Singular periodic problem for nonlinear ordinary differential equations with ϕ-Laplacian

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Abstract We investigate the singular periodic boundary value problem ϕ-Laplacian

$$(\phi(u'))' = f(t, u, u'),$$

$$u(0) = u(T), \quad u'(0) = u'(T),$$

where ϕ is an increasing homeomorphism, $\phi(\mathbb{R}) = \mathbb{R}$, $\phi(0) = 0$. We assume that f satisfies the Carathéodory conditions on each set $[a, b] \times \mathbb{R}^2$, $[a, b] \subset (0, T)$ and f does not satisfy the Carathéodory conditions on $[0, T] \times \mathbb{R}^2$, which means that f has time singularities at $t = 0$, $t = T$.

We provide sufficient conditions for the existence of solutions to the above problem belonging to $C^1[0, T]$. We also find conditions which guarantee the existence of a sign-changing solution to the problem.
Keywords singular periodic problem, ϕ-Laplacian, smooth sign-changing solutions, lower and upper functions
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1 Introduction

The growth in the theory of singular nonlinear boundary value problems has been strongly influenced by the rich and large number of applications that occur particulary in the sciences. For example the singular differential equation with the time singularity at $t = 0$

$$u'' + \frac{2}{t}u' = f(t, u)$$

arises in the study of steady-state oxygen diffusion in a cell with Michaelis-Menten Kinetics [1], [13].

Here we will investigate the singular nonlinear periodic problem with ϕ-Laplacian

$$(\phi(u'))' = f(t, u, u'), \tag{1.1}$$

$$u(0) = u(T), \quad u'(0) = u'(T). \tag{1.2}$$

We assume that ϕ is an increasing homeomorphism with $\phi(\mathbb{R}) = \mathbb{R}$, $\phi(0) = 0$, $[0, T] \subset \mathbb{R}$. The function f is supposed to satisfy the Carathéodory conditions on each set $[a, b] \times \mathbb{R}^2$, $[a, b] \subset (0, T)$, but f does not satisfy the Carathéodory conditions on $[0, T] \times\ \mathbb{R}^2$. We will write it $f \in Car((0, T) \times \mathbb{R}^2)$.

Definition 1.1 The function f satisfies the Carathéodory conditions on the set $[a, b] \times \mathbb{R}^2$, $[a, b] \subset (0, T)$ if

(i) $f(\cdot, x, y) : [a, b] \to R$ is measurable for all $(x, y) \in \mathbb{R}^2$,

(ii) $f(t, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ is continuous for a.e. $t \in [a, b]$,

(iii) for each compact set $K \subset \mathbb{R}^2$ there is a function $m_K \in L_1[a,b]$ such that $|f(t,x,y)| \le m_K(t)$ for a.e. $t \in [a,b]$ and all $(x,y) \in K$.

Further we assume that f has time singularities at the endpoints 0 and T.

Definition 1.2 We say that f has *time singularities at the points* 0 *and* T, respectively, if there exist $x, y \in \mathbb{R}$ such that

$$\int_0^{\varepsilon} |f(t,x,y)|dt = \infty \quad \text{and} \quad \int_{T-\varepsilon}^{T} |f(t,x,y)|dt = \infty$$

for each sufficiently small $\varepsilon > 0$. The points 0 and T are called *singular points of* f.

In order to prove the existence of solutions for periodic problem (1.1), (1.2) we start with the proof of the existence of solutions of auxiliary Dirichlet problems. For that reason we will consider boundary conditions

$$u(0) = u(T) = C, \tag{1.3}$$

where $C \in \mathbb{R}$.

Definition 1.3 Let $i \in \{2,3\}$. A function $u : [0,T] \to \mathbb{R}$ with $\phi(u') \in AC[0,T]$ is called *a solution* of problem (1.1), (1.i) if u satisfies

$$(\phi(u'(t)))' = f(t,u(t),u'(t))$$

for a.e. $t \in [0,T]$ and fulfils (1.i) .

Note that the condition $\phi(u') \in AC[0,T]$ implies $u \in C^1[0,T]$. Therefore we will seek solutions of problems (1.1), (1.2) and (1.1), (1.3) in the space of functions having continuous first derivatives on $[0,T]$, in particular at the singular points 0 and T. In the most works studying Dirichlet problems with time singularities the existence of so called w-solutions has been proved. See e.g. [9], [10], [11], [15].

Definition 1.4 A function $u \in C[0,T]$ is called *a w-solution* of problem (1.1), (1.3) if $\phi(u') \in AC_{loc}(0,T)$, u satisfies

$$(\phi(u'(t)))' = f(t,u(t),u'(t))$$

for a.e. $t \in [0,T]$ and fulfils (1.3) .

Since the condition $\phi(u') \in AC_{loc}(0,T)$ implies that a w-solution u belongs only to $C^1(0,T)$, we do not know the behaviour of u' at the singular endpoints 0, T. The notion of w-solutions can not be used for periodic problem (1.1), (1.2), where condition (1.2) requires $u \in C^1[0,T]$. That is why existence results for periodic problem (1.1), (1.2) with time singularities have not beeen proved up to now and in literature we can find only existence results for periodic problems with space singularities (i.e. $f(t,x,y)$ has singularities at x or at y). See e.g [4], [5], [6], [7], [8], [12], [14], [17], [18], [20], where the existence of positive periodic solutions was proved.

It seems worth to fill in this gap and to present existence results for problem (1.1), (1.2) with time singularities, which is the main goal of our paper (Theorem 4.1). Moreover we show conditions giving sign-changing solutions of (1.1), (1.2) (Corollary 4.2).

