T]$, u satisfies the boundary conditions (10.3) and $(\phi(u'(t)))' = f(t,u(t),u'(t))$ holds for almost all $t \in [0,T]$.

Special cases of the boundary conditions (10.3) are the Dirichlet (Neumann; mixed; periodic and Sturm-Liouville type) boundary conditions which we get setting $\alpha(x) = x(0)$, $\beta(x) = x(T)$ ($\alpha(x) = x'(0)$, $\beta(x) = x'(T)$; $\alpha(x) = x(0)$, $\beta(x) = x'(T)$; $\alpha(x) = x(0) - x(T)$, $\beta(x) = x'(0) - x'(T)$ and $\alpha(x) = a_0 x(0) + a_1 x'(0)$, $\beta(x) = b_0\, x(T) + b_1\, x'(T)$).

Existence principles

In order to give an existence result for problem (10.1), (10.3), we use regularization and sequential techniques. For this purpose consider the sequence of regular differential equations

$$(\phi(u'))' = f_n(t,u,u') \tag{10.5}$$

where $f_n \in Car([0,T] \times \mathbb{R}^2)$, $n \in \mathbb{N}$. Each function f_n is constructed in such a way that

$$f_n(t,x,y) = f(t,x,y) \qquad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathcal{Q}_n$$

where $\mathcal{Q}_n \subset \mathcal{D}$ and, roughly speaking, $\mathcal{Q}_n$ converges to $\mathcal{D}$ as $n \to \infty$.

Let $h \in Car([0,T] \times \mathbb{R}^2)$ and consider the regular differential equation

$$(\phi(u'))' = h(t,u,u'). \tag{10.6}$$

The next result is an existence principle which can be used for solving the nonlocal regular problem (10.6), (10.3).

Theorem 10.2 (**Existence principle for nonlocal regular problems**).
Assume (10.2), $h \in Car([0,T] \times \mathbb{R}^2)$ *and* $\alpha,\ \beta \in \mathcal{A}$. *Suppose there exist positive constants* S_0 *and* S_1 *such that*

$$\|u\|_\infty < S_0, \quad \|u'\|_\infty < S_1$$

for each $\lambda \in [0,1]$ *and each solution* u *to the problem*

$$\begin{cases} (\phi(u'))' = \lambda\, h(t,u,u'), \\ \alpha(u) = 0, \quad \beta(u) = 0. \end{cases} \tag{10.7}$$

Also assume that there exist positive constants Λ_0 *and* Λ_1 *such that*

$$|A| < \Lambda_0, \quad |B| < \Lambda_1 \tag{10.8}$$

for each $\mu \in [0,1]$ *and each solution* $(A,B) \in \mathbb{R}^2$ *of system* (10.4).

Then problem (10.6), (10.3) *has a solution.*

Proof. Set

$$\Omega = \Big\{ x \in C^1[0,T] : \|x\|_\infty < \max\{S_0, \Lambda_0 + \Lambda_1 T\},\ \|x'\|_\infty < \max\{S_1, \Lambda_1\} \Big\}.$$

Then Ω is an open, bounded and symmetric with respect to $0 \in C^1[0,T]$ subset of the space $C^1[0,T]$. Define an operator $\mathcal{P}\colon [0,1] \times \overline{\Omega} \to C^1[0,T]$ by the formula

$$\begin{cases} \mathcal{P}(\lambda,x)(t) = x(0) + \alpha(x) \\ \qquad + \displaystyle\int_0^t \phi^{-1}\Big(\phi(x'(0) + \beta(x)) + \lambda \int_0^s h(v, x(v), x'(v))\, \mathrm{d}v\Big)\, \mathrm{d}s. \end{cases} \tag{10.9}$$

It follows from $h \in Car([0,T] \times \mathbb{R}^2)$, the continuity of α, β, ϕ and from the Lebesgue dominated convergence theorem that $\mathcal{P}$ is a continuous operator. We claim that the set $\mathcal{P}([0,1] \times \overline{\Omega})$ is relatively compact in $C^1[0,T]$. Indeed, since $\overline{\Omega}$ is bounded in $C^1[0,T]$, we have

$$|\alpha(x)| \le r, \quad |\beta(x)| \le r, \quad |h(t,x(t),x'(t))| \le \varrho(t)$$

for a.e. $t \in [0,T]$ and all $x \in \overline{\Omega}$, where $r > 0$ is a constant and $\varrho \in L_1[0,T]$. Then

$$|\mathcal{P}(\lambda,x)(t)| \le \max\{S_0, \Lambda_0 + \Lambda_1 T\} + r + T\,\phi^{-1}\left(\phi\left(\max\{S_1,\Lambda_1\} + r\right) + \|\varrho\|_1\right),$$

$$|\mathcal{P}(\lambda,x)'(t)| \le \phi^{-1}\left(\phi\left(\max\{S_1,\Lambda_1\}+r\right)+\|\varrho\|_1\right)$$

and

$$|\phi[\mathcal{P}(\lambda,x)'(t_2)]-\phi[\mathcal{P}(\lambda,x)'(t_1)]| \le \left|\int_{t_1}^{t_2}\varrho(t)\,\mathrm{d}t\right|$$

for $t,t_1,t_2\in[0,T]$ and $(\lambda,x)\in[0,1]\times\overline{\Omega}$. Here $\mathcal{P}(\lambda,x)'(t)=\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}(\lambda,x)(t)$. Hence the set $\mathcal{P}([0,1]\times\overline{\Omega})$ is bounded in $C^1[0,T]$ and the set

$$\{\phi(\mathcal{P}(\lambda,x)'):(\lambda,x)\in[0,1]\times\overline{\Omega}\}$$

is equicontinuous on $[0,T]$. Using the fact that ϕ^{-1} is an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and

$$|\mathcal{P}(\lambda,x)'(t_2)-\mathcal{P}(\lambda,x)'(t_1)|=|\phi^{-1}\left(\phi(\mathcal{P}(\lambda,x)'(t_2))\right)-\phi^{-1}\left(\phi(\mathcal{P}(\lambda,x)'(t_1))\right)|,$$

we deduce that $\{\mathcal{P}(\lambda,x)':(\lambda,x)\in[0,1]\times\overline{\Omega}\}$ is also equicontinuous on $[0,T]$. Now the Arzelà-Ascoli theorem shows that $\mathcal{P}([0,1]\times\overline{\Omega})$ is relatively compact in $C^1[0,T]$. Thus $\mathcal{P}$ is a compact operator.

Suppose that x_0 is a fixed point of the operator $\mathcal{P}(1,\cdot)$. Then

$$x_0(t)=x_0(0)+\alpha(x_0)+\int_0^t\phi^{-1}\left(\phi(x_0'(0)+\beta(x_0))+\int_0^s h(v,x_0(v),x_0'(v))\,\mathrm{d}v\right)\mathrm{d}s.$$

Hence $\alpha(x_0)=0$, $\beta(x_0)=0$ and x_0 is a solution of the differential equation (10.6). Therefore x_0 is a solution of problem (10.6), (10.3) and to prove our theorem, it suffices to show that

$$\deg(\mathcal{I}-\mathcal{P}(1,\cdot),\Omega)\ne 0 \tag{10.10}$$

where $\mathcal{I}$ is the identity operator on $C^1[0,T]$. To see this let us define a compact operator $\mathcal{K}:[0,1]\times\overline{\Omega}\to C^1[0,T]$ by

$$\mathcal{K}(\mu,x)(t)=x(0)+\alpha(x)-\mu\,\alpha(-x)+[x'(0)+\beta(x)-\mu\,\beta(-x)]\,t.$$

Then $\mathcal{K}(1,\cdot)$ is odd (i.e. $\mathcal{K}(1,-x)=-\mathcal{K}(1,x)$ for $x\in\overline{\Omega}$) and

$$\mathcal{K}(0,\cdot)=\mathcal{P}(0,\cdot). \tag{10.11}$$

If $\mathcal{K}(\mu_1, x_1) = x_1$ for some $\mu_1 \in [0,1]$ and $x_1 \in \overline{\Omega}$, then

$$x_1(t) = x_1(0) + \alpha(x_1) - \mu_1\, \alpha(-x_1) + [x_1'(0) + \beta(x_1) - \mu_1\, \beta(-x_1)]\, t$$

for $t \in [0,T]$. Thus $x_1(t) = A_1 + B_1\, t$ where $A_1 = x_1(0) + \alpha(x_1) - \mu_1\, \alpha(-x_1)$ and $B_1 = x_1'(0) + \beta(x_1) - \mu_1\, \beta(-x_1)$, so

$$\alpha(x_1) - \mu_1\, \alpha(-x_1) = 0 \quad \text{and} \quad \beta(x_1) - \mu_1\, \beta(-x_1) = 0.$$

Hence

$$\alpha(A_1 + B_1\, t) - \mu_1\, \alpha(-A_1 - B_1\, t) = 0,$$

$$\beta(A_1 + B_1\, t) - \mu_1\, \beta(-A_1 - B_1\, t) = 0.$$

Therefore $|A_1| < \Lambda_0, |B_1| < \Lambda_1$ and $\|x_1\|_\infty < \Lambda_0 + \Lambda_1 T$, $\|x_1'\|_\infty < \Lambda_1$, which gives $x_1 \notin \partial\Omega$. Now, by the Borsuk antipodal theorem and the homotopy property (see the Leray-Schauder degree theorem with $\mathcal{U} = \overline{\Omega}$),

$$\deg(\mathcal{I} - \mathcal{K}(0,\cdot), \Omega) = \deg(\mathcal{I} - \mathcal{K}(1,\cdot), \Omega) \neq 0. \tag{10.12}$$

Finally, assume that $\mathcal{P}(\lambda_*, x_*) = x_*$ for some $\lambda_* \in [0,1]$ and $x_* \in \overline{\Omega}$. Then x_* is a solution of problem (10.7) with $\lambda = \lambda_*$ and, by our assumptions, $\|x_*\|_\infty < S_0$ and $\|x_*'\|_\infty < S_1$. Hence $x_* \notin \partial\,\Omega$ and the homotopy property yields

$$\deg(\mathcal{I} - \mathcal{P}(0,\cdot), \Omega) = \deg(\mathcal{I} - \mathcal{P}(1,\cdot), \Omega).$$

This together with (10.11) and (10.12) implies (10.10). We have proved that problem (10.6), (10.3) has a solution. □

Remark 10.3. If functionals $\alpha, \beta \in \mathcal{A}$ are linear, then they satisfy the compatibility condition. Indeed, system (10.4) has the form

$$A\, \alpha(1) + B\, \alpha(t) = 0,$$
$$A\, \beta(1) + B\, \beta(t) = 0$$

for each $\mu \in [0,1]$ and we see that it is always solvable in $\mathbb{R}^2$. The set of all its solutions (A, B) is bounded if and only if $\alpha(1)\, \beta(t) - \alpha(t)\, \beta(1) \neq 0$. In such a case system (10.4) has only the trivial solution $(A, B) = (0, 0)$. This is satisfied for example for the Dirichlet conditions but not for the periodic conditions.

Let us consider the singular problem (10.1), (10.3). By regularization and sequential techniques we construct an approximate sequence of the regular problems (10.5), (10.3) for whose solvability Theorem 10.2 can be used. Existence results for problem (10.1), (10.3) can be proved by the following existence principle which is based on a combination of the Lebesgue dominated convergence theorem with the Fatou lemma.

Let I and J be intervals containing 0. Assume that

$$\begin{cases} f \in Car([0,T] \times \mathcal{D}) \quad \text{where } \mathcal{D} = (I \setminus \{0\}) \times (J \setminus \{0\}) \\ \quad \text{and } f \text{ may have space singularities at } x = 0 \text{ and } y = 0. \end{cases} \tag{10.13}$$

Theorem 10.4 (**Existence principle for nonlocal singular problems**). *Assume* (10.2) *and* (10.13). *Let* $f_n \in Car([0,T] \times \mathbb{R}^2)$ *satisfy*

$$\begin{cases} 0 \le f_n(t,x,y) \le p(t,|x|,|y|) \\ \qquad \textit{for a.e. } t \in [0,T] \textit{ and each } x,y \in \mathbb{R} \setminus \{0\},\ n \in \mathbb{N}, \\ \qquad \textit{where } p \in Car([0,T] \times (0,\infty)^2). \end{cases} \tag{10.14}$$

Suppose that for each $n \in \mathbb{N}$ *the regular problem* (10.5), (10.3) *has a solution* u_n *and there exists a subsequence* $\{u_{k_n}\}$ *of* $\{u_n\}$ *converging in* $C^1[0,T]$ *to some* u. *Then* u *is a solution of problem* (10.1), (10.3) *if* u *and* u' *have a finite number of zeros and*

$$\lim_{n\to\infty} f_{k_n}(t, u_{k_n}(t), u'_{k_n}(t)) = f(t, u(t), u'(t)) \quad \textit{for a.e. } t \in [0,T]. \tag{10.15}$$

Proof. Assume that (10.15) is true and $0 \le \xi_1 < \xi_2 < \cdots < \xi_m \le T$ are all zeros of u and u'. We have $\|u_{k_n}\|_\infty \le L$ and $\|u'_{k_n}\|_\infty \le L$ for each $n \in \mathbb{N}$, where L is a positive constant, and

$$\phi(u'_{k_n}(T)) - \phi(u'_{k_n}(0)) = \int_0^T f_{k_n}(t, u_{k_n}(t), u'_{k_n}(t))\,\mathrm{d}t, \quad n \in \mathbb{N}.$$

It follows from assumptions (10.14), (10.15) and from the Fatou lemma that

$$\int_0^T f(t,u(t),u'(t))\,\mathrm{d}t \le 2\phi(L).$$

Hence $f(t,u(t),u'(t)) \in L_1[0,T]$. Set $\xi_0 = 0$ and $\xi_{m+1} = T$. We claim that for all $j \in \{0,1,\dots,m\}$ such that $\xi_j < \xi_{j+1}$, the equality

$$\phi(u'(t)) = \phi\Big(u'(\frac{\xi_j + \xi_{j+1}}{2})\Big) + \int_{(\xi_j+\xi_{j+1})/2}^{t} f(s,u(s),u'(s))\,\mathrm{d}s \tag{10.16}$$

is satisfied for $t \in [\xi_j, \xi_{j+1}]$. Indeed, let $j \in \{0,1,\dots,m\}$ and $\xi_j < \xi_{j+1}$. Let us look at the interval $[\xi_j + \delta, \xi_{j+1} - \delta]$ where $\delta \in (0, \frac{\xi_j+\xi_{j+1}}{2})$. We know that $|u| > 0$ and $|u'| > 0$ on (ξ_j, ξ_{j+1}) and consequently, there exists a positive ε such that $|u(t)| \ge \varepsilon$, $|u'(t)| \ge \varepsilon$ for $t \in [\xi_j + \delta, \xi_{j+1} - \delta]$. Hence there exists $n_0 \in \mathbb{N}$ such that $|u_{k_n}(t)| \ge \frac{\varepsilon}{2}$, $|u'_{k_n}(t)| \ge \frac{\varepsilon}{2}$ for $t \in [\xi_j + \delta, \xi_{j+1} - \delta]$ and $n \ge n_0$. This yields (see (10.14))

$$0 \le f_{k_n}(t, u_{k_n}(t), u'_{k_n}(t)) \le \psi(t)$$

for a.e. $t \in [\xi_j + \delta, \xi_{j+1} - \delta]$ and all $n \ge n_0$, where

$$\psi(t) = \sup\{p(t,u,v) : t \in [\xi_j+\delta, \xi_{j+1}-\delta],\ u,v \in [\frac{\varepsilon}{2}, L]\} \in L_1[\xi_j+\delta, \xi_{j+1}-\delta].$$

Letting $n \to \infty$ in

$$\phi(u'_{k_n}(t)) = \phi\Big(u'_{k_n}(\frac{\xi_j + \xi_{j+1}}{2})\Big) + \int_{(\xi_j+\xi_{j+1})/2}^{t} f_{k_n}(s, u_{k_n}(s), u'_{k_n}(s))\,\mathrm{d}s$$

gives (10.16) for $t \in [\xi_j + \delta, \xi_{j+1} - \delta]$ by the Lebesgue dominated convergence theorem. Since $\delta \in (0, \frac{\xi_j+\xi_{j+1}}{2})$ is arbitrary, equality (10.16) is true on the interval (ξ_j, ξ_{j+1}) and using the fact that $f(t,u(t),u'(t)) \in L_1[0,T]$, (10.16) holds also at $t = \xi_j$ and ξ_{j+1}. From equality (10.16) for $t \in [\xi_j, \xi_{j+1}]$ and $0 \le j \le m$ it follows that $\phi(u') \in AC[0,T]$ and that

$$(\phi(u'(t)))' = f(t,u(t),u'(t)) \quad \text{for a.e.} \quad t \in [0,T].$$

Finally, $\alpha(u_{k_n}) = 0$ and $\beta(u_{k_n}) = 0$ and the continuity of α and β yields $\alpha(u){=}0$ and $\beta(u){=}0$. Hence u is a solution of problem (10.1), (10.3). $\square$

Application of existence principles

The next part of this chapter is devoted to an application of the above existence principles. We consider equation (10.1) where f satisfies the Carathéodory conditions on a subset of $[0,T] \times \mathbb{R}^2$ and $f(t,x,y)$ may have space singularities at $x = 0$ and $y = 0$. Along with equation (10.1) we discuss the nonlocal boundary conditions

$$\min\{u(t) : t \in [0,T]\} = 0, \ \gamma(u') = 0, \ \gamma \in \mathcal{B}, \tag{10.17}$$

where $\mathcal{B}$ denotes the set of functionals $\gamma : C[0,T] \to \mathbb{R}$ which are

(a) continuous, $\gamma(0) = 0$, and

(b) increasing, i.e. $x, y \in C[0,T]$ and $x < y$ on $(0,T) \implies \gamma(x) < \gamma(y)$.

EXAMPLE. Let $n \in \mathbb{N}$, $0 \le a < b \le T$, $\xi \in (0,T)$ and $0 < t_1 < \ldots < t_n < T$. Then the functionals

$$\gamma_1(x) = x(\xi) + \max\{x(t) : t \in [a,b]\}, \quad \gamma_2(x) = \int_a^b x^{2n+1}(t)\,\mathrm{d}t,$$

$$\gamma_3(x) = \int_0^T e^{x(t)}\,\mathrm{d}t - T, \qquad \gamma_4(x) = \sum_{j=1}^n x(t_j)$$

belong to the set $\mathcal{B}$. The functionals

$$\gamma_5(x) = x(0) \ \text{ and } \ \gamma_6(x) = x(0) + x(T)$$

satisfy condition (a) of $\mathcal{B}$ but do not satisfy condition (b). Hence $\gamma_5, \gamma_6 \notin \mathcal{B}$.

Notice that the boundary conditions (10.17) satisfy the compatibility condition. Indeed, if we put $\alpha(x) = \min\{x(t) : t \in [0,T]\}$ and $\beta(x) = \gamma(x')$ in (10.4), we obtain the system

$$\max\{A+Bt : t \in [0,T]\} - \mu \max\{-A-Bt : t \in [0,T]\} = 0,$$
$$\gamma(B) - \mu\,\gamma(-B) = 0,$$

having the solution $(A,B) = (0,0) \in \mathbb{R}^2$ for each $\mu \in [0,1]$.

We are interested in conditions on the functions ϕ and f in (10.1) which guarantee solvability of problem (10.1), (10.17) for each $\gamma \in \mathcal{B}$. Notice that if f is positive, then solutions of problem (10.1), (10.17) have singular points of type II.

We will need the following result.

Lemma 10.5. *Let* $\gamma \in \mathcal{B}$ *and let* $\gamma(u) = 0$ *for some* $u \in C[0,T]$*. Then* u *vanishes at some point of* $(0,T)$*.*

Proof. To obtain a contradiction, suppose that $u(t) \neq 0$ for all $t \in (0,T)$. Then $u > 0$ or $u < 0$ on $(0,T)$. Therefore $\gamma(u) > \gamma(0) = 0$ or $\gamma(u) < \gamma(0) = 0$, contrary to $\gamma(u) = 0$. Consequently, $u(\xi) = 0$ for some $\xi \in (0,T)$. □

We state an existence result for problem (10.1), (10.17).

Theorem 10.6. *Let* (10.2) *hold. Further, assume that* $f \in Car([0,T] \times \mathcal{D})$*, where* $\mathcal{D} = (0,\infty) \times (\mathbb{R} \setminus \{0\})$*, and that the following conditions are satisfied:*

$$\begin{cases} \varphi(t) \le f(t,x,y) \le (h_1(x) + h_2(x))[\omega_1(\phi(|y|)) + \omega_2(\phi(|y|))] \\ \qquad \text{for a.e. } t \in [0,T] \text{ and each } (x,y) \in \mathcal{D}, \text{ where} \\ \qquad \varphi \in L_\infty[0,T] \text{ is positive,} \\ \qquad h_1, \omega_1 \in C[0,\infty) \text{ are positive and nondecreasing,} \\ \qquad h_2, \omega_2 \in C(0,\infty) \text{ are positive and nonincreasing,} \\ \qquad \displaystyle\int_0^1 h_2(s)\,\mathrm{d}s < \infty \end{cases} \tag{10.18}$$

and

$$\liminf_{x \to \infty} \frac{V(x)}{H(Tx)} > 1 \tag{10.19}$$

where

$$V(x) = \int_0^{\phi(x)} \frac{\phi^{-1}(s)}{\omega_1(s+1) + \omega_2(s)}\,\mathrm{d}s, \; H(x) = \int_0^x [h_1(s+1) + h_2(s)]\,\mathrm{d}s \tag{10.20}$$

for $x \in [0,\infty)$*.*

Then for each $\gamma \in \mathcal{B}$*, problem* (10.1), (10.17) *has a solution* u *such that* $\phi(u') \in AC[0,T]$*.*

In order to prove Theorem 10.6 we use regularization and sequential techniques. To this end, for each $n \in \mathbb{N}' = \{n \in \mathbb{N} : \phi(\frac{1}{n}) \le 1\}$, define

$f_n \in Car([0,T]\times\mathbb{R}^2)$ by the formula

$$f_n(t,x,y) = \begin{cases} f(t,x,y) & \text{for } \ t\in[0,T],\ x\ge \frac{1}{n},\ |y|\ge \frac{1}{n},\\ f(t,\frac{1}{n},y) & \text{for } \ t\in[0,T],\ x< \frac{1}{n},\ |y|\ge \frac{1}{n},\\ \frac{n}{2}\,\left[f_n(t,x,\frac{1}{n})\,(y+\frac{1}{n}) - f_n(t,x,-\frac{1}{n})\,(y-\frac{1}{n})\right] & \\ & \text{for } \ t\in[0,T],\ x\in\mathbb{R},\ |y|<\frac{1}{n}. \end{cases}$$

Then assumption (10.18) gives

$$\begin{cases} \varphi(t) \le f_n(t,x,y) \\ \qquad \le [h_1(|x|+1)+h_2(|x|)]\,[\omega_1(\phi(|y|)+1)+\omega_2(\phi(|y|))] \end{cases} \tag{10.21}$$

for a.e. $t\in[0,T]$ and each $x,y\in\mathbb{R}\setminus\{0\}$, $n\in\mathbb{N}'$.

Consider the regular differential equation

$$(\phi(u'))' = f_n(t,u,u') \tag{10.22}$$

where $n\in\mathbb{N}'$.

For the proof of Theorem 10.6 the following lemma is essential.

Lemma 10.7. *Let the assumptions of Theorem 10.6 be satisfied and let $\gamma\in\mathcal{B}$. Then for each $n\in\mathbb{N}'$, problem (10.22), (10.17) has a solution u_n such that $\phi(u_n')\in AC[0,T]$ and*

$$\begin{cases} -u_n'(t) \ge \phi^{-1}\Big(\displaystyle\int_t^{\xi_n}\varphi(s)\,\mathrm{d}s\Big), \quad u_n(t) \ge \displaystyle\int_t^{\xi_n}\phi^{-1}\Big(\displaystyle\int_s^{\xi_n}\varphi(v)\,\mathrm{d}v\Big)\,\mathrm{d}s \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{for } \ t\in[0,\xi_n], \\ u_n'(t) \ge \phi^{-1}\Big(\displaystyle\int_{\xi_n}^{t}\varphi(s)\,\mathrm{d}s\Big), \quad u_n(t) \ge \displaystyle\int_{\xi_n}^{t}\phi^{-1}\Big(\displaystyle\int_{\xi_n}^{s}\varphi(v)\,\mathrm{d}v\Big)\,\mathrm{d}s \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{for } \ t\in[\xi_n,T], \end{cases} \tag{10.23}$$

where $\xi_n\in(0,T)$ is the unique zero both of u_n and of u_n'. In addition, the sequence $\{u_n\}_{n\in\mathbb{N}'}$ is bounded in $C^1[0,T]$ and $\{u_n'\}_{n\in\mathbb{N}'}$ is equicontinuous on $[0,T]$.

