Two-point higher order BVPs with singularities in phase variables *

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Abstract: The existence of solutions for singular higher order differential equations with the Lidstone or the (n,p) boundary conditions is proved. The right-hand sides of differential equations can have singularities in the zero value of their phase variables and so higher derivatives of solutions changing their signs can pass through these singularities. Proofs are based on the method of a priori estimates, the degree theory arguments and on the Vitali's convergence theorem.
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1 Introduction

Let T be a positive constant, $J=[0,T]$ and $\mathbb{R}_-=(-\infty,0)$, $\mathbb{R}_+=(0,\infty)$, $\mathbb{R}_0=\mathbb{R}\setminus\{0\}$.

We will consider two types of singular boundary value problems for higher order differential equations. The first one is the singular Lidstone boundary value problem (BVP for short)

$$(-1)^n x^{(2n)}(t)=f(t,x(t),\dots,x^{(2n-2)}(t)), \tag{1.1}$$

$$x^{(2j)}(0)=x^{(2j)}(T)=0, \quad 0\le j\le n-1 \tag{1.2}$$

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where $n \geq 1$ and f satisfies the local Carathéodory conditions on $J \times D$ ($f \in Car(J \times D)$) with

$$D = \begin{cases} \underbrace{\mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 \times \cdots \times \mathbb{R}_+}_{4k-3} & \text{if } n = 2k-1 \\ \underbrace{\mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 \times \cdots \times \mathbb{R}_-}_{4k-1} & \text{if } n = 2k \end{cases}$$

(for $n = 1, 2$ and 3, we have $D = \mathbb{R}_+$, $D = \mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_-$ and $D = \mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 \times \mathbb{R}_+$, respectively). In our considerations the function $f(t, x_0, \ldots, x_{2n-2})$ may be singular at the points $x_i = 0$, $0 \leq i \leq 2n-2$, of the phase variables $x_0, \ldots, x_{2n-2}$.

The second one is the singular (n, p) boundary value problem

$$-x^{(n)}(t) = f(t, x(t), \ldots, x^{(n-1)}(t)), \tag{1.3}$$

$$x^{(i)}(0) = 0, \ 0 \leq i \leq n-2, \ x^{(p)}(T) = 0, \ p \ \text{ fixed}, \ 0 \leq p \leq n-1, \tag{1.4}$$

where $n \geq 2$ and $f \in Car(J \times X)$ with

$$X = \mathbb{R}_+ \times \mathbb{R}_0^{n-2} \times \mathbb{R}.$$

In this case the function $f(t, x_0, \ldots, x_{n-1})$ may be singular at the points $x_i = 0$, $0 \leq i \leq n-2$ of the phase variables $x_0, \ldots, x_{n-2}$.

We will prove the existence of solutions to problems (1.1), (1.2) and (1.3), (1.4).

Definition 1.1. A function $x \in AC^{2n-1}(J)$ (i.e., x has absolutely continuous the $(2n-1)^{\text{st}}$ derivative on J) is said to be a *solution of BVP* (1.1), (1.2) if $(-1)^j x^{(2j)}(t) > 0$ for $t \in (0, T)$ and $0 \leq j \leq n-1$, x satisfies the boundary conditions (1.2) and (1.1) holds a.e. on J.

Definition 1.2. By *a solution of BVP* (1.3), (1.4) we understand a function $x \in AC^{n-1}(J)$ which is positive on $(0, T)$, satisfies conditions (1.4) and for a.e. $t \in J$ fulfils (1.3).

From now on, $\|x\| = \max\{|x(t)| : 0 \leq t \leq T\}$, $\|x\|_L = \int_0^T |x(t)|\, dt$ and $\|x\|_\infty = \operatorname{ess\,max}\{|x(t)| : 0 \leq t \leq T\}$ stands for the norm in $C^0(J)$, $L_1(J)$ and $L_\infty(J)$, respectively. For a subset Ω of a Banach space, $cl(\Omega)$ and $\partial\Omega$ stands for the closure and the boundary of Ω, respectively. Finally, for any measurable set $\mathcal{M}$, $\mu(\mathcal{M})$ denotes the Lebesgue measure of $\mathcal{M}$.

The fact that a BVP is singular means that the right hand side f of the differential equation does not fulfil the Carathéodory conditions on the region where we seek for solutions, i.e. on $J \times cl(D)$ if we work with equation (1.1) or on $J \times cl(X)$ if we study equation (1.3). In singular problems the Carathéodory conditions can be broken in the time variable t or in the phase variables or in the both types of variables. The first type of singularities where f need not be integrable on J for fixed phase variables was studied by many authors. For BVPs of the n-th order differential equations such problems were considered for the first time by Kiguradze in [18]. The second type of singularities where f need not be continuous in its phase variables $x_0, x_1, \ldots$ for fixed $t \in J$ was mainly solved for BVPs of the second order differential equations. One of the first papers concerning the second order Dirichlet BVP with a singularity at $x = 0$ of the right hand side of the differential equation $x''(t) = f(t, x(t))$ was written by Taliaferro in [26], where necessary and sufficient conditions for the existence of a concave solution $x > 0$ on $(0, 1)$ were found. Then a lot of papers extending or generalizing Taliaferro's result appeared. Let us mention [19] by Lomtatidze and Torres and [5] by Agarwal and O'Regan dealing with sign-changing right hand sides f of singular second order equations and proving the existence of a solution which is nonconcave and positive on $(0, 1)$. The existence of nonconcave and sign-changing solutions of the above problem was proved by the authors in [25].

Problems $(1.1), (1.2)$ and $(1.3), (1.4)$ have received a lot of attention in the literature. For $n = 1$, the Lidstone boundary conditions (1.2) are equal to the Dirichlet conditions and conditions (1.4) with $n = 2$ contain the Dirichlet ones as the special case $p = 0$. The Lidstone BVP (with a general n) was studied in the regular case e.g. by Agarwal and Wong [1], [7], [31] and for the singular (n, p) BVP with a special case of the right hand side f in (1.3) we can refer to the papers [6], [30] by Agarwal, O'Regan, Lakshmikantham and Wong.

In this paper we extend the citied results on the case of a general Carathéodory right-hand side f which may depend on higher derivatives up to the order $2n-2$ in (1.1) and the order $n-1$ in (1.3). Let us note that conditions (1.2) imply that odd derivatives of any solution of $(1.1), (1.2)$ are sign-changing functions on J. Similarly, if x is a solution of $(1.3), (1.4)$ with $0 \le p \le n-2$, then $x^{(i)}$ changes its sign inside of J for $p+1 \le i \le n-1$.

So, the main common feature of problems $(1.1), (1.2)$ and $(1.3), (1.4)$ is the fact that some derivatives of solutions go through singularities of f somewhere inside of J. This is the substantional difference of our problems $(1.1), (1.2)$ and $(1.3), (1.4)$ from all the problems citied above. As we know, such situation has not been considered, yet.

The following assumptions [1] will be used in the study of problem $(1.1), (1.2)$:

(H_1) $f \in Car(J \times D)$ and there exists $\psi \in L_1(J)$ such that

$$0 < \psi(t) \le f(t, x_0, \ldots, x_{2n-2})$$

for a.e. $t \in J$ and each $(x_0, \ldots, x_{2n-2}) \in D$;

[1] Throughout the paper, we set $\sum_{i=0}^{n-2} = 0$ if $n = 1$.

(H_2) For a.e. $t \in J$ and for each $(x_0, \dots, x_{2n-2}) \in D$,

$$f(t, x_0, \dots, x_{2n-2}) \le \phi(t) + \sum_{j=0}^{2n-2} q_j(t)\omega_j(|x_j|) + \sum_{j=0}^{2n-2} h_j(t)|x_j|$$

where ϕ, $h_j \in L_1(J)$ and $q_j \in L_\infty(J)$ are nonnegative, $\omega_j : \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing,

$$\begin{aligned} S = \sum_{i=0}^{n-1} \frac{T^{2(n-i)-3}}{6^{n-i-1}} \int_0^T t(T-t)h_{2i}(t)\,dt \\ + \sum_{i=0}^{n-2} \frac{T^{2(n-i-2)}}{6^{n-i-2}} \int_0^T t(T-t)h_{2i+1}(t)\,dt < 1 \end{aligned} \tag{1.5}$$

and

$$\int_0^T \omega_j(s)\,ds < \infty, \quad \omega_j(uv) \le \Lambda\omega_j(u)\omega_j(v) \tag{1.6}$$

for $0 \le j \le 2n-2$ and u, $v \in \mathbb{R}_+$ with a positive constant Λ.

In the study of problem (1.3), (1.4) we will work with assumptions:

(H_3) $f \in Car(J \times X)$ and there exist positive $\psi \in L_1(J)$ and $K \in \mathbb{R}_+$ such that

$$\psi(t) \le f(t, x_0, \dots, x_{n-1})$$

for a.e. $t \in J$ and each $(x_0, \dots, x_{n-1}) \in (0, K] \times \mathbb{R}_0^{n-2} \times \mathbb{R} \subset X$;

(H_4) For a.e. $t \in J$ and for each $(x_0, \dots, x_{n-1}) \in X$,

$$0 < f(t, x_0, \dots, x_{n-1}) \le \phi(t) + \sum_{i=0}^{n-2} q_i(t)\omega_i(|x_i|) + \sum_{j=0}^{n-1} h_j(t)|x_j|,$$

where ϕ, $h_k \in L_1(J)$ and $q_i \in L_\infty(J)$ are nonnegative, $\omega_i : \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing,

$$H = \sum_{j=0}^{n-1} \frac{1}{(n-j-1)!} \int_0^T h_j(s)s^{n-j-1}ds < 1 \tag{1.7}$$

and

$$\int_0^T \omega_i(s^{n-i-1})ds < \infty \tag{1.8}$$

for $0 \le i \le n-2$, $0 \le k \le n-1$.

Remark 1.3. Since $\omega_j : \mathbb{R}_+ \to \mathbb{R}_+$ in (H_2) are nonincreasing, the assumption $\int_0^T \omega_j(s)ds < \infty$ implies that $\int_0^V \omega_j(s)ds < \infty$ for each $V \in \mathbb{R}_+$, $0 \le j \le 2n-2$. The same is true for integrals in (1.8) and $0 \le i \le n-2$.

The paper is organized as follows. Section 2 presents properties of the Green's function $G_j(t,s)$ for the problem $x^{(2j)} = 0$, $x^{(2i)}(0) = x^{(2i)}(T) = 0$, $0 \le i \le j-1$ and of the Green's function $G(t,s)$ for the problem $-x^{(n)} = 0$, (1.4), which are necessary for our next considerations. Section 3 deals with auxiliary regular BVPs to problems (1.1), (1.2) and (1.3), (1.4). We give a priori bounds for their solutions and prove their existence by the theory of homotopy and the topological degree. In addition, we prove that some sets of functions containing solutions of our auxiliary regular BVPs are uniformly absolutely continuous on J. The main results about the existence of solutions to BVPs (1.1), (1.2) and (1.3), (1.4) are given in Section 4. Proofs are based on the Arzelà-Ascoli theorem and the Vitali's convergence theorem, see e.g. [8], [21].

2 Green's functions and a priori estimates

2.1 Problem (1.1), (1.2)

Given $j \in \mathbb{N}$. From now on, $G_j(t,s)$ denotes the Green's function of BVP

$$x^{(2j)}(t) = 0,$$

$$x^{(2i)}(0) = x^{(2i)}(T) = 0, \quad 0 \le i \le j-1.$$

Then

$$G_1(t,s) = \begin{cases} \dfrac{s}{T}(t-T) & \text{for } 0 \le s \le t \le T \\ \dfrac{t}{T}(s-T) & \text{for } 0 \le t < s \le T. \end{cases} \tag{2.1}$$

The Green's function $G_j(t,s)$ can be expressed as ([1], [7], [28])

$$G_j(t,s) = \int_0^T G_1(t,u)G_{j-1}(u,s)\,du, \quad j > 1 \tag{2.2}$$

and it is known that ([7], [28])

$$(-1)^j G_j(t,s) > 0 \quad \text{for } (t,s) \in (0,T) \times (0,T). \tag{2.3}$$

Lemma 2.1. *For $(t,s) \in J \times J$ and $j \in \mathbb{N}$, we have*

$$(-1)^j G_j(t,s) \le \frac{T^{2j-3}}{6^{j-1}} s(T-s). \tag{2.4}$$

Proof. For $(t,s) \in J \times J$, we see from (2.1) that

$$|G_1(t,s)| \leq \frac{s}{T}(T-s) \tag{2.5}$$

i.e. (2.4) is true for $j=1$. Assume that (2.4) holds for $j = i\,(\geq 1)$. Then it follows from (2.2)-(2.5) that

$$\begin{aligned}
|G_{i+1}(t,s)| &= \int_0^T |G_1(t,u)||G_i(u,s)|\,du \\
\leq \frac{T^{2i-4}}{6^{i-1}} s(T-s) &\int_0^T u(T-u)\,du = \frac{T^{2i-1}}{6^i} s(T-s)
\end{aligned}$$

for $(t,s) \in J \times J$. Thus (2.4) is true for $j = i+1$. □

Remark 2.2. If $T = 1$, Lemma 2.1 gives the result proved in [31].

