Existence of two positive solutions of a singular nonlinear periodic boundary value problem

Irena Rachůnková *

Department of Mathematics, Palacký University, Tomkova 40, 77900 Olomouc, Czech Republic

1 Introduction

This paper deals with the singular boundary value problem

$$x'' = f(t, x, x'), \tag{1}$$

$$x(0) = x(T),\ x'(0) = x'(T), \tag{2}$$

where $f \in \mathrm{Car}(J \times (\mathbf{R}^+ \times \mathbf{R}))$, $J = [0, T] \subset \mathbf{R}$, $\mathbf{R}^+ = (0, \infty)$, and f has a repulsive singularity at $x = 0^+$, i.e.

$$\lim_{x \to 0^+} f(t, x, y) = +\infty \text{ for a.e. } t \in J \text{ and each } y \in \mathbf{R}.$$

We are interested in positive solutions of (1), (2), because we have been motivated by a problem from the Theory of Nonlinear Elasticity modelling radial oscillations of an elastic spherical membrane made up of a Neo-Hookean material, and subjected to an internal pressure. The oscillations are governed by the scalar equation

$$x'' = p(t)x^2 - x + x^{-5}, \tag{3}$$

where the pressure $p : \mathbf{R} \to (0, \infty)$ is continuous and T-periodic and $x(t)$ is the ratio of the deformed radius of the membrane at time t with respect to the undeformed radius, and hence $x(t) > 0$ on $\mathbf{R}$. See [3] or [1]. Here, we prove the existence of at least two different positive solutions to the more general problem (1), (2) using theorems of [2] which are based on a connection between lower and upper solutions and the topological degree of an operator associated to (1), (2). We generalize results of [1], where the equation

$$x'' = g(t, x), \tag{4}$$

with $g : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ continuous and T-periodic in t, is considered and where the authors prove the existence of two positive periodic solutions for (4) by a variational method. Such an approach cannot be used for (1), (2).

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2 Notation and definitions

For $k \in \mathbf{N} \cup \{0\}$, $\mathrm{C}^k(J)$ is the Banach space of functions having continuous k-th derivatives on J with the norm

$$\|x\|_{C^k} = \sum_{i=0}^{k} \max\{|x^{(i)}(t)| : t \in J\} \text{ for } x \in \mathrm{C}^k(J);$$

$\mathrm{L}_1(J)$ is the Banach space of Lebesque integrable functions on J with the norm

$$\|x\|_{L_1} = \int_0^T |x(t)| dt \text{ for } x \in \mathrm{L}_1(J).$$

By $\mathrm{AC}^k(J)$ we denote the set of functions having absolutely continuous k-th derivatives on J, and we use $\mathrm{C}(J)$ or $\mathrm{AC}(J)$ instead of $\mathrm{C}^0(J)$ or $\mathrm{AC}^0(J)$. For $B \subset \mathbf{R}$, the symbol $\mathrm{Car}(J \times (B \times \mathbf{R}))$ denotes the set of functions satisfying the Carathéodory conditions on $J \times (B \times \mathbf{R})$, i.e. $f(\cdot, x, y) : J \to \mathbf{R}$ is measurable for all $x \in B, y \in \mathbf{R}$, $f(t, \cdot, \cdot) : B \times \mathbf{R} \to \mathbf{R}$ is continuous for a.e. $t \in J$ and $\sup\{|f(t, x, y)| : (x, y) \in K\} \in \mathrm{L}_1(J)$ for any compact set $K \subset (B \times \mathbf{R})$. For $\varphi \in \mathrm{L}_1(J)$ we denote by $\bar{\varphi}$ the mean value $(1/T) \int_0^T \varphi(s) ds$.

By a solution of (1), (2) we understand a function $u \in \mathrm{AC}^1(J)$ satisfying (1) for a.e. $t \in J$, as well as the boundary conditions (2). By a lower solution of (1), (2) we mean a function $\sigma_1 \in \mathrm{AC}^1(J)$ satisfying

$$\sigma_1''(t) \geq f(t, \sigma_1(t), \sigma_1'(t)) \text{ for a.e. } t \in J \tag{5}$$

and the boundary conditions

$$\sigma_1(0) = \sigma_1(T), \;\; \sigma_1'(0) \geq \sigma_1'(T). \tag{6}$$

Similarly, by an upper solution of (1), (2) we mean a function $\sigma_2 \in \mathrm{AC}^1(J)$ satisfying

$$\sigma_2''(t) \leq f(t, \sigma_2(t), \sigma_2'(t)) \text{ for a.e. } t \in J \tag{7}$$