We will investigate singular problem (1.1), (1.2) by means of Dirichlet problem (1.1), (1.3). To establish the existence of a solution of the singular problem (1.1), (1.3) we introduce a sequence of approximating regular Dirichlet problems which are solvable. Then we pass to the limit in the sequence of approximate solutions to get a solution (a w-solution) of the problem (1.1), (1.3) and finally of the original problem (1.1), (1.2). In the next theorem we provide an existence principle which contains the main rules for the construction of such approximating sequences.

For $n \in \mathbb{N}$ consider equations

$$(\phi(u'))' = f_n(t, u, u'), \tag{1.4}$$

where $f_n \in Car([0,T] \times R^2)$. Denote

$$J_n = [\frac{1}{n}, T - \frac{1}{n}] \cap [0, T]. \tag{1.5}$$

Theorem 1.5 [16, Theorem 2.1] *Assume that*

$$f \in Car((0,T) \times R^2) \text{ has time singularities at } t = 0 \text{ and } t = T, \tag{1.6}$$

$$f_n(t, x, y) = f(t, x, y) \text{ for a.e. } t \in J_n \text{ and all } x, y \in \mathbb{R}, n \in \mathbb{N}, \tag{1.7}$$

$$\begin{gathered} \text{there exists a bounded set } \Omega \subset C^1[0,T] \text{ such that for each } n \in \mathbb{N} \\ \text{the regular problem } (1.4), (1.3) \text{ has a solution } u_n \in \Omega. \end{gathered} \tag{1.8}$$

Then

- *there exist* $u \in C[0,T] \cap C^1(0,T)$ *and a subsequence* $\{u_{n_l}\} \subset \{u_n\}$ *such that* $\lim_{l\to\infty} \|u_{n_l} - u\|_{C[0,T]} = 0, \quad \lim_{l\to\infty} u'_{n_l}(t) = u'(t)$ *locally uniformly on* $(0,T)$,
- u *is a w-solution of* (1.1), (1.3).

Moreover, assume that there exist $\eta \in (0, \frac{T}{2})$, $\lambda_1, \lambda_2 \in \{-1, 1\}$, $d \in R$ *and* $\psi \in L_1[0,T]$ *such that for each* $n \in \mathbb{N}$

$$\begin{cases} \lambda_1 \operatorname{sign}(u'_n(t) - d) f_n(t, u_n(t), u'_n(t)) \geq \psi(t) & \text{a.e. on } (0, \eta), \\ \lambda_2 \operatorname{sign}(u'_n(t) - d) f_n(t, u_n(t), u'_n(t)) \geq \psi(t) & \text{a.e. on } (T - \eta, T). \end{cases} \tag{1.9}$$

Then u *is a solution of* (1.1), (1.3).

2 Lemmas

Consider a sequence of functions $v_n : [0,T] \to R$, $n \in \mathbb{N}$.

Definition 2.1 We say that the sequence $\{v_n\}$ is equicontinuos in $t_0 \in [0,T]$, if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall t \in (t_0 - \delta, t_0 + \delta) \cap [0,T], \forall n \in \mathbb{N} : |v_n(t) - v_n(t_0)| < \varepsilon$$

We will show conditions which imply the equicontinuity of $\{v_n\}$ at the points 0, T. This result will be used in Section 4.

Lemma 2.2 *Assume that there exist* $\eta \in (0, \frac{T}{2})$ *and nonnegative functions* $\alpha \in C[0,\eta]$, $\beta \in C(0,\eta]$, $\alpha(0) = 0$, $\beta(0+) = 0$ *such that for each* $n \in \mathbb{N}$, $n > \frac{1}{\eta}$

$$|v_n(t)| \leq \beta(t) \quad \text{for } t \in [\frac{1}{n}, \eta], \tag{2.1}$$

$$|v_n(t) - v_n(0)| \leq \alpha(t) \quad \text{for } t \in [0, \frac{1}{n}). \tag{2.2}$$

Then $\{v_n\}$ *is equicontinuous in* 0 *and* $\lim_{n\to\infty} v_n(0) = 0$.

Proof

Choose an arbitrary $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$t \in [0, \delta) \Rightarrow |\alpha(t)| < \frac{\varepsilon}{3}, \quad t \in (0, \delta) \Rightarrow |\beta(t)| < \frac{\varepsilon}{3}.$$

Choose an arbitrary $t \in [0, \delta)$ an and arbitrary $n \in \mathbb{N}$.

(*a*) Let $t \in [0, \frac{1}{n}]$. Then by (2.2), $|v_n(t) - v_n(0)| \le \alpha(t) < \frac{\varepsilon}{3}$.

(*b*) Let $t \in [\frac{1}{n}, \delta)$. Then by (2.1), (2.2), $|v_n(t) - v_n(0)| \le |v_n(t)| + |v_n(0) - v_n(\frac{1}{n})| + |v_n(\frac{1}{n})| \le \beta(t) + \alpha(\frac{1}{n}) + \beta(\frac{1}{n}) < \varepsilon$.

Hence, we have proved that $\{v_n\}$ is equicontinuous in 0. Further, $|v_n(0)| \le |v_n(0) - v_n(\frac{1}{n})| + |v_n(\frac{1}{n})| \le \alpha(\frac{1}{n}) + \beta(\frac{1}{n}) \to 0$ for $n \to \infty$.

□

Lemma 2.3 *Assume that there exist* $\eta \in (0, \frac{T}{2})$ *and nonnegative functions* $\alpha \in C[T-\eta, T]$, $\beta \in C[T-\eta, T)$, $\alpha(T) = 0$, $\beta(T-) = 0$ *such that for each* $n \in \mathbb{N}$, $n > \frac{1}{\eta}$

$$|v_n(t)| \le \beta(t) \quad \textit{for } t \in [T-\eta, T-\frac{1}{n}], \tag{2.3}$$

$$|v_n(t) - v_n(T)| \le \alpha(t) \quad \textit{for } t \in (T-\frac{1}{n}, T]. \tag{2.4}$$

Then $\{v_n\}$ *is equicontinuous in* T *and* $\lim_{n\to\infty} v_n(T) = 0$.