Lemma 2.3. *For $(t,s) \in J \times J$ and $j \in \mathbb{N}$, we have*

$$|G_j(t,s)| \geq \frac{T^{2j-5}}{30^{j-1}} st(T-t)(T-s). \tag{2.6}$$

Proof. We see that

$$|G_1(t,s)| = \begin{cases} \frac{s}{T}(T-t) \geq \frac{st(T-t)(T-s)}{T^3} & \text{for } 0 \leq s \leq t \leq T \\ \frac{t}{T}(T-s) \geq \frac{st(T-t)(T-s)}{T^3} & \text{for } 0 \leq t < s \leq T, \end{cases} \tag{2.7}$$

and so (2.6) holds for $j=1$. Assume that (2.6) is true for $j = i\,(\geq 1)$. Then (2.2), (2.3), (2.6) and (2.7) give

$$\begin{aligned}
|G_{i+1}(t,s)| &= \int_0^T |G_1(t,u)||G_i(u,s)|\,du \\
\geq \frac{T^{2i-8}}{30^{i-1}} st(T-t)(T-s) &\int_0^T u^2(T-u)^2\,du = \frac{T^{2i-3}}{30^i} st(T-t)(T-s)
\end{aligned}$$

for $(t,s) \in J \times J$ and so (2.6) is valid for $j = i+1$. □

Lemma 2.4. *Let $x \in AC^{2n-1}(J)$ satisfies (1.2) and $(-1)^n x^{(2n)}(t) \geq \psi(t)$ for a.e. $t \in J$ with ψ given by (H_1). Set*

$$\Omega = \int_0^T s(T-s)\psi(s)\,ds. \tag{2.8}$$

Then

(*a*) $(-1)^j x^{(2j)}(t) > 0$ *for* $t \in (0,T)$, $0 \leq j \leq n-1$, (*b*) $(-1)^j x^{(2j+1)}$ *is decreasing on* J *and vanishes at a unique point* $\xi_j \in (0,T)$ *for* $0 \le j \le n-1$.

In addition,

$$|x^{(2j)}(t)| \ge \frac{T^{2(n-j)-5}}{30^{n-j-1}}\Omega t(T-t) \quad \text{for } t \in J,\ 0 \le j \le n-1 \tag{2.9}$$

and if $n > 1$ *then*

$$|x^{(2j+1)}(t)| \ge \frac{T^{2(n-j)-7}}{30^{n-j-2}}\Omega\Big|\int_{\xi_j}^{t} s(T-s)\,ds\Big| \quad \text{for } t \in J,\ 0 \le j \le n-2.\ ^{2} \tag{2.10}$$

Proof. From the equalities

$$(-1)^j x^{(2j)}(t) = \int_0^T (-1)^{n-j} G_{n-j}(t,s)(-1)^n x^{(2n)}(s)\,ds, \quad 0 \le j \le n-1,$$

(2.3) and $(-1)^n x^{(2n)}(t) \ge \psi(t) > 0$ for a.e. $t \in J$ we deduce (a). Then $(-1)^j x^{(2j+1)}$ is decreasing on J for $0 \le j \le n-2$ and $(-1)^n x^{(2n)} > 0$ a.e. on J implies $(-1)^{n-1} x^{(2n-1)}$ is decreasing on J. Now from (1.2) we deduce that $x^{(2j+1)}(\xi_j) = 0$ for a unique $\xi_j \in (0,T)$ with $0 \le j \le n-1$ which finishes the proof of (b).

Further, from the inequalities

$$|x^{(2j)}(t)| \ge \int_0^T |G_{n-j}(t,s)|\psi(s)\,ds, \quad t \in J,\ 0 \le j \le n-1$$

and (2.6) it follows that

$$|x^{(2j)}(t)| \ge \frac{T^{2(n-j)-5}}{30^{n-j-1}} t(T-t) \int_0^T s(T-s)\psi(s)\,ds$$

for $t \in J$, $0 \le j \le n-1$, and so (2.9) holds.

Finally, let $0 \le j \le n-2$. Then $x^{(2j+1)}(t) = \int_{\xi_j}^t x^{(2j+2)}(s)\,ds$ and, by (2.9),

$$|x^{(2j+1)}(t)| \ge \frac{T^{2(n-j)-7}}{30^{n-j-2}}\Omega\Big|\int_{\xi_j}^{t} s(T-s)\,ds\Big|, \quad t \in J$$

which proves (2.10). □

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[2] If some statements depend on j with $0 \le j \le n-2$, then $n > 1$ is assumed throughout the paper.

2.2 Problem (1.3), (1.4)

Let us consider the equation

$$-x^{(n)}(t) = 0. \tag{2.11}$$

Lemma 2.5. *The Green's function of problem* (2.11), (1.4) *has the form*

$$G(t,s) = \frac{1}{(n-1)!}\begin{cases} \left(t^{n-1}\left(\frac{T-s}{T}\right)^{n-p-1} - (t-s)^{n-1}\right) & \text{for } 0 \le s \le t \le T \\ t^{n-1}\left(\frac{T-s}{T}\right)^{n-p-1} & \text{for } 0 \le t < s \le T. \end{cases}$$

Proof. See e.g. [1]. □

Lemma 2.6. *The Green's function of problem* (2.11), (1.4) *fulfils*

$$G(T,s) > 0 \text{ for } s \in (0,T) \text{ and for } p > 0, \tag{2.12}$$

$$\frac{\partial^i G(t,s)}{\partial t^i} > 0 \quad \text{for } (t,s) \in (0,T)\times(0,T),\ 0 \le i \le \min\{p, n-2\}. \tag{2.13}$$

Proof. Condition (2.12) follows from the inequality

$$\left(1 - \frac{s}{T}\right)^{n-p-1} > \left(1 - \frac{s}{T}\right)^{n-1}$$

which is true for $s \in (0,T)$ and for $p > 0$. Let us suppose $0 \le i \le \min\{p, n-2\}$ and prove (2.13). We have

$$\frac{\partial^i G(t,s)}{\partial t^i} = \frac{1}{(n-i-1)!}\begin{cases} \left(t^{n-i-1}\left(\frac{T-s}{T}\right)^{n-p-1} - (t-s)^{n-i-1}\right) & \text{for } 0 \le s \le t \le T \\ t^{n-i-1}\left(\frac{T-s}{T}\right)^{n-p-1} & \text{for } 0 \le t < s \le T, \end{cases}$$

and therefore it is sufficient to show that

$$\left(\frac{T-s}{T}\right)^{n-p-1} > \left(\frac{t-s}{t}\right)^{n-i-1} \quad \text{for } \ 0 < s \le t < T. \tag{2.14}$$

Since inequalities

$$\left(1 - \frac{s}{T}\right)^{n-p-1} > \left(1 - \frac{s}{t}\right)^{n-p-1} \ge \left(1 - \frac{s}{t}\right)^{n-i-1}$$

are valid for $0 < s \le t < T$, condition (2.14) is true, as well. □

Lemma 2.7. *Let $x \in AC^{(n-1)}(J)$ satisfy* (1.4) *and let*

$$-x^{(n)}(t) > 0 \quad \text{for a.e. } \ t \in J. \tag{2.15}$$

Then, if $p > 0$, we have

$$x^{(i)}(t) > 0 \text{ on } (0,T],\ 0 \le i \le p-1,\ x^{(p)}(t) > 0 \text{ on } (0,T). \tag{2.16}$$

If $p = 0$, we have

$$x(t) > 0 \quad \text{for} \quad t \in (0,T). \tag{2.17}$$

Proof. We will consider two cases, namely (i) $p = n-1$ and (ii) $0 \le p \le n-2$.

Case (i). Let $p = n-1$. Then, by (1.4) and (2.15), we have

$$0 < -\int_t^T x^{(n)}(s)ds = x^{(n-1)}(t) \quad \text{for} \quad t \in [0,T). \tag{2.18}$$

Thus, integrating (2.18) from 0 to $t \in (0,T]$ and using (1.4), we get step by step

$$x^{(i)}(t) > 0 \quad \text{for} \quad t \in (0,T],\ 0 \le i \le n-2. \tag{2.19}$$

Inequalities (2.18) and (2.19) give the assertion of Lemma 2.7.

Case (ii). Suppose that $0 \le p \le n-2$. Then, using the formula

$$x(t) = -\int_0^T G(t,s)x^{(n)}(s)ds, \tag{2.20}$$

we can see that the assertion of Lemma 2.7 follows from (2.15) and Lemma 2.6.
□

In the study of problems having singularities in zero values of phase variables it is necessary to find a priori estimates of solutions below. The following three lemmas give a priori estimates below for functions satisfying conditions (1.4) and (2.15). We consider the cases $p = n-1$, $p = 0$ and $1 \le p \le n-2$ separately.

Lemma 2.8. *Let $p = n-1$ and let $x \in AC^{n-1}(J)$ satisfy* (1.4), (2.15). *Then the inequalities*

$$x^{(i)}(t) \ge \frac{\|x\|}{T^{n-1}} t^{n-i-1} \quad \text{for} \quad t \in J, \tag{2.21}$$

$0 \le i \le n-2$, are fulfilled.

Proof. Put

$$p_0(t) = \|x\| \left(\frac{t}{T}\right)^{n-1} \quad \text{for } t \in J. \tag{2.22}$$

Then $p_0(0) = \ldots = p_0^{(n-2)}(0) = 0$, $p_0(T) = \|x\|$. According to (2.16) we have $\|x\| = x(T)$. So, if $h(t) = x(t) - p_0(t)$ for $t \in J$, then h satisfies the boundary conditions $h(0) = \ldots = h^{n-2}(0) = 0$, $h(T) = 0$ and moreover $h^{(n)}(t) = x^{(n)}(t) - p_0^{(n)}(t) = x^{(n)}(t) < 0$ for a.e. $t \in J$. Therefore Lemma 2.7 (with h instead of x) gives $h(t) > 0$ for $t \in (0,T)$, i.e.

$$x(t) \ge p_0(t) \quad \text{for} \quad t \in J. \tag{2.23}$$

Further, put

$$p_1(t) = \|x'\| \left(\frac{t}{T}\right)^{n-2} \quad \text{for} \quad t \in J. \tag{2.24}$$

Then $p_1(0) = \ldots = p_1^{(n-3)}(0) = 0$, $p_1(T) = \|x'\|$. Since $\|x'\| = x'(T)$, the function $h = x' - p_1$ satisfies $h(0) = \ldots = h^{(n-3)}(0) = 0$, $h(T) = 0$, and moreover $h^{(n-1)} = x^{(n)} - p_1^{(n-1)} = x^{(n)} < 0$ a.e. on J. So, by Lemma 2.7, where we use h and $n-1$ instead of x and n, respectively, we have $h > 0$ on $(0,T)$, i.e.

$$x'(t) \geq p_1(t) \quad \text{for} \quad t \in J. \tag{2.25}$$

Similarly, for $2 \leq i \leq n-2$, we put $p_i(t) = \|x^{(i)}\| \left(\frac{t}{T}\right)^{n-i-1}$ and $h(t) = x^{(i)}(t) - p_i(t)$ for $t \in J$. Using Lemma 2.7 (with h and $n-i$ instead of x and n, resp.), we get $h > 0$ on $(0,T)$ and so

$$x^{(i)}(t) \geq p_i(t) \quad \text{for} \quad t \in J, \ 2 \leq i \leq n-2. \tag{2.26}$$

Having (2.22) - (2.26) together with the inequalities

$$\|x^{(i)}\| \geq \frac{\|x\|}{T^i}, \ 1 \leq i \leq n-2, \tag{2.27}$$

we obtain (2.21) for $0 \leq i \leq n-2$. □

Lemma 2.9. *Let $p = 0$ and let $x \in AC^{n-1}(J)$ satisfy* (1.4), (2.15). *Then we have on J for $0 \leq i \leq n-2$*

$$x^{(i)}(t) \geq \begin{cases} \dfrac{\|x\|}{T^{n-1}} t^{n-i-1} & \textit{for} \ \ 0 \leq t \leq \xi_{i+1} \\[2mm] \dfrac{\|x\|}{T^{i+1}} (\xi_i - t) & \textit{for} \ \ \xi_{i+1} \leq t \leq \xi_i, \end{cases} \tag{2.28}$$

$$x^{(i)}(t) \leq \frac{\|x\|}{T^{i+1}} (\xi_i - t) \ \ \textit{for} \ \ \xi_i \leq t \leq T,$$

with

$$\begin{cases} 0 < \xi_{n-1} < \xi_{n-2} < \ldots < \xi_2 < \xi_1 < \xi_0 = T, \quad \textit{where} \\[2mm] \xi_j \ \ \textit{is a unique zero of} \ \ x^{(j)} \ \ \textit{in} \ \ (0,T), \ 1 \leq j \leq n-1. \end{cases} \tag{2.29}$$

Proof. In view of (1.4) and (2.17) we have $x(0) = x(T) = 0$, $x(t) > 0$ for $t \in (0,T)$. Further, there is a unique $\xi_1 \in (0,T)$ such that $x'(\xi_1) = 0$. (Otherwise we get a contradiction to (2.15).) Similarly, in $(0,T)$ there is a unique $\xi_i < \xi_{i-1}$ such that $x^{(i)}(\xi_i) = 0$, $2 \leq i \leq n-1$. According to (2.15) we get

$$x^{(i)}(t) > 0 \text{ on } (0, \xi_i), \ x^{(i)}(t) < 0 \text{ on } (\xi_i, T], \ 1 \leq i \leq n-1. \tag{2.30}$$

Therefore

$$x^{(i)} \text{ is concave on } [\xi_{i+2}, T] \text{ and convex on } [0, \xi_{i+2}],\ 0 \le i \le n-2, \tag{2.31}$$

where $\xi_n = 0$. Let us prove (2.28) for $i = 0$. Put

$$p_0(t) = \|x\| \left(\frac{t}{\xi_1}\right)^{n-1} \text{ for } t \in [0, \xi_1].$$

Then $p_0(0) = \ldots = p_0^{(n-2)}(0) = 0$, $p_0(\xi_1) = \|x\|$. Since $\|x\| = x(\xi_1)$, the function $h = x - p_0$ fulfils boundary conditions $h(0) = \ldots = h^{(n-2)}(0) = 0$, $h(\xi_1) = 0$, and $h^{(n)}(t) < 0$ for a.e. $t \in (0, \xi_1)$. Therefore, by Lemma 2.7 (where we use h and ξ_1 instead of x and T, respectively), we deduce that the inequality $h > 0$ holds on $(0, \xi_1)$, which gives

$$x(t) \ge \frac{\|x\|}{T^{n-1}} t^{n-1} \text{ for } t \in [0, \xi_1]. \tag{2.32}$$

By (2.31), x is concave on $[\xi_1, T] \subset [\xi_2, T]$. Thus $x(t) \ge x(\xi_1)\frac{T-t}{T-\xi_1}$ on $[\xi_1, T]$, and so

$$x(t) \ge \frac{\|x\|}{T}(T - t) \text{ for } t \in [\xi_1, T]. \tag{2.33}$$

Estimates (2.32) and (2.33) lead to (2.28) for $i = 0$.

