Singular Dirichlet BVP for second order ODE

Irena Rachůnková and Jakub Stryja

Abstract. This paper investigates the singular Dirichlet problem

$$-u'' = f(t, u, u') \ , \quad u(0) = 0 \ , \quad u(T) = 0 \ ,$$

where f satisfies the Carathéodory conditions on the set $(0, T) \times \mathbb{R}_0^2$ and $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$.

The function $f(t, x, y)$ can have time singularities at $t = 0$ and $t = T$ and space singularities at $x = 0$ and $y = 0$. The existence principle for the above problem is given and its application is presented here. The paper provides conditions which guarantee the existence of a solution which is positive on $(0, T)$ and which has the absolutely continuous first derivative on $[0, T]$.

2000 Mathematics Subject Classification. 34B16, 34B15, 34B18

Keywords: singular Dirichlet problem, existence, smooth positive solution, time and space singularities

1 Introduction

Let $[0, T] \subset \mathbb{R}$ and $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. We will investigate the solvability of the problem

$$-u'' = f(t, u, u') \ , \tag{1.1}$$

$$u(0) = 0 \ , \quad u(T) = 0 \ , \tag{1.2}$$

where f satisfies the Carathéodory conditions on $(0, T) \times \mathbb{R}_0^2$ and $f(t, x, y)$ can have time singularities at $t = 0$, $t = T$ and space singularities at $x = 0$ and $y = 0$.

Definition 1.1 We say that f has *a time singularity* at $t = 0$ ($t = T$) if there exist $x, y \in \mathbb{R}_0$ such that

$$\int_0^\varepsilon |f(t, x, y)| \, \mathrm{d}t = \infty \quad \left(\int_{T-\varepsilon}^T |f(t, x, y)| \, \mathrm{d}t = \infty \right)$$

for any sufficiently small $\varepsilon > 0$.

Definition 1.2 We say that f has *a space singularity* at $x = 0$ ($y = 0$) if there exists a set $J \subset [0, T]$ with a positive Lebesgue measure such that the condition

$$\limsup_{x \to 0} |f(t, x, y)| = \infty \quad \left(\limsup_{y \to 0} |f(t, x, y)| = \infty \right)$$

holds for a.e. $t \in J$ and some $y \in \mathbb{R}_0$ ($x \in \mathbb{R}_0$).

In what follows we will use the notation:

$[a, b] \subset \mathbb{R}$; $\Sigma \subset (a, b)$ - a finite set; $\mathcal{M} \subset \mathbb{R}^2$;

$C[a, b]$ - the Banach space of functions continuous on $[a, b]$ with the norm $||x||_C = \max\{|x(t)|;\ t \in [a, b]\}$;

$C^1[a, b]$ - the Banach space of functions having continuous first derivatives on $[a, b]$ with the norm $||x||_{C^1} = ||x||_C + ||x'||_C$;

$AC^1[a, b]$ - the set of functions having absolutely continuous derivatives on $[a, b]$;

$AC^1_{loc}((a, b) \setminus \Sigma)$ - the set of functions $x \in AC^1[c, d]$ for each $[c, d] \subset [a, b] \setminus \Sigma$;

$L[a, b]$ - the Banach space of functions Lebesgue integrable on $[a, b]$ with the norm $||x||_L = \int_a^b |x(t)|\, \mathrm{d}t$;

$Car([a, b] \times \mathcal{M})$ - the set of functions $f:\ [a, b] \times \mathcal{M} \to \mathbb{R}$ satisfying the Carathéodory conditions on $[a, b] \times \mathcal{M}$, i.e.

$f(\cdot, x, y):\ [a, b] \to \mathbb{R}$ is measurable for all $(x, y) \in \mathcal{M}$;

$f(t, \cdot, \cdot):\ \mathcal{M} \to \mathbb{R}$ is continuous for a.e. $t \in [a, b]$;

for each compact set $\mathcal{K} \subset \mathcal{M}$ there is a function $m_{\mathcal{K}} \in L[a, b]$ such that

$$|f(t, x, y)| \le m_{\mathcal{K}}(t) \text{ for a.e. } t \in [a, b] \text{ and all } (x, y) \in \mathcal{K}\ .$$

$Car((a, b) \times \mathcal{M})$ - the set of function $f \in Car([c, d] \times \mathcal{M})$ for each $[c, d] \subset (a, b)$;

$\operatorname{meas} \mathcal{A}$ - the Lebegue measure of $\mathcal{A} \subset \mathbb{R}$.