Proof

Choose an arbitrary $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$t \in (T-\delta, T] \Rightarrow |\alpha(t)| < \frac{\varepsilon}{3}, \quad t \in (T-\delta, T) \Rightarrow |\beta(t)| < \frac{\varepsilon}{3}.$$

Choose an arbitrary $t \in (T-\delta, T]$ and an arbitrary $n \in \mathbb{N}$.

(*a*) Let $t \in [T - \frac{1}{n}, T]$. Then by (2.4), $|v_n(t) - v_n(T)| \le \alpha(t) < \frac{\varepsilon}{3}$.

(*b*) Let $t \in (T-\delta, T-\frac{1}{n})$. Then by (2.3), (2.4), $|v_n(t) - v_n(T)| \le |v_n(t)| + |v_n(T) - v_n(T-\frac{1}{n})| + |v_n(T-\frac{1}{n})| \le \beta(t) + \alpha(T-\frac{1}{n}) + \beta(T-\frac{1}{n}) < \varepsilon$.

Hence, we have proved that $\{v_n\}$ is equicontinuous in T. Further, $|v_n(T)| \le |v_n(T) - v_n(T - \frac{1}{n})| + |v_n(T-\frac{1}{n})| \le \alpha(T-\frac{1}{n}) + \beta(T-\frac{1}{n}) \to 0$ for $n \to \infty$.

□

Lemma 2.4 *Assume that* $\eta \in (0, \frac{T}{2})$, $\beta_0 \in (0, \infty)$, $\gamma \in L_1[0,T]$, $g^* \in L_1[0,T]$ *and that* $h^* \in L_{1_{loc}}(0,T)$ *is nonnegative. Further let for each* $n \in \mathbb{N}$, $n > \frac{1}{\eta}$, *a function* $v_n \in AC[0,T]$ *fulfil conditions*

$$|v_n(\eta)| \le \beta_0, \tag{2.5}$$

$$v_n'(t) \operatorname{sign} v_n(t) \ge h^*(t)|v_n(t)| + g^*(t) \quad \textit{for a.e. } t \in [\frac{1}{n}, \eta], \tag{2.6}$$

$$v_n'(t) = \gamma(t) \quad \textit{for a.e. } t \in (0, \frac{1}{n}), \tag{2.7}$$

where

$$\int_0^\varepsilon h^*(s)ds = \infty \quad \textit{for each sufficiently small } \varepsilon > 0. \tag{2.8}$$

Then the sequence $\{v_n\}$ *is equicontinuous in* 0 *and* $\lim_{n\to\infty} v_n(0) = 0$.

Proof

We will construct functions α and β of Lemma 2.2. Consider the auxiliary problem

$$\beta'(t) = h^*(t)\beta(t) + g^*(t), \quad \beta(\eta) = \beta_0. \tag{2.9}$$

Problem (2.9) has a unique solution of the form

$$\beta(t) = e^{-\int_t^\eta h^*(s)ds}\big[\beta_0 - \int_t^\eta g^*(\tau)e^{\int_\tau^\eta h^*(s)ds}d\tau\big] \quad \text{for } t \in (0,\eta].$$

By (2.8) we get

$$\lim_{t\to 0+} \beta(t) = \beta_0 e^{-\int_0^\eta h^*(s)ds} - \int_0^\eta g^*(\tau)e^{-\int_0^\tau h^*(s)ds}d\tau = 0$$

because $\int_0^\tau h^*(s)ds = \infty$ for each $\tau \in (0,\eta]$.

Choose an arbitrary $n \in \mathbb{N}$. Let us prove that (2.1) is satisfied. In contrary assume that there exist $t_1 \in (\frac{1}{n},\eta)$ and $t_2 \in (t_1,\eta]$ such that

$$|v_n(t_2)| = \beta(t_2), \quad |v_n(t)| > \beta(t) \quad \text{for all } t \in [t_1,t_2).$$

Then, by (2.6) and (2.9), we get

$$0 < |v_n(t_1)| - \beta(t_1) = -\int_{t_1}^{t_2}(v_n'(t)\,\mathrm{sign}\,v_n(t) - \beta'(t))dt \le \int_{t_1}^{t_2} -h^*(t)(|v_n(t)| - \beta(t))dt \le 0,$$

a contradiction. Further, due to (2.7), we have

$$|v_n(t) - v_n(0)| \le |\int_0^t \gamma(s)ds| = \alpha(t) \quad \text{for } t \in [0,\frac{1}{n}).$$

It means that (2.2) is satisfied and, using Lemma 2.2, Lemma 2.4 is proved.

□

Lemma 2.5 *Assume that $\eta \in (0,\frac{T}{2})$, $\beta_0 \in (0,\infty)$, $\gamma \in L_1[0,T]$, $g^* \in L_1[0,T]$ and that $h^* \in L_{1_{loc}}(0,T)$ is nonnegative. Further let for each $n \in \mathbb{N}$, $n > \frac{1}{\eta}$, a function $v_n \in AC[0,T]$ fulfil conditions*

$$|v_n(T-\eta)| \le \beta_0, \tag{2.10}$$

$$-v_n'(t)\,\mathrm{sign}\,v_n(t) \ge h^*(t)|v_n(t)| + g^*(t) \quad \textit{for a.e. } t \in [T-\eta, T-\frac{1}{n}], \tag{2.11}$$

$$v_n'(t) = \gamma(t) \quad \textit{for a.e. } t \in (T-\frac{1}{n}, T), \tag{2.12}$$

where

$$\int_{T-\varepsilon}^T h^*(s)ds = \infty \quad \textit{for each sufficiently small } \varepsilon > 0. \tag{2.13}$$

Then the sequence $\{v_n\}$ is equicontinuous in T and $\lim_{n\to\infty} v_n(T) = 0$.