For $1 \le i \le n-2$, we put on $[0, \xi_{i+1}]$

$$p_i(t) = x^{(i)}(\xi_{i+1}) \left(\frac{t}{\xi_{i+1}}\right)^{n-i-1} \quad \text{and} \quad h(t) = x^{(i)}(t) - p_i(t).$$

Since

$$x^{(i)}(\xi_{i+1}) \ge \frac{\|x\|}{T^i},\ 1 \le i \le n-2, \tag{2.34}$$

we get as before

$$x^{(i)}(t) \ge \frac{\|x\|}{T^{n-1}} t^{n-i-1} \quad \text{for} \quad t \in [0, \xi_{i+1}]. \tag{2.35}$$

Further, using (2.31), we see that $x^{(i)}$ is concave on $[\xi_{i+1}, T] \subset [\xi_{i+2}, T]$. Thus we get the following two inequalities

$$\begin{aligned} x^{(i)}(t) &\ge x^{(i)}(\xi_{i+1}) \frac{\xi_i - t}{\xi_i - \xi_{i+1}} \ge 0 \quad \text{for} \quad t \in [\xi_{i+1}, \xi_i], \\ x^{(i)}(t) &\le x^{(i)}(\xi_{i+1}) \frac{\xi_i - t}{\xi_i - \xi_{i+1}} \le 0 \quad \text{for} \quad t \in [\xi_i, T]. \end{aligned} \tag{2.36}$$

According to (2.34) the above inequalities yield

$$|x^{(i)}(t)| \ge \frac{\|x\|}{T^{i+1}} |\xi_i - t| \quad \text{for} \quad t \in [\xi_{i+1}, T]. \tag{2.37}$$

Estimates (2.36) and (2.37) imply (2.28) for $1 \le i \le n-2$. $\square$

Lemma 2.10. *Let $1 \le p \le n-2$ and let $x \in AC^{n-1}(J)$ satisfy* (1.4), (2.15). *Then, for $0 \le i \le p-1$, inequality* (2.21) *is true and for $p \le i \le n-2$, conditions* (2.28) *are valid on J, with $0 < \xi_{n-1} < \xi_{n-2} < \ldots < \xi_{p+1} < \xi_p = T$, where ξ_j is a unique zero of $x^{(j)}$ in $(0,T)$, $p+1 \le j \le n-1$.*
Proof. For $0 \le i \le p-1$ we use arguments of the proof of Lemma 2.8 and for $p \le i \le n-2$ we argue as in the proof of Lemma 2.9. $\square$

Lemma 2.11. *Let $\psi \in L_1(J)$ be positive. Then there is a positive constant $c = c(\psi)$ such that for each function $x \in AC^{n-1}(J)$ satisfying* (1.4) *and*

$$\psi(t) \le -x^{(n)}(t) \quad \textit{for a.e. } t \in J, \tag{2.38}$$

the estimate

$$\|x\| \ge c \tag{2.39}$$

holds.
Proof. Let G be the Green's function of problem (2.11), (1.4). There are two cases to consider, namely (i) $1 \le p \le n-1$ and (ii) $p = 0$.

Case (i). Let us suppose $1 \le p \le n-1$ and define a function

$$\Phi(t,s) = \frac{G(t,s)}{t^{n-1}} \quad \text{for} \quad (t,s) \in (0,T] \times (0,T].$$

In view of Lemma 2.6, Φ is continuous and positive on $(0,T] \times (0,T)$. Further, for any $s \in (0,T)$ we have

$$\frac{\partial^{n-1} G(t,s)}{\partial t^{n-1}}\Big|_{(t,s)=(0,s)} = \left(\frac{T-s}{T}\right)^{n-p-1} > 0.$$

Choose an arbitrary $s \in (0,T)$. Then

$$\lim_{t \to 0^+} \Phi(t,s) = \frac{1}{(n-1)!} \frac{\partial^{n-1} G(t,s)}{\partial t^{n-1}}\Big|_{(t,s)=(0,s)} > 0,$$

which means that for any $s \in (0,T)$ we can extend $\Phi(\cdot, s)$ at $t = 0$ as a continuous and positive function on J. Thus the function $F(t) = \int_0^T \Phi(t,s)\psi(s)ds$ is continuous and positive on J, as well. Therefore we can find $d > 0$ such that $F(t) \ge d$ on J. So,

$$x(t) = -\int_0^T G(t,s)x^{(n)}(s)ds \ge \int_0^T G(t,s)\psi(s)ds$$

$$= t^{n-1}\int_0^T \Phi(t,s)\psi(s)ds = t^{n-1}F(t) \ge t^{n-1}d \quad \text{for} \quad t \in J.$$

This implies $\|x\| = x(T) \ge T^{n-1}d = c$.

Case (ii). Let $p = 0$. Define a function

$$\Phi(t,s) = \frac{G(t,s)}{t^{n-1}(T-t)} \quad \text{for} \quad (t,s) \in (0,T) \times (0,T).$$

In view of Lemma 2.6, Φ is continuous and positive on $(0,T) \times (0,T)$. For any $s \in (0,T)$ we get

$$\lim_{t \to 0+} \Phi(t,s) = \frac{1}{T(n-1)!} \frac{\partial^{n-1} G(t,s)}{\partial t^{n-1}}\Big|_{(t,s)=(0,s)} = \frac{1}{T(n-1)!} \left(\frac{T-s}{T}\right)^{n-1} > 0,$$

and

$$\begin{aligned} \lim_{t \to T-} \Phi(t,s) &= \frac{-1}{T^{n-1}} \frac{\partial G(t,s)}{\partial t}\Big|_{(t,s)=(T,s)} \\ &= \frac{-1}{T(n-2)!} \left[\left(1 - \frac{s}{T}\right)^{n-1} - \left(1 - \frac{s}{T}\right)^{n-2} \right] > 0, \end{aligned}$$

which means that for any $s \in (0,T)$ we can extend $\Phi(\cdot, s)$ on J as a continuous and positive function. Further we can argue as in *Case* (i).

□

3 Auxiliary regular BVPs

3.1 Problem (1.1), (1.2)

For each $m \in \mathbb{N}$, define χ_m, $\varphi_m, \tau_m \in C^0(\mathbb{R})$, $\mathbb{R}_m \subset \mathbb{R}$ and $f_m \in Car(J \times \mathbb{R}^{2n-1})$ by the formulas

$$\chi_m(u) = \begin{cases} u & \text{for } u \geq \frac{1}{m} \\ \frac{1}{m} & \text{for } u < \frac{1}{m}, \end{cases}$$

$$\varphi_m(u) = \begin{cases} -\frac{1}{m} & \text{for } u > -\frac{1}{m} \\ u & \text{for } u \leq -\frac{1}{m}, \end{cases}$$

$$\tau_m = \begin{cases} \chi_m & \text{if } n = 2k-1 \\ \varphi_m & \text{if } n = 2k, \end{cases}$$

$$\mathbb{R}_m = (-\infty, -\frac{1}{m}) \cup (\frac{1}{m}, \infty)$$

and

$$f_m(t,x_0,x_1,x_2,x_3,\dots,x_{2n-2}) =$$

$$\begin{cases} f(t,\chi_m(x_0),x_1,\varphi_m(x_2),x_3,\dots,\tau_m(x_{2n-2})) \\ \quad \text{for } (t,x_0,x_1,x_2,x_3,\dots,x_{2n-2}) \in J\times\mathbb{R}\times\mathbb{R}_m\times\mathbb{R}\times\mathbb{R}_m\times\cdots\times\mathbb{R} \\ \frac{m}{2}[f_m(t,x_0,\frac{1}{m},x_2,x_3,\dots,x_{2n-2})(x_1+\frac{1}{m}) \\ -f_m(t,x_0,-\frac{1}{m},x_2,x_3,\dots,x_{2n-2})(x_1-\frac{1}{m})] \\ \quad \text{for } (t,x_0,x_1,x_2,x_3,\dots,x_{2n-2}) \in J\times\mathbb{R}\times[-\frac{1}{m},\frac{1}{m}]\times\mathbb{R}\times\mathbb{R}_m\times\cdots\times\mathbb{R} \\ \frac{m}{2}[f_m(t,x_0,x_1,x_2,\frac{1}{m},\dots,x_{2n-2})(x_3+\frac{1}{m}) \\ -f_m(t,x_0,x_1,x_2,-\frac{1}{m},\dots,x_{2n-2})(x_3-\frac{1}{m})] \\ \quad \text{for } (t,x_0,x_1,x_2,x_3,\dots,x_{2n-2}) \in J\times\mathbb{R}^3\times[-\frac{1}{m},\frac{1}{m}]\times\cdots\times\mathbb{R} \\ \quad \vdots \\ \frac{m}{2}[f_m(t,x_0,x_1,x_2,\dots,\frac{1}{m},x_{2n-2})(x_{2n-3}+\frac{1}{m}) \\ -f_m(t,x_0,x_1,x_2,\dots,-\frac{1}{m},x_{2n-2})(x_{2n-3}-\frac{1}{m})] \\ \quad \text{for } (t,x_0,x_1,x_2,\dots,x_{2n-3},x_{2n-2}) \in J\times\mathbb{R}^{2n-3}\times[-\frac{1}{m},\frac{1}{m}]\times\mathbb{R}. \end{cases}$$

Then

$$\begin{aligned} 0 < \psi(t) &\le f_m(t,x_0,\dots,x_{2n-2}) \\ &\le \phi(t) + \sum_{j=0}^{2n-2} q_j(t)\omega_j(\frac{1}{m}) + \sum_{j=0}^{2n-2} h_j(t)(\frac{1}{m}+|x_j|) \\ &\qquad \text{for a.e. } t\in J \text{ and each } (x_0,\dots,x_{2n-2}) \in \mathbb{R}^{2n-1} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} 0 < \psi(t) &\le f_m(t,x_0,\dots,x_{2n-2}) \\ &\le \phi(t) + \sum_{j=0}^{2n-2} q_j(t)\omega_j(|x_j|) + \sum_{j=0}^{2n-2} h_j(t)(1+|x_j|) \\ &\qquad \text{for a.e. } t\in J \text{ and each } (x_0,\dots,x_{2n-2}) \in \mathbb{R}_0^{2n-1} \end{aligned} \tag{3.2}$$

provided f satisfies assumptions (H_1) and (H_2).

Consider auxiliary regular differential equations

$$(-1)^n x^{(2n)}(t) = f_m(t,x(t),\dots,x^{(2n-2)}(t)) \tag{3.3}$$

and

$$(-1)^n x^{(2n)}(t) = \lambda f_m(t, x(t), \ldots, x^{(2n-2)}(t)), \quad \lambda \in [0,1] \tag{3.4}$$

depending on the parameter $m \in \mathbb{N}$.

Proposition 3.1. *Let $m \in \mathbb{N}$. If there exists a positive constant K such that*

$$\|x^{(j)}\| \le K, \qquad 0 \le j \le 2n-2 \tag{3.5}$$

for any solution x of BVPs (3.4), (1.2) *with $\lambda \in [0,1]$, then BVP* (3.3), (1.2) *has a solution x satisfying* (3.5).

Proof. Solving BVP (3.4), (1.2) is equivalent to finding $x \in C^{2n-2}(J)$ to

$$x(t) = (-1)^n \lambda \int_0^T G_n(t,s) f_m(s, x(s), \ldots, x^{(2n-2)}(s))\, ds. \tag{3.6}$$

It is easy to see that $\mathcal{S} : C^{2n-2}(J) \to C^{2n-2}(J)$,

$$(\mathcal{S}x)(t) = (-1)^n \int_0^T G_n(t,s) f_m(s, x(s), \ldots, x^{(2n-2)}(s))\, ds$$

is a completely continuous operator. Since we can rewrite (3.6) as

$$x = \lambda \mathcal{S} x, \quad \lambda \in [0,1] \tag{3.7}$$

and, by our assumption, (3.5) holds for any solution x of (3.7), there exists a solution x of the operator equation $x = \mathcal{S}x$ by [17]. Of course, x is a solution of BVP (3.3), (1.2) satisfying (3.5). $\square$

Lemma 3.2. *Let assumptions* (H_1) *and* (H_2) *be satisfied. Then for for each $m \in \mathbb{N}$ there exists a solution of BVP* (3.3), (1.2).