We say that the sequence $\{v_n\} \subset C[0, T]$ is equicontinuous on $[0, T]$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $t_1, t_2 \in [0, T]$

$$|t_1 - t_2| < \delta \quad \Rightarrow \quad |v_n(t_1) - v_n(t_2)| < \varepsilon$$

for each $n \in \mathbb{N}$.

Definition 1.3 By *a solution* of problem (1.1), (1.2) we understand a function $u \in AC^1[0, T]$ satisfying equation (1.1) a.e. on $[0, T]$ and fulfilling conditions (1.2).

In literature we can find an alternative approach to solvability of singular problems where solutions are defined as continuous functions whose first derivatives can have discontinuities at some points in $[0, T]$. Here we will call such functions w-solutions and according to [4] or [11] we will define them as follows:

Definition 1.4 We say that $u \in C[0,T]$ is *a w-solution* of (1.1), (1.2) if there exists a finite set $\Sigma \subset (0,T)$ such that $u \in AC^1_{loc}((0,T) \setminus \Sigma)$ satisfies equation (1.1) a.e. on $[0,T]$ and satisfies conditions (1.2).

A systematic study of solvability of Dirichlet problems having both time and space singularities was initiated by Taliaferro [21]. Now, we can find a large groups of works which focused their attention on the existence of w-solutions, e.g. [1] – [4], [10] – [15], and a less number of works which provide also conditions for the existence of solutions, e.g. [5], [7], [8], [16], [17], [22]. All the above works deal with differential equations where the nonlinearity $f(t,x,y)$ has a space singularity at $x = 0$ and/or time singularities at $t = 0, t = T$. The first existence result for the Dirichlet problem where $f(t,x,y)$ has singularities at both variables x and y was reached by Staněk [20]. He assumed that f is strictly positive and its behaviour on a right neighbourhood of the singular point $x = 0$ is controlled by a function $\omega_0(x)$ which is integrable. Then we say that f has a weak space singularity at $x = 0$. Here, we will extend the existence result of [20] to f with a strong space singularity at $x = 0$, i.e. we will admit f which is controlled by a nonintegrable function $\omega_0(x)$. Our main result is contained in Theorem 2.2.

In our proofs we will need the following Fredholm type existence theorem:

Theorem 1.5 (Fredholm type existece theorem, [23]) *Let $h \in Car([a,b] \times \mathbb{R}^2)$ and $m \in L[a,b]$ be such that*

$$|h(t,x,y)| \le m(t) \text{ for a.e. } t \in [a,b] \text{ and all } x,y \in \mathbb{R} .$$

Then the problem

$$-u'' = h(t,u,u') , \quad u(a) = u(b) = 0$$

has a solution $u \in AC^1[a,b]$.

We will approximate the singular equation (1.1) by a sequence of regular equations

$$-u'' = f_n(t,u,u') , \tag{1.3}$$

where $f_n \in Car([0,T] \times \mathbb{R}^2)$, $n \in \mathbb{N}$.

Having a sequence $\{u_n\}$ of solutions of problems (1.3), (1.2) we will need to prove the existence of its converging subsequence. A type of this convergence will determine properties of its limit u and, at the same time, it will determine if u is a solution (or a w-solution) of the original problem (1.1), (1.2). The investigations of convergence will be based on next two theorems:

Theorem 1.6 (Arzelà - Ascoli theorem in $C[a,b]$ and $C^1[a,b]$, [9]) *$A \subset C[a,b]$ is relatively compact if and only if A is bounded in $C[a,b]$ and functions in A are equicontinuous on $[a,b]$.*

$B \subset C^1[a,b]$ is relatively compact if and only if B is bounded in $C^1[a,b]$ and the first derivates of functions in B are equicontinuous on $[a,b]$.