Proof

We will construct functions α and β of Lemma 2.3. Consider the auxiliary problem

$$\beta'(t) = -h^*(t)\beta(t) - g^*(t), \quad \beta(T-\eta) = \beta_0. \tag{2.14}$$

Problem (2.14) has a unique solution of the form

$$\beta(t) = e^{-\int_{T-\eta}^t h^*(s)ds}\big[\beta_0 - \int_{T-\eta}^t g^*(\tau)e^{\int_{T-\eta}^\tau h^*(s)ds}d\tau\big] \quad \text{for } t \in [T-\eta, T).$$

By (2.13) we get

$$\lim_{t\to T-} \beta(t) = \beta_0 e^{-\int_{T-\eta}^T h^*(s)ds} - \int_{T-\eta}^T g^*(\tau)e^{-\int_\tau^T h^*(s)ds}d\tau = 0$$

because $\int_\tau^T h^*(s)ds = \infty$ for each $\tau \in [T-\eta, T)$.
Choose an arbitrary $n \in \mathbb{N}$. Let us prove that (2.3) is satisfied. In contrary assume that there exist $t_1 \in [T-\eta, T-\frac{1}{n})$ and $t_2 \in (t_1, T-\frac{1}{n})$ such that

$$|v_n(t_1)| = \beta(t_1), \quad |v_n(t)| > \beta(t) \quad \text{for all } t \in (t_1, t_2].$$

Then, by (2.11) and (2.14), we get

$$0 < |v_n(t_2)| - \beta(t_2) = \int_{t_1}^{t_2} (v_n'(t) \operatorname{sign} v_n(t) - \beta'(t))dt \le \int_{t_1}^{t_2} -h^*(t)(|v_n(t)| - \beta(t))dt \le 0,$$

a contradiction. Further, due to (2.12), we have

$$|v_n(t) - v_n(T)| \le |\int_t^T \gamma(s)ds| = \alpha(t) \quad \text{for } t \in (T - \frac{1}{n}, T].$$

It means that (2.4) is satisfied and, by Lemma 2.3, Lemma 2.5 is proved.

□

3 Regular Dirichlet BVP's

In order to fulfil the basic condition (1.8) in Theorem 1.5 we need existence results for regular problems (1.4), (1.3) and a priori estimates for their solutions. To this aim we consider a regular equation

$$(\phi(u'))' = h(t, u, u'), \tag{3.1}$$

$h \in Car([0,T] \times R^2)$, and use the lower and upper functions method to get solvability of problem (3.1), (1.3).

Definition 3.1 Functions $\sigma_1, \sigma_2 : [0,T] \to R$ are respectively *lower and upper functions of problem* (3.1), (1.3) if $\phi(\sigma_i') \in AC[0,T]$ for $i \in \{1,2\}$ and

$$\begin{array}{lll} (\phi(\sigma_1'(t)))' \ge f(t, \sigma_1(t), \sigma_1'(t)), & (\phi(\sigma_2'(t)))' \le f(t, \sigma_2(t), \sigma_2'(t)) & \text{for a.e. } t \in [0,T], \\ \sigma_1(0) \le C, \sigma_1(T) \le C, & \sigma_2(0) \ge C, \sigma_2(T) \ge C. & \end{array}$$

Since the lower and upper functions method for regular problems with ϕ-Laplacian can be found in literature (see e.g. [3], [2], [19], [16]), we only cite the results without their proofs..

Lemma 3.2 [2, Theorem 2.1] *Let σ_1 and σ_2 be respectively lower and upper functions of problem* (3.1), (1.3) *and let $\sigma_1 \le \sigma_2$ on $[0,T]$. Further assume that there is $h_0 \in L_1[0,T]$ such that*

$$|h(t,x,y)| \le h_0(t) \quad \text{for a.e. } t \in [0,T] \text{ and for all } (x,y) \in [\sigma_1(t), \sigma_2(t)] \times R.$$

Then problem (3.1), (1.3) has a solution $u \in C^1[0,T]$ with $\phi(u') \in AC[0,T]$ such that

$$\sigma_1 \le u \le \sigma_2 \text{ on } [0,T]. \tag{3.2}$$

Lemma 3.2 gives the existence result for (3.1), (1.3) provided the function h has a Lebesgue integrable majorant h_0. The method of a priori estimates enables us to extend this result to more general right-hand sides h.

Lemma 3.3 [16, Lemma 3.3]*(An a priori estimate)*
Assume that $a, b \in [0,T], a \le b$, $d \in R$, $c_0 \in (0,\infty)$. Let $g_0 \in L_1[0,T]$ be nonnegative and let $\omega \in C[0,\infty)$ be positive and

$$\int_0^\infty \frac{ds}{\omega(s)} = \infty. \tag{3.3}$$

Then there exists $\varrho_0 \in (c_0, \infty)$ *such that for each function* $u \in C^1[0,T]$ *satisfying the conditions*

$$\phi(u') \in AC[0,T],$$

$$|u(t)| \le c_0 \quad \textit{for each } t \in [0,T], \tag{3.4}$$

$$|u'(\xi)| \le c_0 \quad \textit{for some } \xi \in [a,b], \tag{3.5}$$

$$(\phi(u'(t)))' \operatorname{sign}(u'(t)-d) \ge -\omega(|\phi(u'(t)) - \phi(d)|)(g_0(t) + |u'(t)-d|)$$
$$\textit{for a.e. } t \in [0,b] \textit{ and for } |\phi(u'(t))| > |\phi(d)| \tag{3.6}$$

and

$$(\phi(u'(t)))' \operatorname{sign}(u'(t)-d) \le \omega(|\phi(u'(t)) - \phi(d)|)(g_0(t) + |u'(t)-d|)$$
$$\textit{for a.e. } t \in [a,T] \textit{ and for } |\phi(u'(t))| > |\phi(d)|, \tag{3.7}$$

the estimate

$$|u'(t)| \le \varrho_0 \quad \textit{for each } t \in [0,T] \tag{3.8}$$

is valid.