Proof. Let $n \in \mathbb{N}'$. First, using Theorem 10.2 with

$$\alpha(u) = \min\{u(t) : t \in [0,T]\} \quad \text{and} \quad \beta(u) = \gamma(u') \quad \text{for} \quad u \in C^1[0,T]$$

we prove existence of a solution of problem (10.22), (10.17). To this end, we consider the family of regular differential equations

$$(\phi(u'))' = \lambda f_n(t, u, u') \tag{10.24}$$

depending on the parameter $\lambda \in [0,1]$. Let u be a solution of problem (10.24), (10.17). If $\lambda = 0$ then $(\phi(u'))' = 0$ a.e. on $[0,T]$ and consequently, $u(t) = A + Bt$ where $A, B \in \mathbb{R}$. Since $\gamma(u') = 0$, Lemma 10.5 shows that $u'(\xi) = 0$ for some $\xi \in (0,T)$ and therefore $B = 0$. Now the condition $\min\{u(t) : t \in [0,T]\} = 0$ gives $A = 0$. Hence $u = 0$. Let $\lambda \in (0,1]$. Then $(\phi(u'(t)))' \geq \lambda \varphi(t) > 0$ for a.e. $t \in [0,T]$. Therefore $\phi(u')$ is increasing on $[0,T]$ and since ϕ is increasing on $\mathbb{R}$, u' is increasing on $[0,T]$. Due to Lemma 10.5, $u'(\xi) = 0$ for a unique $\xi \in (0,T)$ and from $\min\{u(t) : 0 \leq t \leq T\} = 0$ we see that $u(\xi) = 0$. Obviously, $u > 0$ on $[0,T] \setminus \{\xi\}$, $u' < 0$ on $[0,\xi)$, $u' > 0$ on $(\xi, T]$ and (see inequality (10.21))

$$(\phi(u'(t)))' \leq [h_1(u(t)+1) + h_2(u(t))]\,[\omega_1(\phi(|u'(t)|)+1) + \omega_2(\phi(|u'(t)|))]$$

for a.e. $t \in [0,T]$. Integrating

$$\frac{(\phi(u'(t)))' u'(t)}{\omega_1(1-\phi(u'(t))) + \omega_2(-\phi(u'(t)))} \geq [h_1(u(t)+1) + h_2(u(t))]u'(t) \tag{10.25}$$

over $[t,\xi] \subset [0,\xi]$ and

$$\frac{(\phi(u'(t)))' u'(t)}{\omega_1(\phi(u'(t))+1) + \omega_2(\phi(u'(t)))} \leq [h_1(u(t)+1) + h_2(u(t))]u'(t) \tag{10.26}$$

over $[\xi, t] \subset [\xi, T]$, we get

$$V(|u'(t)|) \leq H(u(t)) \quad \text{for} \quad t \in [0,\xi] \tag{10.27}$$

and

$$V(u'(t)) \leq H(u(t)) \quad \text{for} \quad t \in [\xi, T], \tag{10.28}$$

respectively, where the functions V and H are given in formula (10.20). From $u(t) = \int_\xi^t u'(s)\,\mathrm{d}s$ for $t \in [0,T]$ it follows that $\|u\|_\infty \leq T\|u'\|_\infty$ and therefore (10.27) and (10.28) imply $V(|u'(t)|) \le H(T\|u'\|_\infty)$ for $t \in [0,T]$. Hence

$$V(\|u'\|_\infty) \le H(T\|u'\|_\infty). \tag{10.29}$$

By assumption (10.19) we can find a positive constant S such that

$$V(x) > H(T\,x) \quad \text{whenever} \quad x \ge S.$$

This together with relation (10.29) implies that $\|u'\|_\infty < S$ and consequently, $\|u\|_\infty \le T\|u'\|_\infty < S\,T$. We have proved that $\|u\|_\infty < S\,T$ and $\|u'\|_\infty < S$ for all solutions of problem (10.24), (10.17) and each $\lambda \in [0,1]$.

We are now looking for all solutions $(A,B) \in \mathbb{R}^2$ of the system

$$\min\{A+Bt\colon t\in[0,T]\} - \mu\,\min\{-A-Bt\colon t\in[0,T]\} = 0, \tag{10.30}$$

$$\gamma(B) - \mu\,\gamma(-B) = 0, \tag{10.31}$$

where $\mu \in [0,1]$. Fix $\mu \in [0,1]$ and suppose that $(A,B) \in \mathbb{R}^2$ is a solution of system (10.30), (10.31). If $B \neq 0$ then Lemma 10.5 shows that $\gamma(B) \neq 0$ and since γ is an increasing functional and $\gamma(0) = 0$, we have $\gamma(-B)\,\gamma(B) < 0$, contrary to (see (10.31)) $\gamma(-B)\,\gamma(B) = \mu\,\gamma^2(-B) \ge 0$. Hence $B = 0$ and then $A = 0$, which follows immediately from (10.30). We have proved that $(A,B) = (0,0)$ is the unique solution of system (10.30), (10.31) for each $\mu \in [0,1]$.

By Theorem 10.2, for each $n \in \mathbb{N}'$ there exists a solution u_n of problem (10.22), (10.17). From the above consideration we have $u_n(\xi_n) = u_n'(\xi_n) = 0$ for a unique $\xi_n \in (0,T)$. Furthermore, $\{u_n\}_{n\in\mathbb{N}'}$ is bounded in $C^1[0,T]$ since $\|u_n\|_\infty < S\,T$ and $\|u_n'\|_\infty < S$ for $n \in \mathbb{N}'$. Integrating for each $n \in \mathbb{N}'$ the inequality $(\phi(u_n'(t)))' \ge \varphi(t)$ which holds for a.e. $t \in [0,T]$ and having in mind that $u_n(\xi_n) = u_n'(\xi_n) = 0$, we obtain (10.23).

It remains to verify that $\{u_n'\}_{n\in\mathbb{N}'}$ is equicontinuous on $[0,T]$. We know that $\{u_n\}_{n\in\mathbb{N}'}$ is bounded in $C^1[0,T]$. Thus $\{u_n\}_{n\in\mathbb{N}'}$ is equicontinuous on $[0,T]$ and so is $\{H(u_n)\}_{n\in\mathbb{N}'}$ since $H \in C[0,\infty)$. Hence for each $\varepsilon > 0$ we can find $\delta > 0$ such that

$$|H(u_n(t_2)) - H(u_n(t_1))| < \varepsilon, \quad n \in \mathbb{N}',$$

whenever $0 \le t_1 < t_2 \le T$ and $t_2 - t_1 < \delta$. Put

$$V^*(v) = \begin{cases} V(v) & \text{for } v \in [0,\infty) \\ -V(-v) & \text{for } v \in (-\infty,0). \end{cases}$$

Let $0 \le t_1 < t_2 \le T$ and $t_2 - t_1 < \delta$. If $t_2 \le \xi_n$, then integrating the inequality

$$\frac{(\phi(u_n'(t)))' u_n'(t)}{\omega_1(1-\phi(u_n'(t))) + \omega_2(-\phi(u_n'(t)))} \ge [h_1(u_n(t)+1)+h_2(u_n(t))]u_n'(t) \quad (10.32)$$

(see (10.25)) from t_1 to t_2 yields

$$0 < V^*(u_n'(t_2)) - V^*(u_n'(t_1)) \le H(u_n(t_1)) - H(u_n(t_2)) < \varepsilon$$

and if $t_1 \ge \xi_n$ then integrating the inequality

$$\frac{(\phi(u_n'(t)))' \, u_n'(t)}{\omega_1(\phi(u_n'(t))+1) + \omega_2(\phi(u_n'(t)))} \le [h_1(u_n(t)+1)+h_2(u_n(t))]u_n'(t) \quad (10.33)$$

(see (10.26)) from t_1 to t_2 gives

$$0 < V^*(u_n'(t_2)) - V^*(u_n'(t_1)) \le H(u_n(t_2)) - H(u_n(t_1)) < \varepsilon.$$

Finally, if $t_1 < \xi_n < t_2$ then integrating inequality (10.32) over the interval $[t_1, \xi_n]$ and inequality (10.33) over the interval $[\xi_n, t_2]$, we obtain

$$0 < -V^*(u_n'(t_1)) \le H(u_n(t_1)) = H(u_n(t_1)) - H(u_n(\xi_n)) < \varepsilon$$

and

$$0 < V^*(u_n'(t_2)) \le H(u_n(t_2)) = H(u_n(t_2)) - H(u_n(\xi_n)) < \varepsilon.$$

We have proved that

$$0 < V^*(u_n'(t_2)) - V^*(u_n'(t_1)) < 2\,\varepsilon \quad \text{for all} \quad n \in \mathbb{N}'.$$

Consequently, the sequence $\{V^*(u_n')\}_{n\in\mathbb{N}'}$ is equicontinuous on $[0,T]$ and since $V^* \in C(\mathbb{R})$ is increasing and the sequence $\{u_n'\}_{n\in\mathbb{N}'}$ is bounded in $C[0,T]$, we conclude that $\{u_n'\}_{n\in\mathbb{N}'}$ is equicontinuous on $[0,T]$. $\square$

We are now in a position to prove Theorem 10.6.

Proof of Theorem 10.6. Fix $\gamma \in \mathcal{B}$. Due to Lemma 10.7, for each $n \in \mathbb{N}'$ there exists a solution u_n of problem (10.22), (10.17) satisfying inequalities (10.23) where $\xi_n \in (0,T)$ is the unique zero both of u_n and of u_n', the sequence $\{u_n\}_{n\in\mathbb{N}'}$ is bounded in $C^1[0,T]$ and $\{u_n'\}_{n\in\mathbb{N}'}$ is equicontinuous on $[0,T]$. By the Arzelà-Ascoli theorem and the Bolzano-Weierstrass Theorem, we may assume without loss of generality that $\{u_n\}_{n\in\mathbb{N}'}$ is convergent in $C^1[0,T]$ and $\{\xi_n\}_{n\in\mathbb{N}'}$ is convergent in $\mathbb{R}$. Let $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} \xi_n = \xi$. Then $u\in C^1[0,T]$ satisfies the nonlocal boundary conditions (10.17) and letting $n\to\infty$ in inequalities (10.23) we get

$$|u'(t)|\ge\phi^{-1}\Big(\int_t^\xi \varphi(s)\,\mathrm{d}s\Big),\quad u(t)\ge\int_t^\xi \phi^{-1}\Big(\int_s^\xi \varphi(v)\,\mathrm{d}v\Big)\,\mathrm{d}s \quad\text{for}\quad t\in[0,\xi]$$

and

$$u'(t)\ge\phi^{-1}\Big(\int_\xi^t \varphi(s)\,\mathrm{d}s\Big),\quad u(t)\ge\int_\xi^t \phi^{-1}\Big(\int_\xi^s \varphi(v)\,\mathrm{d}v\Big)\,\mathrm{d}s \quad\text{for}\quad t\in[\xi,T].$$

Hence ξ is the unique zero both of u and of u' and since $\gamma(u')=0$, Lemma 10.5 yields $\xi\in(0,T)$. Moreover,

$$\lim_{n\to\infty} f_n(t,u_n(t),u_n'(t)) = f(t,u(t),u'(t)) \quad\text{for a.e.}\quad t\in[0,T]$$

and (see inequality (10.21))

$$0\le f_n(t,x,y)\le p(t,|x|,|y|) \quad\text{for a.e.}\quad t\in[0,T] \quad\text{and all}\quad x,\,y\in\mathbb{R}\setminus\{0\},$$

where $p(t,z,v) = (h_1(z+1)+h_2(z))\,[\omega_1(\phi(v)+1)+\omega_2(\phi(v))]$ is continuous on $[0,T]\times(0,\infty)^2$. Hence Theorem 10.4 guarantees that $\phi(u')\in AC[0,T]$ and u is a solution of problem (10.1), (10.17). $\square$

EXAMPLE. Let $p\in(1,\infty)$, $\beta\in(0,1)$, $\alpha,\,\mu,\,\lambda,\,c_j\in(0,\infty)$, $j=1,2,3,4$, $\alpha+\mu<p-1$ and let $\varphi\in L_\infty[0,\infty)$ be positive. By Theorem 10.6, for each $\gamma\in\mathcal{B}$ the differential equation

$$(|u'|^{p-2}\,u')' = \varphi(t)\Big(1+c_1\,u^\alpha+\frac{c_2}{u^\beta}\Big)\Big(1+c_3\,|u'|^\mu+\frac{c_4}{|u'|^\lambda}\Big)$$

has a solution u satisfying conditions (10.17) and $|u'|^{p-2}u'\in AC[0,T]$.

Bibliographical notes

Theorem 10.2 was taken from Agarwal, O'Regan and Staněk [20] and from Rachůnková, Staněk and Tvrdý [163]. Theorem 10.4 was adapted from [163] and Theorem 10.6 from Staněk [186]. Other singular nonlocal problems for equation (10.1) may be found in [20] and Staněk [184], [185]. The paper [184] deals with the nonlocal boundary conditions

$$u(0) = u(T), \quad \max\{u(t) : t \in [0,T]\} = c \quad (c \in \mathbb{R}),$$

whereas [185] discusses conditions

$$u(0) = u(T) = -\gamma \, \min\{u(t) : t \in [0,T]\} \quad (\gamma \in (0,\infty)).$$

In [20] conditions $\min\{u(t) : t \in [0,T]\} = 0, \quad \alpha(u) = 0$ are considered where α belongs to the set of functionals $\alpha : C^1[0,T] \to \mathbb{R}$ which are (i) continuous, (ii) bounded and satisfy (iii) $x \in C^1[0,T], \quad \varepsilon\, x' > 0$ on $[0,T]$ for $\varepsilon \in \{-1,1\} \quad \Rightarrow \varepsilon\, \alpha(x) > 0$.

Chapter 11

Problems with a parameter

This chapter is devoted to a class of singular boundary value problems with the ϕ-Laplacian

$$(\phi(u'))' = \mu f(t, u, u'), \tag{11.1}$$

$$u \in \mathcal{S} \tag{11.2}$$

depending on the parameter μ. Here ϕ is an increasing homomorphism from $\mathbb{R}$ onto $\mathbb{R}$, f is a Carathéodory function on a set $[0,T]\times\mathcal{D}$, $\mathcal{D}\subset\mathbb{R}^2$, f may have singularities in both its space variables and $\mathcal{S}$ is a closed subset in $C^1[0,T]$. Usually the set $\mathcal{S}$ is described by three boundary conditions. Such conditions have for example the form

$$u(0) = 0, \quad u(T) = 0, \quad \max\{u(t) : 0 \le t \le T\} = A, \tag{11.3}$$

or

$$u(0) = 0, \quad u(T) = 0, \quad \int_0^T \sqrt{1 + (u'(t))^2}\,\mathrm{d}t = B, \tag{11.4}$$

where A, $B \in \mathbb{R}$. We note that problems (11.1), (11.3) and (11.1), (11.4) are singular boundary value problems depending on the parameter μ and we are looking for a value μ_* of the parameter μ for which the Dirichlet problem (11.1), $u(0) = u(T) = 0$ has a solution $u \in C^1[0,T]$ satisfying the third (nonlocal) condition in (11.3) or (11.4), $\phi(u') \in AC[0,T]$ and $(\phi(u'(t)))' = \mu_* f(t, u(t), u'(t))$ for a.e. $t \in [0,T]$. If problem (11.1), $u(0) = u(T) = 0$ has a unique solution for each μ from a subset of $\mathbb{R}$, then the shooting method can be applied for solving problems (11.1), (11.3) and (11.1), (11.4). However, in our considerations such assumption is not introduced. Our method for establishing the solvability of problem (11.1), (11.2) is based on a regularization and a sequential technique. We present an existence principle for solving problem (11.1), (11.2) and give its application to problem (11.1), (11.3).

Existence principle

Consider the family of auxiliary regular differential equations

$$(\phi(u'))' = \mu f_n(t, u, u') \tag{11.5}$$

depending on the parameters $\mu \in \mathbb{R}$ and $n \in \mathbb{N}$. Here $f_n \in Car([0,T] \times \mathbb{R}^2)$. The next existence principle for solving problem (11.1), (11.2) is closely related to the principle which is presented in Theorem 10.4.

Definition 11.1. A function $u : [0,T] \to \mathbb{R}$ with $\phi(u') \in AC[0,T]$ is called a *solution of problem* (11.1), (11.2) if there exists $\mu_u \in \mathbb{R}$ such that $(\phi(u'(t)))' = \mu_u f(t, u(t), u'(t))$ for a.e. $t \in [0,T]$ and $u \in \mathcal{S}$.

Let I and J be intervals containing 0. Assume that

$$\begin{cases} f \in Car([0,T] \times \mathcal{D}) \text{ where } \mathcal{D} = (I \setminus \{0\}) \times (J \setminus \{0\}) \\ \quad \text{and } f \text{ may have space singularities at } x = 0 \text{ and } y = 0. \end{cases} \tag{11.6}$$

Theorem 11.2 (**Existence principle for singular problems with a parameter**). *Let f satisfy (11.6) and let $f_n \in Car([0,T] \times \mathbb{R}^2)$ satisfy the inequality*

$$0 \le -f_n(t, x, y) \le p(t, |x|, |y|), \quad n \in \mathbb{N}, \tag{11.7}$$

for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R} \setminus \{0\}$, where $p \in Car([0,T] \times (0,\infty)^2)$. Suppose that there exist positive constants μ_*, μ^*, $\mu_* < \mu^*$, such that for each $n \in \mathbb{N}$, the regular problem (11.5), (11.2) has a solution $u_n \in C^1[0,T]$, $\phi(u_n') \in AC[0,T]$, with $\mu = \mu_n \in [\mu_*, \mu^*]$. Let $\{u_n\}$ be bounded in $C^1[0,T]$ and $\{u_n'\}$ be equicontinuous on $[0,T]$. Then

(i) *there exist $u \in C^1[0,T]$, $\mu_0 \in [\mu_*, \mu^*]$ and subsequences $\{u_{k_n}\}$, $\{\mu_{k_n}\}$ such that $\|u_{k_n} - u\|_{C^1} \to 0$ and $|\mu_{k_n} - \mu_0| \to 0$ as $n \to \infty$,*

(ii) *if u and u' have a finite number of zeros and*

$$\lim_{n\to\infty} f_{k_n}(t, u_{k_n}(t), u_{k_n}'(t)) = f(t, u(t), u'(t)) \text{ for a.e. } t \in [0,T], \tag{11.8}$$

then $\phi(u') \in AC[0,T]$ and u is a solution of problem (11.1), (11.2) with $\mu = \mu_0$.

Proof. Assertion (i) follows from the Arzelà-Ascoli theorem and the Bolzano-Weierstrass Theorem.

In order to prove assertion (ii) assume that equality (11.8) is true and that $0 \le \xi_1 < \cdots < \xi_m \le T$ are all zeros of u and u' and put $\xi_0 = 0$, $\xi_{m+1} = T$. Since the next part of the proof uses similar procedures as the proof of Theorem 10.4, we show only the main differences. We have $\|u_{k_n}\|_{C^1} \le L$ for each $n \in \mathbb{N}$, where L is a positive constant, and

$$\phi(u'_{k_n}(T)) = \phi(u'_{k_n}(0)) + \mu_{k_n} \int_0^T f_{k_n}(t, u_{k_n}(t), u'_{k_n}(t))\,\mathrm{d}t, \quad n \in \mathbb{N}.$$

It follows from $\mu_n \in [\mu_*, \mu^*]$, conditions (11.7), (11.8) and from the Fatou lemma that

$$-\int_0^T f(t, u(t), u'(t))\,\mathrm{d}t \le \frac{\phi(L) - \phi(-L)}{\mu_*}.$$

Hence $f(t, u(t), u'(t)) \in L_1[0,T]$. We can also verify that

$$\phi(u'(t)) = \phi\Big(u'\Big(\frac{\xi_j + \xi_{j+1}}{2}\Big)\Big) + \mu_0 \int_{(\xi_j+\xi_{j+1})/2}^t f(s, u(s), u'(s))\,\mathrm{d}s$$

for $t \in [\xi_j, \xi_{j+1}]$ provided $j \in \{0, \dots, m\}$ and $\xi_j < \xi_{j+1}$. Hence $\phi(u') \in AC[0,T]$ and

$$(\phi(u'(t)))' = \mu_0\, f(t, u(t), u'(t)) \quad \text{for a.e.} \quad t \in [0,T].$$

Since $\{u_{k_n}\} \subset \mathcal{S}$ and $\mathcal{S}$ is closed in $C^1[0,T]$ we have $u \in \mathcal{S}$. Therefore u is a solution of problem (11.1), (11.2) for $\mu = \mu_0$. $\square$

Application of the existence principle

We now present an application of Theorem 11.2 to the singular problem (11.1), (11.2).

Definition 11.3. A function $u : [0,T] \to \mathbb{R}$ with $\phi(u') \in AC[0,T]$ is called a *solution of problem* (11.1), (11.3) if there exists $\mu_u \in \mathbb{R}$ such that $(\phi(u'(t)))' = \mu_u\, f(t, u(t), u'(t))$ for a.e. $t \in [0,T]$ and u fulfils the boundary conditions (11.3).

We will use the following assumptions:

$$\begin{cases} \phi : \mathbb{R} \to \mathbb{R} \quad \text{is an increasing and odd homeomorphism,} \\ \phi(\mathbb{R}) = \mathbb{R} \quad \text{and there exists } \beta > 0 \text{ such that} \\ \phi(v) \le v^\beta \quad \text{for } v \in [0, \infty), \end{cases} \tag{11.9}$$

$$\begin{cases} f \in Car([0,T] \times \mathcal{D}),\ \mathcal{D} = (0,\infty) \times (\mathbb{R} \setminus \{0\}), \\ \text{and there exists } \ a > 0 \ \text{ such that} \\ a \le -f(t,x,y) \quad \text{for a.e. } t \in [0,T] \text{ and each } \ (x,y) \in \mathcal{D}, \end{cases} \tag{11.10}$$

$$\begin{cases} -f(t,x,y) \le [h_1(x) + h_2(x)]\,[\omega_1(\phi(|y|)) + \omega_2(\phi(|y|))] \\ \text{for a.e. } \ t \in [0,T] \text{ and each } (x,y) \in \mathcal{D}, \\ \text{where } \ h_1,\, \omega_1 \in C[0,\infty) \text{ are positive and nondecreasing,} \\ h_2,\, \omega_2 \in C(0,\infty) \ \text{ are positive and nonincreasing, and} \\ \displaystyle\int_0^1 h_2(s)\,\mathrm{d}s < \infty, \qquad \int_0^\infty \frac{\sqrt[\beta]{s}}{\omega_1(s)}\,\mathrm{d}s = \infty. \end{cases} \tag{11.11}$$

For each $n \in \mathbb{N}$ define $\varrho_n \in C(\mathbb{R})$ and $f_n \in Car([0,T] \times \mathbb{R}^2)$ by

$$\varrho_n(v) = \begin{cases} v & \text{for } \ v \ge \frac{1}{n}, \\ \frac{1}{n} & \text{for } \ v < \frac{1}{n}, \end{cases}$$

$$f_n(t,x,y) = \begin{cases} f(t,\varrho_n(x),y) \quad \text{for } \ (t,x,y) \in [0,T] \times \mathbb{R} \times (\mathbb{R} \setminus [-\frac{1}{n},\frac{1}{n}]), \\ \frac{n}{2}\,[f(t,\varrho_n(x),\frac{1}{n})(y+\frac{1}{n}) - f(t,\varrho_n(x),-\frac{1}{n})(y-\frac{1}{n})] \\ \qquad\qquad\qquad \text{for } \ (t,x,y) \in [0,T] \times \mathbb{R} \times [-\frac{1}{n},\frac{1}{n}]. \end{cases}$$

By assumptions (11.10) and (11.11),

$$a \le -f_n(t,x,y) \tag{11.12}$$

and

$$-f_n(t,x,y) \le [h_1(x+1) + h_2(x)]\,[\omega_1(\phi(|y|)+1) + \omega_2(\phi(|y|))] \tag{11.13}$$

hold for a.e. $t \in [0,T]$ and each $(x,y) \in \mathcal{D}$, $n \in \mathbb{N}$.