Theorem 1.7 (Fatou lemma, [19]) *Let* $\varphi_n \in L[a,b]$ *for* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} \varphi_n(t) = \varphi(t)$ *a.e. on* $[a,b]$. *Assume that there exists* $c \in (0,\infty)$ *such that*

$$\int_a^b |\varphi_n(t)|\,\mathrm{d}t \le c \text{ for each } n \in \mathbb{N}\ .$$

Then $|\varphi| \in L[a,b]$ *and* $\int_a^b |\varphi(t)|\,\mathrm{d}t \le c$. *(Clearly* $\varphi \in L[a,b]$*, as well.)*

The paper is organized as follows. In Section 2 we provide the existence principle (Theorem 2.2) giving the properties of approximating regular functions f_n in order to find a sequence of approximate solutions converging to a solution (or to a w-solution) of problem (1.1), (1.2). In Section 3 we apply the existence principle of Section 2 to get conditions which are imposed to f directly and which yields the solvability of problem (1.1), (1.2).

2 Existence principle for singular Dirichlet problems

Theorem 2.1 *Assume that* $f \in Car\,((0,T)\times\mathbb{R}_0^2)$, $f_n \in Car([0,T]\times\mathbb{R}^2)$,

$$\begin{gathered} f_n(t,x,y) = f(t,x,y) \text{ for a.e. } t \in \Delta_n \text{ and each } |x| \ge \tfrac{1}{n},\ |y| \ge \tfrac{1}{n},\ n \in \mathbb{N}\ , \\ \text{where } \Delta_n = \left[\tfrac{1}{n}, T-\tfrac{1}{n}\right] \cap [0,T]\ ; \end{gathered} \tag{2.1}$$

$$\begin{gathered} \text{there exists a bounded set } \Omega \subset C^1[0,T] \text{ such that for each} \\ n \in \mathbb{N},\ \text{problem (1.3), (1.2) has a solution } u_n \in \Omega\ . \end{gathered} \tag{2.2}$$

Then there exist $u \in C[0,T]$ *and a subsequence* $\{u_k\} \subset \{u_n\}$ *such that*

$$\lim_{k\to\infty} ||u_k - u||_C = 0\ . \tag{2.3}$$

Assume in addition that there exists a finite set $\Sigma = \{s_1, \cdots, s_\nu\} \subset (0,T)$ *such that*

$$\text{on each interval } [a,b] \subset (0,T)\setminus\Sigma \text{ the sequence } \{u_k'\} \text{ is equicontinuous.} \tag{2.4}$$

Then $u \in C^1\,((0,T)\setminus\Sigma)$ *and*

$$\lim_{k\to\infty} u_k'(t) = u'(t) \text{ locally uniformly on } (0,T)\setminus\Sigma\ . \tag{2.5}$$

Proof. By (2.2) there exist $r > 0$ and a sequence $\{u_n\}$ of solutions of (1.3), (1.2) such that

$$||u_n||_{C^1} \le r \text{ for each } n \in \mathbb{N}\ . \tag{2.6}$$

Therefore the sequence $\{u_n\}$ is bounded in $C[0,T]$ and equicontinuous on $[0,T]$. By Theorem 1.6 (Arzelà - Ascoli) we can choose a subsequence $\{u_l\}$ such that

$$\lim_{l\to\infty} ||u_l - u||_C = 0 \ , \quad u \in C[0,T] \ . \tag{2.7}$$

Now assume also (2.4) and choose an interval $[a,b] \subset (0,T) \setminus \Sigma$ arbitrarily. Then $\{u_l'\}$ is equicontinuous on $[a,b]$. By (2.6) the sequence $\{u_l\}$ is bounded in $C^1[a,b]$. Theorem 1.6 implies that we can choose a subsequence $\{u_k\} \subset \{u_l\}$ such that

$$\lim_{k\to\infty} u_k'(t) = u'(t) \text{ uniformly on } [a,b] \ .$$

By the virtue of (2.7) the sequence $\{u_k\}$ satisfies (2.3). Using the diagonalization method we can choose such $\{u_k\}$ that (2.5) holds, as well. □