Using Lemma 3.2 and Lemma 3.3 we get the existence result for (3.1), (1.3) under one-sided growth restrictions of the Nagumo type (3.12), (3.13).

Theorem 3.4 [16, Theorem 3.4] *Assume that the following conditions are fulfilled:*

$$\sigma_1 \textit{ and } \sigma_2 \textit{ are respectively lower and upper functions of } (3.1), (1.3) \quad \textit{and} \quad \sigma_1 \le \sigma_2 \textit{ on } [0,T], \tag{3.9}$$

$$a, b \in [0,T],\ a < b,\ d \in R,\ c_0 \ge 2\frac{1+b-a}{b-a}(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty), \tag{3.10}$$

$$g \in L_1[0,T] \textit{ is nonnegative, } \omega \in C[0,\infty) \textit{ is positive and fulfils } (3.3), \tag{3.11}$$

$$h(t,x,y) \operatorname{sign} y \ge -\omega(|\phi(y) - \phi(d)|)(g(t) + |y|)$$
$$\textit{for a.e. } t \in [0,b], \forall x \in [\sigma_1(t), \sigma_2(t)], \forall y \in R \textit{ such that } |\phi(y)| > |\phi(d)| \tag{3.12}$$

and

$$h(t,x,y) \operatorname{sign} y \le \omega(|\phi(y) - \phi(d)|)(g(t) + |y|)$$
$$\textit{for a.e. } t \in [a,T], \forall x \in [\sigma_1(t), \sigma_2(t)], \forall y \in R \textit{ such that } |\phi(y)| > |\phi(d)|. \tag{3.13}$$

Then problem (3.1), (1.3) *has a solution* u *satisfying*

$$\sigma_1 \le u \le \sigma_2 \quad \textit{on } [0,T] \tag{3.14}$$

and

$$|u'(t)| \le \varrho_0 \quad \textit{for } t \in [0,T], \tag{3.15}$$

where $\varrho_0 \in (0, \infty)$ *is the constant from Lemma 3.3 with* $g_0 = g + |d|$.

4 Main result

In this section we prove our main result about the solvability of the singular periodic boundary value problem (1.1), (1.2).

Theorem 4.1 *(Existence of a solution of the periodic problem)*
Let $a, b \in [0,T], a < b$. Let there exist $r_1, r_2, d \in R$, such that

$$\begin{cases} r_1 + td \le C, \ \ r_2 + td \ge C \quad \text{for } t \in [0,T], \\ f(t, r_1 + td, d) \le 0, \quad f(t, r_2 + td, d) \ge 0 \quad \text{for a.e. } t \in [0,T]. \end{cases} \tag{4.1}$$

Further, let there exist nonnegative function $g \in L_1[0,T]$ and positive function $\omega \in C[0,\infty)$ satisfying (3.3),

$$f(t,x,y)\,\mathrm{sign}\,y \ge -\omega(|\phi(y) - \phi(d)|)(g(t) + |y|)$$
$$\text{for a.e. } t \in [0,b], \forall x \in [r_1 + td, r_2 + td], \forall y \in R \text{ such that } |\phi(y)| > |\phi(d)| \tag{4.2}$$

and

$$f(t,x,y)\,\mathrm{sign}\,y \le \omega(|\phi(y) - \phi(d)|)(g(t) + |y|)$$
$$\text{for a.e. } t \in [a,T], \forall x \in [r_1 + td, r_2 + td], \forall y \in R \text{ such that } |\phi(y)| > |\phi(d)|. \tag{4.3}$$

Then there exists a function u which is w-solution of problem (1.1), (1.3) *and satisfies*

$$r_1 + td \le u(t) \le r_2 + td \qquad \text{for } t \in [0,T] \tag{4.4}$$

and

$$|u'(t)| \le \varrho_0 \quad \text{for each } t \in (0,T), \tag{4.5}$$

where ϱ_0 is the constant from Lemma 3.3 with $g_0 = g + |d|$.

Moreover, let there exist $\eta \in (0, \frac{T}{2})$, $g^* \in L_1[0,T]$ and nonnegative function $h^* \in L_{1_{loc}}(0,T)$ such that

$$\mathrm{sign}(y-d) f(t,x,y) \ge h^*(t)|\phi(y) - \phi(d)| + g^*(t)$$
$$\text{for a.e. } t \in (0,\eta), \forall x \in [r_1 + td, r_2 + td], \forall y \in [-\varrho_0, \varrho_0], \tag{4.6}$$

$$-\,\mathrm{sign}(y-d) f(t,x,y) \ge h^*(t)|\phi(y) - \phi(d)| + g^*(t)$$
$$\text{for a.e. } t \in (T-\eta,T), \forall x \in [r_1 + td, r_2 + td], \forall y \in [-\varrho_0, \varrho_0]. \tag{4.7}$$

Then the function u is a solution of problem (1.1), (1.2) *and $u'(0) = u'(T) = d$.*