Proof. Fix $m \in \mathbb{N}$. By Proposition 3.1, it is sufficient to show that there exists a positive constant K such that (3.5) is satisfied for any solution x of BVPs (3.4), (1.2) with $\lambda \in [0,1]$. We see that $x = 0$ is the unique solution of BVP (3.4), (1.2) with $\lambda = 0$. Let $\lambda \in (0,1]$ and x be a solution of BVP (3.4), (1.2). By (3.1) and (2.4), for $t \in J$ and $0 \le j \le n-1$ we have

$$\begin{aligned}
|x^{(2j)}(t)| &= \lambda \int_0^T |G_{n-j}(t,s)| f_m(s, x(s), \ldots, x^{(2n-2)}(s))\, ds \\
\le \frac{T^{2(n-j)-3}}{6^{n-j-1}} &\int_0^T s(T-s)\Big(\phi(s) + \sum_{i=0}^{2n-2} q_i(s)\omega_i(\frac{1}{m}) + \sum_{i=0}^{2n-2} h_i(s)(\frac{1}{m} + |x^{(i)}(s)|)\Big)\, ds \\
&\le \frac{T^{2(n-j)-3}}{6^{n-j-1}} \Big(M + \sum_{i=0}^{2n-2} H_i \|x^{(i)}\|\Big)
\end{aligned}$$

where

$$H_i = \int_0^T s(T-s) h_i(s)\, ds, \quad 0 \le i \le 2n-2 \tag{3.8}$$

and

$$M=\int_0^T s(T-s)\phi(s)\,ds+\sum_{i=0}^{2n-2}\|q_i\|_\infty\omega_i(\frac{1}{m})\int_0^T s(T-s)\,ds+\frac{1}{m}\sum_{i=0}^{2n-2}H_i.$$

By (1.2), $x^{(2j+1)}(\xi_j)=0$ for some $\xi_j\in(0,T)$, $0\le j\le n-1$, and so the equalities $x^{(2j+1)}(t)=\int_{\xi_j}^t x^{(2j+2)}(s)\,ds$ for $t\in J$, $0\le j\le n-2$, imply

$$|x^{(2j+1)}(t)|\le\frac{T^{2(n-j-2)}}{6^{n-j-2}}\Big(M+\sum_{i=0}^{2n-2}H_i\|x^{(i)}\|\Big).$$

We have proved that

$$\|x^{(2j)}\|\le\frac{T^{2(n-j)-3}}{6^{n-j-1}}\Big(M+\sum_{i=0}^{2n-2}H_i\|x^{(i)}\|\Big),\quad 0\le j\le n-1,\tag{3.9}$$

$$\|x^{(2j+1)}\|\le\frac{T^{2(n-j-2)}}{6^{n-j-2}}\Big(M+\sum_{i=0}^{2n-2}H_i\|x^{(i)}\|\Big),\quad 0\le j\le n-2.\tag{3.10}$$

Assume that $\sum_{i=0}^{2n-2}H_i>0$ and set $A=\sum_{i=0}^{2n-2}H_i\|x^{(i)}\|$. Then (3.9) and (3.10) yield $A\le S(M+A)$, where $S<1$ is defined by (1.5). Then $A\le\frac{SM}{1-S}$ and so (see (3.9) and (3.10))

$$\|x^{(2j)}\|\le\frac{T^{2(n-j)-3}M}{6^{n-j-1}(1-S)},\quad 0\le j\le n-1,$$

$$\|x^{(2j+1)}\|\le\frac{T^{2(n-j-2)}M}{6^{n-j-2}(1-S)},\quad 0\le j\le n-2.$$

From (3.9) and (3.10) we see that the last two inequalities hold also in the case of $\sum_{i=0}^{2n-2}H_i=0$ where $S=0$. Consequently, there exists a positive constant K for which (3.5) holds. □

Lemma 3.3. *Let assumptions* (H_1) *and* (H_2) *be satisfied. Then there exists a positive constant* V *such that*

$$\|x^{(j)}\|\le V,\qquad 0\le j\le 2n-2\tag{3.11}$$

for any solution x *of BVP* (3.3), (1.2) *with* $m\in\mathbb{N}$.

Proof. Let $m\in\mathbb{N}$ and x be a solution of BVP (3.3), (1.2). Then inequalities (2.9) and (2.10) hold with Ω defined by (2.8) and where $\xi_j\in(0,T)$ is a zero of $x^{(2j+1)}$, $0\le j\le n-2$. Set

$$\mu_j=\int_0^T s(T-s)\omega_{2j}(s(T-s))\,ds,\quad 0\le j\le n-1.$$

From (2.9), (2.10) and the properties of ω_j we conclude for $0 \le j \le n-1$ that

$$\int_0^T s(T-s)\omega_{2j}(|x^{(2j)}(s)|)\,ds \le \int_0^T s(T-s)\omega_{2j}\Big(\frac{T^{2(n-j)-5}}{30^{n-j-1}}\Omega s(T-s)\Big)\,ds$$
$$\le \Lambda\omega_{2j}\Big(\frac{T^{2(n-j)-5}}{30^{n-j-1}}\Omega\Big)\mu_j$$

and for $0 \le j \le n-2$ that

$$\int_0^T s(T-s)\omega_{2j+1}(|x^{(2j+1)}(s)|)\,ds$$
$$\le \int_0^T s(T-s)\omega_{2j+1}\Big(\frac{T^{2(n-j)-7}}{30^{n-j-2}}\Omega\Big|\int_{\xi_j}^s u(T-u)\,du\Big|\Big)\,ds$$
$$\le 2\Lambda\omega_{2j+1}\Big(\frac{T^{2(n-j)-7}}{30^{n-j-2}}\Omega\Big)\int_0^{\frac{T^3}{6}}\omega_{2j+1}(u)\,du$$

since

$$\int_0^T s(T-s)\omega_{2j+1}\Big(\Big|\int_{\xi_j}^s u(T-u)\,du\Big|\Big)\,ds$$
$$= \int_0^{\xi_j} s(T-s)\omega_{2j+1}\Big(\int_s^{\xi_j} u(T-u)\,du\Big)\,ds$$
$$+ \int_{\xi_j}^T s(T-s)\omega_{2j+1}\Big(\int_{\xi_j}^s u(T-u)\,du\Big)\,ds$$
$$= \int_0^{\int_0^{\xi_j} u(T-u)\,du}\omega_{2j+1}(s)\,ds + \int_0^{\int_{\xi_j}^T u(T-u)\,du}\omega_{2j+1}(s)\,ds$$
$$\le 2\int_0^{\int_0^T u(T-u)\,du}\omega_{2j+1}(s)\,ds = 2\int_0^{\frac{T^3}{6}}\omega_{2j+1}(u)\,du.$$

Consequently, by (2.4) and (3.2),

$$|x^{(2j)}(t)| = \int_0^T |G_{n-j}(t,s)|f_m(s,x(s),\ldots,x^{(2n-2)}(s))\,ds$$
$$\le \frac{T^{2(n-j)-3}}{6^{n-j-1}}\int_0^T s(T-s)\Big(\phi(s) + \sum_{i=0}^{2n-2} q_i(s)\omega_i(|x^{(i)}(s)|)$$
$$+ \sum_{i=0}^{2n-2} h_i(s)(1+|x^{(i)}(s)|)\}\Big)\,ds$$
$$\le \frac{T^{2(n-j)-3}}{6^{n-j-1}}\Big(W + \sum_{i=0}^{2n-2} H_i\|x^{(i)}\|\Big)$$

for $t \in J$ and $0 \le j \le n-1$, where H_i is given by (3.8) and

$$W = \int_0^T s(T-s)\phi(s)\,ds + \Lambda\sum_{i=0}^{n-1}\|q_{2i}\|_\infty\omega_{2i}\Big(\frac{T^{2(n-i)-5}}{30^{n-i-1}}\Omega\Big)\mu_i$$
$$+2\Lambda\sum_{i=0}^{n-2}\|q_{2i+1}\|_\infty\omega_{2i+1}\Big(\frac{T^{2(n-i)-7}}{30^{n-i-2}}\Omega\Big)\int_0^{\frac{T^3}{6}}\omega_{2i+1}(u)\,du + \sum_{j=0}^{2n-2} H_j$$

is independent of $m \in \mathbb{N}$. Therefore

$$|x^{(2j+1)}(t)| = \Big| \int_{\xi_j}^{t} x^{(2j+2)}(s)\,ds \Big| \le \frac{T^{2(n-j-2)}}{6^{n-j-2}} \Big(W + \sum_{i=0}^{2n-2} H_i \|x^{(i)}\| \Big)$$

for $t \in J$ and $0 \le j \le n-2$. Hence

$$\|x^{(2j)}\| \le \frac{T^{2(n-j)-3}}{6^{n-j-1}} \Big(W + \sum_{i=0}^{2n-2} H_i \|x^{(i)}\| \Big), \quad 0 \le j \le n-1,$$

$$\|x^{(2j+1)}\| \le \frac{T^{2(n-j-2)}}{6^{n-j-2}} \Big(W + \sum_{i=0}^{2n-2} H_i \|x^{(i)}\| \Big), \quad 0 \le j \le n-2.$$

Now applying the same procedure as in the proof of Lemma 3.2, we get

$$\|x^{(2j)}\| \le \frac{T^{2(n-j)-3} W}{6^{n-j-1}(1-S)} \quad 0 \le j \le n-1, \tag{3.12}$$

$$\|x^{(2j+1)}\| \le \frac{T^{2(n-j-2)} W}{6^{n-j-2}(1-S)} \quad 0 \le j \le n-2 \tag{3.13}$$

where $S < 1$ is given in (H_2). From (3.12) and (3.13) we see that there exists a positive constant V independent of m such that (3.11) is true. □

Lemma 3.4. *Let assumptions (H_1) and (H_2) be satisfied. Let $\{x_m\}$ be a sequence of solutions to BVPs* (3.3), (1.2) *with $m \in \mathbb{N}$ and $\xi_{m,j}$ be a (unique) zero of $x_m^{(2j+1)}$ in $(0,T)$, $0 \le j \le n-2$. Then there exist $0 < \alpha < \beta < T$ independent of m such that*

$$\alpha \le \xi_{m,j} \le \beta \quad \textit{for } m \in \mathbb{N},\ 0 \le j \le n-2. \tag{3.14}$$

Proof. If not, there exist a subsequence $\{m_k\}$ of $\mathbb{N}$ and $\tau \in \{0,1,\dots,n-2\}$ such that either $\lim_{k\to\infty} \xi_{m_k,\tau} = 0$ or $\lim_{k\to\infty} \xi_{m_k,\tau} = T$. Suppose $\lim_{k\to\infty} \xi_{m_k,\tau} = 0$. By Lemma 2.4,

$$(-1)^\tau x_{m_k}^{(2\tau)}(t) \ge \frac{T^{2(n-\tau)-5}}{30^{n-\tau-1}} \Omega t(T-t), \quad t \in J,$$

and so

$$(-1)^\tau x_{m_k}^{(2\tau+1)}(0) = (-1)^\tau \lim_{t\to 0^+} \frac{x_{m_k}^{(2\tau)}(t)}{t} \ge \lim_{t\to 0^+} \frac{T^{2(n-\tau)-5}}{30^{n-\tau-1}} \Omega(T-t) = \frac{T^{2(n-\tau-2)}}{30^{n-\tau-1}} \Omega.$$

Therefore

$$\begin{aligned} \frac{T^{2(n-\tau-2)}}{30^{n-\tau-1}} \Omega &\le (-1)^\tau x_{m_k}^{(2\tau+1)}(0) \\ &= (-1)^\tau (x_{m_k}^{(2\tau+1)}(0) - x_{m_k}^{(2\tau+1)}(\xi_{m_k,\tau})) = (-1)^{\tau+1} x_{m_k}^{(2\tau+2)}(\nu_k)\xi_{m_k,\tau} \end{aligned}$$

where $\nu_k \in (0, \xi_{m_k,\tau})$. Hence $\lim_{k\to\infty} |x_{m_k}^{(2\tau+2)}(\nu_k)| = \infty$, contrary to $\|x_{m_k}^{(2\tau+2)}\| \le V$ for $k \in \mathbb{N}$ by (3.11). The case where $\lim_{k\to\infty} \xi_{m_k,\tau} = T$ can be treated analogously. $\square$

Lemma 3.5. *Let assumptions* (H_1) *and* (H_2) *be satisfied. Then there exists a positive constant* V_* *such that*

$$\|x^{(2n-1)}\| \le V_* \tag{3.15}$$

for any solution x *of BVP* (3.3), (1.2) *with* $m \in \mathbb{N}$.