and the boundary conditions

$$\sigma_2(0) = \sigma_2(T), \;\; \sigma_2'(0) \leq \sigma_2'(T). \tag{8}$$

3 Assumptions

We will use the following assumptions in our lemmas and theorems:

$$f \in \mathrm{Car}(J \times (\mathbf{R}^+ \times \mathbf{R})); \tag{9}$$

$$\exists m \in \mathrm{L}_1(J) : f(t,x,y) > m(t) \text{ for a.e. } t \in J, \forall (x,y) \in \mathbf{R}^+ \times \mathbf{R}; \tag{10}$$

$$\exists g \in \mathrm{C}(\mathbf{R}^+) : f(t,x,y) > g(x) \text{ for a.e. } t \in J, \forall (x,y) \in \mathbf{R}^+ \times [0,\infty), \tag{11}$$

where $g(x) > 0$ for all $x \in (0,r)$ and for some $r \in \mathbf{R}^+$;

$$G(0^+) = -\infty, G(\infty) = -\infty, \text{ where } G(x) = \int_r^x g(s)ds \text{ for } x \in \mathbf{R}^+; \tag{12}$$

$$\exists \sigma \in (r,\infty) : f(t,\sigma,0) \leq 0 \text{ for a.e. } t \in J; \tag{13}$$

$$\exists \varepsilon \in \mathbf{R}^+ : f(t,\sigma+\varepsilon,0) \leq 0 \text{ for a.e. } t \in J; \tag{14}$$

$$\begin{gathered} \exists \varphi \in \mathrm{L}_1(J), \exists A \in \mathbf{R}^+ : \bar{\varphi} \geq 0, \ f(t,x,y) \geq \varphi(t) \\ \text{for a.e. } t \in J, \forall x > A, \forall y \in [-\|m\|_{L_1}, \|m\|_{L_1}]. \end{gathered} \tag{15}$$

Suppose, that the function p on the right-hand side of (3) satisfies

$$0 < p_1 = \min_{t \in J} p(t) < \max_{t \in J} p(t) = p_2 < 6/7^{7/6} \tag{16}$$

and put

$$b(t,x) = p(t)x^2 - x + x^{-5}.$$

Then (16) guarantees that b has both signs which is necessary for the existence of periodic solutions to (3).

Lemma 3.1. *Under the assumption (16) $f = b$ fulfils all conditions (9) — (15).*

Proof. The condition (9) follows from the fact that $b \in \mathrm{C}(J \times \mathbf{R}^+)$. The condition (10): $b(t,x) > p_1 x^2 - x \geq -1/4p_1$ on $J \times \mathbf{R}^+$. The condition (11): $b(t,x) > -x + x^{-5} = g(x)$ on $J \times \mathbf{R}^+$, where $g \in \mathrm{C}(\mathbf{R}^+)$ and $g(x) > 0$ for all $x \in (0,1)$. The condition (12): $G(x) = \int_1^x g(s)ds = -\frac{x^{-4}}{4} - \frac{x^2}{2} + \frac{3}{4}$ and $G(0^+) = -\infty, G(\infty) = -\infty$. The conditions (13) and (14): $b(t,x) \leq p_2 x^2 - x + x^{-5} = \psi_{p_2}(x)$ on $J \times \mathbf{R}^+$. For $p_2 = 6/7^{7/6}$ the function ψ_{p_2} is nonnegative on $\mathbf{R}^+$ and it has just one minimal zero value at $x = 7^{1/6}$. For $p_2 \in (0, 6/7^{7/6})$ we can find an interval $(\alpha,\beta) \subset (1,\infty)$ such that $\psi_{p_2}(x) < 0$ for all $x \in (\alpha,\beta)$. Thus we can choose numbers $\sigma, \sigma+\varepsilon \in (\alpha,\beta)$ satisfying (13) and (14). The condition (15): $b(t,x) > p_1 x^2 - x > 2/p_1$ for all $t \in J, x > A$, where $A = 2/p_1$. □

4 A priori estimates

Lemma 4.1. *Suppose (9) and (10) hold. Then for any positive solution u of the problem (1), (2) the estimate*

$$\|u'\|_C < \|m\|_{L_1} \tag{17}$$

is true.