Consider the family of regular differential equations

$$(\phi(u'))' = \mu f_n(t,u,u') \tag{11.14}$$

depending on the parameters $\mu \in \mathbb{R}$ and $n \in \mathbb{N}$ along with the boundary conditions

$$u(0) = 0, \quad u(T) = 0, \tag{11.15}$$

$$\max\{u(t) \colon 0 \le t \le T\} = A. \tag{11.16}$$

A priori bounds for solutions of problem (11.14)–(11.16) and the corresponding values of the parameter μ are given in the next three lemmas.

Lemma 11.4. *Let assumptions* (11.9) *and* (11.10) *hold. Let* $A > 0$ *and let* u *be a solution of problem* (11.14)–(11.16) *with some* $\mu = \mu_u$. *Then* $\mu_u > 0$, u' *is decreasing on* $[0,T]$,

$$u'(t) \begin{cases} \ge \quad \phi^{-1}(a\,\mu_u(\xi - t)) & \text{for } t \in [0,\xi], \\ \le -\phi^{-1}(a\,\mu_u(t-\xi)) & \text{for } t \in [\xi,T], \end{cases} \tag{11.17}$$

where $\xi \in (0,T)$ *is the unique zero of* u',

$$u(t) \ge \begin{cases} \dfrac{A}{\xi}\, t & \text{for } t \in [0,\xi], \\ \dfrac{A}{T-\xi}\,(T-t) & \text{for } t \in (\xi,T] \end{cases} \tag{11.18}$$

and

$$\mu_u \le \frac{1}{a}\Big(A\Big(1+\frac{1}{\beta}\Big)\Big)^{\beta}\Big(\frac{2}{T}\Big)^{1+\beta}. \tag{11.19}$$

Proof. If $\mu_u \le 0$ then $(\phi(u'))' \ge -a\,\mu_u \ge 0$ a.e. on $[0,T]$. Hence $\phi(u')$ is nondecreasing on $[0,T]$ which implies that of u'. Due to (11.15), $u'(t_0) = 0$ for $t_0 \in (0,T)$ and therefore $u' \le 0$ on $[0,t_0]$ and $u' \ge 0$ on $[t_0,T]$. This and (11.15) yield $u \le 0$ on $[0,T]$, contrary to equality (11.16). Hence

$\mu_u > 0$ and then from $(\phi(u'))' \le -a\mu_u < 0$ a.e. on $[0,T]$ we see that u' is decreasing on $[0,T]$ and u' has a unique zero $\xi \in (0,T)$. Using $\phi(0)=0$, $u'(\xi)=0$ and integrating $(\phi(u'))' \le -a\,\mu_u$ we obtain inequality (11.17).

Since $u(0)=u(T)=0$, $u(\xi)=A$ and u is concave on $[0,T]$, which follows from the fact that u' is decreasing on $[0,T]$, we see that (11.18) holds.

It remains to prove inequality (11.19). By (11.9), we have $\phi(v) \le v^\beta$ for $v \in [0,\infty)$ and consequently

$$\phi^{-1}(v) \ge \sqrt[\beta]{v} \quad \text{for} \quad v \in [0,\infty). \tag{11.20}$$

This and inequality (11.17) give

$$\begin{aligned}
A = u(\xi) &= \int_0^\xi u'(t)\,\mathrm{d}t \ge \int_0^\xi \phi^{-1}(a\,\mu_u\,(\xi - t))\,\mathrm{d}t \\
&= \frac{1}{a\,\mu_u}\int_0^{a\,\mu_u\,\xi} \phi^{-1}(s)\,\mathrm{d}s \ge \frac{1}{a\mu_u}\int_0^{a\,\mu_u\,\xi} \sqrt[\beta]{s}\,\mathrm{d}s \\
&= \frac{\beta\,\sqrt[\beta]{a\,\mu_u}}{1+\beta}\,\xi^{1+\frac{1}{\beta}},
\end{aligned}$$

$$\begin{aligned}
A = u(\xi) &= \int_T^\xi u'(t)\,\mathrm{d}t \ge \int_\xi^T \phi^{-1}(a\,\mu_u\,(t-\xi))\,\mathrm{d}t \\
&= \frac{1}{a\,\mu_u}\int_0^{a\,\mu_u\,(T-\xi)} \phi^{-1}(s)\,\mathrm{d}s \ge \frac{1}{a\,\mu_u}\int_0^{a\,\mu_u\,(T-\xi)} \sqrt[\beta]{s}\,\mathrm{d}s \\
&= \frac{\beta\,\sqrt[\beta]{a\,\mu_u}}{1+\beta}(T-\xi)^{1+\frac{1}{\beta}}.
\end{aligned}$$

Hence

$$A \ge \frac{\beta\,\sqrt[\beta]{a\,\mu_u}}{1+\beta}\max\Big\{\xi^{1+\frac{1}{\beta}},\,(T-\xi)^{1+\frac{1}{\beta}}\Big\} \ge \frac{\beta\,\sqrt[\beta]{a\,\mu_u}}{1+\beta}\Big(\frac{T}{2}\Big)^{1+\frac{1}{\beta}}$$

and then we see from the inequality

$$\sqrt[\beta]{a\,\mu_u} \le A\Big(1+\frac{1}{\beta}\Big)\Big(\frac{2}{T}\Big)^{1+\frac{1}{\beta}}$$

that inequality (11.19) is true. □

Lemma 11.5. *Let assumptions (11.9)–(11.11) hold and let* $A > 0$. *Then there exists a positive constant* P *independent of* $n \in \mathbb{N}$ *and* $\lambda \in (0,1]$ *such that for any solution* u *of problem (11.14), (11.15) with some* $\mu = \mu_u$ *satisfying*

$$\max\{u(t)\colon 0 \le t \le T\} = \lambda A, \quad \lambda \in (0,1], \tag{11.21}$$

the inequalities

$$\|u'\|_\infty < P \tag{11.22}$$

and $0 < \mu_u \le \mu^*$ *are valid where*

$$\mu^* = \frac{1}{a}\Big(A\Big(1+\frac{1}{\beta}\Big)\Big)^\beta \Big(\frac{2}{T}\Big)^{1+\beta}. \tag{11.23}$$

Proof. Let u be a solution of problem (11.14), (11.15) with some $\mu = \mu_u$. Let u satisfy condition (11.21) for some $\lambda \in (0,1]$. Then it follows from Lemma 11.4 (with λA instead of A) that u is positive on $(0,T)$, u' is decreasing on $[0,T]$, u' has a unique zero $\xi \in (0,T)$ and

$$0 < \mu_u \le \frac{1}{a}\Big(\lambda A\Big(1+\frac{1}{\beta}\Big)\Big)^\beta \Big(\frac{2}{T}\Big)^{1+\beta} \le \mu^*.$$

Hence

$$\|u'\|_\infty = \max\{u'(0),\, -u'(T)\} \tag{11.24}$$

and $u(\xi) = \lambda A$. In addition, by inequality (11.13),

$$(\phi(u'(t)))' \ge -\mu_u\,[h_1(u(t)+1) + h_2(u(t))]\,[\omega_1(\phi(|u'(t)|)+1) + \omega_2(\phi(|u'(t)|))]$$

for a.e. $t \in [0,T]$. Thus

$$\frac{(\phi(u'(t)))'u'(t)}{\omega_1(\phi(u'(t))+1) + \omega_2(\phi(u'(t)))} \ge -\mu_u[h_1(u(t)+1) + h_2(u(t))]u'(t) \tag{11.25}$$

for a.e. $t \in [0,\xi]$ and

$$\frac{(\phi(u'(t)))'u'(t)}{\omega_1(1-\phi(u'(t)))+\omega_2(-\phi(u'(t)))} \le -\mu_u[h_1(u(t)+1)+h_2(u(t))]u'(t) \tag{11.26}$$

for a.e. $t \in [\xi, T]$. Integrating (11.25) over $[0, \xi]$ and (11.26) over $[\xi, T]$, we get

$$\begin{cases} \displaystyle\int_0^{\phi(u'(0))} \frac{\phi^{-1}(s)}{\omega_1(s+1)+\omega_2(s)}\,\mathrm{d}s \\ \displaystyle\leq \mu_u \int_0^{u(\xi)} (h_1(s+1)+h_2(s))\,\mathrm{d}s \leq \mu_u \int_0^A (h_1(s+1)+h_2(s))\,\mathrm{d}s \\ \displaystyle\leq \mu^* \int_0^A (h_1(s+1)+h_2(s))\,\mathrm{d}s \end{cases} \tag{11.27}$$

and

$$\begin{cases} \displaystyle\int_0^{\phi(-u'(T))} \frac{\phi^{-1}(s)}{\omega_1(s+1)+\omega_2(s)}\,\mathrm{d}s \leq \mu_u \int_0^{u(\xi)} (h_1(s+1)+h_2(s))\,\mathrm{d}s \\ \displaystyle\qquad \leq \mu^* \int_0^A (h_1(s+1)+h_2(s))\,\mathrm{d}s. \end{cases} \tag{11.28}$$

We now show that

$$\int_0^\infty \frac{\phi^{-1}(s)}{\omega_1(s+1)+\omega_2(s)}\,\mathrm{d}s = \infty. \tag{11.29}$$

Due to assumption (11.11) we have

$$\int_0^\infty \frac{\sqrt[\beta]{s}}{\omega_1(s)}\,\mathrm{d}s = \infty \quad \text{and, consequently,} \quad \int_2^\infty \frac{\sqrt[\beta]{s}}{\omega_1(s)}\,\mathrm{d}s = \infty.$$

From assumption (11.9) and from the properties of the functions ω_1 and ω_2 if follows that $\phi^{-1}(s) \geq \sqrt[\beta]{s}$ for $s \in [0, \infty)$ and

$$\omega_1(s+1)+\omega_2(s) \leq \omega_1(s+1)+\omega_2(1) \leq L\,\omega_1(s+1) \quad \text{for} \quad s \in [1, \infty),$$

where

$$L = 1 + \frac{\omega_2(1)}{\omega_1(2)}.$$

Hence

$$\int_2^\infty \frac{\sqrt[\beta]{s}}{\omega_1(s)}\,\mathrm{d}s = \int_2^\infty \sqrt[\beta]{\frac{s}{s-1}}\,\frac{\sqrt[\beta]{s-1}}{\omega_1(s)}\,\mathrm{d}s \le \sqrt[\beta]{2}\int_2^\infty \frac{\sqrt[\beta]{s-1}}{\omega_1(s)}\,\mathrm{d}s$$
$$= \sqrt[\beta]{2}\int_1^\infty \frac{\sqrt[\beta]{s}}{\omega_1(s+1)}\,\mathrm{d}s \le \sqrt[\beta]{2}\,L\int_1^\infty \frac{\sqrt[\beta]{s}}{\omega_1(s+1)+\omega_2(s)}\,\mathrm{d}s$$
$$\le \sqrt[\beta]{2}\,L\int_1^\infty \frac{\phi^{-1}(s)}{\omega_1(s+1)+\omega_2(s)}\,\mathrm{d}s.$$

Therefore

$$\int_1^\infty \frac{\phi^{-1}(s)}{\omega_1(s+1)+\omega_2(s)}\,\mathrm{d}s = \infty$$

and consequently, (11.29) holds. Equality (11.29) guarantees the existence of a positive constant Q such that

$$\int_0^Q \frac{\phi^{-1}(s)}{\omega_1(s+1)+\omega_2(s)}\,\mathrm{d}s > \mu^* \int_0^A (h_1(s+1)+h_2(s))\,\mathrm{d}s.$$

Now inequalities (11.27) and (11.28) give $\max\{\phi(u'(0)),\,\phi(-u'(T))\} < Q$ and from (11.24) we see that (11.22) holds with $P = \phi^{-1}(Q)$. □

Lemma 11.6. *Let conditions* (11.9)–(11.11) *hold and let* $A > 0$. *Then there exists a positive constant* μ_* *independent of* $n \in \mathbb{N}$ *such that for any solution* u *of problem* (11.14)–(11.16) *with some* $\mu = \mu_u$ *the inequality*

$$\mu_u \ge \mu_* \tag{11.30}$$

is satisfied.

Proof. Let u be a solution of problem (11.14)–(11.16) with some $\mu = \mu_u$. Then $u(\xi) = A$, where $\xi \in (0,T)$ is the unique zero of u', and therefore

$$A = u(\xi) - u(0) = u'(\eta_1)\,\xi, \quad A = u(\xi) - u(T) = -u'(\eta_2)\,(T-\xi),$$

where $0 < \eta_1 < \xi < \eta_2 < T$. Hence $u'(\eta_1) = \frac{A}{\xi}$, $-u'(\eta_2) = \frac{A}{T-\xi}$ and since $\min\{\xi, T-\xi\} \le \frac{T}{2}$, we have $\max\{u'(\eta_1), -u'(\eta_2)\} \ge \frac{2A}{T}$. Thus $\|u'\|_\infty \ge \frac{2A}{T}$

and it follows from (11.24), (11.27) and (11.28)) that

$$\int_0^{\phi(2A/T)} \frac{\phi^{-1}(s)}{\omega_1(s+1)+\omega_2(s)}\,\mathrm{d}s \le \int_0^{\phi(\|x'\|_\infty)} \frac{\phi^{-1}(s)}{\omega_1(s+1)+\omega_2(s)}\,\mathrm{d}s$$
$$\le \mu_u \int_0^A (h_1(s+1)+h_2(s))\,\mathrm{d}s.$$

We see that (11.30) holds with

$$\mu_* = \frac{\displaystyle\int_0^{\phi(2A/T)} \frac{\phi^{-1}(s)}{\omega_1(s+1)+\omega_2(s)}\,\mathrm{d}s}{\displaystyle\int_0^A (h_1(s+1)+h_2(s))\,\mathrm{d}s}.$$

□

We are now in a position to show that the regular problem (11.14)–(11.16) has a solution for each $n \in \mathbb{N}$.

Lemma 11.7. *Let conditions* (11.9)–(11.11) *hold and let* $A>0$. *Then problem* (11.14)–(11.16) *has a solution for each* $n \in \mathbb{N}$.

Proof. Fix $n \in \mathbb{N}$ and let $P>0$ be given by Lemma 11.5. Set

$$\Omega = \Big\{(u,\mu) \in C^1[0,T] \times \mathbb{R} \colon \|u\|_\infty < A+1,\ \|u'\|_\infty < P,$$
$$|\mu| < \frac{1}{a}\Big(A\Big(1+\frac{1}{\beta}\Big)\Big)^\beta \Big(\frac{2}{T}\Big)^{1+\beta} + 1\Big\}.$$

Then Ω is an open, bounded and symmetric with respect to (0,0) subset of the Banach space $C^1[0,T]\times\mathbb{R}$.

Define an operator $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2) \colon [0,1] \times \overline{\Omega} \to C^1[0,T] \times \mathbb{R}$ by

$$\mathcal{H}(\lambda,u,\mu) \ = (\mathcal{H}_1(\lambda,u,\mu),\ \mathcal{H}_2(\lambda,u,\mu)),$$
$$\mathcal{H}_1(\lambda,u,\mu) = \int_0^t \phi^{-1}\Big(B + \mu\Big((\lambda-1)s + \lambda \int_0^s f_n(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\Big)\,\mathrm{d}s,$$
$$\mathcal{H}_2(\lambda,u,\mu) = \lambda\,[\max\{u(t)\colon 0\le t\le T\} + \min\{u(t)\colon 0 \le t \le T\}]$$
$$+ (1-\lambda)\,u(\tfrac{T}{2}) + \mu\,,$$

where the constant $B = B(\lambda, u, \mu)$ is the unique solution of the equation

$$p(B; \lambda, u, \mu) = 0 \tag{11.31}$$

with

$$p(B; \lambda, u, \mu) = \int_0^T \phi^{-1}\Big(B + \mu\Big((\lambda - 1)\, t + \lambda \int_0^t f_n(s, u(s), u'(s))\, \mathrm{d}s\Big)\Big)\, \mathrm{d}t. \tag{11.32}$$

The existence and uniqueness of a solution for equation (11.31) follows from the fact that $p(\cdot\,; \lambda, u, \mu)$ is continuous and increasing on $\mathbb{R}$ and

$$\lim_{B \to \pm\infty} p(B; \lambda, u, \mu) = \pm\infty$$

for each $(\lambda, u, \mu) \in [0, 1] \times \overline{\Omega}$.

Since

$$\mathcal{H}(0, u, \mu) = \Big(\int_0^t \phi^{-1}(B - \mu\, s)\, \mathrm{d}s,\; u(\tfrac{T}{2}) + \mu\Big),$$

where B is the unique solution of the equation $\int_0^T \phi^{-1}(B - \mu\, t)\, \mathrm{d}t = 0$, the Mean Value Theorem for integrals gives $B = \mu\, t_0$ for some $t_0 \in (0, T)$. Hence

$$\mathcal{H}(0, u, \mu) = \Big(\int_0^t \phi^{-1}(\mu(t_0 - s))\, \mathrm{d}s,\; u(\tfrac{T}{2}) + \mu\Big)$$

and therefore $\mathcal{H}(0, -u, -\mu) = -\mathcal{H}(0, u, \mu)$ for $(u, \mu) \in \overline{\Omega}$, which shows that $\mathcal{H}(0, \cdot, \cdot)$ is an odd operator.

We claim that $\mathcal{H}$ is a compact operator. To this aim let

$$\{(\lambda_m, u_m, \mu_m)\} \subset [0, 1] \times \overline{\Omega}$$

and

$$\lim_{m \to \infty} (\lambda_m, u_m, \mu_m) = (\lambda_0, u_0, \mu_0) \text{ in } [0, 1] \times C^1[0, T] \times \mathbb{R}.$$

Let B_m be the solution of the equation $p(B; \lambda_m, u_m, \mu_m) = 0$. Since the sequence $\{u_m\}$ is bounded in $C^1[0, T]$ and $f_n \in Car([0, T] \times \mathbb{R}^2)$, there exists

$q \in L_1[0,T]$ such that $|f_n(t, u_m(t), u_m'(t))| \le q(t)$ for a.e. $t \in [0,T]$ and each $m \in \mathbb{N}$. Consequently, $\{B_m\}$ is bounded, otherwise

$$\limsup_{m\to\infty} |p(B_m;\, \lambda_m, u_m, \mu_m)| = \infty,$$

a contradiction.

We will show that $\{B_m\}$ is convergent. Let $\{B_{k_m}\}$ be a convergent subsequence of $\{B_m\}$ and $\varkappa = \lim_{m\to\infty} B_{k_m}$. Then

$$0 = \lim_{m\to\infty} p(B_{k_m};\, \lambda_{k_m}, u_{k_m}, \mu_{k_m}) = p(\varkappa;\, \lambda_0, u_0, \mu_0)$$

by the Lebesgue dominated convergence theorem, and consequently $\varkappa = B_0$ where B_0 is the unique solution of the equation $p(B; \lambda_0, u_0, \mu_0) = 0$. We have proved that any convergent subsequence of $\{B_m\}$ has the same limit B_0. Therefore $\lim_{m\to\infty} B_m = B_0$. Then

$$\lim_{m\to\infty} \int_0^t \phi^{-1}\Big(B_m + \mu_m\Big((\lambda_m - 1)s + \lambda_m \int_0^s f_n(\tau, u_m(\tau), u_m'(\tau))\, \mathrm{d}\tau\Big)\Big)\, \mathrm{d}s$$

$$= \int_0^t \phi^{-1}\Big(B_0 + \mu_0\Big((\lambda_0 - 1)s + \lambda_0 \int_0^s f_n(\tau, u_0(\tau), u_0'(\tau))\, \mathrm{d}\tau\Big)\Big)\, \mathrm{d}s$$

in $C^1[0,T]$. This, together with

$$\lim_{m\to\infty} \Big(\lambda_m[\max\{u_m(t)\colon 0 \le t \le T\} + \min\{u_m(t)\colon 0 \le t \le T\}]$$

$$+ (1-\lambda_m) u_m(\tfrac{T}{2}) + \mu_m\Big)$$

$$= \lambda_0[\max\{u_0(t)\colon 0 \le t \le T\} + \min\{u_0(t)\colon 0 \le t \le T\}]$$

$$+ (1-\lambda_0) u_0(\tfrac{T}{2}) + \mu_0,$$

implies that $\mathcal{H}$ is a continuous operator.

In order to verify that the set $\mathcal{H}([0,T] \times \overline{\Omega})$ is relatively compact in $C^1[0,T] \times \mathbb{R}$, let us consider a sequence $\{(\lambda_j, u_j, \mu_j)\} \subset [0,1] \times \overline{\Omega}$. Then the sequence

$$\{\lambda_j[\max\{u_j(t)\colon 0 \le t \le T\} + \min\{u_j(t)\colon 0 \le t \le T\}] + (1-\lambda_j)\, u_j(\tfrac{T}{2}) + \mu_j\}$$

is bounded in $\mathbb{R}$ and there exists $r \in L_1[0,T]$ such that the inequality

$$|f_n(t, u_j(t), u_j'(t))| \le r(t) \qquad \text{for a.e. } t \in [0,T] \text{ and all } j \in \mathbb{N}.$$

holds. Let $p(B_j; \lambda_j, u_j, \mu_j) = 0$ for $j \in \mathbb{N}$. Then the sequence $\{B_j\}$ is bounded in $\mathbb{R}$ and the sequence

$$\left\{ \int_0^t \phi^{-1}\Big(B_j + \mu_j\Big((\lambda_j - 1)\, s + \lambda_j \int_0^s f_n(\tau, u_j(\tau), u_j'(\tau))\, \mathrm{d}\tau\Big)\Big)\, \mathrm{d}s \right\}$$

is bounded in $C^1[0,T]$. Moreover, the sequence

$$\left\{ \mu_j\Big((\lambda_j - 1)\, t + \lambda_j \int_0^t f_n(s, u_j(s), u_j'(s)\, \mathrm{d}s\Big) \right\}$$

is equicontinuous on $[0,T]$. Therefore $\{\mathcal{H}(\lambda_j, u_j, \mu_j)\}$ is relatively compact in $C^1[0,T] \times \mathbb{R}$ by the Arzelà-Ascoli theorem and the Bolzano-Weierstrass Theorem.