Proof

For each $n \in \mathbb{N}$ define J_n by (1.5),

$$f_n(t,x,y) = \begin{cases} f(t,x,y) & \text{for a.e. } t \in J_n, \forall x, y \in R, \\ 0 & \text{for a.e. } t \in [0, \frac{1}{n}) \cup (T - \frac{1}{n}, T], \forall x, y \in R. \end{cases} \tag{4.8}$$

Then $f_n \in Car([0,T] \times R^2)$ for each $n \in \mathbb{N}$. Choose $n \in \mathbb{N}$ and show that problem (1.4), (1.3) satisfies the assumptions of Theorem 3.4. Let us put $\sigma_1(t) = r_1 + td$ and $\sigma_2(t) = r_2 + td$ for

$t \in [0,T]$. Then $\sigma_1 \le \sigma_2$ on $[0,T]$ and according to (4.1), σ_1 and σ_2 are lower and upper function of problem (1.4), (1.3), i.e. (3.9) holds. From inequalities (4.2) and (4.3) we get

$$\begin{aligned}
f_n(t,x,y)\operatorname{sign} y &= f(t,x,y)\operatorname{sign} y \ge -\omega(|\phi(y)-\phi(d)|)(g(t)+|y|)\\
&\quad \text{for a.e. } t\in[0,b]\cap J_n, \forall x\in[r_1+td, r_2+td], \forall y\in R, |\phi(y)|>|\phi(d)|,\\
f_n(t,x,y)\operatorname{sign} y &= 0 \ge -\omega(|\phi(y)-\phi(d)|)(g(t)+|y|)\\
&\quad \text{for a.e. } t\in[0,b]\backslash J_n, \forall x\in[r_1+td, r_2+td], \forall y\in R,\\
f_n(t,x,y)\operatorname{sign} y &= f(t,x,y)\operatorname{sign} y \le \omega(|\phi(y)-\phi(d)|)(g(t)+|y|)\\
&\quad \text{for a.e. } t\in[a,T]\cap J_n, \forall x\in[r_1+td, r_2+td], \forall y\in R, |\phi(y)|>|\phi(d)|,\\
f_n(t,x,y)\operatorname{sign} y &= 0 \le \omega(|\phi(y)-\phi(d)|)(g(t)+|y|)\\
&\quad \text{for a.e. } t\in[a,T]\backslash J_n, \forall x\in[r_1+td, r_2+td], \forall y\in R.
\end{aligned}$$

It means that conditions (3.12) and (3.13) are fulfilled for $h = f_n$. By Theorem 3.4, problem (1.4), (1.3) has a solution $u_n \in C^1[0,T]$ with $\phi(u_n') \in AC[0,T]$. Moreover, u_n satisfies (4.4) and

$$|u_n'(t)| \le \varrho_0 \qquad \text{for } t\in[0,T], \tag{4.9}$$

where $\varrho_0 \in (0,\infty)$ is the constant from Lemma 3.3 with $g_0 = g + |d|$. By virtue of Lemma 3.3, ϱ_0 does not depend on u_n. Therefore condition (1.8) is fulfilled, where

$$\Omega = \{x \in C^1([0,T]) : \|x\|_\infty \le \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + \varrho_0\}.$$

Hence, by Theorem 1.5, problem (1.4), (1.3) has a w-solution u which satisfies (4.4) and (4.5). Now, furthermore, assume (4.6), (4.7). Let us define

$$\psi(t) = \min\{g^*(t), 0\} \quad \text{for } t\in[0,T].$$

Then $\psi \in L_1[0,T]$. Let us put $\lambda_1 = 1$ and $\lambda_2 = -1$. We can see that

$$\begin{aligned}
&\lambda_1 \operatorname{sign}(u_n'(t)-d) f_n(t,u_n(t),u_n'(t)) = \operatorname{sign}(u_n'(t)-d) f(t,u_n(t),u_n'(t)) \ge\\
&\qquad h^*(t)|\phi(u_n'(t))-\phi(d)| + g^*(t) \ge \psi(t) \text{ for a.e. } t\in[0,\eta]\cap J_n,\\
&\lambda_1 \operatorname{sign}(u_n'(t)-d) f_n(t,u_n(t),u_n'(t)) = 0 \ge \psi(t) \text{ for a.e. } t\in[0,\eta]\backslash J_n,\\
&\lambda_2 \operatorname{sign}(u_n'(t)-d) f_n(t,u_n(t),u_n'(t)) = -\operatorname{sign}(u_n'(t)-d) f(t,u_n(t),u_n'(t)) \ge\\
&\qquad h^*(t)|\phi(u_n'(t))-\phi(d)| + g^*(t) \ge \psi(t) \text{ for a.e. } t\in[0,\eta]\cap J_n,\\
&\lambda_2 \operatorname{sign}(u_n'(t)-d) f_n(t,u_n(t),u_n'(t)) = 0 \ge \psi(t) \text{ for a.e. } t\in[0,\eta]\backslash J_n.
\end{aligned}$$

Hence, by Theorem 1.5, u is the solution of the problem (1.1), (1.3). Moreover there exists a subsequence $\{u_{n_l}\} \subset \{u_n\}$, which uniformly converges to u on $[0,T]$ and $\{u_{n_l}'\}$ converges locally uniformly to u' on $(0,T)$.