Proof. Let u be a positive solution of (1), (2). Then there exists $t_0 \in J$ such that $u'(t_0) = 0$. (10) implies that $m(t) < u''(t)$ for a.e. $t \in J$. Therefore

$$-\int_{t_0}^{T} |m(t)|dt < u'(t) \text{ for any } t \in [t_0, T] \text{ and for } t = 0.$$

Thus $-\|m\|_{L_1} < u'(t)$ on J. Similarly

$$u'(t) < \int_0^{t_0} |m(t)|dt \text{ for any } t \in [0, t_0] \text{ and for } t = T.$$

So, $u'(t) < \|m\|_{L_1}$ on J, and (17) is proved. □

Lemma 4.2. *Suppose (9), (10), (11) and (12) hold. Then there exists $c^* \in \mathbf{R}^+$ such that any positive solution u of (1), (2) satisfies*

$$0 < c^* < u(t) \text{ for any } t \in J. \tag{18}$$

Proof. Let u be a positive solution of (1), (2). According to (2) we can extend u periodically on $\mathbf{R}$. Suppose that there exists $s \in J$ such that

$$0 < u(s) \le r. \tag{19}$$

Using Lemma 4.1 we get (17) which together with (19) gives the estimate

$$\|u\|_C < \|m\|_{L_1} T + r = r^*. \tag{20}$$

Suppose that

$$\min_{t \in J} u(t) = u(t_0) \in (0, r). \tag{21}$$

Then $u'(t_0) = 0$. Let $\varepsilon > 0$ be such that $u'(t) = 0$ for all $t \in I_\varepsilon = [t_0, t_0 + \varepsilon]$. Then $u(t) = u(t_0)$ and $u''(t) = 0$ for all $t \in I_\varepsilon$. So, by (11), we get

$$0 = \int_{I_\varepsilon} f(t, u(t_0), 0)dt > \int_{I_\varepsilon} g(u(t_0))dt > 0,$$

a contradiction. Suppose that we can find a sequence $\{t_n\} \subset \mathbf{R}$ such that $t_n \to t_{0+}$ and $u'(t_n) = 0$ for all $n \in \mathbf{N}$. Then there exists $m \in \mathbf{N}$ such that $u(t) \in (0, r)$ and $u'(t) > 0$ for all $t \in (t_m, t_{m+1})$. Therefore

$$0 = \int_{t_m}^{t_{m+1}} u''(t)dt = \int_{t_m}^{t_{m+1}} f(t, u(t), u'(t))dt > \int_{t_m}^{t_{m+1}} g(u(t))dt > 0,$$

a contradiction. So, we have proved that u' is positive on some right neighbourhood of t_0. Let us show that there exists t_1 such that

$$0 < t_1 - t_0 < T, u'(t_1) = 0 \text{ and } u'(t) > 0 \text{ for all } t \in (t_0, t_1). \tag{22}$$

If $u'(t) < 0$ on $[0, t_0)$ and $u'(t) > 0$ on $(t_0, T]$, then this contradicts (2). Hence $t_1 \in (t_0, T]$ or $u'(\tau) = 0$ for some $\tau \in [0, t_0)$ and we set $t_1 = \tau + T$.

Now, using (11) and (22), we have

$$u'' \cdot u' = f(t, u(t), u'(t)) \cdot u'(t) > g(u(t)) \cdot u'(t) \text{ for a.e. } t \in (t_0, t_1)$$

and

$$\int_{t_0}^{t_1} u'(t)u''(t)dt > \int_{t_0}^{t_1} g(u(t))u'(t)dt.$$

Thus $0 = \frac{1}{2}(u'^2(t_1) - u'^2(t_0)) > G(u(t_1)) - G(u(t_0))$, i.e.

$$G(u(t_0)) > G(u(t_1)). \tag{23}$$

We can suppose without loss of generality that $g(x) < 0$ for all $x > r$. Then G is increasing on $(0, r)$ and decreasing on (r, ∞), which together with (21) and (23) imply that $u(t_1) > r$. In view of (20) $G(r^*) < G(u(t_1))$ and there exists just one number $c^* \in (0, r)$ such that

$$G(c^*) = G(r^*). \tag{24}$$

Therefore $G(c^*) < G(u(t_0))$, which gives (18). We have proved (18) for any positive solution satisfying (19). If some positive solution v of (1), (2) has not the property (19), then $\min_{t \in J} v(t) > r$, and so we get (18) immediately. □