Let $\mathcal{H}(\lambda_0, u_0, \mu_0) = (u_0, \mu_0)$ for some $\lambda_0 \in [0,1]$ and $(u_0, \mu_0) \in \partial\Omega$. Then

$$(\phi(u_0'(t)))' = \mu_0[\lambda_0 - 1 + \lambda_0\, f_n(t, u_0(t), u_0'(t))] \quad \text{for a.e. } t \in [0,T], \tag{11.33}$$

$$u_0(0) = 0, \quad u_0(T) = 0, \tag{11.34}$$

$$\begin{cases} \lambda_0\, [\max\{u_0(t) \colon 0 \le t \le T\} + \min\{u_0(t) \colon 0 \le t \le T\}] \\ \qquad\qquad\qquad\qquad\qquad\qquad + (1 - \lambda_0)\, u_0(\frac{T}{2}) = 0. \end{cases} \tag{11.35}$$

If $\mu_0 > 0$ then (11.12) and (11.33) give $(\phi(u_0'))' < 0$ a.e. on $[0,T]$ and (11.34) implies that $u_0 > 0$ on $(0,T)$. Therefore $\min\{u_0(t) : 0 \le t \le T\} = 0$ and by virtue of (11.35)

$$0 = \lambda_0\, \max\{u_0(t) \colon 0 \le t \le T\} + (1 - \lambda_0)\, u_0(\tfrac{T}{2}) > 0,$$

which is impossible. Let $\mu_0 < 0$. Then (11.12) and (11.33) yield $(\phi(u_0'))' > 0$ a.e. on $[0,T]$, which together with (11.34) implies that $u_0 < 0$ on $(0,T)$ and

$$0 = \lambda_0\, \min\{u_0(t) \colon 0 \le t \le T\} + (1 - \lambda_0)\, u_0(\tfrac{T}{2}) < 0,$$

a contradiction. Hence $\mu_0 = 0$ and then we see from $(\phi(u_0'))' = 0$ a.e. on $[0,T]$ and (11.34) that $u_0 = 0$. We have proved that $(u_0,\mu_0) \notin \partial\Omega$ and therefore $\mathcal{H}(\lambda,u,\mu) \neq (u,\mu)$ for $\lambda \in [0,1]$ and $(u,\mu) \in \partial\Omega$. Now, by the Borsuk antipodal theorem, $\deg(\mathcal{I} - \mathcal{H}(0,\cdot,\cdot),\Omega) \neq 0$, where $\mathcal{I}$ is the identity operator on $C^1[0,T] \times \mathbb{R}$. In addition,

$$\deg(\mathcal{I} - \mathcal{H}(1,\cdot,\cdot),\Omega) = \deg(\mathcal{I} - \mathcal{H}(0,\cdot,\cdot),\Omega)$$

by the homotopy property (see the Leray-Schauder degree theorem with $\mathcal{U} = \overline{\Omega}$). Consequently,

$$\deg(\mathcal{I} - \mathcal{H}(1,\cdot,\cdot),\Omega) \neq 0. \tag{11.36}$$

Finally, define an operator $\mathcal{K} = (\mathcal{K}_1,\mathcal{K}_2)\colon [0,1] \times \overline{\Omega} \to C^1[0,T] \times \mathbb{R}$ by the formulas

$$\mathcal{K}_1(\lambda,u,\mu) = \int_0^t \phi^{-1}\Big(D + \mu \int_0^s f_n(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s,$$

$$\mathcal{K}_2(\lambda,u,\mu) = \max\{u(t)\colon 0 \le t \le T\} + \min\{u(t)\colon 0 \le t \le T\} - \lambda\,A + \mu,$$

where the constant $D = D(u,\mu)$ is the unique solution of the equation

$$r(D;u,\mu) = 0 \tag{11.37}$$

with

$$r(D;u,\mu) = \int_0^T \phi^{-1}\Big(D + \mu \int_0^t f_n(s,u(s),u'(s))\,\mathrm{d}s\Big)\,\mathrm{d}t. \tag{11.38}$$

Essentially the same reasoning as for equation (11.31) and for the operator $\mathcal{H}$ shows that there exists a unique solution of equation (11.37) and that $\mathcal{K}$ is a compact operator. Assume that $\mathcal{K}(\lambda_*,u_*,\mu_*) = (u_*,\mu_*)$ for some $\lambda_* \in [0,1]$ and $(u_*,\mu_*) \in \partial\Omega$. Then

$$(\phi(u_*'(t)))' = \mu_* f_n(t,u_*(t),u_*'(t))) \quad \text{for a.e. } t \in [0,T], \tag{11.39}$$

$$u_*(0) = 0, \quad u_*(T) = 0, \tag{11.40}$$

$$\max\{u_*(t)\colon 0 \le t \le T\} + \min\{u_*(t)\colon 0 \le t \le T\} = \lambda_*\,A. \tag{11.41}$$

If $\mu_* \le 0$ then $(\phi(u'_*))' \ge 0$ a.e. on $[0,T]$ and from (11.40) we deduce that $u_* \le 0$ on $[0,T]$. Then (11.41) gives

$$0 \le \lambda_* A = \max\{u_*(t) : 0 \le t \le T\} + \min\{u_*(t) \colon 0 \le t \le T\}$$
$$= \min\{u_*(t) \colon 0 \le t \le T\},$$

which leads to $u_* = 0$. Consequently, by (11.39), $\mu_* = 0$ and therefore $(u_*, \mu_*) = (0,0)$, contrary to $(u_*, \mu_*) \in \partial\Omega$. It follows that $\mu_* > 0$ and then $(\phi(u'_*))' < 0$ a.e. on $[0,T]$. From this inequality and from (11.40) we get $u_* > 0$ on $(0,T)$ and (11.41) gives $\max\{u_*(t) \colon 0 \le t \le T\} = \lambda_* A$. Thus u_* is a solution of problem (11.14),(11.15),(11.21) (with $\mu = \mu_*$ in (11.14) and $\lambda = \lambda_*$ in (11.21)). Therefore $\|u_*\|_\infty = \lambda_* A$ and, by Lemma 11.5,

$$\|u'_*\|_\infty < P, \quad 0 < \mu_* \le \frac{1}{a}\Big(A\Big(1 + \frac{1}{\beta}\Big)\Big)^\beta \Big(\frac{2}{T}\Big)^{1+\beta}.$$

Hence $(u_*, \mu_*) \notin \partial\Omega$ and we have proved that $\mathcal{K}(\lambda, u, \mu) \ne (u, \mu)$ for all $\lambda \in [0,1]$ and $(u, \mu) \in \partial\Omega$. By the homotopy property,

$$\deg(\mathcal{I} - \mathcal{K}(0, \cdot, \cdot), \Omega) = \deg(\mathcal{I} - \mathcal{K}(1, \cdot, \cdot), \Omega).$$

Since $\mathcal{H}(1, \cdot, \cdot) = \mathcal{K}(0, \cdot, \cdot)$, relation (11.36) gives $\deg(\mathcal{I} - \mathcal{K}(1, \cdot, \cdot), \Omega) \ne 0$. Therefore there exists a fixed point $(\hat{u}, \hat{\mu})$ of the operator $\mathcal{K}(1, \cdot, \cdot)$ and it is easy to check that $\hat{u}$ is a solution of problem (11.14)-(11.16) with $\mu = \hat{\mu}$. $\square$

Our next result is needed for applying Theorem 11.2 to the solvability of problem (11.1), (11.3).

Lemma 11.8. *Let conditions* (11.9)–(11.11) *hold and let* $A > 0$. *Let* u_n *be a solution of problem* (11.14)–(11.16) *with some* $\mu = \mu_n$, $n \in \mathbb{N}$.

Then the sequence $\{u'_n\}$ *is equicontinuous on* $[0,T]$.

Proof. By Lemmas 11.4–11.6, for each $n \in \mathbb{N}$ we have $0 \le u_n(t) \le A$ for $t \in [0,T]$, u'_n is decreasing on $[0,T]$ and u'_n vanishes at a unique $\xi_n \in (0,T)$. Furthermore, there exist positive constants P, μ_* and μ^* such that

$$\|u'_n\|_\infty < P, \quad n \in \mathbb{N}, \tag{11.42}$$

$$\mu_* \le \mu_n \le \mu^*, \quad n \in \mathbb{N}. \tag{11.43}$$

Put

$$G(v) = \int_0^{\phi(v)} \frac{\phi^{-1}(s)}{\omega_1(s+1)+\omega_2(s)}\,\mathrm{d}s, \quad H(v) = \int_0^v (h_1(s+1)+h_2(s))\,\mathrm{d}s$$

for $v \in [0,\infty)$ and

$$G^*(v) = \begin{cases} G(v) & \text{for } v \in [0,\infty), \\ -G(-v) & \text{for } v \in (-\infty,0). \end{cases}$$

Since $\{u_n\}$ is bounded in $C^1[0,T]$, the sequence $\{H(u_n)\}$ is equicontinuous on $[0,T]$ and therefore for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|H(u_n(t_2)) - H(u_n(t_1))| < \varepsilon \tag{11.44}$$

whenever $0 \le t_1 < t_2 \le T$ and $t_2 - t_1 < \delta$. Choose $0 \le t_1 < t_2 \le T$. If $t_2 \le \xi_n$ then integrating (see (11.25))

$$\begin{cases} \dfrac{\phi(u_n'(t))\,u_n'(t)}{\omega_1(\phi(u_n'(t))+1)+\omega_2(\phi(u_n'(t)))} \\ \qquad \ge -\mu_n\,[h_1(u_n(t)+1)+h_2(u_n(t))]\,u_n'(t) \end{cases} \tag{11.45}$$

from t_1 to t_2 yields

$$\begin{cases} 0 < G(u_n'(t_1)) - G(u_n'(t_2)) \le \mu_n\,[H(u_n(t_2)) - H(u_n(t_1))] \\ \quad \le \mu^*\,[H(u_n(t_2)) - H(u_n(t_1))], \end{cases} \tag{11.46}$$

while if $\xi_n \le t_1$ then integrating (see (11.26))

$$\begin{cases} \dfrac{\phi(u_n'(t))\,u_n'(t)}{\omega_1(-\phi(u_n'(t))+1)+\omega_2(-\phi(u_n'(t)))} \\ \qquad \le -\mu_n\,[h_1(u_n(t)+1)+h_2(u_n(t))]\,u_n'(t), \end{cases} \tag{11.47}$$

over $[t_1,t_2]$ gives

$$\begin{cases} 0 < G(-u_n'(t_2)) - G(-u_n'(t_1)) \le \mu_n\,[H(u_n(t_1)) - H(u_n(t_2))] \\ \quad \le \mu^*\,[H(u_n(t_1)) - H(u_n(t_2))]. \end{cases} \tag{11.48}$$

Finally, if $t_1 < \xi_n < t_2$ then integrating (11.45) over $[t_1, \xi_n]$ and (11.47) over $[\xi_n, t_2]$ gives

$$\begin{cases} 0 < G(u_n'(t_1)) \le \mu_n \, [H(u_n(\xi_n)) - H(u_n(t_1))] \\ \quad \le \mu^* \, [H(u_n(\xi_n)) - H(u_n(t_1))], \end{cases} \tag{11.49}$$

$$\begin{cases} 0 < G(-u_n'(t_2)) \le \mu_n \, [H(u_n(\xi_n)) - H(u_n(t_2))] \\ \quad \le \mu^* \, [H(u_n(\xi_n)) - H(u_n(t_2))]. \end{cases} \tag{11.50}$$

Now inequalities (11.46) and (11.48)–(11.50) imply that

$$0 < G^*(u_n'(t_1)) - G^*(u_n'(t_2)) \le \mu^* \, |H(u_n(t_1)) - H(u_n(t_2))|$$

if $0 \le t_1 < t_2 \le \xi_n$ or $\xi_n \le t_1 < t_2 \le T$ and

$$0 < G^*(u_n'(t_1)) - G^*(u_n'(t_2)) \le \mu^* \, [2\, H(u_n(\xi_n)) - H(u_n(t_1)) - H(u_n(t_2))]$$

if $0 \le t_1 < \xi_n < t_2 \le T$. This and inequality (11.44) give

$$0 < G^*(u_n'(t_1)) - G^*(u_n'(t_2)) \le 2\, \mu^* \, \varepsilon$$

whenever $0 \le t_1 < t_2 \le T$ and $t_2 - t_1 < \delta$. Hence $\{G^*(u_n')\}$ is equicontinuous on $[0,T]$ and since $G^* \in C(\mathbb{R})$ is increasing and $\{u_n'\}$ is bounded in $C[0,T]$, we see that $\{u_n'\}$ is equicontinuous on $[0,T]$. $\square$

The following theorem gives an existence result for problem (11.1), (11.3).

Theorem 11.9. *Let assumptions (11.9)–(11.11) hold. Then for each $A > 0$ there exists $\mu > 0$ such that problem (11.1), (11.3) has a solution $u \in C^1[0,T]$ such that $\phi(u') \in AC[0,T]$ and $u > 0$ on $(0,T)$.*

Proof. Fix $A > 0$. By Lemma 11.7, for each $n \in \mathbb{N}$ there exists a solution u_n of problem (11.14)–(11.16) with some $\mu = \mu_n$. Lemmas 11.4–11.6 yield that

$$0 \le u_n(t) \le A \quad \text{for } t \in [0,T], \tag{11.51}$$

u_n' is decreasing on $[0,T]$ and vanishes at a unique $\xi_n \in (0,T)$,

$$u_n(t) \ge \begin{cases} \dfrac{A}{\xi_n}\, t & \text{for } t \in [0, \xi_n], \\ \dfrac{A}{T - \xi_n}\,(T - t) & \text{for } t \in [\xi_n, T], \end{cases} \tag{11.52}$$

$$u_n'(t)\begin{cases}\geq \ \ \phi^{-1}(a\,\mu_n\,(\xi_n-t)) & \text{for } \ t\in[0,\xi_n],\\[2ex] \leq -\phi^{-1}(a\,\mu_n\,(t-\xi_n)) & \text{for } \ t\in[\xi_n,T],\end{cases} \tag{11.53}$$

and there exist positive constants P, μ_* and μ^* such that inequalities (11.42) and (11.43) are satisfied for all $n\in\mathbb{N}$. In addition, by Lemma 11.8, $\{u_n'\}$ is equicontinuous on $[0,T]$. Using the Arzelà-Ascoli theorem and the Bolzano-Weierstrass Theorem we can assume without loss of generality that $\{u_n\}$ is convergent in $C^1[0,T]$ and $\{\mu_n\}$ and $\{\xi_n\}$ are convergent in $\mathbb{R}$. Let $\lim_{n\to\infty} u_n = u$, $\lim_{n\to\infty}\mu_n=\mu$ and $\lim_{n\to\infty}\xi_n=\xi$. Then $u\in C^1[0,T]$ fulfils (11.3), $u'(\xi)=0$ and letting $n\to\infty$ in inequalities (11.43) and (11.51)–(11.53), we get $0\leq u(t)\leq A$ for $t\in[0,T]$,

$$u(t)\geq\begin{cases}\dfrac{A}{\xi}\,t & \text{for } \ t\in[0,\xi],\\[2ex] \dfrac{A}{T-\xi}\,(T-t) & \text{for } \ t\in[\xi,T],\end{cases}$$

$$u'(t)\begin{cases}\geq \ \ \phi^{-1}(a\,\mu_*\,(\xi-t)) & \text{for } \ t\in[0,\xi],\\[2ex] \leq -\phi^{-1}(a\,\mu_*\,(t-\xi)) & \text{for } \ t\in[\xi,T],\end{cases}$$

and $\mu_*\leq\mu\leq\mu^*$. Hence $\xi\in(0,T)$ is the unique zero of u', $u>0$ on $(0,T)$ and

$$\lim_{n\to\infty} f_n(t,u_n(t),u_n'(t)) = f(t,u(t),u'(t)) \quad \text{for a.e. } \ t\in[0,T].$$

By inequality (11.13),

$$0<a\leq -f_n(t,x,y)\leq [h_1(|x|+1)+h_2(|x|)]\,[\omega_1(\phi(|y|)+1)+\omega_2(\phi(|y|))]$$

for a.e. $t\in[0,T]$ and all $x,y\in\mathbb{R}\setminus\{0\}$. Put

$$p(t,x,y)=[h_1(x+1)+h_2(x)]\,[\omega_1(\phi(y)+1)+\omega_2(\phi(y))]$$

for $(t,x,y)\in[0,T]\times(0,\infty)^2$. Then f_n satisfies inequality (11.7) and, consequently, Theorem 11.2 guarantees that $\phi(u')\in AC[0,T]$ and u is a solution of problem (11.1), (11.3). □

EXAMPLE. Let $p \in (1,\infty)$, γ_1, η_1, $\eta_2 \in (0,\infty)$, γ_2, $\gamma_3 \in (0,1)$ and $\eta_3 \in (0,p)$. By Theorem 11.9, for all $A>0$ there exist $\mu>0$ and a solution u of the differential equation

$$(|u'|^{p-2}\,u')' + \mu\Big(1 + u^{\gamma_1} + \frac{1}{u^{\gamma_2}} + \frac{1}{u^{\gamma_3}|u'|^{\eta_1}} + \frac{1}{|u'|^{\eta_2}} + |u'|^{\eta_3}\Big) = 0$$

satisfying the boundary conditions (11.3) and $u>0$ on $(0,T)$.

Bibliographical notes

Theorem 11.2 is taken from Staněk [187], Theorem 11.9 was adapted from Agarwal, O'Regan and Staněk [19]. Another singular problems for equation (11.1) depending on a parameter were considered in Staněk [187] and Staněk and Přibyl [188]. The paper [187] deals with the boundary conditions $u(0)=0$, $u(T)=0$, $\varphi(u')=A\,(A>0)$ where $\varphi \in \mathcal{A}$. Here $\mathcal{A}$ is the set of functionals $\varphi : C[0,T] \to \mathbb{R}$ which are (i) continuous, $\varphi(0)=0$, $\varphi(x)=\varphi(|x|)$ for $x \in C[0,T]$, (ii) increasing and (iii) unbounded in the following sense: $\lim_{\mu\to\infty}\varphi(\mu\,x)=\infty$ for each $x \in C[0,T]$, $x \not\equiv 0$. We note that the boundary conditions (11.4) are a special case of the conditions discussed in [187]. In [188] the authors considered the boundary conditions $u(0)+u(T)=0$, $u'(0)=u'(T)=0$ and $\max\{u(t): 0 \le t \le T\} = A\,(A>0)$. The method of implementation of parameters to a singular Lidstone problem for higher order differential equations with the extra condition $\max\{u(t): 0 \le t \le T\} = A$ was studied in Agarwal, O'Regan and Staněk [17].

Appendix A

Uniform integrability, equicontinuity

Here we present three criteria guaranteeing uniform integrability of sequences in $L_1[0,T]$ which are applied in our proofs.

A sequence $\{\varphi_m\} \subset L_1[0,T]$ is called *uniformly integrable on* $[0,T]$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{M} \subset [0,T]$ and $\operatorname{meas}(\mathcal{M}) < \delta$, then

$$\int_{\mathcal{M}} |\varphi_m(t)| \, \mathrm{d}t < \varepsilon \quad \text{for } m \in \mathbb{N}.$$

An immediate consequence of the definition is the following simple criterion.

Criterion A.1. *Let* $\varphi_m,\ \alpha \in L_1[0,T]$ *be such that*

$$|\varphi_m(t)| \le \alpha(t) \quad \textit{for a.e. } t \in [0,T] \textit{ and all } m \in \mathbb{N}.$$

Then $\{\varphi_m\}$ *is uniformly integrable on* $[0,T]$.

In order to prove more sophisticated criteria the following auxiliary result is useful.

Lemma A.2. *Let* $\{\varphi_m\} \subset L_1[0,T]$. *Suppose that for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any at most countable set* $\{(a_i,b_i)\}_{i\in\mathbb{J}}$ *of mutually disjoint intervals* $(a_i,b_i) \subset [0,T]$, $\sum_{i\in\mathbb{J}}(b_i - a_i) < \delta$, *we have*

$$\sum_{i\in\mathbb{J}} \int_{a_i}^{b_i} |\varphi_m(t)| \, \mathrm{d}t < \varepsilon \quad \textit{for } \ m \in \mathbb{N}.$$

Then $\{\varphi_m\}$ *is uniformly integrable on* $[0,T]$.

Proof. Fix $\varepsilon > 0$ and let $\delta > 0$ be from the assumption. Let $\mathcal{M} \subset [0,T]$ be a measurable set, $\operatorname{meas}(\mathcal{M}) < \delta/2$. Then there exists an open set $\mathcal{M}_1 \subset [0,T]$,

$\mathcal{M}\cap(0,T)\subset\mathcal{M}_1$ such that $\operatorname{meas}(\mathcal{M}_1)<\delta$. From the structure of open and bounded subsets in $\mathbb{R}$ it follows that $\mathcal{M}_1$ is the union of at most countable set $\{(\alpha_j,\beta_j)\}_{j\in\mathbb{J}_*}$ of mutually disjoint intervals $(\alpha_j,\beta_j)\subset[0,T]$. Then

$$\int_{\mathcal{M}_1}|\varphi_m(t)|\,\mathrm{d}t=\sum_{j\in\mathbb{J}_*}\int_{\alpha_j}^{\beta_j}|\varphi_m(t)|\,\mathrm{d}t<\varepsilon,\quad m\in\mathbb{N},$$

by our assumptions. Hence

$$\int_{\mathcal{M}}|\varphi_m(t)|\,\mathrm{d}t\le\int_{\mathcal{M}_1}|\varphi_m(t)|\,\mathrm{d}t<\varepsilon,\quad m\in\mathbb{N}.$$

Consequently, $\{\varphi_m\}$ is uniformly integrable on $[0,T]$. ☐

Criterion A.3. *Let* $\{u_m\}\subset C[0,T]$ *and* $\ell\in\mathbb{N}$. *Let there exist* ℓ_m+1 *disjoint intervals* $(d_{m,k},d_{m,k+1})$, $0\le k\le\ell_m$, $\ell_m\le\ell$, *such that*

$$\bigcup_{k=0}^{\ell_m}[d_{m,k},d_{m,k+1}]=[0,T],$$

and for $k\in\{0,\dots,\ell_m\}$ *and* $m\in\mathbb{N}$ *one of the inequalities*

$$|u_m(t)|\ge b(t-d_{m,k})^{r_{m,k}}\quad\textit{for}\ \ t\in[d_{m,k},d_{m,k+1}]$$

or

$$|u_m(t)|\ge b(d_{m,k+1}-t)^{r_{m,k}}\quad\textit{for}\ \ t\in[d_{m,k},d_{m,k+1}]$$

is satisfied where $b>0$, $1\le r_{m,k}\le r$. *In addition, assume that* g *is a nonincreasing and positive function on* $(0,\infty)$ *and*

$$\int_0^1 g(s^r)\,\mathrm{d}s<\infty.$$

Then the sequence $\{g(|u_m(t)|)\}$ *is uniformly integrable on* $[0,T]$.

Proof. Put $c=\min\{\frac{1}{T},\min\{b^{1/r_{m,k}}:0\le k\le\ell_m,\ m\in\mathbb{N}\}\}$. Then

$$b\,(t-d_{m,k})^{r_{m,k}}\ge[c\,(t-d_{m,k})]^r,\quad b\,(d_{m,k+1}-t)^{r_{m,k}}\ge[c\,(d_{m,k+1}-t)]^r$$

for $t\in[d_{m,k},d_{m,k+1}]$. Therefore for $k\in\{0,\dots,\ell_m\}$ and $m\in\mathbb{N}$ one of the inequalities

$$|u_m(t)|\ge[c\,(t-d_{m,k})]^r\quad\text{for}\ \ t\in[d_{m,k},d_{m,k+1}]\tag{A.1}$$

or

$$|u_m(t)| \geq [c\,(d_{m,k+1}-t)]^r \quad \text{for} \quad t \in [d_{m,k}, d_{m,k+1}] \tag{A.2}$$

is satisfied.

Let $\{(a_i,b_i)\}_{i\in\mathbb{J}}$ be an at most countable set of mutually disjoint intervals $(a_i,b_i)\subset[0,T]$. Put

$$\mathbb{J}_{m,k} = \{i\in\mathbb{J}\colon (a_i,b_i)\subset(d_{m,k},d_{m,k+1})\}$$

for $m\in\mathbb{N}$ and $k\in\{0,\dots,\ell_m\}$. If $i\in\mathbb{J}_{m,k}$ then

$$\int_{a_i}^{b_i} g(|u_m(t)|)\,\mathrm{d}t \leq \int_{a_i}^{b_i} g([c(t-d_{m,k})]^r)\,\mathrm{d}t = \frac{1}{c}\int_{c(a_i-d_{m,k})}^{c(b_i-d_{m,k})} g(t^r)\,\mathrm{d}t$$

if (A.1) holds or

$$\int_{a_i}^{b_i} g(|u_m(t)|)\,\mathrm{d}t \leq \int_{a_i}^{b_i} g([c(d_{m,k+1}-t)]^r)\,\mathrm{d}t = \frac{1}{c}\int_{c(d_{m,k+1}-b_i)}^{c(d_{m,k+1}-a_i)} g(t^r)\,\mathrm{d}t$$

if (A.2) holds. Hence

$$\sum_{i\in\mathbb{J}_{m,k}} \int_{a_i}^{b_i} g(|u_m(t)|)\,\mathrm{d}t \leq \frac{1}{c}\int_{\mathcal{M}_{m,k}} g(t^r)\,\mathrm{d}t, \quad 0\leq k\leq\ell_m,\ m\in\mathbb{N}, \tag{A.3}$$

where $\mathcal{M}_{k,m}\subset[0,cT]$ and $\operatorname{meas}(\mathcal{M}_{m,k})\leq c\sum_{i\in\mathbb{J}}(b_i-a_i)$.