Proof. Let $m \in \mathbb{N}$ and x be a solution of BVP (3.3), (1.2). Let ξ_j be a (unique) zero of $x^{(2j+1)}$ in $(0,T)$, $0 \le j \le n-1$ (see Lemma 2.4). By Lemma 3.4, there exist $0 < \alpha < \beta < T$ independent of m such that

$$\alpha \le \xi_j \le \beta, \qquad 0 \le j \le n-2. \tag{3.16}$$

By (3.2),

$$\begin{aligned}|x^{(2n-1)}(t)| &= \Big| \int_{\xi_{n-1}}^t f_m(s, x(s), \ldots, x^{(2n-2)}(s))\, ds\Big| \\ \le \int_0^T \Big(\phi(t) &+ \sum_{i=0}^{2n-2} q_i(t)\omega_i(|x^{(i)}(t)|) + \sum_{i=0}^{2n-2} h_i(t)(1 + |x^{(i)}(t)|\Big)\, dt.\end{aligned} \tag{3.17}$$

Using the properties od ω_i given in (H_2), (2.9) and the inequality

$$t(T-t) \ge \begin{cases} \frac{T}{2}t & \text{for } 0 \le t \le \frac{T}{2} \\ \frac{T}{2}(T-t) & \text{for } \frac{T}{2} < t \le T, \end{cases}$$

we get

$$\begin{aligned} &\int_0^T \omega_{2i}(|x^{(2i)}(t)|)\, dt \\ \le &\int_0^{\frac{T}{2}} \omega_{2i}\Big(\frac{T^{2(n-i-2)}}{2\cdot 30^{n-i-1}}\Omega t\Big)\, dt + \int_{\frac{T}{2}}^T \omega_{2i}\Big(\frac{T^{2(n-i-2)}}{2\cdot 30^{n-i-1}}\Omega(T-t)\Big)\, dt \\ &\le \Lambda\omega_{2i}\Big(\frac{T^{2(n-i-2)}}{2\cdot 30^{n-i-1}}\Omega\Big)\Big[\int_0^{\frac{T}{2}} \omega_{2i}(t)\, dt + \int_{\frac{T}{2}}^T \omega_{2i}(T-t)\, dt\Big] \\ &\qquad = 2\Lambda\omega_{2i}\Big(\frac{T^{2(n-i-2)}}{2\cdot 30^{n-i-1}}\Omega\Big)\int_0^{\frac{T}{2}} \omega_{2i}(t)\, dt \end{aligned} \tag{3.18}$$

for $0 \le i \le n-1$. Next we claim that

$$\Big|\int_{\xi_j}^t s(T-s)\, ds\Big| \ge \frac{\alpha(T-\beta)}{2}|t-\xi_j| \quad \text{for } t \in J,\ 0 \le j \le n-2. \tag{3.19}$$

Indeed, since ξ_j satisfies (3.16), we have

$$\begin{aligned}\int_t^{\xi_j} s(T-s)\,ds &\ge (T-\beta)\int_t^{\xi_j} s\,ds = \frac{T-\beta}{2}(\xi_j^2-t^2)\\ &= \frac{T-\beta}{2}(\xi_j+t)(\xi_j-t) \ge \frac{\alpha(T-\beta)}{2}(\xi_j-t)\end{aligned}$$

for $t\in[0,\xi_j]$ and

$$\begin{aligned}\int_{\xi_j}^t s(T-s)\,ds &\ge \alpha\int_{\xi_j}^t (T-s)\,ds = \frac{\alpha}{2}((T-\xi_j)^2-(T-t)^2)\\ &= \frac{\alpha}{2}(2T-\xi_j-t)(t-\xi_j) \ge \frac{\alpha(T-\beta)}{2}(t-\xi_j)\end{aligned}$$

for $t\in(\xi_j,T]$. Consequently, using (2.10) and (3.19) we obtain

$$\begin{aligned}&\int_0^T \omega_{2i+1}(|x^{(2i+1)}(t)|)\,dt\\ &\le \int_0^T \omega_{2i+1}\Big(\frac{T^{2(n-i)-7}}{30^{n-i-2}}\Omega\Big|\int_{\xi_i}^t s(T-s)\,ds\Big|\Big)\,dt\\ &\le \int_0^T \omega_{2i+1}\Big(\frac{\alpha(T-\beta)T^{2(n-i)-7}}{2\cdot 30^{n-i-2}}\Omega|t-\xi_i|\Big)\,dt\\ &\le \Lambda\omega_{2i+1}\Big(\frac{\alpha(T-\beta)T^{2(n-i)-7}}{2\cdot 30^{n-i-2}}\Omega\Big)\int_0^T \omega_{2i+1}(|t-\xi_i|)\,dt\\ &< 2\Lambda\omega_{2i+1}\Big(\frac{\alpha(T-\beta)T^{2(n-i)-7}}{2\cdot 30^{n-i-2}}\Omega\Big)\int_0^T \omega_{2i+1}(t)\,dt.\end{aligned} \tag{3.20}$$

Applying (3.11), (3.17), (3.18) and (3.20) yields

$$\begin{aligned}|x^{(2n-1)}(t)| \;\le\; &\|\phi\|_L + 2\Lambda\sum_{i=0}^{n-1}\|q_{2i}\|_\infty\omega_{2i}\Big(\frac{T^{2(n-i-2)}}{2\cdot 30^{n-i-1}}\Omega\Big)\int_0^{\frac{T}{2}}\omega_{2i}(t)\,dt\\ &+2\Lambda\sum_{i=0}^{n-2}\|q_{2i+1}\|_\infty\omega_{2i+1}\Big(\frac{\alpha(T-\beta)T^{2(n-i)-7}}{2\cdot 30^{n-i-2}}\Omega\Big)\int_0^T\omega_{2i+1}(t)\,dt\\ &+(1+V)\sum_{i=0}^{2n-2}\|h_i\|_L = V_*\end{aligned}$$

for $t\in J$. Here V_* is a positive constant independent of m. □

Lemma 3.6. *Let assumptions* (H_1) *and* (H_2) *be satisfied. Let* $\{x_m\}$ *be a sequence of solutions to BVPs* (3.3), (1.2), $m\in\mathbb{N}$. *Then the sequence*

$$\{f_m(t,x_m(t),\dots,x_m^{(2n-2)}(t))\}\subset L_1(J)$$

is uniformly absolutely continuous on J, that is for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{\mathcal{M}} f_m(t, x_m(t), \dots, x_m^{(2n-2)}(t))\, dt < \varepsilon$$

for any measurable $\mathcal{M} \subset J$, $\mu(\mathcal{M}) < \delta$.

Proof. With respect to (3.2) and properties of measurable sets, it is sufficient to verify that to every $\varepsilon > 0$ there exists $\delta > 0$ such that for any at most countable set $\{(a_j, b_j)\}_{j\in\mathbb{J}}$ of mutually disjoint intervals $(a_j, b_j) \subset J$ with $\sum_{j\in\mathbb{J}}(b_j - a_j) < \delta$, we have for each $m \in \mathbb{N}$

$$\sum_{j\in\mathbb{J}} \int_{a_j}^{b_j} \Big[\phi(t) + \sum_{i=0}^{2n-2} q_i(t)\omega_i(|x_m^{(i)}(t)|) + \sum_{i=0}^{2n-2} h_i(t)(1 + |x_m^{(i)}(t)|)\Big]\, dt < \varepsilon. \qquad (3.21)$$

By Lemmas 2.4, 3.3 and 3.4, we know that there exist $V > 0$ and $0 < \alpha < \beta < T$ such that

$$|x_m^{(2j)}(t)| \geq \frac{T^{2(n-j)-5}}{30^{n-j-1}} \Omega t(T - t), \quad t \in J,\ 0 \leq j \leq n-1,\ m \in \mathbb{N},$$

$$|x_m^{(2j+1)}(t)| \geq \frac{T^{2(n-j)-7}}{30^{n-j-2}} \Omega \Big|\int_{\xi_{m,j}}^{t} s(T - s)\, ds\Big|, \quad t \in J,\ 0 \leq j \leq n-2,\ m \in \mathbb{N}$$

where $\xi_{m,j} \in (0, T)$ is a (unique) zero of $x_m^{(2j+1)}$ in J,

$$\alpha \leq \xi_{m,j} \leq \beta, \quad m \in \mathbb{N},\ 0 \leq j \leq n-2$$

and

$$\|x_m^{(j)}\| \leq V, \quad m \in \mathbb{N},\ 0 \leq j \leq 2n-2.$$

In addition, by (3.19),

$$\Big|\int_{\xi_{m,j}}^{t} s(T - s)\, ds\Big| \geq \frac{\alpha(T - \beta)}{2}|t - \xi_{m,j}|, \quad m \in \mathbb{N},\ 0 \leq j \leq n-2.$$

Hence, for $0 \leq t_1 < t_2 \leq T$, we have

$$\sum_{i=0}^{n-1} \int_{t_1}^{t_2} q_{2i}(t)\omega_{2i}(|x_m^{(2i)}(t)|)\, dt \leq \Lambda \sum_{i=0}^{n-1} \|q_{2i}\|_\infty \omega_{2i}\Big(\frac{T^{2(n-i)-5}}{30^{n-i-1}}\Omega\Big) \int_{t_1}^{t_2} \omega_{2i}(t(T - t))\, dt,$$

$$\sum_{i=0}^{2n-2} \int_{t_1}^{t_2} h_i(t)(1 + |x_m^{(i)}(t)|)\, dt \leq (1 + V) \sum_{i=0}^{2n-2} \int_{t_1}^{t_2} h_i(t)\, dt,$$

and, for $0 \leq i \leq n-2$, we have

$$\begin{aligned} &\int_{t_1}^{t_2} q_{2i+1}(t)\omega_{2i+1}(|x_m^{(2i+1)}(t)|)\, dt \\ &\leq \|q_{2i+1}\|_\infty \int_{t_1}^{t_2} \omega_{2i+1}\Big(\frac{\alpha(T-\beta)T^{2(n-i)-7}}{2 \cdot 30^{n-i-2}}\Omega|t - \xi_{m,i}|\Big)\, dt \\ &\leq \Lambda\|q_{2i+1}\|_\infty \omega_{2i+1}\Big(\frac{\alpha(T-\beta)T^{2(n-i)-7}}{2 \cdot 30^{n-i-2}}\Omega\Big) \int_{t_1}^{t_2} \omega_{2i+1}(|t - \xi_{m,i}|)\, dt. \end{aligned}$$

Obviously,

$$\int_{t_1}^{t_2} \omega_{2i+1}(|t-\xi_{m,i}|)\,dt =$$

$$\begin{cases} \displaystyle\int_{t_1-\xi_{m,i}}^{t_2-\xi_{m,i}} \omega_{2i+1}(t)\,dt & \text{if } \xi_{m,i} \le t_1 \\ \displaystyle\int_0^{\xi_{m,i}-t_1} \omega_{2i+1}(t)\,dt + \int_0^{t_2-\xi_{m,i}} \omega_{2i+1}(t)\,dt & \text{if } t_1 < \xi_{m,i} < t_2 \\ \displaystyle\int_{\xi_{m,i}-t_2}^{\xi_{m,i}-t_1} \omega_{2i+1}(t)\,dt & \text{if } t_2 \le \xi_{m,i}. \end{cases} \tag{3.22}$$

Let now $\{(a_j,b_j)\}_{j\in\mathbb{J}}$ be at most countable set of mutually disjoint intervals $(a_j,b_j)\subset J$. Set

$$\mathbb{J}^1_{m,i} = \{j : j\in\mathbb{J},\, \xi_{m,i}\le a_j\},\quad \mathbb{J}^2_{m,i} = \{j : j\in\mathbb{J},\, \xi_{m,i}\ge b_j\}$$

for $m\in\mathbb{N}$ and $0\le i\le n-2$. Then, by (3.22),

$$\sum_{j\in\mathbb{J}} \int_{a_j}^{b_j} \omega_{2i+1}(|t-\xi_{m,i}|)\,dt = \sum_{j\in\mathbb{J}^1_{m,i}} \int_{a_j-\xi_{m,i}}^{b_j-\xi_{m,i}} \omega_{2i+1}(t)\,dt + \sum_{j\in\mathbb{J}^2_{m,i}} \int_{\xi_{m,i}-b_j}^{\xi_{m,i}-a_j} \omega_{2i+1}(t)\,dt + E$$

where

$$E = \begin{cases} 0 & \text{if } \mathbb{J} = \mathbb{J}^1_{m,i}\cup\mathbb{J}^2_{m,i} \\ \displaystyle\int_0^{\xi_{m,i}-a_\tau} \omega_{2i+1}(t)\,dt + \int_0^{b_\tau-\xi_{m,i}} \omega_{2i+1}(t)\,dt & \text{if } a_\tau < \xi_{m,i} < b_\tau,\ \tau\in\mathbb{J}. \end{cases}$$

Set

$$\mathcal{M}^1_{m,i} = \mathcal{N}^1_{m,i} \cup \bigcup_{j\in\mathbb{J}^1_{m,i}} (a_j-\xi_{m,i}, b_j-\xi_{m,i}),$$
$$\mathcal{M}^2_{m,i} = \mathcal{N}^2_{m,i} \cup \bigcup_{j\in\mathbb{J}^2_{m,i}} (\xi_{m,i}-b_j, \xi_{m,i}-a_j)$$

for $m\in\mathbb{N}$ and $0\le i\le n-2$ where

$$\mathcal{N}^1_{m,i} = \begin{cases} \emptyset & \text{if } \mathbb{J} = \mathbb{J}^1_{m,i}\cup\mathbb{J}^2_{m,i} \\ (0, b_\tau-\xi_{m,i}) & \text{if } a_\tau < \xi_{m,i} < b_\tau,\ \tau\in\mathbb{J}, \end{cases}$$

$$\mathcal{N}^2_{m,i} = \begin{cases} \emptyset & \text{if } \mathbb{J} = \mathbb{J}^1_{m,i}\cup\mathbb{J}^2_{m,i} \\ (0, \xi_{m,i}-a_\tau) & \text{if } a_\tau < \xi_{m,i} < b_\tau,\ \tau\in\mathbb{J}. \end{cases}$$

Then $\mathcal{M}^k_{m,i}\subset J$ are measurable, $\mu(\mathcal{M}^k_{m,i}) \le \sum_{j\in\mathbb{J}}(b_j-a_j)$, $1\le k\le 2$, and

$$\sum_{j\in\mathbb{J}} \int_{a_j}^{b_j} \omega_{2i+1}(|t-\xi_{m,i}|)\,dt \le \int_{\mathcal{M}^1_{m,i}} \omega_{2i+1}(t)\,dt + \int_{\mathcal{M}^2_{m,i}} \omega_{2i+1}(t)\,dt.$$

Setting

$$\Phi = \max\{1 + V, P, Q\}$$

where

$$P = \Lambda \max\Big\{\|q_{2i}\|_\infty \omega_{2i}\Big(\frac{T^{2(n-i)-5}}{30^{n-i-1}}\Omega\Big) : 0 \le i \le n-1\Big\},$$

$$Q = \Lambda \max\Big\{\|q_{2i+1}\|_\infty \omega_{2i+1}\Big(\frac{\alpha(T-\beta)T^{2(n-i)-7}}{2\cdot 30^{n-i-2}}\Omega\Big) : 0 \le i \le n-2\Big\}$$

we get

$$\begin{aligned}
&\sum_{j\in\mathbb{J}} \int_{a_j}^{b_j} \Big[\phi(t) + \sum_{i=0}^{2n-2} q_i(t)\omega_i(|x_m^{(i)}(t)|) + \sum_{i=0}^{2n-2} h_i(t)(1 + |x_m^{(i)}(t)|)\Big]\,dt\\
&\le \sum_{j\in\mathbb{J}} \Big[\int_{a_j}^{b_j} \phi(t)\,dt + \Phi\Big(\sum_{i=0}^{n-1}\int_{a_j}^{b_j} \omega_{2i}(t(T-t))\,dt + \sum_{i=0}^{2n-2}\int_{a_j}^{b_j} h_i(t)\,dt\Big)\Big]\\
&\qquad + \Phi\sum_{i=0}^{n-2}\Big(\int_{\mathcal{M}^1_{m,i}} \omega_{2i+1}(t)\,dt + \int_{\mathcal{M}^2_{m,i}} \omega_{2i+1}(t)\,dt\Big).
\end{aligned}$$

By (H_2), ϕ, h_i, $\omega_i \in L_1(J)$ for $0 \le i \le 2n-2$ and ω_i are nonincreasing on $\mathbb{R}_+$. Consequently, for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any at most countable set $\{(a_j, b_j)\}_{j\in\mathbb{J}}$ of mutually disjoint intervals $(a_j, b_j) \subset J$ with $\sum_{j\in\mathbb{J}}(b_j - a_j) < \delta$, (3.21) holds. This completes the proof. $\square$

3.2 Problem (1.3), (1.4)

We would argue similarly as in Section 3.1 but we would like to show the reader a different approach to this type of singular BVPs which seems to be more comfortable for problem (1.3), (1.4). Therefore, first we prove some a priori estimates above, then we discuss the uniform absolute continuity of certain function sets and at the end of this section we prove the existence principle for auxiliary regular BVPs corresponding to problem (1.3), (1.4).