Let us show that u is also a solution of the periodic problem (1.1), (1.2). Without lost of generality, let us denote $\{u_{n_l}\}$ as $\{u_n\}$. We will verify the assumptions of Lemmas 2.4 and 2.5 to show that $\{u_n'\}$ is equicontinuous in 0 and T. Note that $\operatorname{sign}(\phi(y)-\phi(d)) = \operatorname{sign}(y-d)$ for all $y\in\mathbb{R}$ and that $f_n = f$ for $t \ge \frac{1}{n}$. Let us put $\phi(u_n'(t)) - \phi(d) = v_n(t)$. By (4.9) there exists $\beta_0 > 0$ such that $|v_n(\eta)| \le \beta_0$ and $|v_n(T-\eta)| \le \beta_0$. From (4.6) we have

$$[\phi(u_n'(t))]' \operatorname{sign}(\phi(u_n'(t))-\phi(d)) \ge h^*(t)|\phi(u_n'(t))-\phi(d)| + g^*(t)$$

for all $n\in\mathbb{N}$ and a.e. $t\in[\frac{1}{n},\eta]$ and from (4.7)

$$-[\phi(u_n'(t))]' \operatorname{sign}(\phi(u_n'(t))-\phi(d)) \ge h^*(t)|\phi(u_n'(t))-\phi(d)| + g^*(t)$$

for all $n\in\mathbb{N}$ and a.e. $t\in[T-\eta, T-\frac{1}{n}]$. Then

$$v_n'(t)\operatorname{sign} v_n(t) \ge h^*(t)|v_n(t)| + g^*(t) \quad \text{for a.e. } t\in[\frac{1}{n},\eta].$$

and

$$-v_n'(t)\operatorname{sign} v_n(t) \ge h^*(t)|v_n(t)| + g^*(t) \quad \text{for a.e. } t \in [T-\eta, T-\frac{1}{n}].$$

Further, $v_n'(t) = 0$ for a.e. $t \in (0, \frac{1}{n})$ and a.e. $t \in (T-\frac{1}{n}, T)$. According to Lemmas 2.4 and 2.5, $\{v_n\}$ is equicontinuous in 0 and T and $\lim_{n\to\infty} v_n(0) = 0$ and $\lim_{n\to\infty} v_n(T) = 0$. It means that also $\{\phi(u_n')\}$ and $\{u_n'\}$ are equicontinuous in 0 and T. The equicontinuity in 0 means that for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for each $t \in [0, \delta)$ and all $n \in \mathbb{N}$ the inequality $|u_n'(t) - u_n'(0)| < \varepsilon$ is valid. Moreover, by Lemmas 2.4 and 2.5, $\lim_{n\to\infty} u_n'(0) = d$ and $\lim_{n\to\infty} u_n'(T) = d$. According to the first limit we can find $n_0 \in \mathbb{N}$ such that for each $n \ge n_0$ the inequality $|u_n'(0) - d| < \varepsilon$ holds. From locally uniform convergence of $\{u_{n_l}'\}$ on $(0, T)$ there exists $n_t \in \mathbb{N}$, $n_t \ge n_0$, such that $|u'(t) - u_{n_t}'(t)| < \varepsilon$. Therefore we have

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall t \in (0, \delta): \ |u'(t) - d| \le |u'(t) - u_{n_t}'(t)| + |u_{n_t}'(t) - u_{n_t}'(0)| + |u_{n_t}'(0) - d| < 3\varepsilon,$$

which yields $\lim_{t\to 0+} u'(t) = d$. The property $\lim_{t\to T-} u'(t) = d$ can be proved similarly. Hence $u'(0) = u'(T)$ and u is a solution of the periodic problem (1.1), (1.2).

□

Corollary 4.2 *Let all assumptions of Theorem 4.1 be fulfiled and let $C = 0$, $d \ne 0$. Then problem* (1.1), (1.2) *has a sign-changing solution.*

Example 4.3 (Existence of a periodic sign changing solution)
Let $p > 1$ and $\phi_p(y) = |y|^{p-2}y$ for $y \in R$. Consider the equation

$$(\phi_p(u'))' = q(t)(u^k - r^k) + c\phi_p(u')u' + (\frac{1}{t^\alpha} - \frac{1}{(T-t)^\beta})(\phi_p(u') - \phi_p(d)), \qquad (4.10)$$

where $r, c, d \in \mathbb{R}$, $k \in \mathbb{N}$ is odd, α, $\beta \in (1, \infty)$, $q \in L_1[0, T]$ is nonnegative. Choose an arbitrary $C \in \mathbb{R}$ and show that all the conditions of Theorem 4.1 are satisfied. Let $r_1, r_2 \in \mathbb{R}$. Then

$$f(t, r_i + td, d) = q(t)((r_i + td)^k - r^k) + c\phi_p(d)d \quad \text{for a.e. } t \in [0, T].$$

Since q is nonnegative on [0, T], we can find a large positive r_2 and a negative r_1 with large absolute value such that (4.1) holds. Denote

$$q_1(t) = q(t)\max\{|x^k - r^k| : r_1 + td \le x \le r_2 + td\} \quad \text{for a.e. } t \in [0, T],$$

$$q_2(t) = \begin{cases} (T-t)^{-\beta} & \text{for a.e. } t \in [0, a), \\ (T-t)^{-\beta} + t^{-\alpha} & \text{for a.e. } t \in [a, b], \\ t^{-\alpha} & \text{for a.e. } t \in (b, T]. \end{cases}$$

Then for a.e. $t \in [0, b]$, each $x \in [r_1 + td, r_2 + td]$ and each $y \in R$, $|\phi_p(y)| > |\phi_p(d)|$ we have

$$f(t,x,y)\operatorname{sign} y = f(t,x,y)\operatorname{sign}(\phi_p(y) - \phi_p(d)) > -q_1(t) -$$
$$-|c||\phi_p(y) - \phi_p(d)||y| - |c||\phi_p(d)||y| - \frac{1}{(T-t)^\beta}|\phi_p(y) - \phi_p(d)| >$$
$$> -(|\phi_p(y) - \phi_p(d)| + 1)((|c| + 1)(|\phi_p(d)| + 1))(|q_1(t)| + |q_2(t)| + |y|).$$