Let $i_0\in\mathbb{J}\setminus\bigcup_{k=0}^{\ell_m}\mathbb{J}_{m,k}$ for some $m\in\mathbb{N}$. Then

$$d_{m,l_0} \leq a_{i_0} \leq d_{m,l_0+1} < \dots < d_{m,l_*} \leq b_{i_0} \leq d_{m,l_*+1},$$

where $l_0, l_*\in\{0,\dots,\ell_m\}$, $l_0+1\leq l_*$ and

$$d_{m,l_*} - d_{m,l_0+1} < b_{i_0} - a_{i_0} < d_{m,l_*+1} - d_{m,l_0}.$$

Notice that there exist at most ℓ_m positive integers i_0 having the above property. Thus

$$\int_{a_{i_0}}^{b_{i_0}} g(|u_m(t)|)\,\mathrm{d}t = \int_{a_{i_0}}^{d_{m,l_0+1}} g(|u_m(t)|)\,\mathrm{d}t + \sum_{k=l_0+1}^{l_*-1}\int_{d_{m,k}}^{d_{m,k+1}} g(|u_m(t)|)\,\mathrm{d}t$$
$$+ \int_{d_{m,l_*}}^{b_{i_0}} g(|u_m(t)|)\,\mathrm{d}t$$

(here $\sum_{k=l_0+1}^{l_*-1} = 0$ if $l_0 + 1 = l_*$). Since

$$\int_{a_{i_0}}^{d_{m,l_0+1}} g(|u_m(t)|)\,\mathrm{d}t \le \begin{cases} \dfrac{1}{c}\displaystyle\int_{c(a_{i_0}-d_{m,l_0})}^{c\,(d_{m,l_0+1}-d_{m,l_0})} g(t^r)\,\mathrm{d}t \\ \qquad\qquad \text{if } |u_m(t)| \ge [c\,(t-d_{m,l_0})]^r \\ \dfrac{1}{c}\displaystyle\int_{0}^{c(d_{m,l_0+1}-a_{i_0})} g(t^r)\,\mathrm{d}t \\ \qquad\qquad \text{if } |u_m(t)| \ge [c\,(d_{m,l_0+1}-t)]^r, \end{cases}$$

$$\int_{d_{m,k}}^{d_{m,k+1}} g(|u_m(t)|)\,\mathrm{d}t \le \int_0^{c(d_{m,k+1}-d_{m,k})} g(t^r)\,\mathrm{d}t, \quad l_0+1 \le k \le l_*-1$$

$$\text{if } l_* \ge l_0 + 2,$$

and

$$\int_{d_{m,l_*}}^{b_{i_0}} g(|u_m(t)|)\,\mathrm{d}t \le \begin{cases} \dfrac{1}{c}\displaystyle\int_{0}^{c(b_{i_0}-d_{m,l_*})} g(t^r)\,\mathrm{d}t \\ \qquad\qquad \text{if } |u_m(t)| \ge [c\,(t-d_{m,l_*})]^r \\ \dfrac{1}{c}\displaystyle\int_{c\,(d_{m,l_*+1}-b_{i_0})}^{c\,(d_{m,l_*+1}-d_{m,l_*})} g(t^r)\,\mathrm{d}t \\ \qquad\qquad \text{if } |u_m(t)| \ge [c\,(d_{m,l_*+1}-t)]^r, \end{cases}$$

it follows that

$$\begin{cases} \displaystyle\int_{a_{i_0}}^{b_{i_0}} g(|u_m(t)|)\,\mathrm{d}t & < \dfrac{l_*-l_0+1}{c}\displaystyle\int_0^{c(b_{i_0}-a_{i_0})} g(t^r)\,\mathrm{d}t \\ & < \dfrac{\ell}{c}\displaystyle\int_0^{c(b_{i_0}-a_{i_0})} g(t^r)\,\mathrm{d}t < \dfrac{\ell}{c}\displaystyle\int_{\mathcal{M}_*} g(t^r)\,\mathrm{d}t \end{cases} \tag{A.4}$$

where $\mathcal{M}_* \subset [0, cT]$ and $\operatorname{meas}(\mathcal{M}_*) \le c \sum_{i\in\mathbb{J}}(b_i - a_i)$. Due to (A.3) and (A.4) we have that

$$\sum_{i\in\mathbb{J}} \int_{a_i}^{b_i} g(|u_m(t)|)\,\mathrm{d}t < \frac{1}{c}\sum_{k=0}^{\ell_m} \int_{\mathcal{M}_{m,k}} g(t^r)\,\mathrm{d}t + \frac{\ell^2}{c}\int_{\mathcal{M}_*} g(t^r)\,\mathrm{d}t. \tag{A.5}$$

Since $g(t^r) \in L_1[0,1]$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{\mathcal{M}} g(t^r)\,\mathrm{d}t < \frac{c\,\varepsilon}{\ell\,(\ell+1)} \tag{A.6}$$

whenever $\mathcal{M} \subset [0,1]$ is measurable and $\operatorname{meas}(\mathcal{M}) < \delta$. Hence for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any at most countable set $\{(a_i,b_i)\}_{i\in\mathbb{J}}$ of mutually disjoint intervals $(a_i,b_i) \subset [0,T]$, $\sum_{i\in\mathbb{J}}(b_i - a_i) < \frac{\delta}{c}$, we have (see (A.5) and (A.6))

$$\sum_{i\in\mathbb{J}} \int_{a_i}^{b_i} g(|u_m(t)|)\,\mathrm{d}t < \left(\frac{\ell}{c} + \frac{\ell^2}{c}\right)\frac{c\,\varepsilon}{\ell\,(\ell+1)} = \varepsilon, \quad m \in \mathbb{N}.$$

So, $\{g(|u_m(t)|)\}$ is uniformly integrable on $[0,T]$ by Lemma A.2, where we put $r_m(t) = g(|u_m(t)|)$). $\square$

In particular, for $\ell_m = 1$ and $r_{m,k} = r$ we get

Criterion A.4. *Let* $\{u_m\} \subset C[0,T]$ *and let there exist* $\{\xi_m\} \subset (0,T)$ *and* $b > 0$, $r \geq 1$ *such that*

$$|u_m(t)| \geq b\,|t - \xi_m|^r \quad \textit{for} \quad t \in [0,T].$$

Suppose that $g\colon (0,\infty) \to (0,\infty)$ *is nonincreasing and*

$$\int_0^1 g(s^r)\,\mathrm{d}s < \infty.$$

Then the sequence $\{g(|u_m(t)|)\}$ *is uniformly integrable on* $[0,T]$.

Equicontinuity

Consider a sequence of functions $v_k \in C[a,b]$, $k \in \mathbb{N}$, $[a,b] \subset \mathbb{R}$. We say that $\{v_k\}$ is *equicontinuous on* $[a,b]$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $t_1, t_2 \in [a,b]$ and each $k \in \mathbb{N}$

$$|t_1 - t_2| < \delta \quad \Longrightarrow \quad |v_k(t_1) - v_k(t_2)| < \varepsilon.$$

Similarly, we say that the sequence $\{v_k\}$ is *equicontinous at a point* $t_0 \in [a,b]$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $t \in (t_0 - \delta, t_0 + \delta) \cap [a,b]$

and each $k \in \mathbb{N}$ the inequality $|v_k(t) - v_k(t_0)| < \varepsilon$ holds. If $t_0 = 0\,(t_0 = T)$ we talk about equicontinuity at 0 from the right (at T from the left).

It is well known that if $\{v_k\} \subset C^1[a,b]$ and there exists $c > 0$ such that $|v_k'(t)| \le c$ on $[a,b]$ for $k \in \mathbb{N}$, then $\{v_k\}$ is equicontinuous on $[a,b]$.

Here we provide conditions which imply the equicontinuity of $\{v_k\}$ at the singular point $t_0 \in [0,T]$ and which are not generally available in literature.

Lemma A.5. *Let $t_0 \in (0,T)$. Assume that there exist $\eta > 0$ such that $[t_0 - \eta, t_0 + \eta] \subset [0,T]$ and nonnegative functions $\alpha \in C[t_0 - \eta, t_0 + \eta]$, $\beta \in C[t_0 - \eta, t_0)$ such that $\alpha(t_0) = 0$, $\beta(t_0-) = 0$. Further assume that for each $k \in \mathbb{N}$, $k > \frac{1}{\eta}$*

$$|v_k(t)| \le \beta(t) \qquad \text{for } t \in [t_0 - \eta, t_0 - \tfrac{1}{k}], \tag{A.7}$$

$$|v_k(t) - v_k(t_0)| \le \alpha(t) \qquad \text{for } t \in [t_0 - \tfrac{1}{k}, t_0 + \tfrac{1}{k}], \tag{A.8}$$

$$\begin{cases} |v_k(t)| \le \beta(t_0 - \tfrac{1}{k}) + \alpha(t_0 - \tfrac{1}{k}) + \alpha(t_0 + \tfrac{1}{k}) + \alpha(t) \\ \qquad\qquad\qquad \text{for } t \in [t_0 + \tfrac{1}{k}, t_0 + \eta]. \end{cases} \tag{A.9}$$

Then $\lim_{k\to\infty} v_k(t_0) = 0$ and the sequence $\{v_k\}$ is equicontinuous at t_0.

Proof. Choose an arbitrary $\varepsilon > 0$. Then there exists $\delta \in (0,\eta)$ such that

$$t \in (t_0 - \delta, t_0 + \delta) \Longrightarrow |\alpha(t)| < \frac{\varepsilon}{6}, \quad t \in (t_0 - \delta, t_0) \Longrightarrow |\beta(t)| < \frac{\varepsilon}{6}.$$

Choose an arbitrary $k \in \mathbb{N}, k \ge \frac{\varepsilon}{6}$. Let $t \in [t_0 - \frac{1}{k}, t_0 + \frac{1}{k}]$. Then by (A.8),

$$|v_k(t) - v_k(t_0)| \le \alpha(t) < \frac{\varepsilon}{6} < \varepsilon.$$

Let $t \in (t_0 - \delta, t_0 - \frac{1}{k})$. Then by (A.7) and (A.8),

$$\begin{aligned} |v_k(t) - v_k(t_0)| &\le |v_k(t)| + |v_k(t_0) - v_k(t_0 - \tfrac{1}{k})| + |v_k(t_0 - \tfrac{1}{k})| \\ &\le \beta(t) + \alpha(t_0 - \tfrac{1}{k}) + \beta(t_0 - \tfrac{1}{k}) < \frac{3\,\varepsilon}{6} < \varepsilon. \end{aligned}$$

Let $t \in (t_0 + \frac{1}{k}, t_0 + \delta)$. Then by (A.8) and (A.9),

$$\begin{aligned} |v_k(t) - v_k(t_0)| &\le |v_k(t)| + |v_k(t_0) - v_k(t_0 - \tfrac{1}{k})| + |v_k(t_0 - \tfrac{1}{k})| \\ &\le \beta(t_0 - \tfrac{1}{k}) + \alpha(t_0 - \tfrac{1}{k}) + \alpha(t_0 + \tfrac{1}{k}) \\ &\qquad + \alpha(t) + \alpha(t_0 - \tfrac{1}{k}) + \beta(t_0 - \tfrac{1}{k}) < \varepsilon. \end{aligned}$$

Hence, we have proved that $\{v_k\}$ is equicontinuous at t_0. Further,

$$|v_k(t_0)| \le |v_k(t_0) - v_k(t_0 - \tfrac{1}{k})| + |v_k(t_0 - \tfrac{1}{k})| \le \alpha(t_0 - \tfrac{1}{k}) + \beta(t_0 - \tfrac{1}{k}).$$

Therefore $\lim_{k\to\infty} v_k(t_0) = 0$. $\square$

Similarly we can prove

Lemma A.6. *Let* $t_0 \in (0,T)$. *Assume that there exist* $\eta > 0$ *such that* $[t_0 - \eta, t_0 + \eta] \subset [0,T]$ *and nonnegative functions* $\alpha \in C[t_0 - \eta, t_0 + \eta]$, $\beta \in C(t_0, t_0 + \eta]$ *such that* $\alpha(t_0) = 0$, $\beta(t_0+) = 0$. *Further assume that for each* $k \in \mathbb{N}$, $k > \frac{1}{\eta}$

$$\begin{aligned}
&|v_k(t)| \le \beta(t) && \text{for } t \in [t_0 + \tfrac{1}{k}, t_0 + \eta],\\
&|v_k(t) - v_k(t_0)| \le \alpha(t) && \text{for } t \in [t_0 - \tfrac{1}{k}, t_0 + \tfrac{1}{k}],\\
&\begin{cases} |v_k(t)| \le \beta(t_0 + \tfrac{1}{k}) + \alpha(t_0 + \tfrac{1}{k}) + \alpha(t_0 - \tfrac{1}{k}) + \alpha(t) \\ \qquad\qquad\qquad\qquad\qquad \text{for } t \in [t_0 - \eta, t_0 - \tfrac{1}{k}]. \end{cases}
\end{aligned}$$

Then $\lim_{k\to\infty} v_k(t_0) = 0$ *and the sequence* $\{v_k\}$ *is equicontinuous at* t_0.

In particular, for $t_0 = T$ and $t_0 = 0$ arguing as before we get the following two lemmas.

Lemma A.7. *Assume that there exist* $\eta \in (0,T)$ *and nonnegative functions* $\alpha \in C[T - \eta, T]$, $\beta \in C[T - \eta, T)$ *such that* $\alpha(T) = 0$, $\beta(T-) = 0$. *Further assume that for* $k \in \mathbb{N}$, $k > \frac{1}{\eta}$

$$\begin{aligned}
&|v_k(t)| \le \beta(t) && \text{for } t \in [T - \eta, t_0 - \tfrac{1}{k}],\\
&|v_k(t) - v_k(T)| \le \alpha(t) && \text{for } t \in [T - \tfrac{1}{k}, T].
\end{aligned}$$

Then $\lim_{k\to\infty} v_k(T) = 0$ *and the sequence* $\{v_k\}$ *is equicontinuous at* T *from the left.*

Lemma A.8. *Assume that there exist* $\eta \in (0,T)$ *and nonnegative functions* $\alpha \in C[0,\eta]$, $\beta \in C(0,\eta]$ *such that* $\alpha(0) = 0$, $\beta(0+) = 0$. *Further assume that for* $k \in \mathbb{N}$, $k > \frac{1}{\eta}$

$$\begin{aligned}
&|v_k(t)| \le \beta(t) && \text{for } t \in [\tfrac{1}{k}, \eta],\\
&|v_k(t) - v_k(T)| \le \alpha(t) && \text{for } t \in [0, \tfrac{1}{k}].
\end{aligned}$$

Then $\lim_{k\to\infty} v_k(0) = 0$ *and the sequence* $\{v_k\}$ *is equicontinuous at* 0 *from the right.*

Now we provide criteria of equicontinuity of $\{v_k\}$ at the point $t_0 \in (0,T)$.

Criterion A.9. *Let* $t_0 \in (0,T)$, $\beta_0, \eta \in (0,\infty)$ *be such that* $[t_0-\eta, t_0+\eta] \subset [0,T]$. *Assume that there exist nonnegative functions* $h^*, g^* \in L_1[0,T]$ *and a nonnegative function* $h \in L_{loc}([0,T]\setminus\{t_0\})$ *such that for each* $k \in \mathbb{N}$, $k > \frac{1}{\eta}$ *there is a function* $v_k \in AC[0,T]$ *fulfilling conditions*

$$|v_k(t_0-\eta)| \le \beta_0, \tag{A.10}$$

$$\begin{cases} v_k'(t)\,\operatorname{sign} v_k(t) \le -h(t)\,|v_k(t)| + g^*(t) \\ \quad \textit{for a.e. } t \in [t_0-\eta, t_0+\eta] \setminus (t_0-\frac{1}{k}, t_0+\frac{1}{k}), \end{cases} \tag{A.11}$$

$$|v_k'(t)| \le h^*(t) \quad \textit{for a.e. } t \in [t_0-\tfrac{1}{k}, t_0+\tfrac{1}{k}], \tag{A.12}$$

where

$$\int_{t_0-\varepsilon}^{t_0} h(s)\,\mathrm{d}s = +\infty \quad \textit{for each sufficiently small } \varepsilon > 0. \tag{A.13}$$

Then $\lim_{k\to\infty} v_k(t_0) = 0$ *and the sequence* $\{v_k\}$ *is equicontinuous at* t_0.

Proof. We will construct functions α and β of Lemma A.5. Consider the auxiliary problem

$$\beta'(t) = -h(t)\beta(t) + g^*(t), \quad \beta(t_0-\eta) = \beta_0. \tag{A.14}$$

Problem (A.14) has a unique solution and this solution has the form

$$\beta(t) = \exp\left(-\int_{t_0-\eta}^{t} h(s)\,\mathrm{d}s\right)\left(\beta_0 + \int_{t_0-\eta}^{t} g^*(\tau)\,\exp\left(\int_{t_0-\eta}^{\tau} h(s)\,\mathrm{d}s\right)\mathrm{d}\tau\right)$$

for $t \in [t_0-\eta, t_0)$. Then $\beta \in C[t_0-\eta, t_0)$ and, by (A.13), we get

$$\begin{aligned}\lim_{t\to t_0-} \beta(t) = \beta_0 \exp\left(-\int_{t_0-\eta}^{t_0} h(s)\,\mathrm{d}s\right) \\ + \int_{t_0-\eta}^{t_0} g^*(\tau)\,\exp\left(-\int_{\tau}^{t_0} h(s)\,\mathrm{d}s\right)\mathrm{d}\tau = 0,\end{aligned}$$

because

$$\int_{\tau}^{t_0} h(s)\,\mathrm{d}s = \infty \quad \text{for each} \quad \tau \in [t_0 - \eta, t_0).$$

Let us prove that (A.7) is satisfied. On the contrary, assume that there exist $t_1 \in [t_0 - \eta, t_0 - \frac{1}{k})$ and $t_2 \in (t_1, t_0 - \frac{1}{k}]$ such that

$$|v_k(t_1)| = \beta(t_1), \quad |v_k(t)| > \beta(t) \quad \text{for all } t \in (t_1, t_2].$$

Then, by (A.11) and (A.14), we get

$$\begin{aligned}
0 < |v_k(t_2)| - \beta(t_2) &= \int_{t_1}^{t_2} (v_k'(t)\,\mathrm{sign}\, v_k(t) - \beta'(t))\,\mathrm{d}t \\
&\le -\int_{t_1}^{t_2} h(t)(|v_k(t)| - \beta(t))\,\mathrm{d}t \le 0,
\end{aligned}$$

a contradiction. So, (A.7) is proved.

Further, due to (A.12), we have

$$|v_k(t) - v_k(t_0)| \le \left| \int_{t_0}^{t} h^*(s)\,\mathrm{d}s \right| \quad \text{for} \quad t \in [t_0 - \tfrac{1}{k}, t_0 + \tfrac{1}{k}] \tag{A.15}$$

and integrating (A.11) we obtain

$$|v_k(t)| \le |v_k(t_0 + \tfrac{1}{k})| + \int_{t_0 + \frac{1}{k}}^{t} g^*(s)\,\mathrm{d}s \quad \text{for} \quad t \in [t_0 + \tfrac{1}{k}, t_0 + \eta]. \tag{A.16}$$

Let us put

$$\alpha(t) = \max\left\{ \left| \int_{t_0}^{t} h^*(s)\,\mathrm{d}s \right|, \left| \int_{t_0}^{t} g^*(s)\,\mathrm{d}s \right| \right\} \quad \text{for} \quad t \in [t_0 - \eta, t_0 + \eta].$$

Then $\alpha \in C[t_0 - \eta, t_0 + \eta]$ and $\alpha(t_0) = 0$. Moreover, (A.15) and (A.16) imply

$$|v_k(t) - v_k(t_0)| \le \alpha(t) \quad \text{for} \quad t \in [t_0 - \tfrac{1}{k}, t_0 + \tfrac{1}{k}]$$

and

$$\begin{aligned}
|v_k(t)| &\le |v_k(t_0 + \tfrac{1}{k})| + \alpha(t) \\
&\le |v_k(t_0 + \tfrac{1}{k}) - v_k(t_0)| + |v_k(t_0) - v_k(t_0 - \tfrac{1}{k})| + |v_k(t_0 - \tfrac{1}{k})| + \alpha(t) \\
&\le \alpha(t_0 + \tfrac{1}{k}) + \alpha(t_0 - \tfrac{1}{k}) + \beta(t_0 - \tfrac{1}{k}) + \alpha(t) \quad \text{for} \quad t \in [t_0 + \tfrac{1}{k}, t_0 + \eta].
\end{aligned}$$

Thus (A.8) and (A.9) are satisfied and, by Lemma A.5, the proof is completed. $\square$

Using Lemma A.6 instead of Lemma A.5 we get a modified form of Criterion A.9.

Criterion A.10. *Let* $t_0 \in (0,T)$, $\beta_0 \in (0,\infty)$ *and* $\eta > 0$ *be such that* $[t_0-\eta, t_0+\eta] \subset [0,T]$. *Assume that there exist nonnegative functions* h^*, $g^* \in L_1[0,T]$ *and a nonnegative function* $h \in L_{loc}([0,T] \setminus \{t_0\})$ *such that for each* $k \in \mathbb{N}$, $k > \frac{1}{\eta}$ *there is a function* $v_k \in AC[0,T]$ *fulfilling conditions*

$$|v_k(t_0+\eta)| \le \beta_0,$$

$$\begin{cases} v_k'(t) \operatorname{sign} v_k(t) \ge h(t)|v_k(t)| - g^*(t) \\ \quad \textit{for a.e. } t \in [t_0-\eta, t_0+\eta] \setminus (t_0-\frac{1}{k}, t_0+\frac{1}{k}), \end{cases}$$

$$|v_k'(t)| \le h^*(t) \quad \textit{for a.e. } t \in [t_0-\tfrac{1}{k}, t_0+\tfrac{1}{k}],$$

where

$$\int_{t_0}^{t_0+\varepsilon} h(s)\,\mathrm{d}s = +\infty \quad \textit{for each sufficiently small } \varepsilon > 0.$$

Then $\lim_{k\to\infty} v_k(t_0) = 0$ *and the sequence* $\{v_k\}$ *is equicontinuous at* t_0.

In particular, Lemmas A.7 and A.8 yield criteria which are used in our proofs and which guarantee the equicontinuity of $\{v_k\}$ at T from the left and at 0 from the right, respectively.

Criterion A.11. *Let* $\beta_0 \in (0,\infty)$ *and* $\eta \in (0,T)$. *Assume that there exist nonnegative functions* h^*, $g^* \in L_1[0,T]$ *and a nonnegative function* $h \in L_{loc}[0,T)$ *such that for each* $k \in \mathbb{N}$, $k > \frac{1}{\eta}$, *there exists a function* $v_k \in AC[0,T]$ *fulfilling conditions*

$$|v_k(T-\eta)| \le \beta_0, \tag{A.17}$$

$$\begin{cases} v_k'(t) \operatorname{sign} v_k(t) \le -h(t)|v_k(t)| + g^*(t) \\ \quad \textit{for a.e. } t \in [T-\eta, T-\frac{1}{k}], \end{cases} \tag{A.18}$$

$$|v_k'(t)| \le h^*(t) \quad \textit{for a.e. } t \in [T-\tfrac{1}{k}, T], \tag{A.19}$$

where

$$\int_{T-\varepsilon}^{T} h(s)\,\mathrm{d}s = +\infty \quad \textit{for each sufficiently small} \ \ \varepsilon > 0. \tag{A.20}$$

Then $\lim_{k\to\infty} v_k(T) = 0$ *and the sequence* $\{v_k\}$ *is equicontinuous at* T *from the left.*

Criterion A.12. *Let* $\beta_0 \in (0,\infty)$ *and* $\eta \in (0,T)$. *Assume that there exist nonnegative functions* $h^*, g^* \in L_1[0,T]$ *and a nonnegative function* $h \in L_{loc}(0,T]$ *such that for each* $k \in \mathbb{N}$, $k > \frac{1}{\eta}$ *there exists a function* $v_k \in AC[0,T]$ *fulfilling conditions*

$$|v_k(\eta)| \le \beta_0, \tag{A.21}$$

$$v_k'(t)\ \mathrm{sign}\, v_k(t) \ge h(t)|v_k(t)| - g^*(t) \quad \textit{for a.e.} \ \ t \in [\tfrac{1}{k}, \eta], \tag{A.22}$$

$$|v_k'(t)| \le h^*(t) \quad \textit{for a.e.}\ t \in [0, \tfrac{1}{k}], \tag{A.23}$$

where

$$\int_0^{\varepsilon} h(s)\,\mathrm{d}s = +\infty \quad \textit{for each sufficiently small} \ \ \varepsilon > 0. \tag{A.24}$$

Then $\lim_{k\to\infty} v_k(0) = 0$ *and the sequence* $\{v_k\}$ *is equicontinuous at* 0 *from the right.*

Appendix B

Convergence theorems

The main tool for proving solvability of singular problems is a regularization and a sequential technique. In this way, solutions of singular problems are obtained by limit processes. Classical arguments here are convergence theorems in spaces of integrable functions and differentiable functions.

Integrable functions

The following three theorems for integrable functions can be found e.g. in Bartle [30], Hewitt and Stromberg [105], Lang [119], Natanson [143], Shilov and Gurevich [178].

Theorem B.1 (**Lebesgue dominated convergence theorem**).
Let $\varphi_m,\ \alpha \in L_1[0,T]$ *be such that*

$$|\varphi_m(t)| \le \alpha(t) \quad \textit{for a.e. } t \in [0,T] \textit{ and all } m \in \mathbb{N},$$
$$\lim_{m\to\infty} \varphi_m(t) = \varphi(t) \quad \textit{for a.e. } t \in [0,T].$$

Then $\varphi \in L_1[0,T]$ *and*

$$\lim_{m\to\infty} \int_0^T \varphi_m(t)\,\mathrm{d}t = \int_0^T \varphi(t)\,\mathrm{d}t.$$

If the sequence is bounded by a Lebesgue integrable function only from one side, we often use the theorem which is known in literature as the Fatou lemma.