Lemma 3.7. *Let $K > 0$, $\psi \in L_1(J)$ be positive, $h^*, h_j \in L_1(J)$, $q_i \in L_\infty(J)$ be nonnegative, $\omega_i : \mathbb{R}_+ \to \mathbb{R}_+$ be nonincreasing, $\int_0^T \omega_i(s^{n-i-1})ds < \infty$, $0 \le i \le n-2$, $0 \le j \le n-1$. Then there exist constants $r^* > 0$ and $\alpha \in (0, K]$ such that for each function $x \in AC^{n-1}(J)$ satisfying* (1.4),

$$\psi(t) \le -x^{(n)}(t) \quad \text{for a.e.} \quad t \in J \quad \text{provided } \|x\| \le K, \tag{3.23}$$

and

$$\begin{aligned}
0 < -x^{(n)}(t) &\le h^*(t) + \sum_{j=0}^{n-1} h_j(t)|x^{(j)}(t)|\\
&+ \sum_{i=0}^{n-2} q_i(t)\omega_i(|x^{(i)}(t)|), \quad \text{for a.e.} \quad t \in J,
\end{aligned} \tag{3.24}$$

the estimates

$$\|x^{(n-1)}\| < r^*, \quad \|x\| \geq \alpha \tag{3.25}$$

are valid.

Proof. Consider a function $x \in AC^{n-1}(J)$ satisfying (1.4), (3.23) and (3.24). Let $\|x\| \leq K$. Then, by (3.23) and Lemma 2.11, there is a positive constant $c = c(\psi)$ such that $\|x\| \geq c$. Otherwise $\|x\| > K$. If we put $\alpha = \min\{c, K\}$, then the second inequality in (3.25) is satisfied.

Let us prove the first estimate in (3.25). Put $\|x^{(n-1)}\| = \rho$. Then $-\rho \leq x^{(n-1)}(t) \leq \rho$ on J and if we integrate this inequality from 0 to $t \in (0, T]$ and use (1.4), we get step by step

$$|x^{(i)}(t)| \leq \rho \frac{t^{n-i-1}}{(n-i-1)!} \quad \text{for} \quad t \in J, \ 0 \leq i \leq n-1. \tag{3.26}$$

Lemmas 2.9 and 2.10 guarantee the existence of the unique zero ξ_{n-1} of $x^{(n-1)}$ with $\xi_{n-1} \in (0, T)$ for $0 \leq p \leq n-2$ and $\xi_{n-1} = T$ for $p = n-1$. Integrate (3.24) from t to ξ_{n-1}. Then

$$0 < x^{(n-1)}(t) \leq \int_t^{\xi_{n-1}} h^*(s)ds + \sum_{i=0}^{n-1} \int_t^{\xi_{n-1}} h_i(s)|x^{(i)}(s)|ds$$

$$+ \sum_{i=0}^{n-2} \int_t^{\xi_{n-1}} q_i(s)\omega_i(|x^{(i)}(s)|)ds \quad \text{for} \quad t \in [0, \xi_{n-1}).$$

If $p < n-1$ and thus $\xi_{n-1} < T$, we integrate (3.24) from ξ_{n-1} to t and get

$$0 < -x^{(n-1)}(t) \leq \int_{\xi_{n-1}}^t h^*(s)ds + \sum_{i=0}^{n-1} \int_{\xi_{n-1}}^t h_i(s)|x^{(i)}(s)|ds$$

$$+ \sum_{i=0}^{n-2} \int_{\xi_{n-1}}^t q_i(s)\omega_i(|x^{(i)}(s)|)ds \quad \text{for} \quad t \in (\xi_{n-1}, T].$$

Hence, the inequality (see (3.26))

$$|x^{(n-1)}(t)| \leq \Big| \int_t^{\xi_{n-1}} h^*(s)ds \Big| + \rho \sum_{i=0}^{n-1} \Big| \int_t^{\xi_{n-1}} h_i(s) \frac{s^{n-i-1}}{(n-i-1)!} ds \Big|$$

$$+ \sum_{i=0}^{n-2} \Big| \int_t^{\xi_{n-1}} q_i(s)\omega_i(|x^{(i)}(s)|)ds \Big|$$

is true for $t \in J$. Therefore we have

$$\begin{aligned} &\rho \left(1 - \sum_{i=0}^{n-1} \frac{1}{(n-i-1)!} \int_0^T h_i(s)s^{n-i-1}ds \right) \\ &\quad \leq \|h^*\|_L + \sum_{i=0}^{n-2} \|q_i\|_\infty \int_0^T \omega_i(|x^{(i)}(s)|)ds. \end{aligned} \tag{3.27}$$

It remains to estimate the integrals

$$\int_0^T \omega_i(|x^{(i)}(s)|)ds, \ 0 \le i \le n-2.$$

We will consider three cases.

Case (i). Let $p = n-1$. Then, by Lemma 2.8, for $0 \le i \le n-2$

$$\omega_i(|x^{(i)}(s)|) \le \omega_i\left(\frac{\|x\|}{T^{n-1}} s^{n-i-1}\right) \quad \text{for} \quad s \in (0,T].$$

Thus

$$\omega_i(|x^{(i)}(s)|) \le \omega_i((c_i s)^{n-i-1}), \ 0 \le i \le n-2, \quad \text{for } s \in (0,T], \tag{3.28}$$

where $c_i^{n-i-1} = \alpha T^{1-n}$. Inequality (3.28) implies

$$\int_0^T \omega_i(|x^{(i)}(s)|)ds \le \frac{1}{c_i}\int_0^{c_i T} \omega_i(t^{n-i-1})dt = B_i$$

and so we have for $0 \le i \le n-2$

$$\int_0^T \omega_i(|x^{(i)}(s)|)ds \le B_i. \tag{3.29}$$

Case (ii). Let $p = 0$. Then, by Lemma 2.9, for $0 \le i \le n-2$

$$\omega_i(|x^{(i)}(s)|) \le \begin{cases} \omega_i((c_i s)^{n-i-1}) & \text{if } \ 0 \le s \le \xi_{i+1} \\ \omega_i(k_i|\xi_i - s|) & \text{if } \ \xi_{i+1} \le s \le T, \end{cases} \tag{3.30}$$

where

$$c_i^{n-i-1} = \alpha T^{1-n}, \ k_i = \alpha T^{-i-1}, \tag{3.31}$$

and ξ_i, ξ_{i+1} fulfil (2.29). Therefore

$$\int_0^T \omega_i(|x^{(i)}(s)|)ds \le \int_0^{\xi_{i+1}} \omega_i((c_i s)^{n-i-1})ds + \int_{\xi_{i+1}}^{\xi_i} \omega_i(k_i(\xi_i - s))ds$$

$$+ \int_{\xi_i}^T \omega_i(k_i(s-\xi_i))ds \le B_i + \frac{1}{k_i}\int_0^{k_i(\xi_i-\xi_{i+1})} \omega_i(t)dt + \frac{1}{k_i}\int_0^{k_i(T-\xi_i)} \omega_i(t)dt \le B_i + C_i,$$

with

$$C_i = \frac{2}{k_i}\int_0^{k_i T} \omega_i(t)dt.$$

Therefore, we have for $0 \le i \le n-2$

$$\int_0^T \omega_i(|x^{(i)}(s)|)ds \le B_i + C_i. \tag{3.32}$$

Case (iii). Let $1 \le p \le n-2$. Then, for $0 \le i \le p-1$, we have estimate (3.29) and, for $p \le i \le n-2$, estimate (3.32) holds, where ξ_j, $p+1 \le j \le n+1$ are from Lemma 2.10.

In view of (1.7), (3.27), (3.29) and (3.32) we deduce that in all these three cases

$$\rho(1-H) \le \|h^*\|_L + \sum_{i=0}^{n-2} \|q_i\|_\infty (B_i + C_i) = D.$$

So, if we put $r^* = 1 + D(1-H)^{-1}$, we get the first inequality in (3.25). $\square$

Now we will consider the uniform absolute continuity of the following function sets. Let us choose $\alpha > 0$ and define

$$\mathcal{B} = \{x \in AC^{n-1}(J) : \|x\| \ge \alpha,\ x \text{ fulfils (1.4) and (2.15)}\}. \tag{3.33}$$

Lemma 3.8. *Suppose that $\omega_i : \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing, $\int_0^T \omega_i(s^{n-i-1})ds < \infty$, $0 \le i \le n-2$. Let us put*

$$\mathcal{A} = \{\omega_i(|x^{(i)}|) : x \in \mathcal{B},\ 0 \le i \le n-2\}. \tag{3.34}$$

Then the functions of $\mathcal{A}$ are uniformly absolutely continuous on J, that is for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{\mathcal{M}} \omega_i(|x^{(i)}(s)|)ds < \varepsilon$$

for each $x \in \mathcal{B}$, $0 \le i \le n-2$ and for any $\mathcal{M} \subset J$, $\mu(\mathcal{M}) < \delta$.

Proof. It is sufficient to prove that for each $\varepsilon > 0$ there is $\delta > 0$ such that for any system $\{(\tau_j, t_j)\}_1^\infty$ of mutually disjoint intervals $(\tau_j, t_j) \subset J$ the condition

$$\sum_{j=1}^{\infty}(t_j - \tau_j) < \delta \Longrightarrow \sum_{j=1}^{\infty} \int_{\tau_j}^{t_j} \omega_i(|x^{(i)}(s)|)ds < \varepsilon \tag{3.35}$$

is valid for $x \in \mathcal{B}$ and $0 \le i \le n-2$. We will distinguish three cases, which will be denoted by (i), (ii) and (iii).

Case (i). Let $p = n-1$. Choose $i \in \{0, 1, \dots, n-2\}$ and put $c = c_i$, where c_i is given by (3.31). Then, in view of (3.28), for each $\tau_j, t_j \in J$, $\tau_j < t_j$

$$\int_{\tau_j}^{t_j} \omega_i(|x^{(i)}(s)|)ds \le \int_{\tau_j}^{t_j} \omega_i((cs)^{n-i-1})ds = \frac{1}{c}\int_{c\tau_j}^{ct_j} \omega_i(t^{n-i-1})dt$$
$$= \frac{1}{c}\int_{c\tau_j}^{ct_j} \gamma_i(t)dt,$$

with $\gamma_i(t) = \omega_i(t^{n-i-1})$ for $t \in \mathbb{R}_+$. Thus

$$\int_{\tau_j}^{t_j} \omega_i(|x^{(i)}(s)|)ds \le \frac{1}{c}\left(\Gamma_i(ct_j) - \Gamma_i(c\tau_j)\right), \tag{3.36}$$

with $\Gamma_i(t) = \int_0^t \gamma_i(s)ds$. According to Remark 1.3 we can see that Γ_i is absolutely continuous on $[0, A]$ for an arbitrary $A \in \mathbb{R}_+$.

Let us choose an $\varepsilon > 0$ and put $\varepsilon_1 = c\varepsilon$. Then there is a $\delta_1 > 0$ such that for any system $\{(a_j, b_j)\}_1^\infty$ of mutually disjoint intervals in $[0, cT]$ the condition

$$\sum_{j=1}^{\infty}(b_j - a_j) < \delta_1 \Longrightarrow \sum_{j=1}^{\infty}(\Gamma_i(b_j) - \Gamma_i(a_j)) < \varepsilon_1 \tag{3.37}$$

is true. Put $\delta = \frac{\delta_1}{c}$ and suppose $\sum_{j=1}^{\infty}(t_j - \tau_j) < \delta$. Then $\sum_{j=1}^{\infty}(ct_j - c\tau_j) < \delta_1$, $(c\tau_j, ct_j) \subset [0, cT]$, $j \in \mathbb{N}$, which implies by (3.37)

$$\sum_{j=1}^{\infty}(\Gamma_i(ct_j) - \Gamma_i(c\tau_j)) < \varepsilon_1,$$

wherefrom, by (3.36), we get

$$\sum_{j=1}^{\infty}\int_{\tau_j}^{t_j}\omega_i(|x^{(i)}(s)|)ds < \varepsilon.$$

Case (ii). Let $p = 0$. Then, by Lemma 2.9, for $0 \le i \le n-2$ the inequalities (3.30) are satisfied, where c_i and k_i are given by (3.31) and ξ_i, ξ_{i+1} fulfil (2.29). Let us choose an arbitrary $i \in \{0, 1, \ldots, n-2\}$ and put $c = c_i$ and $k = k_i$. Now, for a $j \in \mathbb{N}$, we will discuss five possible locations of τ_j, t_j with respect to ξ_i, ξ_{i+1}.