Therefore, if we put

$$\omega(s) = (1+s)c_0, \quad c_0 = (|c|+1)(|\phi_p(d)|+1), \quad g(t) = |q_1(t)| + |q_2(t)|,$$

we get (4.2). Similarly we can derive (4.3). Let us put

$$h^*(t) = \begin{cases} t^{-\alpha} & \text{for a.e. } t \in (0, \eta), \\ (T-t)^{-\beta} & \text{for a.e. } t \in (T-\eta, T), \end{cases}$$

$h^* \in L_1[\eta, T-\eta]$. For a.e. $t \in (0,\eta)$, each $x \in [r_1+td, r_2+td]$ and each $y \in [-\varrho_0, \varrho_0]$ we get

$$f(t,x,y)\,\mathrm{sign}(y-d) = f(t,x,y)\,\mathrm{sign}(\phi_p(y)-\phi_p(d)) > -q_1(t) - |c|\phi_p(\varrho_0)\varrho_0$$
$$-q_2(t)(\phi_p(\varrho_0)+|\phi_p(d)|) + \frac{1}{t^\alpha}|\phi_p(y)-\phi_p(d)| = g^*(t) + h^*(t)|\phi_p(y)-\phi_p(d)|,$$

where $g^* \in L_1[0,T]$, which means that (4.6) is satisfied. Identically we can derive (4.7). Therefore, by Theorem 4.1, problem (4.10), (1.2) has a solution u. Moreover $u(0) = u(T) = C$ and $u'(0) = u'(T) = d$. If we choose $C = 0$ and $d \neq 0$, we get by Corollary 4.2 that u changes its sign on $(0,T)$.

References

[1] Anderson, N., Arthurs, A., M.: Analytic bounding functions for diffusion problems with Michaelis-Menten kinetics. *Bull. Math. Biol.*, **47**, 145-153 (1985)

[2] Cabada, A., Pouso, L., R.: Existence results for the problem $(\phi(u'))' = f(t,u,u')$ with nonlinear boundary conditions. *Nonlinear Anal.*, **35**, 221 - 231 (1999)

[3] De Coster, C.: Pairs of positive solutions for the one-dimensional $p-$Laplacian. *Nonlinear Analysis, Theory, Methods and Applications*, **23**, 669 - 681 (1994)

[4] Del Pino, M., Manásevich, R., Montero, A.: T-periodic solutions for some second order differential equations with singularities. *Proc. Royal Soc. Edinburgh*, **120A**, 231-244 (1992)

[5] Fonda, A.: Periodic solutions of scalar second order differential equations with a singularity. *Mémoire de la Classe de Sciences*, Acad. Roy. Belgique, **8-IV**, 1-39 (1993)

[6] Fonda, A., Manásevich, R., Zanolin, F.: Subharmonic solutions for some second-order differential equations with singularities. *SIAM J. Math. Anal.*, **24**, 1294-1311 (1993)

[7] Ge, W. G., Mawhin, J.: Positive solutions to boundary value problems for second order ordinary differential equations with singular nonlinearities. *Result. Math.*, **34**, 108-119 (1998)

[8] Habets, P., Sanchez L.: Periodic solutions of some Linéard equations with singularities. *Proc. Amer. Math. Soc.*, **109**, 1035 - 1044 (1990)

[9] Kiguradze, I. T.: Some Singular Boundary Value Problems for Ordinary Differential Equations, Izdat. Tbilis. Univ., Tbilisi, 1975

[10] Kiguradze, I. T.: Some optimal conditions for solvability of two-point singular boundary value problem. *Functional Differential Equations*, **10**, 259 - 281 (2003)

[11] Kiguradze, I. T., Shekhter, B. L.: Singular boundary value problems for second order ordinary differential equations. *Itogi Nauki Tekh., Ser. Sovrm. Probl. Mat., Viniti*, **30**, 105 - 201 (1987)

[12] Lazer, A. C., Solimini S.: On periodic solutions of nonlinear differential equations with singularities. *Proc. Amer. Math. Soc.*, **99**, 109-114 (1987)

[13] Lin, S., H.: Oxygen diffusion in a spherical cell with nonlinear oxygen uptake kinetics. *J. Theor. Biol.*, **60**, 449-457 (1976)

[14] Omari, P., YE, W.: Necesary and sufficient conditions for the existence of periodic solutions of second order ordinary differential equations with singular nonlinearities. *Differential Integral Equations* , **8**, 1843 - 1858 (1995)

[15] O'Regan, D.: Theory of Singular Boundary Value Problems, World Scientific, Singapore, 1995

[16] Polášek, V., Rachůnková, I.: Singular Dirichlet problem for ordinary differential equations with $\phi-$Laplacian. *Mathematica Bohemica*, **130**, 409 - 425 (2005)

[17] Rachůnková, I., Tvrdý, M.: Construction of lower and upper functions and their application to regular and singular boundary value problems . *Nonlinear Analysis, Theory, Methods and Applications*, **47**, 3937 - 3948 (2001)

[18] Rachůnková, I., Tvrdý, M., Vrkoč, I.: Existence nonnegative and nonpositive solutions for second order periodic boundary value problems. *J. Differential Equations*, **176**, 445 - 469 (2001)

[19] Wang, J., Gao, W., Lin, Z.: Boundary value problems for general second order equations and similarity solutions to the Rayleigh problem. *Tôhoku Math. J.*, **47**, 327 - 344 (1995)

[20] Zhang, M.: A relationship between the periodic and the Dirichlet BVP's of singular differential equations. *Proc. Royal Soc. Edinburgh*, **128A**, 1099-1114 (1998)