Theorem B.2 (**Fatou lemma**).
Let $c \in (0,\infty)$ *and* $\varphi_m, \alpha \in L_1[0,T]$ *be such that*

$$\alpha(t) \le \varphi_m(t) \quad \textit{for a.e. } t \in [0,T] \textit{ and all } m \in \mathbb{N},$$
$$\int_0^T \varphi_m(t)\,\mathrm{d}t \le c \quad \textit{for all } \ m \in \mathbb{N}$$

and

$$\lim_{m\to\infty} \varphi_m(t) = \varphi(t) \qquad \textit{for a.e. } t \in [0,T].$$

Then $\varphi \in L_1[0,T]$.

If we do not know the localization in $[0,T]$ of singular points corresponding to solutions of singular problems, i.e. problems have singular points of type II, then it often happens that we cannot find a Lebesgue integrable majorant function. In such cases, the Vitali convergence theorem is used in limit processes since the existence of a Lebesgue integrable majorant function is replaced in this theorem by a more general assumption about the uniform integrability.

Theorem B.3 (**Vitali convergence theorem**).
Let $\varphi_m \subset L_1[0,T]$ *for* $m \in \mathbb{N}$ *and let*

$$\lim_{m\to\infty} \varphi_m(t) = \varphi(t) \qquad \textit{for a.e. } t \in [0,T].$$

Then the following statements are equivalent:

(i) $\varphi \in L_1[0,T]$ *and* $\lim_{m\to\infty} \|\varphi_m - \varphi\|_1 = 0$,

(ii) *the sequence* $\{\varphi_m\}$ *is uniformly integrable on* $[0,T]$.

Differentiable functions

First, we will consider the space $C([a,b];\mathbb{R}^m)$, $m \in \mathbb{N}$, which is the space of continuous m-vector valued functions on the interval $[a,b]$. It is well known that all norms on $\mathbb{R}^m$ are equivalent (see e.g. Lang [119]), that is, if $\|\cdot\|_*$ and $\|\cdot\|_{**}$ are two norms on $\mathbb{R}^m$ then there exist positive constants C_1, C_2 such that for all $x \in \mathbb{R}^m$, $x = (x_1, \dots, x_m)$, we have

$$C_1\,|x|_* \le |x|_{**} \le C_2\,|x|_*.$$

Hence without loss of generality we will use in $\mathbb{R}^m$ the norm

$$|x| = \max\{|x_j| : 1 \le j \le m\}.$$

We say that a subset H of $C([a,b];\mathbb{R}^m)$ is *relatively compact* if from each sequence $\{f_n\} \subset H$ we can select a subsequence $\{f_{k_n}\}$ converging in $C([a,b];\mathbb{R}^m\}$, that is, we can select a subsequence which is uniformly convergent on $[a,b]$.

In order to give conditions guaranteeing that a subset H of $C([a,b];\mathbb{R}^m)$ is relatively compact, we introduce the notions of a uniformly bounded on $[a,b]$ and equicontinuous on $[a,b]$ subset of $C([a,b];\mathbb{R}^m)$.

A subset H of $C([a,b];\mathbb{R}^m)$ is said to be *uniformly bounded on* $[a,b]$ if there exists a positive constant L such that

$$|f(t)| \le L \quad \text{for all} \quad f \in H \quad \text{and} \quad t \in [a,b].$$

It is *equicontinuous on* $[a,b]$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $f \in H$ we have

$$|f(t_1) - f(t_2)| < \varepsilon$$

whenever $t_1, t_2 \in [a,b]$ and $|t_1 - t_2| < \delta$.

Sufficient and necessary conditions for a subset H of $C([a,b];\mathbb{R}^m)$ to be relatively compact are given in the following vector version of the Arzelà-Ascoli theorem (see e.g. Hartman [103] or Piccinini, Stampacchia and Vidossich [152]).

Theorem B.4. *A subset H of $C([a,b];\mathbb{R}^m)$ is relatively compact if and only if H is uniformly bounded on $[a,b]$ and equicontinuous on $[a,b]$.*

We use the following scalar version of the Arzelà-Ascoli theorem which describes compact subsets in $C^m[a,b]$.

Theorem B.5 (**Arzelà-Ascoli theorem**).
Let $m \in \mathbb{N}$ be fixed. Assume that $\{u_n\} \subset C^m[a,b]$, the sequence $\{u_n^{(m)}\}$ is equicontinuous on $[a,b]$ and there exists a positive constant S such that

$$\|u_n^{(j)}\|_\infty \le S \quad \textit{for all} \quad n \in \mathbb{N} \quad \textit{and} \quad 0 \le j \le m. \tag{B.1}$$

Then there exist a subsequence $\{u_{k_n}\}$ of $\{u_n\}$ and $u \in C^m[a,b]$ such that

$$\lim_{n\to\infty} \|u_{k_n} - u\|_{C^m} = 0, \tag{B.2}$$

that is $\lim_{n\to\infty} u_{k_n}^{(j)}(t) = u^{(j)}(t)$ uniformly on $[a,b]$ for $0 \le j \le m$.

Proof. Put $f_n(t) = (u_n(t), u_n'(t), \dots, u_n^{(m)}(t))$ for $t \in [a,b]$ and $n \in \mathbb{N}$. Then $\{f_n\} \subset C([a,b];\mathbb{R}^{m+1})$ and since $|f_n(t)| \le S$ for $t \in [a,b]$ by (B.1), the sequence $\{f_n\}$ is uniformly bounded on $[a,b]$. As $\{u_n^{(m)}\}$ is equicontinuous on $[a,b]$ by assumption, for each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for $n \in \mathbb{N}$ we have $|u_n^{(m)}(t_1) - u_n^{(m)}(t_2)| < \varepsilon$ whenever $t_1, t_2 \in [a,b]$ and $|t_1 - t_2| < \delta_\varepsilon$. Due to (B.2), $|u_n^{(j)}(t_1) - u_n^{(j)}(t_2)| \le S|t_1 - t_2|$ for $n \in \mathbb{N}$, $t_1, t_2 \in [a,b]$ and $0 \le j \le m-1$. Choose $\varepsilon > 0$ and let $0 < \delta < \min\{\delta_\varepsilon, \frac{\varepsilon}{S}\}$. Then

$$|f_n(t_1) - f_n(t_2)| < \varepsilon \quad \text{for all } n \in \mathbb{N} \text{ and } t_1, t_2 \in [a,b],\ |t_1 - t_2| < \delta,$$

which shows that the sequence $\{f_n\}$ is equicontinuous on $[a,b]$. Hence $\{f_n\}$ is relatively compact by Theorem B.4 and therefore there exist a subsequence $\{f_{k_n}\}$ of $\{f_n\}$ and $f \in C([a,b];\mathbb{R}^{m+1})$, $f = (f_0, f_1, \dots, f_m)$, such that $\{f_{k_n}\}$ converges in $C([a,b];\mathbb{R}^{m+1})$ to f, which is equivalent to

$$\lim_{n\to\infty} u_{k_n}^{(j)}(t) = f_j(t) \quad \text{uniformly on } [a,b] \text{ for } 0 \le j \le m.$$

We now show that

$$f_j(t) = f_0^{(j)}(t) \quad \text{for } t \in [a,b] \text{ and } 1 \le j \le m. \tag{B.3}$$

Letting $n \to \infty$ in

$$u_{k_n}(t) = u_{k_n}(0) + u_{k_n}'(0)\,t + \dots + \frac{u_{k_n}^{(j-1)}(0)}{(j-1)!}\,t^{j-1} + \frac{1}{(j-1)!}\int_0^t (t-s)^{j-1} u_{k_n}^{(j)}(s)\,\mathrm{d}s$$

yields

$$\begin{cases} f_0(t) = f_0(0) + f_1(0)t + \dots + \dfrac{f_{j-1}(0)}{(j-1)!}\,t^{j-1} \\ \qquad\qquad + \dfrac{1}{(j-1)!}\displaystyle\int_0^t (t-s)^{j-1} f_j(s)\,\mathrm{d}s \end{cases} \tag{B.4}$$

for $t \in [a,b]$ and $1 \le j \le m$. The validity of (B.3) follows from (B.4). Putting $u = f_0$ we see that (B.2) holds. □

The next theorem about locally uniform convergence on an open and bounded interval is proved by means of Cauchy diagonalization principle and, hence, we call it the diagonalization theorem.

Theorem B.6 (**Diagonalization theorem**).
Let $a<\nu_n<\tau_n<b$, where $\{\nu_n\}$ is decreasing and converges to a, $\{\tau_n\}$ is increasing and converges to b. Let $\{u_n\}\subset C^1[\nu_n,\tau_n]$ be a sequence such that for each $\varrho\in(0,\frac{a+b}{2})$ there exist $S_\varrho>0$ and $n_\varrho\in\mathbb{N}$ such that

$$|u_n^{(j)}(t)|\le S_\varrho \quad \textit{for} \quad t\in[a+\varrho,b-\varrho],\ n\ge n_\varrho,\ j=0,1$$

and $\{u_n'\}_{n\ge n_\varrho}$ is equicontinuous on $[a+\varrho,b-\varrho]$.

Then there exist a subsequence $\{u_{k_n}\}$ of $\{u_n\}$ and $u\in C^1(a,b)$ such that

$$\lim_{n\to\infty} u_{k_n}^{(j)}(t)=u^{(j)}(t) \quad \textit{locally uniformly on} \quad (a,b),\ j=0,1. \tag{B.5}$$

Proof. Let $\{\varrho_n\}\subset(0,\frac{a+b}{2})$ be decreasing and $\lim_{n\to\infty}\varrho_n=0$. Then there exists $n_1\in\mathbb{N}$ such that $|u_n^{(j)}(t)|\le S_{\varrho_1}$ for $t\in[a+\varrho_1,b-\varrho_1]$, $n\ge n_1$, $j=0,1$ and, in addition, $\{u_n\}_{n\ge n_1}$ is equicontinuous on $[a+\varrho_1,b-\varrho_1]$. Hence, by Theorem B.5, there is a subsequence $\{u_{k_{1,n}}\}$ of $\{u_n\}_{n\ge n_1}$ for which $\{u_{k_{1,n}}^{(j)}(t)\}$ is uniformly convergent on $[a+\varrho_1,b-\varrho_1]$ for $j=0,1$. Next, there exists a subsequence $\{u_{k_{2,n}}\}$ of $\{u_{k_{1,n}}\}$ such that $\{u_{k_{2,n}}^{(j)}\}$ is uniformly convergent on $[a+\varrho_2,b-\varrho_2]$ for $j=0,1$. We can proceed inductively to obtain a subsequence $\{u_{k_{i,n}}\}$ of $\{u_{k_{i-1,n}}\}$ such that $\{u_{k_{i,n}}^{(j)}\}$ is uniformly convergent on $[a+\varrho_i,b-\varrho_i]$ for $j=0,1$. Put $k_n=k_{n,n}$ for $n\in\mathbb{N}$ and consider the diagonal sequence $\{u_{k_n}\}$. Choose $[\alpha,\beta]\subset(a,b)$. Then $[\alpha,\beta]\subset[a+\varrho_m,b-\varrho_m]$ for some $m\in\mathbb{N}$. Since $\{u_{k_n}\}_{n\ge m}$ is chosen from $\{u_{k_{m,n}}\}$ and we know that $\{u_{k_{m,n}}^{(j)}\}$ is uniformly convergent on $[a+\varrho_m,b-\varrho_m]$ for $j=0,1$ we see that $\{u_{k_n}^{(j)}\}_{n\ge m}$ is uniformly convergent on $[\alpha,\beta]$ for $j=0,1$. We have proved that $\{u_{k_n}^{(j)}\}$ is locally uniformly convergent on (a,b). Let $\lim_{n\to\infty}u_{k_n}(t)=u(t)$ and $\lim_{n\to\infty}u_{k_n}'(t)=v(t)$ for $t\in(a,b)$. Then $u,v\in C(a,b)$ and letting $n\to\infty$ in

$$u_{k_n}(t)=u_{k_n}\left(\tfrac{a+b}{2}\right)+\int_{(a+b)/2}^{t}u_{k_n}'(s)\,\mathrm{d}s, \quad t\in[\nu_{k_n},\tau_{k_n}],\ n\in\mathbb{N},$$

yields

$$u(t)=u\left(\tfrac{a+b}{2}\right)+\int_{(a+b)/2}^{t}v(s)\,\mathrm{d}s, \quad t\in(a,b).$$

Hence $u\in C^1(a,b)$ and $v=u'$ on (a,b), which shows that (B.5) holds. $\square$

Appendix C

Some general existence theorems

We present here the Schauder fixed point theorem (see Deimling [64], Granas and Dugundji [99]), the Leray-Schauder degree theorem and the Borsuk antipodal theorem (see Deimling [64], Mawhin [134]), and the Fredholm type existence theorem (see Lasota [121], Vasiliev and Klokov [194]). These theorems we use in the proofs of solvability of auxiliary regular problems. Since the formulation of Theorem C.5 differs from those in the references cited above we provide its proof.

Let X and Y be Banach spaces. We say that a set $\mathcal{M} \subset X$ is *relatively compact* if from each sequence $\{x_m\} \subset \mathcal{M}$ a convergent subsequence can be chosen.

Let $\mathcal{U}$ be a subset of X. We say that $\mathcal{F} : \mathcal{U} \to Y$ is a *compact operator* if $\mathcal{F}$ is continuous and the set $\mathcal{F}(\mathcal{U})$ is relatively compact.

We say that $\mathcal{F} : \mathcal{U} \to Y$ is *completely continuous* if for each bounded set $\mathcal{V} \subset \mathcal{U}$, the restriction of $\mathcal{F}$ on $\mathcal{V}$ is a compact operator.

Theorem C.1 (**Schauder fixed point theorem**).
Let X be a Banach space, $\Omega \subset X$ a nonempty, closed and convex set and $\mathcal{F} : \Omega \to \Omega$ a compact operator. Then $\mathcal{F}$ has a fixed point.

Theorem C.2 (**Leray-Schauder degree theorem**).
Let X be a Banach space, $\mathcal{U} \subset X$. Let $\Omega \subset \mathcal{U}$ be an open and bounded set. Let $\mathcal{F} : \mathcal{U} \to X$ be a completely continuous operator and $\mathcal{F}(x) \neq x$ for $x \in \partial\Omega$. Let $\mathcal{I}$ be the identity operator on X.

Then there exists an integer $\deg(\mathcal{I} - \mathcal{F}, \Omega)$ which has the following properties:

(i) (Normalization property)
If $0 \in \Omega$, then $\deg(\mathcal{I}, \Omega) = 1$.

(ii) (Existence property)
If $\deg(\mathcal{I} - \mathcal{F}, \Omega) \neq 0$, *then* $\mathcal{F}$ *has a fixed point* $x_0 \in \Omega$.

(iii) (Homotopy property)
If $\mathcal{H} : [0,1] \times \overline{\Omega} \to X$ *is a compact operator and* $\mathcal{H}(\lambda, x) \neq x$ *for* $\lambda \in [0,1]$ *and* $x \in \partial\Omega$, *then*

$$\deg(\mathcal{I} - \mathcal{H}(0,\cdot), \Omega) = \deg(\mathcal{I} - \mathcal{H}(1,\cdot), \Omega).$$

(iv) (Additivity property)
If $\Omega_1 \subset \Omega$ *is an open set and* $\Omega_2 = \Omega \setminus \overline{\Omega}_1$ *and if* $\mathcal{F}(x) \neq x$ *for* $x \in \partial\Omega_1 \cup \partial\Omega_2$, *then*

$$\deg(\mathcal{I} - \mathcal{F}, \Omega) = \deg(\mathcal{I} - \mathcal{F}, \Omega_1) + \deg(\mathcal{I} - \mathcal{F}, \Omega_2).$$

(v) (Excision property)
If $\Omega_1 \subset \Omega$ *is an open set and* $\mathcal{F}(x) \neq x$ *for* $x \in \overline{\Omega \setminus \Omega_1}$, *then*

$$\deg(\mathcal{I} - \mathcal{F}, \Omega) = \deg(\mathcal{I} - \mathcal{F}, \Omega_1).$$

Theorem C.3 (**Borsuk antipodal theorem**).
Let X *be a Banach space, let* $\Omega \subset X$ *be an open, bounded and symmetric set with respect to* $0 \in \Omega$. *Let* $\mathcal{F}$ *be odd in* $\partial\Omega$ (*that is* $\mathcal{F}(-x) = -\mathcal{F}(x)$ *for* $x \in \partial\Omega$). *Then* $\deg(\mathcal{I} - \mathcal{F}, \Omega)$ *is an odd* (*and so nonzero*) *number.*

The integer $\deg(\mathcal{I} - \mathcal{F}, \Omega)$ is the *Leray-Schauder degree* of the operator $\mathcal{F}$ (with respect to the set Ω and the point 0). If $\dim X < \infty$, then the corresponding degree is usually called the *Brouwer degree* (with respect to Ω and 0) and denoted by $\mathrm{d}_B(\mathcal{I} - \mathcal{F}, \Omega)$.

Remark C.4. Let X be a linear normed space with $\dim X = k < \infty$ and let h be an isometrical isomorphism from X onto $\mathbb{R}^k$. Let Ω be a bounded open set in X and $F : \overline{\Omega} \to X$ a continuous mapping. Suppose $F(x) \neq 0$ on $\partial\Omega$. Then

$$\mathrm{d}_B(F, \Omega) = \mathrm{d}_B(h \circ F \circ h^{-1}, h(\Omega)),$$

where $h \circ F \circ h^{-1}$ stands for the composition of mappings h, F and h^{-1}. See e.g. Fučík, Nečas, J. Souček and V. Souček [92] or Deimling [64].

In order to formulate the Fredholm type existence theorem we consider the differential equation

$$u^{(n)} + \sum_{i=0}^{n-1} a_i(t)\, u^{(i)} = g(t, u, \dots, u^{(n-1)}) \tag{C.1}$$

and the corresponding linear homogeneous differential equation

$$u^{(n)} + \sum_{i=0}^{n-1} a_i(t)\, u^{(i)} = 0 \tag{C.2}$$

where $a_i \in L_1[0,T]$, $0 \le i \le n-1$, $g \in Car([0,T] \times \mathbb{R}^n)$. Further, we deal with boundary conditions

$$\mathcal{L}_j(u) = r_j, \quad 1 \le j \le n, \tag{C.3}$$

and with the corresponding homogeneous boundary conditions

$$\mathcal{L}_j(u) = 0, \quad 1 \le j \le n, \tag{C.4}$$

where $\mathcal{L}_j : C^{n-1}[0,T] \to \mathbb{R}$ are linear and continuous functionals and $r_j \in \mathbb{R}$, $1 \le j \le n$.

Theorem C.5 (**Fredholm type existence theorem**).
Let the linear homogeneous problem (*C.2*), (*C.4*) *have only the trivial solution and let there exist a function* $\psi \in L_1[0,T]$ *such that*

$$\begin{cases} |g(t, x_0, \dots, x_{n-1})| \le \psi(t) \\ \qquad \text{for a.e. } t \in [0,T] \text{ and all } (x_0, \dots, x_{n-1}) \in \mathbb{R}^n. \end{cases} \tag{C.5}$$

Then problem (C.1), (C.3) *has a solution* $u \in AC^{n-1}[0,T]$.

Proof. Let $u_1, \dots, u_n$ be the fundamental system of solutions of (C.2). We shall denote by $\Delta_i(t)$ the cofactor of the element $u_i^{(n-1)}$ in the Wronskian $W(t)$ of $u_1, \dots, u_n$. Define $\Gamma : C^{n-1}[0,T] \to C^{n-1}[0,T]$ by the formula

$$(\Gamma x)(t) = \sum_{i=1}^{n} u_i(t) \int_0^t \frac{\Delta_i(s)}{W(s)}\, g(s, x(s), \dots, x^{(n-1)}(s))\, ds.$$

Then

$$(\Gamma x)^{(j)}(t) = \sum_{i=1}^{n} u_i^{(j)}(t) \int_0^t \frac{\Delta_i(s)}{W(s)}\, g(s, x(s), \ldots, x^{(n-1)}(s))\, \mathrm{d}s$$

for $t \in [0,T], x \in C^{n-1}[0,T]$ and $0 \le j \le n-1$. Hence (see (C.5))

$$\|(\Gamma x)^{(j)}\|_\infty \le \sum_{i=1}^{n} \|u_i^{(j)}\|_\infty \int_0^T \frac{|\Delta_i(t)|}{|W(t)|}\, \psi(t)\, \mathrm{d}t, \quad 0 \le j \le n-1,$$

and therefore

$$\|\Gamma x\|_{C^{n-1}} \le \sum_{i=1}^{n} \|u_i\|_{C^{n-1}} \int_0^T \frac{|\Delta_i(t)|}{|W(t)|}\, \psi(t)\, \mathrm{d}t =: V \tag{C.6}$$

for $x \in C^{n-1}[0,T]$. Because of (C.5), Γ is a continuous operator. From the inequalities (for $0 \le t_1 < t_2 \le T$ and $x \in C^{n-1}[0,T]$)

$$\begin{aligned}
&|(\Gamma x)^{(n-1)}(t_2) - (\Gamma x)^{(n-1)}(t_1)| \\
&\quad = \Big| \sum_{i=1}^{n} u_i^{(n-1)}(t_2) \int_0^{t_2} \frac{\Delta_i(s)}{W(s)} g(s, x(s), \ldots, x^{(n-1)}(s))\, \mathrm{d}s \\
&\qquad - \sum_{i=1}^{n} u_i^{(n-1)}(t_1) \int_0^{t_1} \frac{\Delta_i(s)}{W(s)} g(s, x(s), \ldots, x^{(n-1)}(s))\, \mathrm{d}s \Big| \\
&\quad \le \sum_{i=1}^{n} \int_{t_1}^{t_2} |u_i^{(n)}(s)|\, \mathrm{d}s \int_0^T \frac{|\Delta_i(s)|}{|W(s)|} \psi(s)\, \mathrm{d}s \\
&\qquad + \sum_{i=1}^{n} \|u_i^{(n-1)}\|_\infty \int_{t_1}^{t_2} \frac{|\Delta_i(s)|}{|W(s)|}\, \psi(s)\, \mathrm{d}s
\end{aligned}$$

and from $u_i \in AC^{n-1}[0,T]$, $\frac{\Delta_i(t)}{W(t)}\, \psi(t) \in L_1[0,T]$, we see that the set $\{(\Gamma x)^{(n-1)} : x \in C^{n-1}[0,T]\}$ is equicontinous on $[0,T]$. This fact and (C.6) show that the set $\Gamma(C^{n-1}[0,T])$ is compact in $C^{n-1}[0,T]$ by the Arzelà-Ascoli theorem. Hence Γ is a compact operator.

Since, by assumption, problem (C.2), (C.4) has only the trivial solution, the $n \times n$ matrix $(\mathcal{L}_j(u_k))_{j,k=1}^n$ is regular, that is, $\det(\mathcal{L}_j(u_k)) \neq 0$. Consequently, for each $x \in C^{n-1}[0,T]$ the linear system

$$\sum_{i=1}^{n} c_i(x)\,\mathcal{L}_j(u_i) = r_j - \mathcal{L}_j(\Gamma x), \quad 1 \le j \le n,$$

with the unknown vector $(c_1(x), \dots, c_n(x)) \in \mathbb{R}^n$ has the unique solution

$$\left\{\begin{array}{l} c_i(x) = \dfrac{1}{\det(\mathcal{L}_j(u_k))} \begin{vmatrix} \mathcal{L}_1(u_1) & \dots & r_1 - \mathcal{L}_1(\Gamma x) & \dots & \mathcal{L}_1(u_n) \\ \vdots & & \vdots & & \vdots \\ \mathcal{L}_i(u_1) & \dots & r_i - \mathcal{L}_i(\Gamma x) & \dots & \mathcal{L}_i(u_n) \\ \vdots & & \vdots & & \vdots \\ \mathcal{L}_n(u_1) & \dots & r_n - \mathcal{L}_n(\Gamma x) & \dots & \mathcal{L}_n(u_n) \end{vmatrix}, \\ i = 1, 2, \dots, n. \end{array}\right. \tag{C.7}$$

The continuity of $\mathcal{L}_i$ and Γ implies that the functional $c_i \colon C^{n-1}[0,T] \to \mathbb{R}$ is continuous and the inequality (see (C.6))

$$|c_i(x)| \le \frac{n!A^{n-1}B}{|\det(\mathcal{L}_j(u_k))|} \quad \text{for } x \in C^{n-1}[0,T],\ 1 \le i \le n,$$

where

$$A = \max\{|\mathcal{L}_j(u_k)| \colon 1 \le j, k \le n\}$$

and

$$B = \max\{|r_j| \colon 0 \le j \le n\} + \sup\{|\mathcal{L}_j(x)| \colon \|x\|_{C^{n-1}} \le V,\ 1 \le j \le n\},$$

implies that the set $c_j(C^{n-1}[0,T])$ is compact on $\mathbb{R}$ for $1 \le j \le n$. Hence c_j $(0 \le j \le n)$ is a compact functional.