(a) Suppose

$$0 \le \tau_j < t_j \le \xi_{i+1}. \tag{3.38}$$

Then the estimate (3.36) is true.

(b) Let

$$\xi_{i+1} \le \tau_j < t_j \le \xi_i. \tag{3.39}$$

Then, by (3.30), we have

$$\int_{\tau_j}^{t_j}\omega_i(|x^{(i)}(s)|)ds \le \int_{\tau_j}^{t_j}\omega_i(k(\xi_i - s))ds = \frac{1}{k}\int_{k(\xi_i - t_j)}^{k(\xi_i - \tau_j)}\omega_i(t)dt,$$

which yields the estimate

$$\int_{\tau_j}^{t_j}\omega_i(|x^{(i)}(s)|)ds \le \frac{1}{k}\left(\Omega_i(k(\xi_i - \tau_j)) - \Omega_i(k(\xi_i - t_j))\right), \tag{3.40}$$

where $\Omega_i(t) = \int_0^t \omega_i(s)ds$ and, by Remark 1.3, Ω_i is absolutely continuous on $[0, A]$ for an arbitrary $A \in \mathbb{R}_+$.

(c) Let us suppose

$$\xi_i \le \tau_j < t_j \le T. \tag{3.41}$$

(Note that (3.41) can occur only if $i > 0$, because $\xi_0 = T$.) Then, by (3.30), we get as above

$$\int_{\tau_j}^{t_j} \omega_i(|x^{(i)}(s)|)ds \le \frac{1}{k}\left(\Omega_i(k(t_j - \xi_i)) - \Omega_i(k(\tau_j - \xi_i))\right). \tag{3.42}$$

(d) Let us suppose

$$0 < \tau_j \le \xi_{i+1} < t_j < \xi_i. \tag{3.43}$$

Then, by (3.36), (3.38), (3.39) and (3.40),

$$\begin{aligned}\int_{\tau_j}^{t_j} \omega_i(|x^{(i)}(s)|)ds &\le \frac{1}{c}\left(\Gamma_i(c\xi_{i+1}) - \Gamma_i(c\tau_j)\right)\\ &+\frac{1}{k}\left(\Omega_i(k(\xi_i - \xi_{i+1})) - \Omega_i(k(\xi_i - t_j))\right).\end{aligned} \tag{3.44}$$

(e) Finally, suppose

$$\xi_{i+1} < \tau_j \le \xi_i < t_j < T. \tag{3.45}$$

(Note that (3.45) can occur if $i > 0$, only.) Then, by (3.39) - (3.42),

$$\begin{aligned}\int_{\tau_j}^{t_j} \omega_i(|x^{(i)}(s)|)ds &\le \frac{1}{k}\left(\Omega_i(k(\xi_i - \tau_j)) - \Omega_i(0)\right)\\ &+\frac{1}{k}\left(\Omega_i(k(t_j - \xi_i)) - \Omega_i(0)\right).\end{aligned} \tag{3.46}$$

Choose an $\varepsilon > 0$ and put $\varepsilon_1 = \varepsilon\left(\frac{1}{c} + \frac{2}{k}\right)^{-1}$. Then we can find $\delta_1 > 0$ such that for any system of mutually disjoint intervals $\{(a_j, b_j)\}_1^\infty$ in $[0, T\max\{c,k\}]$, the condition

$$\sum_{j=1}^{\infty}(b_j - a_j) < \delta_1 \implies \begin{cases} \sum_{j=1}^{\infty}(\Gamma_i(b_j) - \Gamma_i(a_j)) < \varepsilon_1 \\ \sum_{j=1}^{\infty}(\Omega_i(b_j) - \Omega_i(a_j)) < \varepsilon_1 \end{cases} \tag{3.47}$$

is valid. Put $\delta = \frac{\delta_1}{c+k}$ and take a system $\{(\tau_j, t_j)\}_{j=1}^\infty \subset J$ such that $\sum_1^\infty (t_j - \tau_j) < \delta$. Then

$$\sum_{j=1}^{\infty} c(t_j - \tau_j) < \delta_1 \quad \text{and} \quad \sum_{j=1}^{\infty} k||\xi_i - \tau_j| - |\xi_i - t_j|| < \delta_1.$$

So, by (3.36), (3.40), (3.42), (3.44) and (3.46), we compute that

$$\begin{aligned}\sum_{j=1}^{\infty}\int_{\tau_j}^{t_j} \omega_i(|x^{(i)}(s)|)ds &\le \frac{1}{c}\sum_{j=1}^{\infty}(\Gamma_i(ct_j) - \Gamma_i(c\tau_j))\\ +\frac{2}{k}\sum_{j=1}^{\infty}|\Omega_i(k|\xi_i - \tau_j|) - \Omega_i(k|\xi_i - t_j|)| &< \left(\frac{1}{c} + \frac{2}{k}\right)\varepsilon_1 = \varepsilon.\end{aligned}$$

Case (iii). Let $1 \le p \le n-2$. Then for $0 \le i \le p-1$ we argue as in *Case* (i) and for $p \le i \le n-2$ we follow *Case* (ii). □

Now we present an existence principle for (n,p) BVPs which are regular. Particularly, we consider the equations

$$-x^{(n)}(t) = h(t, x(t), \dots, x^{(n-1)}(t)) \tag{3.48}$$

and

$$-x^{(n)}(t) = \lambda h(t, x(t), \dots, x^{(n-1)}(t)), \ \lambda \in [0,1]. \tag{3.49}$$

Lemma 3.9. *Let $h \in Car(J \times \mathbb{R}^n)$ and let there exist $r > 0$ such that for any $\lambda \in (0,1)$ and any solution x of problem* (3.49), (1.4) *the estimate*

$$\|x^{(n-1)}\| \neq r \tag{3.50}$$

is true. Then problem (3.48), (1.4) *has a solution.*

Proof. Let the operator $\mathcal{S} : C^{n-1}(J) \to C^{n-1}(J)$ be defined by the formula

$$(\mathcal{S}x)(t) = \int_0^T G(t,s) h(s, x(s) \dots, x^{(n-1)}(s)) ds$$

where G is the Green's function of problem $(2.11), (1.4)$. Then $\mathcal{S}$ is a completely continuous operator and we see that a function x is a solution of problem $(3.49), (1.4)$ for some $\lambda \in (0,1)$ if and only if x is a solution of the operator equation

$$x = \lambda \mathcal{S} x. \tag{3.51}$$

Set

$$\Omega = \{y \in C^{n-1}(J) : \|y^{(n-i-1)}\| < rT^i \ \text{ for } 0 \le i \le n-1\}.$$

Then Ω is an open bounded set in $C^{n-1}(J)$. Let x be a solution of $(3.49), (1.4)$ for some $\lambda \in (0,1)$. Then x fulfils (3.50). If $\|x^{(n-1)}\| < r$, then from $x^{(i)}(0) = 0, 0 \le i \le n-2$ (which follows from (1.4)) we deduce

$$\|x^{(n-i-1)}\| < rT^i, \ 0 \le i \le n-1$$

and so $x \in \Omega$. If $\|x^{(n-1)}\| > r$, then $x \notin cl(\Omega)$. So, we have proved that for any $\lambda \in (0,1)$ each solution x of $(3.49), (1.4)$ does not belong to $\partial\Omega$. Further, for $\lambda = 0$ problem $(3.49), (1.4)$ has only the trivial solution which cannot belong to $\partial\Omega$, as well. For $\lambda = 1$ we have two possibilities:
(i) The operator $\mathcal{S}$ has fixed points on $\partial\Omega$.
(ii) The operator $\mathcal{S}$ has no fixed points on $\partial\Omega$. Then the operator $I - \lambda\mathcal{S}$ is a compact homotopy on $cl(\Omega) \times [0,1]$ and

$$1 = deg(I, \Omega) = deg(I - \mathcal{S}, \Omega), \tag{3.52}$$

where deg denotes the Leray-Schauder topological degree and $I : C^{n-1}(J) \to C^{n-1}(J)$ stands for the identity operator $Ix = x$. By (3.52), $\mathcal{S}$ has a fixed point in Ω.

Since fixed points of $\mathcal{S}$ are solutions of $(3.48), (1.4)$, Lemma 3.9 is proved. $\square$

4 Main results

Theorem 4.1. *Let assumptions* (H_1) *and* (H_2) *be satisfied. Then there exists a solution of BVP* (1.1), (1.2).

Proof. For each $m \in \mathbb{N}$, there exists a solution x_m of BVP (3.3), (1.2) by Lemma 3.2. Consider the sequence $\{x_m\}$. Lemmas 3.3 and 3.5 show that $\{x_m\}$ is bounded in $C^{2n-1}(J)$ and, by Lemma 2.4,

$$(-1)^j x_m^{(2j)}(t) \geq \frac{T^{2(n-j)-5}}{30^{n-j-1}} \Omega t(T-t) \quad \text{for } t \in J,\ 0 \leq j \leq n-1 \tag{4.1}$$

where Ω is given by (2.8). The Arzelà-Ascoli theorem guarantees the existence of a subsequence $\{x_{m_k}\}$ converging in $C^{2n-2}(J)$, $\lim_{k\to\infty} x_{m_k} = x$. Then $x \in C^{2n-2}(J)$, (4.1) gives $(-1)^j x^{(2j)}(t) \geq \dfrac{T^{2(n-j)-5}}{30^{n-j-1}} \Omega t(T-t) > 0$ for $t \in (0,T)$, $0 \leq j \leq n-1$ and x satisfies the boundary conditions (1.2). Then $x^{(2j+1)}(\xi_j) = 0$ for a (unique) $\xi_j \in (0,T)$, $0 \leq j \leq n-2$. Now, $f_{m_k} \in Car(J \times \mathbb{R}^{2n-1})$, and from their construction it follows that there exists $\mathcal{U} \subset J$, $\mu(\mathcal{U}) = 0$, such that $f_{m_k}(t, \cdot, \dots, \cdot)$ are continuous on $\mathbb{R}^{2n-1}$ for each $t \in J \setminus \mathcal{U}$ which implies that

$$\lim_{k\to\infty} f_{m_k}(t, x_{m_k}(t), \dots, x_{m_k}^{(2n-2)}(t)) = f(t, x(t), \dots, x^{(2n-2)}(t))$$

for $J \setminus (\mathcal{U} \cup \{0, T, \xi_0, \dots, \xi_{n-2}\})$. By Lemma 3.6, $\{f_{m_k}(t, x_{m_k}(t), \dots, x_{m_k}^{(2n-2)}(t))\}$ is uniformly absolutely continuous on J. Hence $f(t, x(t), \dots, x^{(2n-2)}(t)) \in L_1(J)$ and

$$\lim_{k\to\infty} \int_0^t f_{m_k}(s, x_{m_k}(s), \dots, x_{m_k}^{(2n-2)}(s))\, dt = \int_0^t f(t, x(t), \dots, x^{(2n-2)}(t))\, dt$$

for $t \in J$ by the Vitali's theorem. Since $\{x_{m_k}^{(2n-1)}(0)\}$ is bounded, we can assume that it is convergent, say $\lim_{k\to\infty} x_{m_k}^{(2n-1)}(0) = C$. Then taking the limit as $k \to \infty$ in the equalities

$$x_{m_k}^{(2n-2)}(t) = x_{m_k}^{(2n-1)}(0)t + (-1)^n \int_0^t \int_0^s f_{m_k}(u, x_{m_k}(u), \dots, x_{m_k}^{(2n-2)}(u))\, du\, ds, \quad t \in J$$

we get

$$x^{(2n-2)}(t) = Ct + (-1)^n \int_0^t \int_0^s f(u, x(u), \dots, x^{(2n-2)}(u))\, du\, ds, \quad t \in J.$$

Then $x \in AC^{2n-1}(J)$ and

$$(-1)^n x^{(2n)}(t) = f(t, x(t), \dots, x^{(2n-2)}(t)) \quad \text{for a.e. } t \in J.$$

Therefore x is a solution of BVP (1.1), (1.2). □

Theorem 4.2. *Let assumptions* (H_3) *and* (H_4) *be satisfied. Then there exists a solution of BVP* (1.3), (1.4).