Theorem 2.2 (Existence principle for problem (1.1), (1.2)) *Let all assumptions of Theorem* 2.1 *be fullfiled. Let the finite set* Σ *have the form*

$$\Sigma = \{s \in (0,T): \ u(s) = 0 \textit{ or } u'(s) = 0 \textit{ or } u'(s) \textit{ does not exist.} \tag{2.8}$$

Then $u \in AC^1_{loc}((0,T) \setminus \Sigma)$ *is a* w*-solution of* (1.1), (1.2).

Denote $s_0 = 0$ *and* $s_{\nu+1} = T$*. If* u *and* $\{u_k\}$ *satisfy* (2.3), (2.5) *and moreover there exist* $\eta \in \left(0, \frac{T}{2}\right)$, $\lambda_0, \mu_0, \lambda_1, \mu_1, \cdots, \lambda_{\nu+1}, \mu_{\nu+1} \in \{-1,1\}$ *and* $\psi \in L[0,T]$ *such that*

$$\begin{array}{ll} \lambda_i f_k\left(t, u_k(t), u_k'(t)\right) \geq \psi(t) & \textit{for a.e. } t \in (s_i - \eta, s_i) \cap (0,T) \\ \mu_i f_k(t, u_k(t), u_k'(t)) \geq \psi(t) & \textit{for a.e. } t \in (s_i, s_i + \eta) \cap (0,T) \ , \\ \multicolumn{2}{c}{\textit{for all } i \in \{0, \cdots \nu+1\} \ , \quad k \in \mathbb{N} \ ,} \end{array} \tag{2.9}$$

then $u \in AC^1[0,T]$ *is a solution of* (1.1), (1.2).

Proof.

1. Let (2.1), (2.2), (2.4) and (2.8) be true. Then for $k \in \mathbb{N}$

$$-u_k''(t) = f_k(t, u_k(t), u_k'(t)) \text{ for a.e. } t \in [0,T] \ , \tag{2.10}$$

$$u_k(0) = 0 \ , \quad u_k(T) = 0 \ ,$$

and by Theorem 2.1, there exists $u \in C[0,T]$ such that (2.3) and (2.5) hold. By (2.3), u satisfies (1.2).

Define sets

$$V_1 = \{t \in (0,T): \ f(t,\cdot,\cdot) \ : \ \mathbb{R}_0^2 \to \mathbb{R} \text{ is not continuous}\} \ ,$$

$$V_2 = \{t \in (0,T): \text{ the equality in (2.1) is not satisfied}\}$$

and let

$$U = (0,T) \setminus (\Sigma \cup V_1 \cup V_2) \ .$$

We see that

$$\operatorname{meas}(\Sigma \cup V_1 \cup V_2) = 0 \ . \tag{2.11}$$

Choose an arbitrary $t \in U$. Then there exists $k_0 \in \mathbb{N}$, such that for each $k \in \mathbb{N}$, $k \geq k_0$:

$$t \in \Delta_k \ , \quad |u_k(t)| > \frac{1}{k} \ , \quad |u_k'(t)| > \frac{1}{k}$$

and

$$f_k(t, u_k(t), u_k'(t)) = f(t, u_k(t), u_k'(t)) \ .$$

Since t is an arbitrary element of U, by (2.3), (2.5) and (2.11) we get

$$\lim_{k\to\infty} f_k(t, u_k(t), u_k'(t)) = f(t, u(t), u'(t)) \text{ a.e. on } [0,T] \ . \tag{2.12}$$

Now choose an arbitrary interval $[a,b] \subset (0,T) \setminus \Sigma$ and integrate equation (2.10). We get

$$-u_k'(t) + u_k'(a) = \int_a^t f_k(s, u_k(s), u_k'(s))\, \mathrm{d}s \text{ for each } t \in [a,b] \ . \tag{2.13}$$