Lemma 4.3. *Suppose (9), (10), (11), (12) hold and set* $r^* \geq \|m\|_{L_1} T + r$, $c^* = G^{-1}(G(r^*)) \in (0, r)$. *Further put*

$$f^*(t, x, y) = \begin{cases} f(t, x, y) & \text{if} \quad x > c^* \\ f(t, c^*, y) + c^* - x & \text{if} \quad x \leq c^* \end{cases} \tag{25}$$

for a.e. $t \in J$ *and all* $x, y \in \mathbf{R}$ *and consider the equation*

$$u'' = f^*(t, u, u'). \tag{26}$$

Then any solution u *of the problem (26), (2) is a solution of (1), (2) and satisfies the estimate (18).*

Proof. $f^* \in$Car$(J \times \mathbf{R}^2)$ and f^* satisfies (10) for all $x \in \mathbf{R}$. Therefore the estimate (17) is true for any solution u of (26), (2). The proof of this assertion is similar to that of Lemma 4.1. Let u be a solution of (26), (2) and suppose that $\min_{t\in J} u(t) = u(t_0) < c^*$. Then we can argue like in the proof of Lemma 4.2 and get a point t_1 satisfying (22). If $u(t_1) \le c^*$, then

$$0 = \int_{t_0}^{t_1} u''(s)ds = \int_{t_0}^{t_1} (f(s, c^*, u'(s)) + c^* - u(s))ds > \int_{t_0}^{t_1} g(c^*)ds > 0,$$

a contradiction. Thus, suppose that $u(t_1) > c^*$. Then there exists $t^* \in (t_0, t_1)$ such that $u(t^*) = c^*$ and

$$\begin{aligned} 0 = \int_{t_0}^{t_1} u''(s)u'(s)ds &= \int_{t_0}^{t^*} (f(s, c^*, u'(s)) + c^* - u(s))u'(s)ds + \\ \int_{t^*}^{t_1} f(s, u(s), u'(s))u'(s)ds &> \int_{t^*}^{t_1} g(u(s))u'(s)ds = G(u(t_1)) - G(c^*). \end{aligned}$$

Analogically like in the proof of Lemma 4.2 we deduce that $u(t_1) \in (r, r^*)$ and $G(r^*) < G(u(t_1))$. Since $G(r^*) = G(c^*) > G(u(t_1))$, we get a contradiction. Thus u satisfies (18) and it is a solution of (1), (2), as well. □

5 Main results

First, we will study a regular problem for the equation

$$u'' = h(t, u, u'), \tag{27}$$

with $h \in$Car$(J \times \mathbf{R}^2)$.

Theorem 5.1. *Let σ_1 and σ_2 be lower and upper solutions of (27), (2). Further suppose that for a.e. $t \in J$ and all $x, y \in \mathbf{R}$*

$$h(t, x, y) \ge -\rho(t, |x| + |y|), \tag{28}$$

where $\rho \in$Car$(J \times \mathbf{R}^+)$ is a nonnegative function, which is nondecreasing and sublinear in its second variable, i.e.

$$\lim_{z\to\infty} \frac{1}{z} \int_0^T \rho(t, z)dt = 0. \tag{29}$$

I. Then the problem (27), (2) has at least one solution u.
II. a) If $\sigma_1(t) \le \sigma_2(t)$ for all $t \in J$, then $\sigma_1(t) \le u(t) \le \sigma_2(t)$ for all $t \in J$.
b) If $\sigma_2(t) \le \sigma_1(t)$ for all $t \in J$, then there exists $t_u \in J$ such that $\sigma_2(t_u) \le u(t_u) \le \sigma_1(t_u)$.
c) If σ_1 and σ_2 are not ordered on J, then there exist $t_u, s_u \in J$ such that $\sigma_2(t_u) \le u(t_u)$ and $u(s_u) \le \sigma_1(s_u)$.

Proof. Let us set $H(\mu) = \frac{1}{\mu}\int_0^T \rho(s, \mu(T+1)+r)ds$ for $\mu \in (0,\infty)$ and $r = \|\sigma_1\|_C + \|\sigma_2\|_C$. The condition (29) implies that there exists $\mu^* \in (0,\infty)$ with

$$H(\mu) < 1 \text{ for all } \mu \geq \mu^*. \tag{30}$$

Now, let us consider the auxiliary differential equation

$$u'' = h^*(t, u, u'), \tag{31}$$

where

$$h^*(t,x,y) = \chi(|x|+|y|, r^*)h(t,x,y) \text{ for a.e. } t \in J \text{ and all } x, y \in \mathbf{R},$$

$$r^* = \mu^*(T+1) + r,$$

$$\chi(s,\phi) = \begin{cases} 1 & \text{for} \quad 0 \leq s \leq \phi \\ 2 - s/\phi & \text{for} \quad \phi < s < 2\phi \\ 0 & \text{for} \quad s \geq 2\phi \end{cases} .$$