Finally, define the operator $\mathcal{K} : C^{n-1}[0,T] \to C^{n-1}[0,T]$ by the formula

$$(\mathcal{K}x)(t) = \sum_{i=1}^{n} c_i(x)\,u_i(t) + (\Gamma\, x)(t).$$

Suppose that u is a fixed point of the operator $\mathcal{K}$. Then

$$\mathcal{L}_j(u) = \sum_{i=1}^{n} c_i(u)\,\mathcal{L}_j(u_i) + \mathcal{L}_j(\Gamma u) = r_j, \quad 1 \le j \le n,$$

and

$$
u(t) = \sum_{i=1}^{n} c_i(u)\, u_i(t) + \sum_{i=1}^{n} u_i(t) \int_0^t \frac{\Delta_i(s)}{W(s)}\, g(s, u(s), \ldots, u^{(n-1)}(s))\, \mathrm{d}s, \quad t \in [0,T].
$$

Hence u satisfies the boundary conditions (C.3), $u \in AC^{n-1}[0,T]$,

$$
\begin{aligned}
\left(\sum_{j=0}^{n-1} a_j(t)\, u^{(j)}(t)\right) &= \sum_{i=1}^{n} c_i(u) \left(\sum_{j=0}^{n-1} a_j(t)\, u_i^{(j)}(t)\right) \\
&+ \sum_{i=1}^{n} \int_0^t \frac{\Delta_i(s)}{W(s)}\, g(s, u(s), \ldots, u^{(n-1)}(s))\, \mathrm{d}s \left(\sum_{j=0}^{n-1} a_j(t)\, u_i^{(j)}(t)\right) \qquad \text{(C.8)} \\
&= -\sum_{i=1}^{n} c_i(u)\, u_i^{(n)}(t) - \sum_{i=1}^{n} u_i^{(n)}(t) \int_0^t \frac{\Delta_i(s)}{W(s)}\, g(s, u(s), \ldots, u^{(n-1)}(s))\, \mathrm{d}s
\end{aligned}
$$

for $t \in [0,T]$ and

$$
\begin{cases}
u^{(n)}(t) = \displaystyle\sum_{i=1}^{n} c_i(u)\, u_i^{(n)}(t) \\
\qquad + \displaystyle\sum_{i=1}^{n} u_i^{(n)}(t) \int_0^t \frac{\Delta_i(s)}{W(s)}\, g(s, u(s), \ldots, u^{(n-1)}(s))\, \mathrm{d}s \\
\qquad + g(t, u(t), \ldots, u^{(n-1)}(t))
\end{cases} \qquad \text{(C.9)}
$$

for a.e. $t \in [0,T]$. From (C.8) and (C.9) it follows that

$$
u^{(n)}(t) = -\sum_{j=0}^{n-1} a_j(t)\, u^{(j)}(t) + g(t, u(t), \ldots, u^{(n-1)}(t)) \quad \text{for a.e. } t \in [0,T]
$$

and therefore u is a solution of (C.1). We have verified that any fixed point of $\mathcal{K}$ is a solution of problem (C.1), (C.3). In order to prove our theorem it suffices to show that $\mathcal{K}$ has a fixed point. Since Γ is a compact operator and c_i ($1 \le i \le n$) is a compact functional, the operator $\mathcal{K}$ is compact as well. Therefore there exists a fixed point of $\mathcal{K}$ by the Schauder fixed point theorem since there exists a closed ball Ω in $C^{n-1}[0,T]$ centered at 0 such that $\mathcal{K}(\Omega) \subset \Omega$. □

Sometimes, we can apply Theorem C.5 in the following form.

Corollary C.6. *Let problem* (C.2), (C.4) *have only the trivial solution. Let there exist a positive constant* S *such that* $\|u\|_{C^{n-1}} \le S$ *for all solutions* u *of the problem*

$$\begin{cases} u^{(n)} + \displaystyle\sum_{i=0}^{n-1} a_i(t)\, u^{(i)} \\ \qquad = \gamma\left(\displaystyle\sum_{i=0}^{n-1} |u^{(i)}|\right) \big(g(t,u,\dots,u^{(n-1)}) - \varphi(t)\big) + \varphi(t), \\ \mathcal{L}_j(u) = r_j, \quad 1 \le j \le n, \end{cases} \tag{C.10}$$

where $\varphi \in L_1[0,T]$ *and*

$$\gamma(x) = \begin{cases} 1 & \text{for } 0 \le x \le S, \\ 2 - \dfrac{x}{S} & \text{for } S < x \le 2S, \\ 0 & \text{for } x > 2S. \end{cases}$$

Then problem (C.1), (C.3) *has a solution* $u \in AC^{n-1}[0,T]$ *and* $\|u\|_{C^{n-1}} \le S$.

Proof. Since $g \in Car([0,T] \times \mathbb{R}^n)$ there exists $\psi \in L_1[0,T]$ such that

$$\gamma\Big(\sum_{i=0}^{n-1} |x_i|\Big) |g(t,x_0,\dots,x_{n-1}) - \varphi(t)| + |\varphi(t)| \le \psi(t)$$

for a.e. $t \in [0,T]$ and all $(x_0,\dots,x_{n-1}) \in \mathbb{R}^n$. Hence, by Theorem C.5, there exists a solution $u \in AC^{n-1}[0,T]$ of problem (C.10). Because of our assumption $\|u\|_{C^{n-1}} \le S$ we have $\gamma(\sum_{i=0}^{n-1} |u^{(i)}(t)|) = \gamma(\|u\|_{C^{n-1}}) = 1$, which shows that

$$\begin{aligned} &\gamma\left(\sum_{i=0}^{n-1} |u^{(i)}(t)|\right) \big(g(t,u(t),\dots,u^{(n-1)}(t)) - \varphi(t)\big) + \varphi(t) \\ &\quad = g(t,u(t),\dots,u^{(n-1)}(t)) \end{aligned}$$

for $t \in [0,T]$. Therefore u is a solution of problem (C.1), (C.3). □

Appendix D

Spectrum of the quasilinear Dirichlet problem

Here we recall some basic useful facts from half-linear analysis.

First, let us consider the initial value problem

$$(\phi_p(u'))' + \lambda\,\phi_p(u) = 0, \tag{D.1}$$

$$u(t_0) = 0, \quad u'(t_0) = d, \tag{D.2}$$

where $p \in (1,\infty)$, $t_0 \in \mathbb{R}$, $\lambda \in \mathbb{R}$ and $d \in \mathbb{R}$. As in del Pino, Elgueta and Manásevich [66] (see also e.g. Binding, Drábek and Huang [42], del Pino, Drábek and Manásevich [65], Došlý [75], Došlý and Řehák [76], Manásevich and Mawhin [133] and Zhang [203], [205]), let us put

$$\pi_p = 2\,(p-1)^{1/p} \int_0^1 (1-s^p)^{-1/p}\,\mathrm{d}s.$$

Clearly, $\pi_2 = \pi$. Furthermore, it is known that

$$\pi_p = 2\,(p-1)^{\frac{1}{p}}\,\frac{\frac{\pi}{p}}{\sin(\frac{\pi}{p})} = 2\,\frac{(p-1)^{\frac{1}{p}}}{p}\,B\left(\tfrac{1}{p}, 1-\tfrac{1}{p}\right).$$

(See [76, Sec. 1.1.2], but take into account that our definition differs from that used in [76], where $\pi_p = 2\int_0^1 (1-s^p)^{-1/p}\,\mathrm{d}s$.) It is known (see [76, Theorem 1.1.1]) that for each $t_0 \in \mathbb{R}$, $\lambda \in \mathbb{R}$ and $d \in \mathbb{R}$ problem (D.1), (D.2) has a unique solution u on $\mathbb{R}$ which can be, by [66, sec. 3]), expressed as

$$u(t) = d\,\lambda^{-1/p}\,\sin_p(\lambda^{1/p}\,(t-t_0)) \quad \text{for} \quad t \in \mathbb{R},$$

where the function $\sin_p : \mathbb{R} \to [-(p-1)^{1/p}, (p-1)^{1/p}]$ is defined as follows.

Let $w : [0, \pi_p/2] \to [0, (p-1)^{1/p}]$ be the inverse function to

$$x \in [0, (p-1)^{1/p}] \to \int_0^x \frac{\mathrm{d}s}{(1-\frac{s^p}{p-1})^{1/p}} \in [0, \pi_p/2].$$

Further, put $\widetilde{w}(t) = w(\pi_p - t)$ for $t \in [\pi_p/2, \pi_p]$ and $\widetilde{w}(t) = -\widetilde{w}(-t)$ for $t \in [-\pi_p, 0]$. Finally, define $\sin_p : \mathbb{R} \to \mathbb{R}$ as the $2\,\pi_p-$ periodic extension of $\widetilde{w}$ to the whole $\mathbb{R}$. In particular, if $d = 0$, then $u \equiv 0$ on $\mathbb{R}$. Obviously, we have

$$\begin{aligned} &\sin_p(t) = 0 && \text{if and only if } \; t = n\,\pi_p,\; n \in \mathbb{N} \cup \{0\}, \\ &\sin_p(t) = (p-1)^{1/p} && \text{if and only if } \; t = (2\,n+1)\,\frac{\pi_p}{2},\; n \in \mathbb{N} \cup \{0\}, \end{aligned}$$

and

$$\sin_p(t) > 0 \qquad \text{for } \; t \in (2\,n\,\pi_p, (2\,n+1)\,\pi_p),\; n \in \mathbb{N} \cup \{0\}.$$

As a corollary, we immediately obtain that for given $a, b \in \mathbb{R}, a < b$, the corresponding quasilinear Dirichlet problem

$$(\phi_p(u'))' + \lambda\,\phi_p(u) = 0, \qquad u(a) = u(b) = 0 \tag{D.3}$$

possesses a nontrivial solution, i.e. λ is an eigenvalue for (D.3) if and only if

$$\lambda \in \left\{ \left(\frac{n\,\pi_p}{b-a} \right)^p : n \in \mathbb{N} \cup \{0\} \right\}. \tag{D.4}$$

In particular, $\left(\frac{\pi_p}{T}\right)^p$ is the first eigenvalue for (D.3) with $b - a = T$, wherefrom the following assertion follows.

Lemma D.1. *Let $p \in (1,\infty)$, $a, b \in \mathbb{R}$, $a < b$, and let $\lambda = \left(\frac{\pi_p}{T}\right)^p$. Then problem (D.3) has a nontrivial solution if and only if $b - a \geq T$.*

The following lemma gives the variational definition of the first eigenvalue for (D.3). It follows from the embedding inequalities (cf. e.g.[78, Theorem 5.1], [203, Lemma] or [189]).

Lemma D.2 (**Sharp Poincaré inequality**). *Let $p \in (1,\infty)$. Then*

$$\|u\|_p \leq \frac{T}{\pi_p}\,\|u'\|_p$$

holds for all $u \in AC[0,T]$ such that $u' \in L_p[0,T]$ and $u(0) = u(T) = 0$.

References

[1] R.P. Agarwal, *Boundary Value Problems for Higher Order Differential Equations*, World Scientific, Singapore, 1986.

[2] R.P. Agarwal, *Focal Boundary Value Problems for Differential and Difference Equations*, Kluwer, Dordrecht, 1998.

[3] R.P. Agarwal, H. Lü and D. O'Regan, *An upper and lower solution method for the one-dimensional singular* p*–Laplacian*, Memoirs on Differential Equations and Math. Phys. **28** (2003), 13–31.

[4] R.P. Agarwal and D. O'Regan, *Singular boundary value problems for superlinear second order ordinary and delay differential equations*, J. Differential Equations **130** (1996), 333–355.

[5] R.P. Agarwal and D. O'Regan, *Nonlinear superlinear singular and nonsingular second order boundary value problems*, J. Differential Equations **143** (1998), 60–95.

[6] R.P. Agarwal and D. O'Regan, *Positive solutions for* $(p, n-p)$ *conjugate boundary value problems*, J. Differential Equations **150** (1998), 462–473.

[7] R.P. Agarwal and D. O'Regan, *Twin solutions to singular Dirichlet problems*, J. Math. Anal. Appl. **240** (1999), 433–445.

[8] R.P. Agarwal and D. O'Regan, *Right focal singular boundary value problems*, Z. Angew. Mat. Mech. **79** (1999), 363–373.

[9] R.P. Agarwal and D. O'Regan, *Twin solutions to singular boundary value problems*, Proc. Amer. Math. Soc. **128** (2000), 2085–2094.

[10] R.P. Agarwal and D. O'Regan, *Multiplicity results for singular conjugate, focal, and* (n, p) *problems*, J. Differential Equations **170** (2001), 142–156.

[11] R.P. Agarwal and D. O'Regan, *Singular Differential and Integral Equations with Applications*, Kluwer, Dordrecht 2003.

[12] R.P. Agarwal and D. O'Regan, *A Survey of Recent Results for Initial and Boundary Value Problems Singular in the Dependent Variable*, In: Handbook of Differential Equations, Ordinary Differential Equations, Vol. 1, A. Cañada, P. Drábek, A. Fonda, eds., Elsevier, North Holland, Amsterdam (2004), pp. 1–68.

[13] R.P. Agarwal and D. O'Regan, *An infinite interval problem arising in circularly symmetric deformations of shalow membrane caps*, Int. Journal Nonlin. Mech. **39** (2004), 779–784.

[14] R.P. Agarwal and D. O'Regan, *Singular problems arising in circular membrane theory*, Dyn. Contin. Discrete Impuls. Syst., Ser. A Math. Anal. **10**, No.6, (2003), 965–972.

[15] R.P. Agarwal, D. O'Regan and V. Lakshmikantham, *Singular* $(p, n-p)$ *focal and* (n, p) *higher order boundary value problems*, Nonlinear Anal., Theory Methods Appl. **42** (2000), 215–228.

[16] R.P. Agarwal, D. O'Regan, I. Rachůnková and S. Staněk, *Two-point higher order BVPs with singularities in phase variables*, Comput. Math. Appl. **46** (2003), 1799–1826.

[17] R.P. Agarwal, D. O'Regan and S. Staněk, *Singular Lidstone boundary value problem with given maximal values for solutions*, Nonlinear Anal. Theory Methods Appl. **55** (2003), 859–881.

[18] R.P. Agarwal, D. O'Regan and S. Staněk, *Existence of positive solutions for boundary-value problems with singularities in phase variables*, Proc. Edinb. Math. Soc. **47** (2004), 1–13.

[19] R.P. Agarwal, D. O'Regan and S. Staněk, *Solvability of singular Dirichlet boundary-value problems with given maximal values for positive solutions*, Proc. Edinb. Math. Soc. **48** (2005), 1–19.

[20] R. A. Agarwal, D. O'Regan and S. Staněk, *General existence principles for nonlocal boundary value problems with* ϕ *–Laplacian and their applications*, Abstr. Anal. Appl. **2006** (2006), article ID 968226, 30 pp.

[21] R.P. Agarwal, D. O'Regan and P.J.Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer, Dordrecht, 1999.

[22] R.P. Agarwal and S. Staněk, *Nonnegative solutions of singular boundary value problems with sign changing nonlinearities*, Comp. Math. Appl. **46** (2003), 1827–1837.

[23] R.P. Agarwal and R.A. Usmani, *Iterative methods for solving right focal point boundary value problems*, J. Comput. Appl. Math. **14**(1986), 371–390.

[24] R.P. Agarwal and R.A. Usmani, *On the right focal point boundary value problems for integro-differential equations*, J. Math. Anal. Appl. **126** (1987), 51–69.

[25] R.P. Agarwal and P.J.Y. Wong, *Lidstone polynomials and boundary value problems*, Computers Math. Applic. **17**(1989), 1397–1421.

[26] R.P. Agarwal and P.J.Y. Wong, *Existence of solutions for singular boundary value problems for higher order differential equations*, Rend. del Seminario Math. e Fisico di Milano **65** (1995), 249–264.

[27] A. Ambrosetti, *Critical points and nonlinear variational problems*, Mém. Soc. Math. Fr., Nouv. Sr. 49, 139p. (1992).

[28] R. Aris, *The Mathematical Theory of Diffusion and Reaction in Permeable Catalysts*, Clarendon Press, Oxford 1975.

[29] C. Atkinson and J.E. Bouillet, *Some quantitative properties of solutions of a generalized diffusion equation*, Proc. Camb. Phil. Soc. **86** (1979), 495–510.

[30] R.G. Bartle, *A Modern Theory of Integration*, AMS Providence, Rhode Island 2001.

[31] J.V. Baxley, *Some singular nonlinear boundary value problems*, SIAM J. Math. Anal. **22** (1991), 463–479.

[32] J.V. Baxley, *A singular nonlinear boundary value problem: membrane response of a spherical cap*, SIAM J. Appl. Math. **48** (1988), 497–505.

[33] J.V. Baxley and G.S. Gersdorff, *Singular reaction-diffusion boundary value problem*, J. Differential Equations **115** (1995), 441–457.

[34] J.V. Baxley and S.B. Robinson, *Nonlinear boundary value problems for shallow membrane caps II*, J. Comp. Appl. Math. 88 (1998), 203–224.

[35] J. Bebernes and D. Eberly, *Mathematical Problems from Combustion Theory*, Appl. Math. Sci., Vol. **83**, Springer, Berlin 1989.

[36] H. Berestycki, P.L. Lions and L.A. Peletier, *An ODE approach to the existence of positive solutions for semilinear problems in* $\mathbb{R}^N$, Indiana Univ. Math. J. **30** (1981), 141–157.

[37] F. Bernis, *Nonlinear parabolic equations arising in semiconductor and viscous droplets models*, In: Nonlinear Diffusion Equations and their Equilibrium States, N.G. Lloyd, W.M. Ni, L.A. Peletier and J. Serrin, eds., Birkhäuser, Boston (1992), pp. 77–88.

[38] F. Bernis, *On some nonlinear singular boundary value problems of higher order*, Nonlinear Anal., Theory Methods Appl. **26** (1996), 1061–1078.

[39] F. Bernis, L.A. Peletier and S.M. Williams, *Source-type solutions of a fourth order nonlinear degenerate parabolic equation*, Nonlinear Anal., Theory Methods Appl. **18** (1992), 217–234.

[40] A.L. Bertozzi, M.P. Brenner, T.F. Dupont and L.P. Kadanoff, *Singularities and Similarities in Interface Flows*, In: Trends and Perspectives in Applied Mathematics, L. Sirovich, ed., Appl. Math. Sci., Vol. 100, Springer-Verlag, Berlin (1994), 155–208.

[41] V. Bevc, J.L. Palmer and C. Süskind, *On the design of the transition region of axi-symmetric magnetically beam valves*, J. Br. Inst. Radio Eng. **18** (1958), 697–708.

[42] P. Binding, P. Drábek and Y.X. Huang, *On the Fredholm alternative for the p-Laplacian*, Proc. Amer. Math. Soc. **125** (1997), 3555–3559.

[43] L.E. Bobisud, *Asymptotic dead cores for reacting-diffusion equations*, J. Math. Anal. Appl. **147** (1990), 249–262.

[44] L.E. Bobisud, *Steady-state turbulent flow with reaction*, Rocky Mountain J. Math. **21** (1991), 993-1007.

[45] D. Bonheure, C. De Coster, *Forced singular oscillators and the method of lower and upper functions*, Topol. Methods Nonlinear Anal. **22** (2003), 297–317.

[46] D. Bonheure, C. Fabry and D. Smets, *Periodic solutions of forced isochronous oscillators*, Discrete Contin. Dyn. Syst. **8** (2002), 907–930.

[47] J.E. Bouillet and C. Atkinson, *A generalized diffusion equation: Radial symmetries and comparison theorems*, J. Math. Anal. Appl. **95** (1983), 37–68.

[48] C.J. Budd, G.J. Collins, V.A. Galaktionov, *An asymptotic and numerical description of self-similar blow-up in quasilinear parabolic equations*, J. Comp. Appl. Math. 97 (1998), 51–80.

[49] A. Cabada and R. Pouso, *Existence result for the problem* $(\phi(u'))' = f(t, u, u')$ *with periodic and Neumann boundary conditions*, Nonlinear Anal., Theory Methods Appl. **30** (1997), 1733–1742.

[50] A. Cabada, P. Habets and R. Pouso, *Lower and upper solutions for the periodic problem associated with a* ϕ *–Laplacian equation*, In: EQUADIFF 1999 - International Conference on Differential Equations, Vol. 1, 2 (Berlin, 1999), World Sci. Publishing, River Edge, NJ, 2000, pp. 491–493.

[51] A. Cabada, P. Habets and R. Pouso, *Optimal existence conditions for* ϕ *–Laplacian equations with upper and lower solutions in the reversed order*, J. Differential Equations **166** (2000), 385–401.

[52] A. Cabada, A. Lomtatidze and M. Tvrdý, *Periodic problem with quasilinear differential operator and weak singularity*, to appear.

[53] A. Callegari and M. Friedman, *An analytical solution of a nonlinear singular boundary value problem in the theory of viscous fluids*, J. Math. Anal. Appl. **21** (1968), 510–529.

[54] A. Callegari and A. Nachman, *Some singular, nonlinear differential equations arising in boundary layer theory*, J. Math. Anal. Appl. **64** (1978), 96–105.

[55] A. Callegari and A. Nachman, *A nonlinear singular boundary value problem in the theory of pseudoplastic fluids*, SIAM J. Appl. Math. **38** (1980), 275–281.

[56] A. Capietto, J. Mawhin and F. Zanolin, *A continuation approach to superlinear periodic boundary value problems*, J. Differential Equations **88** (1990), 347–395.

[57] C.Y. Chan, Y.C. Hon, *A constructive solution for a generalized Thomas-Fermi theory of ionized atoms*, Quart. Appl. Math. **45** (1987), 591–599.

[58] M. Cherpion, C. De Coster and P. Habets, *Monotone iterative methods for boundary value problems*, Differ. Integral Equ. **12** (3) (1999), 309–338.

[59] J. Cronin, *Fixed points and topological degree in nonlinear analysis*, AMS Providence, Rhode Island 1964.

[60] C. De Coster and P. Habets, *Lower and Upper Solutions in the Theory of ODE Boundary Value Problems: Classical and Recent Results*, In: Nonlinear Analysis and Boundary Value Problems for Ordinary Differential

Equations, F. Zanolin, ed., CISM Courses and Lectures vol 371, Springer-Verlag, New York (1996), pp. 1-79.

[61] C. De Coster and P. Habets, *The Lower and Upper Solutions Method for Boundary Value Problems*, In: Handbook of Differential Equations, Ordinary Differential Equations, Vol. 1, A. Cañada, P. Drábek, A. Fonda, eds., Elsevier, North Holland, Amsterdam (2004), pp. 69–161.

[62] C. De Coster and P. Habets, *An Overview of the Method of Lower and Upper Solutions for ODEs*, In: Nonlinear Analysis and its Applications to Differential Equations, M.R. Grossinho, M. Ramos, C. Rebelo and L. Sanchez, eds., Progress in Nonlinear Differerential Equations and their Applications, Vol. 43, Birkhäuser, Boston (2001), pp. 3–22.

[63] W. Dambrosio, *Multiple solutions of weakly-coupled systems with p-Laplacian operators*, Result. Math. **36** (1999), 34–54.

[64] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, Heidelberg, 1985.

[65] M. del Pino, P. Drábek and R. Manásevich, *The Fredholm alternative at the first eigenvalue for the one dimensional p-Laplacian*, J. Differential Equations **151** (1999), 386–419.

[66] M. del Pino, M. Elgueta and R. Manásevich, A homotopic deformation along p of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t,u) = 0, \quad u(0) = u(T) = 0, \quad p \in (1,\infty)$. *J. Differential Equations* **80** (1989), 1–13.

[67] M. del Pino and R. Manásevich, *Infinitely many T-periodic solutions for a problem arising in nonlinear elasticity*, J. Differential Equations **103** (1993), 260–277.

[68] M. del Pino, R. Manásevich and A. Montero, *T-periodic solutions for some second order differential equations with singularities*, Proc. Roy. Soc. Edinburgh Sect. A **120** (1992), 231–243.

[69] M. del Pino, R. Manásevich and A. Murúa, *Existence and multiplicity of solutions with prescribed period for a second order quasilinear O.D.E*, Nonlinear Anal., Theory Methods Appl. **18** (1992), 79–92.

[70] L. Derwidué, *Systèmes differérentiels non lineáires ayant des solutions périodiques*, Acad. R. Belgique, Bull. Cl. Sci. (5) **49** (1963), 11–32; (5) **50** (1964), 928–942; (5) **50** (1964), 1130–1142.