Moreover there exists $k^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq k^*$

$$|f_k(t, u_k(t), u_k'(t))| \leq m(t) \text{ for a.e. } t \in [a,b] \ ,$$

where

$$m(t) = \sup\left\{ |f(t,x,y)| : \frac{1}{k^*} \leq |x| \leq r \ , \quad \frac{1}{k^*} \leq |y| \leq r \right\} \in L[a,b] \ .$$

Since $m \in L[a,b]$ we can apply Lebesgue convergence theorem on $[a,b]$ and get $f(\cdot, u(\cdot), u'(\cdot)) \in L[a,b]$. Moreover

$$\lim_{k\to\infty} \int_a^b f_k(s, u_k(s), u_k'(s))\, \mathrm{d}s = \int_a^b f(s, u(s), u'(s))\, \mathrm{d}s \ ,$$

which by (2.13) yields.

$$-u'(t) + u'(a) = \int_a^t f(s, u(s), u'(s))\, \mathrm{d}s \text{ for each } t \in [a,b] \ . \tag{2.14}$$

Since $[a,b]$ is an arbitrary interval in $(0,T) \setminus \Sigma$, we get that $u \in AC^1_{loc}((0,T) \setminus \Sigma)$ is a w-solution of (1.1), (1.2).

2. Now assume also that there exist $\eta \in \left(0, \frac{T}{2}\right)$, $\lambda_0, \mu_0, \lambda_1, \mu_1, \cdots, \lambda_{\nu+1}, \mu_{\nu+1} \in \{-1, 1\}$ and $\psi \in L[0,T]$ such that (2.9) holds. Since u is a w-solution of (1.1), (1.2), it remains to prove that $u \in AC^1[0,T]$.

Choose $i \in \{0, \cdots, \nu+1\}$ and denote $(c_i, d_i) = (s_i - \eta, s_i) \cap (0, T)$. For $k \in \mathbb{N}$ and for a.e. $t \in (c_i, d_i)$ we denote

$$h_k(t) = \lambda_i f_k(t, u_k(t), u_k'(t)) + |\psi(t)| \ , \quad h(t) = \lambda_i f(t, u(t), u'(t)) + |\psi(t)| \ .$$

Then $h_k \in L[c_i, d_i]$ and according to (2.12) we have

$$\lim_{k \to \infty} h_k(t) = h(t) \text{ for a.e. } t \in [c_i, d_i] \ .$$

Integrating (2.10) on $[c_i, d_i]$ we get

$$\int_{c_i}^{d_i} f_k(s, u_k(s), u_k'(s)) \, \mathrm{d}s = -u_k'(d_i) + u_k'(c_i) \ .$$

Therefore, by (2.6) and (2.9)

$$\int_{c_i}^{d_i} |h_k(s)| \, \mathrm{d}s = \int_{c_i}^{d_i} h_k(s) \, \mathrm{d}s = \lambda_i \int_{c_i}^{d_i} f_k(s, u_k(s), u_k'(s)) \, \mathrm{d}s$$

$$+ \int_{c_i}^{d_i} |\psi(s)| \, \mathrm{d}s \le |u_k'(d_i)| + |u_k'(c_i)| + \int_{c_i}^{d_i} |\psi(s)| \, \mathrm{d}s \le c \ ,$$

where $c = 2r + ||\psi||_L$.

Theorem 1.7 (Fatou) implies that $h \in L[c_i, d_i]$ and $f(\cdot, u(\cdot), u'(\cdot)) \in L[c_i, d_i]$.

If $(c_i, d_i) = (s_i, s_i + \eta) \cap (0, T)$ we argue similarly.

Hence $f(\cdot, u(\cdot), u'(\cdot)) \in L[0, T]$ and the equality in (2.14) is fulfilled for each $t \in [0, T]$ and $u \in AC^1[0, T]$. We have proved that u is a solution of (1.1), (1.2). □

3 Aplication of existence principle

The main result of this section is contained in Theorem 3.1, where we present conditions sufficient for the existence of a solution of problem (1.1), (1.2) which is positive on $(0, T)$.