Since there exists $m^* \in \mathrm{L}_1(J)$ such that $|h^*(t,x,y)| < m^*(t)$ for a.e. $t \in J$ and all $x, y \in \mathbf{R}$, we can apply Theorem 6 from [2] onto the problem (31), (2) and get a solution of this problem which fulfills the assertion II of our Theorem 5.1. Let us prove that u is a solution of (27). Putting $\mu = \|u'\|_C$, we get $\|u\|_C \leq \mu T + r$. In view of (2) we can find $t_0 \in J$ such that $u'(t_0) = 0$ and (28) gives

$$u''(t) \geq -\rho(t, |u| + |u'|) \text{ for a.e. } t \in J. \tag{32}$$

In a way similar to the proof of Theorem 8 in [2] we get from (32) by integration

$$|u'(t)| \leq \int_0^T \rho(t, \mu(T+1)+r)dt \text{ for all } t \in J,$$

and so $1 \leq H(\mu)$. The latter inequality together with (30) imply that $\mu < \mu^*$. Thus $\|u\|_C + \|u'\|_C < r^*$ and u satisfies (27). □

Theorem 5.2. *Suppose (9), (10), (11), (12) hold and let a positive function σ_1 be a lower solution of the problem (1), (2). Then this problem has at least one positive solution.*

Proof. Let r^* and c^* be the numbers from Lemma 4.3. Without loss of generality we can suppose $c^* \leq \min_{t\in J}\sigma_1(t)$. (Otherwise we take instead of r^* and c^* numbers r^{**} and c^{**}, which ssatisfy $c^{**} = \min_{t\in J}\sigma_1(t) < c^*$ and $r^{**} = G^{-1}(G(c^{**}) \in (r^*,\infty))$. Now, let us consider the auxiliary problem (26), (2). By (9), (10) and (25), $f^* \in \mathrm{Car}(J \times \mathbf{R}^2)$ and f^* satisfies (28) with $\rho(t,z) = |m(t)|$. Since c^* and σ_1 are upper and lower solution of (26), (2), respectively, Theorem 5.1 implies that (26), (2) has a solution u with

$$c^* \leq u(t_u) \leq \sigma_1(t_u) \text{ for some } t_u \in J. \tag{33}$$

By Lemma 4.3, u is a solution of (1), (2) and satisfies (18). □

Theorem 5.3. *Suppose (9), (10), (15) hold and let a positive function σ_1 be a lower solution of (1), (2). Then this problem has at least one solution u with $\sigma_1(t) \le u(t)$ for all $t \in J$.*

Proof. Let c^* be a positive number satisfying $c^* \le \min_{t\in J} \sigma_1(t)$ and let f^* be given by (25). Let us consider the problem (26), (2). Without loss of generality we can suppose that A in (15) satisfies $A > \max_{t\in J} \sigma_1(t)$ and we can check that

$$\sigma_2(t) = A + 2T\|\varphi\|_{L_1} - \frac{t}{T}\int_0^T \int_0^\tau \varphi(s)dsd\tau + \int_0^t \int_0^\tau \varphi(s)dsd\tau$$

is an upper solution of (26), (2). So, by Theorem 5.1, the problem (26), (2) has a solution u lying between σ_1 and σ_2 on J. In view of (25), u is a solution of (1), as well. □

Theorem 5.4. *Suppose (9), (10), (11), (12), (15) hold and let positive functions σ_1 and $\sigma_1 + \varepsilon$, where $\varepsilon \in \mathbf{R}^+$, be lower solutions of the problem (1), (2). Then this problem has at least two positive solutions.*

Proof. Theorem 5.2 implies the existence of a positive solution u of (1), (2) which satisfies (33). Theorem 5.3 gives the existence of a solution v of (1), (2) which has the property

$$\sigma_1(t) + \varepsilon \le v(t) \text{ for all } t \in J. \tag{34}$$

According to (33) and (34) we see that u and v are different solutions. □

Theorem 5.5. *Suppose (9), (10), (11), (12), (13), (14) and (15) hold. Then the problem (1), (2) has at least two positive solutions.*

Proof. Since (13) and (14) imply that the numbers σ and $\sigma + \varepsilon$ are constant positive lower solutions of (1), (2), the assertion follows from Theorem 5.4. □

Corollary 5.6. *Suppose (16) holds. Then the equation (4) has at least two positive T-periodic solutions.*

Proof. This assertion is a direct consequence of Theorem 5.4 and Lemma 3.1. □

References

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