Proof. Put $h^* = h_0+\phi+\sum_{i=0}^{n-2} q_i\omega_i(1)$ and, by Lemma 3.7, find positive constants r^*, α satisfying (3.25). Further, put $\rho_0 = 1 + r^*T^{n-1} + K$, where K is a constant from (H_3), $\rho_i = 1 + r^*T^{n-i-1}$, $1 \le i \le n-1$,

$$\sigma_i(x) = \begin{cases} x & \text{for } |x| \le \rho_i \\ \rho_i \mathrm{sign} x & \text{for } |x| > \rho_i \end{cases}, \ 0 \le i \le n-1$$

and, for $0 < c < \rho_0$,

$$\sigma_0^*(c,x) = \begin{cases} c & \text{for } x < c \\ x & \text{for } c \le x \le \rho_0 \\ \rho_0 & \text{for } \rho_0 < x. \end{cases}$$

Choose $m \in \mathbb{N}$ and define an auxiliary function h_m by the following recurrent formulas for a.e. $t \in J$ and for $(x_0, \dots, x_{n-1}) \in X$:

$$h_{m,0}(t, x_0, \dots, x_{n-1}) = f(t, x_0, \dots, x_{n-1}),$$

$h_{m,i}(t, x_0, \dots, x_{n-1})$

$$= \begin{cases} h_{m,i-1}(t, x_0 \dots, x_{n-1}) \ \text{ if } \ |x_i| \ge \frac{1}{m} \\ \dfrac{m}{2}[h_{m,i-1}(t, x_0, \dots, x_{i-1}, \dfrac{1}{m}, x_{i+1}, \dots, x_{n-1})(x_i + \dfrac{1}{m}) \\ -h_{m,i-1}(t, x_0, \dots, x_{i-1}, -\dfrac{1}{m}, x_{i+1}, \dots, x_{n-1})(x_i - \dfrac{1}{m})] \ \text{ if } |x_i| < \dfrac{1}{m}, \end{cases}$$

and $1 \le i \le n-2$,

$$h_m(t, x_0, \dots, x_{n-1}) = h_{m,n-2}(t, x_0, \dots, x_{n-1}).$$

Now, for a.e. $t \in J$ and for $(x_0, \dots, x_{n-1}) \in \mathbb{R}^n$ put

$$f_m(t, x_0, \dots, x_{n-1}) = h_m(t, \sigma_0^*(\frac{1}{m}, x_0), \sigma_1(x_1), \dots, \sigma_{n-1}(x_{n-1})). \tag{4.2}$$

Then, by (H_3) and (H_4), $f_m \in Car(J \times \mathbb{R}^n)$ and for $m \ge m_0 \ge \frac{1}{K}$

$$\begin{gathered} \psi(t) \le f_m(t, x_0, \dots, x_{n-1}) \\ \text{for a.e. } t \in J, \ \text{each } (x_0, \dots, x_{n-1}) \in \mathbb{R}^n, \ x_0 \le K, \end{gathered} \tag{4.3}$$

$$\begin{gathered} 0 < f_m(t, x_0, \dots, x_{n-1}) \le h^*(t) + \sum_{i=0}^{n-1} h_i(t)|x_i| + \sum_{i=0}^{n-2} q_i(t)\omega_i(|x_i|), \\ \text{for a.e. } t \in J \text{ and each } \ (x_0, \dots, x_{n-1}) \in \mathbb{R}^n. \end{gathered} \tag{4.4}$$

Inequality (4.4) follows from the fact that $|\sigma_i(x_i)| \le |x_i|$, $1 \le i \le n-1$, $|\sigma_0^*(\frac{1}{m}, x_0)| \le 1 + |x_0|$, $\sigma_0^*(\frac{1}{m}, x_0) \ge \sigma_0(x_0)$, $\omega_i(|\sigma_i(x_i)|) \le \omega_i(|x_i|) + \omega_i(1)$, $0 \le i \le n-2$.

Consider the auxiliary equations

$$-x^{(n)}(t) = f_m(t, x(t), \dots, x^{(n-1)}(t)) \tag{4.5}$$

and

$$-x^{(n)}(t) = \lambda f_m(t, x(t), \dots, x^{(n-1)}(t)),\ \lambda \in [0,1], \tag{4.6}$$

and prove that $(4.5), (1.4)$ has a solution for each $m \in \mathbb{N}$. Fix $m \in \mathbb{N}$ and put

$$g_m(t) = \sup\{|f(t, x_0, \dots, x_{n-1}) : \frac{1}{m} \le |x_i| \le \rho_i, 0 \le i \le n-2,\ |x_{n-1}| \le \rho_{n-1}\}.$$

Then $g_m \in L_1(J)$ and $|\lambda f_m(t, x_0, \dots, x_{n-1})| \le g_m(t)$ for a.e. $t \in J$ and all $\lambda \in (0,1)$, $(x_0, \dots, x_{n-1}) \in \mathbb{R}^n$. Therefore, for any $\lambda \in (0,1)$ and any solution x of $(4.6), (1.4)$, the estimate $\|x^{(n-1)}\| \le \|g_m\|_L$ holds. Thus, if we choose $r > \|g_m\|_L$, we get (3.50). So, Lemma 3.9 guarantees that problem $(4.5), (1.4)$ has a solution x_m. In such a way we get a sequence $\{x_m\}$ of solutions of $(4.5), (1.4)$, $m \in \mathbb{N}$. In view of (4.3) and (4.4) and by Lemma 3.7, we get (for $m \in \mathbb{N}, m \ge m_0,$)

$$\|x_m^{(n-1)}\| < r^* \quad \text{and} \quad \|x_m\| \ge \alpha, \tag{4.7}$$

where r^* and α are positive constants. Conditions (1.4) and (4.7) yield

$$\|x_m^{(n-i-1)}\| < r^* T^i,\ 0 \le i \le n-1. \tag{4.8}$$

Further, by (4.4), we have for $t, \tau \in J$, $\tau < t$

$$|x_m^{(n-1)}(t) - x_m^{(n-1)}(\tau)| \le \int_\tau^t h(s)ds + \sum_{i=0}^{n-2} \|q_i\|_\infty \int_\tau^t \omega_i(|x_m^{(i)}(s)|)ds, \tag{4.9}$$

where

$$h(t) = h^*(t) + \sum_{i=0}^{n-1} \rho_i h_i(t),\ h \in L_1(J).$$

According to (1.4), (4.4) and (4.7), we can use Lemma 3.8 and obtain that the sequence $\{\omega_i(|x_m^{(i)}|)\}_{m_0}^\infty$ is uniformly absolutely continuous on J for $0 \le i \le n-2$. This, by (4.9), implies that the sequence $\{x_m^{(n-1)}\}_{m_0}^\infty$ is equicontinuous on J. Further, by (4.8), we see that the sequence $\{x_m\}_{m_0}^\infty$ is bounded in $C^{n-1}(J)$. Thus, by the Arzelà-Ascoli theorem, we can choose subsequence, which is denoted $\{x_k\}$ and which converges in $C^{n-1}(J)$ to a function $x \in C^{n-1}(J)$. Clearly x satisfies (1.4).

Let $p = 0$ and $0 \le i \le n-2$. Then, in view of Lemma 2.9 and by (1.4), (4.4) and (4.7), we have

$$x(0) = x(T) = 0 \tag{4.10}$$

and

$$x_k^{(i)}(t) \geq \begin{cases} \dfrac{\alpha}{T^{n-1}} t^{n-i-1} & \text{for } 0 \leq t \leq \xi_{i+1,k} \\ \dfrac{\alpha}{T^{i+1}}(\xi_{i,k} - t) & \text{for } \xi_{i+1,k} \leq t \leq \xi_{i,k} \end{cases} \tag{4.11}$$

$$x_k^{(i)}(t) \leq \frac{\alpha}{T^{i+1}}(\xi_{i,k} - t) \quad \text{for } \ \xi_{i,k} \leq t \leq T.$$

Here $\xi_{0,k} = T$, $\xi_{j,k}$ is a (unique) zero of $x_k^{(j)}$ in $(0,T)$, $1 \leq j \leq n-1$, $k \in \mathbb{N}$ and

$$0 < \xi_{n-1,k} < \xi_{n-2,k} < \ldots < \xi_{2,k} < \xi_{1,k} < \xi_{0,k} = T. \tag{4.12}$$

Letting $k \to \infty$ we can choose subsequences which we denote $\{\xi_{j,l}\}_{l=1}^{\infty}$ such that

$$\xi_j = \lim_{l \to \infty} \xi_{j,l}, \ \ 0 \leq j \leq n-1.$$

This limitting proces in (4.11) yields

$$x^{(i)}(t) \geq \begin{cases} \dfrac{\alpha}{T^{n-1}} t^{n-i-1} & \text{for } 0 \leq t \leq \xi_{i+1} \\ \dfrac{\alpha}{T^{i+1}}(\xi_i - t) & \text{for } \xi_{i+1} \leq t \leq \xi_i, \end{cases} \tag{4.13}$$

$$x^{(i)}(t) \leq \frac{\alpha}{T^{i+1}}(\xi_i - t) \quad \text{for } \ \xi_i \leq t \leq T.$$

Now, let us show that estimates (4.13) imply that $x > 0$ in $(0,T)$ and that $x^{(i)}$ has just one zero ξ_i in $(0,T)$ for $1 \leq i \leq n-2$. It suffices to prove that

$$0 < \xi_{n-1} < \xi_{n-2} < \ldots < \xi_2 < \xi_1 < \xi_0 = T. \tag{4.14}$$

According to (4.12) we get (4.14) with nonstrict inequalities. Let us prove that these inequalities must be strict. Suppose the contrary. First, let $\xi_1 = \xi_0$. Then (4.13) gives $x(T) \geq \alpha > 0$, which contradicts (4.10). Now, let $0 = \xi_{n-1}$. In view of (4.13) we can see that the inequality $\xi_{n-1} < \xi_{n-2}$ implies

$$x^{(n-2)}(0) \geq \frac{\alpha \xi_{n-2}}{T^{n-1}} > 0 \tag{4.15}$$

which contradicts (1.4), while the equality $\xi_{n-1} = \xi_{n-2}$ leads to

$$x^{(n-2)}(t) \leq \frac{-\alpha t}{T^{n-2}} < 0 \ \text{ for } t \in (0,T]. \tag{4.16}$$

Integrating (4.16) and using (1.4) repeatedly we obtain $x(t) < 0$ for $t \in (0,T]$, which contradicts (4.10). Finally, let $\xi_{i+1} = \xi_i$, $1 \leq i \leq n-3$. Then (4.13) yields

$$x^{(i)}(\xi_i) \geq \frac{\alpha}{T^{n-1}} \xi_i^{n-i-1} > 0 \ \text{ and } \ x^{(i)}(\xi_i) \leq 0,$$

a contradiction. Hence (4.14) is proved.

If $p > 0$ we can use Lemma 2.8 or Lemma 2.10 and by similar arguments get that $x^{(i)}$ has just one zero ξ_i in $(0,T)$ for $p+1 \le i \le n-2$ and $x^{(i)} > 0$ in $(0,T)$, $0 \le i \le p$.

Finally, let us show that $x \in AC^{n-1}(J)$ and that x fulfils (1.3) a.e. on J. Consider the sequence of equalities

$$x_l^{(n-1)}(t) = x_l^{(n-1)}(0) + \int_0^t f_l(s, x_l(s), \dots, x_l^{(n-1)}(s))ds \quad \text{for } t \in J. \tag{4.17}$$

Denote the set of all $t \in J$ such that $f(t, \cdot, \dots, \cdot) : X \to \mathbb{R}$ is not continuous by $\mathcal{U}$. Then $\mu(\mathcal{U}) = 0$ and

$$\lim_{l \to \infty} f_l(t, x_l(t), \dots, x_l^{(n-1)}(t)) = f(t, x(t), \dots, x^{(n-1)}(t))$$

for all $t \in J \setminus (\mathcal{U} \cup \{0, T, \xi_{p+1}, \dots, \xi_{n-2}\})$, $0 \le p \le n-3$ and for all $t \in J \setminus (\mathcal{U} \cup \{0, T\})$, $n-2 \le p \le n-1$. Using (4.4) and the uniform absolute continuity of $\{\omega_i(|x_l^{(i)}|\}$ on J, $0 \le i \le n-2$, we can deduce that $\{f_l(t, x_l(t) \dots, x_l^{(n-1)}(t))\}$ is also uniformly absolutely continuous on J. Therefore we can use the Vitali's theorem by which $f(t, x(t), \dots, x^{(n-1)}(t)) \in L_1(J)$ and letting $l \to \infty$ in (4.17) we have that

$$x^{(n-1)}(t) = x^{(n-1)}(0) + \int_0^t f(s, x(s), \dots, x^{(n-1)}(s))ds \quad \text{for } t \in J$$

is valid, i.e. $x \in AC^{n-1}(J)$ and x satisfies (1.3) a.e. on J. □

For the continuous function f in equations (1.1) and (1.3) we get immediately from Theorems 4.1 and 4.2 and our previous considerations the following corollaries.

Corollary 4.3. *Let $f \in C^0(J \times D)$ satisfy assumptions (H_1) and (H_2). Then there exists a solution x of BVP (1.1), (1.2) such that $x \in AC^{2n-1}(J) \cap C^{2n}(J \setminus \{0, T, \xi_0, \dots, \xi_{n-2}\})$ and (1.1) holds for each $t \in J \setminus \{0, T, \xi_0, \dots, \xi_{n-2}\}$ where $\xi_j \in (0,T)$ is a unique zero of $x^{(2j+1)}$ in J, $0 \le j \le n-2$.*

Corollary 4.4. *Let $f \in C^0(J \times X)$ satisfy assumptions (H_3) and (H_4). Then BVP (1.3), (1.4) has a solution x such that $x \in AC^{n-1}(J) \cap C^n(J \setminus \{0, T\})$ and (1.3) holds for each $t \in J \setminus \{0, T\}$ provided $n-2 \le p \le n-1$ and $x \in AC^{n-1}(J) \cap C^n(J \setminus \{0, T, \xi_{p+1}, \dots, \xi_{n-2}\})$ and (1.3) holds for each $t \in J \setminus \{0, T, \xi_{p+1}, \dots, \xi_{n-2}\}$ provided $0 \le p \le n-3$ where ξ_j is a unique zero of $x^{(j)}$ in $(0,T)$, $p+1 \le j \le n-2$.*

Remark 4.5. The assertion of Theorem 4.1 remains also valid if the growth condition on f in (H_2) has the form

$$f(t, x_0, \dots, x_{2n-2}) \le \phi(t) + \sum_{j=0}^{2n-2} q_j(t)\omega_j(|x_j|) + \sum_{j=0}^{2n-2} h_j(t)|x_j|^{\alpha_j}$$

with $\alpha_j \in (0,1)$, $0 \le j \le 2n-2$, and (1.5) is omitted. Similarly, if the growth conditions on f in (H_4) has the form

$$0 < f(t, x_0, \ldots, x_{n-1}) \le \phi(t) + \sum_{i=0}^{n-2} q_i(t)\omega_i(|x_i|) + \sum_{j=0}^{n-1} h_j(t)|x_j|^{\alpha_j},$$

with $\alpha_j \in (0,1)$, $0 \le j \le n-1$, and (1.7) is omitted, the assertion of Theorem 4.2 keeps its validity, as well.

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