Now, let us state our assumptions on problem (1.1), (1.2). We will be interested in the existence of a positive solution and hence we will investigate problem (1.1), (1.2) on the set $[0, T] \times [0, \infty) \times \mathbb{R}$. Denote $\mathcal{D} = (0, \infty) \times \mathbb{R}_0$. We will assume that $f \in Car([0, T] \times \mathcal{D})$ has space singularities at $x = 0$ and $y = 0$, particularly

$$\begin{cases} \limsup\limits_{x \to 0+} f(t, x, y) = \infty & \text{for a.e. } t \in [0, T] \text{ and for some } y \in \mathbb{R}_0 \ , \\ \limsup\limits_{y \to 0} f(t, x, y) = \infty & \text{for a.e. } t \in [0, T] \text{ and for some } x \in (0, \infty) \ . \end{cases} \tag{3.1}$$

Theorem 3.1 *Let* (3.1) *hold and let* $c, \gamma, \delta \in (0, \infty)$*,* $\alpha, \beta \in [0, 1]$*. Assume that there exist positive and nonincreasing functions* $\omega_0, \omega_1 \in C(0, \infty)$ *and nonnegative funtions* $h_0, h_1, h_2 \in L[0, T]$ *satisfying*

$$\int_0^T \left(t^\gamma + t^\delta\right) \omega_0(t)\, \mathrm{d}t < \infty \ , \qquad \int_0^T \omega_1(t)\, \mathrm{d}t < \infty \ . \tag{3.2}$$

$$\begin{array}{ll} T||h_1||_L + ||h_2||_L < 1 & \textit{for } \alpha, \beta = 1 \ ; \\ T||h_1||_L < 1 & \textit{for } \alpha = 1, \ \beta < 1 \ ; \\ ||h_2||_L < 1 & \textit{for } \alpha < 1, \ \beta = 1 \ ; \end{array} \tag{3.3}$$

$$\begin{gathered} c \le f(t, x, y) \le t^\gamma (T - t)^\delta \omega_0(x) + \omega_1(|y|) + h_0(t) + h_1(t) x^\alpha + h_2(t)|y|^\beta \\ \textit{for a.e. } t \in [0, T] \ , \quad \textit{and all } x \in (0, \infty) \ , \quad y \in \mathbb{R}_0 \ . \end{gathered} \tag{3.4}$$

Then problem (1.1), (1.2) *has a solution positive on* $(0, T)$*.*

The proof of Theorem 3.1 is based on the Existence principle (Theorem 2.2), where the existence of a bounded set Ω is necessary. Therefore we first prove a priori estimates for a class of functions which will be needed for the construction of such set Ω.

Lemma 3.2 *Let* $c > 0$*. Then there exists* $\eta > 0$ *such that for each* $u \in AC^1[0, T]$ *satisfying* (1.2) *and*

$$c \le -u''(t) \text{ for a.e. } t \in [0, T] \ , \tag{3.5}$$

the estimate $||u||_C \ge \eta$ *is valid.*

Proof. Let $G(t, s)$ be the Green function of problem (1.2), $-u''(t) = 0$. Then

$$G(t, s) = \begin{cases} \frac{t(T-s)}{T} & t \le s \\ \frac{s(T-t)}{T} & s \le t \ . \end{cases}$$

We define

$$\Phi(t, s) = \frac{G(t, s)}{t(T - t)} \text{ for } (t, s) \in (0, T) \times (0, T) \ .$$

For any $s \in (0, T)$ we have

$$\lim_{t \to 0+} \Phi(t, s) = \lim_{t \to 0+} \frac{t(T - s)}{t(T - t)} \frac{1}{T} = \frac{T - s}{T^2} \ ,$$

$$\lim_{t \to T-} \Phi(t, s) = \lim_{t \to T-} \frac{s(T - t)}{t(T - t)} \frac{1}{T} = \frac{s}{T^2} \ ,$$

thus we can extend $\Phi(t, s)$ continuously to $[0, T]$ and for every $s \in (0, T)$ we have $\Phi(t, s) > 0$ for $t \in [0, T]$.