[71] J. Diblík, *The singular Cauchy-Nicoletti problem for the system of two ordinary differential equations*, Math. Bohem. **117** (1992), 55–67.

[72] R.W. Dickey, Rotationally symmetric solutions for shallow membrane caps. *Quart. Appl. Math.* **47** (1989), 571–581.

[73] T. Ding, *A boundary value problem for the periodic Brillouin focusing system*, Acta Sci. Natur. Univ. Pekinensis **11** (1965), 31–38 (in Chinese).

[74] W.Y. Ding, *A generalization of the Poincaré-Birkhoff theorem*, Proc. Amer. Math. Soc. **88** (1983), 341–346.

[75] O. Došlý, *Half-Linear Differential Equations.* In: *Handbook of Differential Equations, Ordinary Differential Equations* Vol. 1, A. Cañada, P. Drábek, A. Fonda, eds., Elsevier, North Holland, Amsterdam (2004), Chapter 3, pp. 161–357.

[76] O. Došlý and Řehák, *Half-Linear Differential Equations.* North-Holland Mathematics Studies 202, Elsevier, Amsterdam (2005).

[77] P. Drábek, *Solvability and bifurcation of nonlinear equations*, Research Notes in Math. Vol. 264, Pitman, Boston, 1992.

[78] P. Drábek and R. Manásevich, *On the closed solution to some nonhomogeneous eigenvalue problems with* p*-Laplacian*, Differ. Integral Equ. **12** (6) (1999), 773–788.

[79] M. Drmota, R. Scheidl, H. Troger, E. Weinmüller, *On the imperfection sensitiviy of complete spherical shells*, Comp. Mech. **2** (1987), 63–74.

[80] P.W. Eloe and J. Henderson, *Singular nonlinear* $(n-1,1)$ *conjugate boundary value problems*, Georgian Math. J. **4**(1997), 401–412.

[81] P.W. Eloe and J. Henderson, *Singular nonlinear* $(k,n-k)$ *conjugate boundary value problems*, J. Differential Equations **133** (1997), 136–151.

[82] J.R. Esteban and J.L. Vazques, *On the equation of turbulent filtration in one-dimensional porous media*, Nonlinear Anal., Theory Methods Appl. **10** (1986), 1303–1325.

[83] C. Fabry and D. Fayyad, *Periodic solutions of second order differential equation with a p-Laplacian and asymmetric nonlinearities*, Rend. Ist. Mat. Univ. Trieste **24** (1992), 207–227.

[84] C. Fabry and P. Habets, *Lower and upper solutions for second-order boundary value problems with nonlinear boundary conditions*, Nonlinear Anal., Theory Methods Appl. **10** (1986), 985–1007.

[85] X.L. Fan and X. Fan, *A Knobloch-type result for p(t)-Laplacian systems*, J. Math. Anal. Appl. **282** (2003), 453–464.

[86] X.L. Fan, H.Q. Wu and F.Z. Wang, *Hartman-type results for p(t)-Laplacian systems*, Nonlinear Anal., Theory Methods Appl. **52** (2003), 585–594.

[87] R. Faure, *Solutions périodiques d'equations différentielles et mèthode de Laray-Schauder* (*Cas des vibrations forceés*), Ann. Inst. Fourier (Grenoble) **14** (1964), 195–204.

[88] A. Fonda, *Periodic solution of scalar second order differential equations with a singularity*, Acad R. Belgique, Mém. Cl. Sci. (3) **4** (1993), 1–39.

[89] A. Fonda, *Periodic solution for a conservative system of differential equations with a singularity of repulsive type*, Nonlinear Anal., Theory Methods Appl. **24** (1995), 667–676.

[90] A. Fonda, R. Manásevich and F. Zanolin, *Subharmonic solutions for some second-order differential equations with singularities*, SIAM J. Math. Anal. **24** (1993), 1294–1311.

[91] N. Forbat and A. Huaux, *Détermination approachée et stabilité locale de la solution périodique d'une equation différentielle non linéaire*, Mém. Public. Soc. Sci. Arts Letters Hainaut **76** (1962), 3–13.

[92] S. Fučík, J. Nečas, J. Souček and V. Souček, *Spectral Analysis of Nonlinear Operators*, Lecture Notes In Mathematics 346, Springer Verlag and Society of Czechoslovak Mathematician and Physicists, Berlin-Heidelberg-New York and Praha, 1973.

[93] S. Gaete and R.F. Manásevich, *Existence of a pair of periodic solutions of an O.D.E. generalizing a problem in nonlinear elasticity via variational methods*, J. Math. Anal. Appl. **134** (1988), 257–271.

[94] J.J. Garcia-Ripoll, V.M. Pérez-Garcia and P. Torres, *Extended parametric resonances in nonlinear Schrödinger systems*, Physical Review Letters **83** (1999), 1715-1718.

[95] W.G. Ge and J. Mawhin, *Positive solutions to boundary value problems for second order ordinary differential equations with singular nonlinearities*, Result. Math. **34** (1998), 108–119.

[96] B. Gidas, W. Ni and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^N$, Adv. Math. Suppl. Studies **7A** (1981), 369–402.

[97] M.A. Goldberg, *An iterative solution for rotationally symmetric nonlinear membrane problems*, Internat. J. Non-Linear Mech. **1** (1966), 169–178.

[98] W.B. Gordon, *Conservative dynamical systems involving strong forces*, Trans. Amer. Math. Soc. **1975** (1975), 113–135.

[99] A. Granas and J. Dugundji, *Fixed point theory.* Springer Monographs in Mathematics. New York, NY: Springer. xv, 690 p.

[100] A. Granas, R.B. Guenther and J.W. Lee, *Some general existence principles in the Carathéodory theory of nonlinear systems*, J. Math. Pure Appl. **70** (1991), 153–196.

[101] P. Habets and L. Sanchez, *Periodic solutions of some Liénard equations with singularities*, Proc. Amer. Math. Soc. **109** (1990), 1035–1044.

[102] P. Habets and L. Sanchez, *Periodic solutions of dissipative dynamical systems with singular potentials*, Differ. Integral Equ. **3** (1990), 1039–1149.

[103] P. Hartman, *Ordinary Differential Equations*, John Wiley, New York - London 1964.

[104] M.A. Herrero and J.L. Vazquez, *On the propagation properties of a nonlinear degenerate parabolic equation*, Comm. Partial Differential Equations **7** (1982), 1381–1402.

[105] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer, New York 1965.

[106] A. Huaux, *Sur l'existence d'une solution périodique de l'equation différentielle non linéaire* $u'' + 0.2\,u' + u/(1-u) = 0.5\,\cos\omega\,t$, Acad. R. Belgique, Bull. Cl. Sci. (5) **48** (1962), 494–504.

[107] P. Jebelean and J. Mawhin, *Periodic solutions of singular nonlinear perturbations of the ordinary p-Laplacian*, Adv. Nonlinear Stud. **2** (2002), 299–312.

[108] P. Jebelean and J. Mawhin, *Periodic solutions of forced dissipative p-Liénard equations with singularities*, Vietnam J. Math. **32** (2004), 97–103.

[109] D.Q. Jiang, *Upper and lower solutions method and a singular superlinear boundary value problem for the one-dimensional p-Laplacian*, Comp. Math. Appl. **42** (2001), 927–940.

[110] D.Q. Jiang, *Upper and lower solutions method and a superlinear singular boundary value problems*, Comp. Math. Appl. **44** (2002), 323–337.

[111] D.Q. Jiang and J.Y. Wang, *A generalized periodic boundary value problem for the one-dimensional p-Laplacian*, Ann. Polon. Math. **65** (1997), 265–270.

[112] K.N. Johnson, *Circularly symmetric deformation of shallow elastic membrane caps.* Quart. Appl. Math. **55** (1997), 537–550.

[113] R. Kannan and D. O'Regan, *Singular and nonsingular boundary value problems with sign changing nonlinearities*, J. Inequal. Appl. **5** (2000), 621–637.

[114] I.T. Kiguradze, *On Some singular boundary value problems for nonlinear ordinary differential equations of the second order* (in Russian), Differencial'nye Uravnenija **4** (1968), 1753–1773.

[115] I.T. Kiguradze, *On Some Singular Boundary Value Problems for Ordinary Differential Equations* (in Russian), Tbilisi Univ. Press, Tbilisi 1975.

[116] I.T. Kiguradze, *Boundary Value Problems for Systems of Ordinary Differential Equations* (in Russian). Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Novejshie Dostizh. 30 (1987), 3-103; English transl.: J. Sov. Math. **43** (1988), 2259-2339.

[117] I. Kiguradze, *Some optimal conditions for the solvability of two-point singular boundary value problems*, Functional Differ. Equations **10** (2003), 259–281.

[118] I.T. Kiguradze and B.L. Shekhter, *Singular boundary value problems for second order ordinary differential equations* (in Russian), Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Noveishie Dostizh. **30** (1987), 105–201; English transl.: J. Sov. Math **43** (1988), 2340–2417.

[119] S. Lang, *Real and Functional Analysis*, Springer, New York 1993.

[120] G.S. Ladde, V. Lakshmikantham and A.S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman, Boston 1995.

[121] A. Lasota, *Sur les problèmes linéaires aux limites pour un système d'équations différentielles ordinaires*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astron. Phys. **10** (1962), 565–570.

[122] A.C. Lazer and S. Solimini, *On periodic solutions of nonlinear differential equations with singularities*, Proc. Amer. Math. Soc. **99** (1987), 109–114.

[123] M.N. Le Roux, H. Wilhelmsson, *Dynamics of coupled temperature and density in a fusion reactor plasma for centrally pronounced heating and particle injection*, Phys. Scripta **47** (1993), 104-108.

[124] H.A. Levine, *The role of critical exponents in blow-up problems*, SIAM Rev. **32** (1990), 262-288.

[125] Bin Liu, *Multiplicity results for periodic solutions of a second order quasilinear ODE with asymmetric nonlinearities*, Nonlinear Anal., Theory Methods Appl. **33** (2) (1998), 139–160.

[126] Bing Liu, *Periodic solutions of dissipative dynamical systems with singular potential and p-Laplacian*, Ann. Pol. Math. **79** (2) (2002), 109–120.

[127] A. Lomtatidze, *Positive solutions of boundary value problems for second order differential equations with singular points* (in Russian), Differentsial'nye Uravneniya **23** (1987), 1685–1692, translated in Differential Equations **23** (1987), 1146–1152.

[128] A. Lomtatidze and L. Malaguti, *On a two-point boundary value problem for the second order ordinary diffeerential equations with singularities*, Nonlinear Anal., Theory Methods Appl. **52** (2003), 1553–1567.

[129] A. Lomtatidze and P. Torres, *On a two-point boundary value problem for second order singular equations*, Czechoslovak Math. J. **53** (2003), 19–43.

[130] L. Lu, D. O'Regan and R.P. Agarwal: *Positive radial solutions for a quasilinear system*, Applicable Analysis **85**(2006), 363-371.

[131] R. Manásevich and J. Mawhin, *Periodic solutions for nonlinear systems with p-Laplacian-like operators*, J. Differential Equations **145** (1998), 367–393.

[132] R. Manásevich and J.Mawhin, *Boundary value problems for nonlinear perturbations of vector p-Laplacian-like operators*, J. Korean Math. Soc. **37** (2000), 665-685.

[133] R. Manásevich and J. Mawhin, The spectrum of p-Laplacian systems under Dirichlet, Neumann and periodic boundary conditions. In: *Morse theory, minimax theory and their applications to nonlinear differential equations*, Proceedings of the workshop, H. Brezis, S.J. Li, J.Q Liu, P.H Rabinowitz (eds.), Beijing, China, April 1-September 30, 1999. Somerville, International Press. New Stud. Adv. Math. 1 (2003) pp. 201–216.

[134] J. Mawhin, *Topological degree methods in nonlinear boundary value problems*, In: Regional Conference Series in Mathematics. No.40. R.I.: The American Mathematical Society (AMS). 1979, 122 p.

[135] J. Mawhin, *Topological Degree and Boundary Value Problems for Nonlinear Differential Equations*, In: Topological methods for ordinary differential equations, M. Furi and P. Zecca, eds., Lecture Notes in Mathematics, Vol. 1537, Springer-Verlag, Berlin (1993), pp. 74–142.

[136] J. Mawhin, *Leray-Schauder continuation theorems in the absence of a priori bounds*, Topol. Methods Nonlinear Anal. **9** (1997), 179–200.

[137] J. Mawhin, *Some boundary value problems for Hartman-type perturbations of ordinary vector p-Laplacian*, Nonlinear Anal., Theory Methods Appl. **40** (2000), 497-503.

[138] J. Mawhin, *Periodic Solutions of Systems with p-Laplacian-like Operators*, In: Nonlinear Analysis and its Applications to Differential Equations, M.R. Grossinho, M. Ramos, C. Rebelo and L. Sanchez, eds., Progress in Nonlinear Differerential Equations and their Applications, Vol. 43, Birkhäuser, Boston (2001), pp. 37–63.

[139] J. Mawhin and A.J. Ureña, *A Hartman-Nagumo inequality for the vector ordinary p-Laplacian and applications to nonlinear boundary value problems*, J. Inequal. Appl. **7** (2002), 701-725.

[140] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonia Systems*, Appl. Math. Sciences, Vol. 74, Springer-Verlag, New York 1989.

[141] P. Martinez-Amores and P.J. Torres, *Dynamics of a periodic differential equation with a singular nonlinearity of attractive type*, J. Math. Anal. Appl. **202** (1996), 1027–1039.

[142] T.Y. Na, *Computational Methods in Engineering Boundary Value Problems*, Academic Press, New York 1979.

[143] I.P. Natanson, *Theorie der Funktionen einer reellen Veränderlichen.* Herausgegeben von Karl Bögel. 4. Aufl. (German). Mathematische Lehrbücher und Monographien. Akademie-Verlag, Berlin 1975; English transl.: Theory of functions of real variable, Ungar, New York 1955.

[144] M.N. Nkashama and J. Santanilla, *Existence of multiple solutions for some nonlinear boundary value problems*, J. Differential Equations **84** (1990), 148–164.

[145] A. Nowakowski and A. Orpel, *Positive solutions for nonlocal boundary-value problem with vector-valued response*, Electronic J. Diff. Equations **2002** (2002), no. 46, 1-15.

[146] P. Omari and W.Y. Ye, *Necessary and sufficient conditions for the existence of periodic solutions of second order ordinary differential equations with singular nonlinearities*, Differ. Integral Equ. **8** (1995), 1843–1858.

[147] D. O'Regan, *Some general existence principles and results for* $(\phi(y'))' = qf(t,y,y'),\quad 0<t<1,$ SIAM J. Math. Anal. **24** (1993), 648–668.

[148] D. O'Regan, *Theory of singular boundary value problems*, World Scientific, Singapore 1994.

[149] S.V. Parter, M.L. Stein, P.R. Stein, *On the multiplicity of solutions of a differential equation arising in chemical reactor theory,* Tech. Rep. **194**, Dept. Comp. Sciences, Univ. of Wisconsin, Madison, 1973.

[150] N. Perrone and R. Kao, *A general nonlinear relaxation iteration technique for solving nonlinear problems in mechanics*, J. Appl. Mech. **38** (1971), 371–378.

[151] N. Phan-Thien, *A method to obtain some similarity solutions to the generalized Newtonian fluid*, Z. Angew. Math. Phys. **32** (1981), 609–615.

[152] L.C. Piccinini, G. Stampacchia and G. Vidossich, *Ordinary Differential Equations in* $\mathbb{R}^n$, Springer, New York Inc., 1984.

[153] V. Polášek and I. Rachůnková, *Singular periodic problem for nonlinear ordinary differential equations with phi-Laplacian*, Electronic J. Differential Equations **2006** (2006),No. 27, 1–12.

[154] I. Rachůnková, *Lower and Upper Solutions and Topological Degree*, J. Math. Anal. Appl. **234** (1999), 311–327.

[155] I. Rachůnková, *Existence of two positive solutions of a singular nonlinear periodic boundary value problems*, J. Comput. Appl. Math. **113** (2000), 27–34.

[156] I. Rachůnková, *On the existence of more positive solutions of periodic BVPs with singularity*, Applicable Analysis **79** (2000), 257–275.

[157] I. Rachůnková, *Singular mixed boundary value problems*, J. Math. Anal. Appl. **320** (2006), 611–618.

[158] I. Rachůnková, O. Koch, G. Pulverer and E. Weinmüller, *On a singular boundary value problem arising in the theory of shallow membrane caps*, J. Math. Anal. Appl., to appear.

[159] I. Rachůnková and S. Staněk, *Sturm-Liouville and focal higher order BVPs with singularities in phase variables*, Georgian Math. J. **10** (2003), 165–191.

[160] I. Rachůnková and S. Staněk, *General existence principle for singular BVPs and its applications*, Georgian Math. J. **11** (2004), 549–565.

[161] I. Rachůnková and S. Staněk, *A singular boundary value problem for odd order differential equations*, J. Math. Anal. Appl. **291** (2004), 741–756.

[162] I. Rachůnková and S. Staněk, *Zeros of derivatives of solutions to singular* $(p, n-p)$ *conjugate BVPs*, Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica **43** (2004), 137–141.

[163] I. Rachůnková, S. Staněk and M. Tvrdý, *Singularities and Laplacians in Boundary Value Problems for Nonlinear Ordinary Differential Equations.* In: Handbook of Differential Equations. Ordinary Differential Equations, vol.**3**, pp. 607–723. Ed. by A. Cañada, P. Drábek, A. Fonda. Elsevier 2006.

[164] I. Rachůnková and J. Stryja, *Singular Dirichlet BVP for second order ODE*, submitted.

[165] I. Rachůnková and J. Stryja, *Dirichlet problem with* ϕ *–Laplacian and mixed singularitis*, submitted.

[166] I. Rachůnková and J. Stryja, *Lower and upper functions in singular Dirichlet problem with* ϕ *–Laplacian*, submitted.

[167] I. Rachůnková and M. Tvrdý, *Nonlinear systems of differential inequalities and solvability of certain nonlinear second order boundary value problems*, J. Inequal. Appl. **6** (2001), 199–226.

[168] I. Rachůnková and M. Tvrdý, *Construction of lower and upper functions and their application to regular and singular boundary value problems*, Nonlinear Anal., Theory Methods Appl. **47** (2001), 3937–3948.

[169] I. Rachůnková and M. Tvrdý, *Periodic problems with ϕ –Laplacian involving non-ordered lower and upper functions*, Fixed Point Theory **6** (2005), 99–112.

[170] I. Rachůnková and M. Tvrdý, *Periodic singular problem with quasilinear differential operator*, Mathematica Bohemica **131** (2006), 321–336.

[171] I. Rachůnková, M. Tvrdý and I. Vrkoč, *Existence of nonnegative and nonpositive solutions for second order periodic boundary value problems*, J. Differential Equations **176** (2001), 445–469.

[172] I. Rachůnková, M. Tvrdý and I. Vrkoč, *Resonance and multiplicity in periodic BVPs with singularity*, Math.Bohem. **128** (2003), 45–70.

[173] G.W. Reddien, *Finite-difference approximation to singular Sturm-Liouville eigenvalue problems,* Math. Comp. **25** (1976), 427–432.

[174] P. Rentrop, *Numerical solution of the singular Ginzburg-Landau equations by multiple shooting,* Computing **16** (1976), 61–67.

[175] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov and A.P. Mikhailov, *Blow-up in Quasilinear Parabolic Equations*, Walter de Gruyter, Berlin, New York, 1995.

[176] L. Sanchez, *Positive solutions for a class of semilinear two-point boundary value problems*, Bull. Austral. Math. Soc. **45** (1992), 439–451.

[177] G. Scorza Dragoni, *Il problema dei valori ai limiti studiato in grande per gli integrali di una equazione differenziale del secondo ordine*, Giorn. di Mat. di Battaglini **69** (1931), 77–112.

[178] G.E. Shilov, B.L. Gurevich, *Integral, measure and derivative: a unified approach.* Rev.Engl. ed. Transl. by R.A. Silverman. Dover Books on Advanced Mathematics, Dover Publ., New York 1977.

[179] S. Solimini, *On forced dynamical systems with a singularity of repulsive type*, Nonlinear Anal., Theory Methods Appl. **14** (1990), 489–500.

[180] R. Srzednicki, *Ważewski method and Conley index*, In: Handbook of Differential Equations, Ordinary Differential Equations, Vol. 1, A. Cañada, P. Drábek, A. Fonda, eds., Elsevier, North Holland, Amsterdam (2004), pp. 591–684.

[181] S. Staněk, *Periodic boundary value problem for second order functional differential equations*, Math. Notes (Miskolc) **1** (2000), 63–81.

[182] S. Staněk, *On solvability of singular periodic boundary value problems*, Nonlinear Oscil. **4** (2001), 529–538.

[183] S. Staněk, *Positive solutions of singular positone Dirichlet boundary value problems*, Math. Comp. Modelling **33** (2001), 341–351.

[184] S. Staněk, *A nonlocal singular boundary value problem for second-order differential equations*, Math. Notes (Miskolc) **5** (2004), 91–104.

[185] S. Staněk, *A nonlocal boundary value problem with singularities in phase variables*, Math. Comput. Modelling **40**(2004), 101–116.

[186] S. Staněk, *Singular nonlocal boundary value problems*, Nonlinear Anal., Theory Methods Appl. **63** (2005), e277–e287.

[187] S. Staněk, *General existence principle for singular BVPs depending on a parameter and its application*, Funct. Differential Equations **13** (2006), 637–656.

[188] S. Staněk and O. Přibyl, *Singular antiperiodic boundary value problem with given maximal values for solutions*, to appear.

[189] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976), 353-372.

[190] S.D. Taliaferro, *A nonlinear singular boundary value problem*, Nonlinear Anal., Theory Methods Appl. **3** (1979), 897–904.

[191] P.J. Torres, *Bounded solutions in singular equations of repulsive type*, Nonlinear Anal., Theory Methods Appl. **32** (1998), 117–125.

[192] P.J. Torres, *Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem*, J. Differential Equations **190** (2003), 643–662.

[193] P.J. Torres, *Existence and uniqueness of elliptic periodic solutions of the Brillouin electron beam focusing system*, Math. Methods Appl. Sci. **23** (2000), 1139–1143.

[194] N.I. Vasiliev and Yu.A. Klokov, *Foundation of the Theory of Boundary Value Problems for Ordinary Differential Equations*, Zinatne, Riga 1978 (in Russian).

[195] J.Y. Wang, *Solvability of singular nonlinear two-point boundary value problems*, Nonlinear Anal., Theory Methods Appl. **24** (1995), 555–561.

[196] J.Y. Wang and J. Jiang, *The existence of positive solutions to a singular nonlinear boundary value problem*, J. Math. Anal. Appl. **176** (1993), 322–329.

[197] J.Y. Wang and W. Gao, *A singular boundary value problem for the one-dimensional p-Laplacian*, J. Math. Anal. Appl. **201** (1996), 851–866.

[198] Z. Wei, *Existence of positive solutuions for* $2n$ *th-order singular boundary problems*, J. Math. Anal. Appl. **306** (2005), 619–636.

[199] P.J.Y. Wong and R.P. Agarwal, *Results and estimates on multiple solutions of Lidstone boundary value problem*, Acta Math. Hungar, **86**(2000), 127–168.

[200] P. Yan, *Nonresonance for one-dimensional p-Laplacian with regular restoring*, J. Math. Anal. Appl. **285** (2003), 141–154.

[201] P. Yan and M.R. Zhang, *Higher order non-resonance for differential equations with singularities*, Math. Meth. Appl. Sci. **26** (2003), 1067–1074.

[202] M.R. Zhang, *Periodic solutions of Liénard equations with singular forces of repulsive type*, J. Math. Anal. Appl. **203** (1996), 254–269.

[203] M.R. Zhang, *Nonuniform nonresonance at the first eigenvalue of the p-Laplacian*, Nonlinear Anal., Theory Methods Appl. **29** (1997), 41–51.

[204] M.R. Zhang, *A relationship between the periodic and the Dirichlet BVPs of singular differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **128A** (1998), 1099–1114.

[205] M.R. Zhang, *Periodic solutions of damped differential systems with repulsive singular forces*, Proc. Amer. Math. Soc. **127** (1999), 401–407.

[206] Z. Zhao, *On the existence of positive solutions for 2n-order singular boundary value problems.* Nonlinear Anal., Theory Methods Appl. **64**(2006), 2553–2561.

[207] N.V. Zmitrenko, S.P. Kurdyumov, A.P. Mikhailov and A.A. Samarski, *Localization of thermonuclear combustion in a plasma with electronic thermal conductivity*, JETF Lett. **26** (1977), 469–472.

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