We can define

$$F(t) = \int_0^T \Phi(t,s)\,\mathrm{d}s \text{ for } t \in [0,T]\ .$$

For every $t \in [0,T]$ there exists $d_0 > 0$ such that $d_0 \le cF(t)$. From equation $-u'' = -u''$ we have

$$u(t) = -\int_0^T G(t,s)u''(s)\,\mathrm{d}s \ge \int_0^T G(t,s)c\,\mathrm{d}s$$

$$= t(T-t)c\int_0^T \Phi(t,s)\,\mathrm{d}s = t(T-t)cF(t) \ge t(T-t)d_0\ .$$

$$||u||_C \ge u\left(\frac{T}{2}\right) \ge \frac{T^2 d_0}{4} = \eta\ .$$

□

Lemma 3.3 *Let* $c,\gamma,\delta > 0$, $\alpha,\beta \in [0,1]$ *functions* $\omega_0,\omega_1,h_0,h_1,h_2$ *satisfy assumptions* (3.2) *and* (3.3). *Then there exists* $r > 1$ *such that for each* $u \in AC^1[0,T]$ *satisfying* (1.2), (3.5) *and*

$$\begin{aligned} -u''(t) \le \left(\omega_0(1) + \omega_0(u(t))\right) t^\gamma (T-t)^\delta + \omega_1(1) + \omega_1(|u'(t)|) + h_0(t) \\ +h_1(t)\left((u(t))^\alpha + 1\right) + h_2(t)\left(|u'(t)|^\beta + 1\right) \end{aligned} \tag{3.6}$$

the estimate $||u||_{C^1} \le r$ *is valid.*

Proof. Condition (3.5) implies that u is nonnegative, concave and that there exists $t_0 \in [0,T]$, such that $u'(t_0) = 0$.

By Lemma 3.2 there exists $\eta > 0$, such that

$$\eta\frac{t}{T} \le \eta\frac{t}{t_0} \le u(t) \text{ for } t \in [0,t_0]\ , \tag{3.7}$$

$$\eta\frac{T-t}{T} \le \eta\frac{T-t}{T-t_0} \le u(t) \text{ for } t \in [t_0,T]\ . \tag{3.8}$$

Since ω_0 is a nonincreasing function, we have

$$\int_0^T t^\gamma (T-t)^\delta \omega_0(u(t))\,\mathrm{d}t \le \int_0^{t_0} t^\gamma (T-t)^\delta \omega_0(u(t))\,\mathrm{d}t + \int_{t_0}^T t^\gamma (T-t)^\delta \omega_0(u(t))\,\mathrm{d}t$$

$$\le \int_0^{t_0} t^\gamma (T-t)^\delta \omega_0\left(\frac{\eta t}{T}\right)\,\mathrm{d}t + \int_{t_0}^T t^\gamma (T-t)^\delta \omega_0\left(\frac{\eta(T-t)}{T}\right)\,\mathrm{d}t$$

$$\le T^\delta \int_0^{t_0} t^\gamma \omega_0\left(\frac{\eta t}{T}\right)\,\mathrm{d}t + T^\gamma \int_{t_0}^T (T-t)^\delta \omega_0\left(\frac{\eta(T-t)}{T}\right)\,\mathrm{d}t\ .$$

Without loss of generality we can assume that $\eta < T$ and, by (3.2), using substitution $z = \frac{\eta t}{T}$ in the first integral and $T - z = \frac{\eta(T-t)}{T}$ in the second integral we get

$$\int_0^T t^\gamma (T-t)^\delta \omega_0(u(t))\,\mathrm{d}t$$

$$\le T^\delta \frac{T}{\eta}\int_0^{\frac{\eta t_0}{T}} \left(\frac{Tz}{\eta}\right)^\gamma \omega_0(z)\,\mathrm{d}z + T^\gamma \frac{T}{\eta}\int_{T-\frac{\eta(T-t_0)}{T}}^T \left(\frac{T}{\eta}(T-z)\right)^\delta \omega_0(T-z)\,\mathrm{d}z$$

$$\le \frac{T^{\delta+1}}{\eta}\left(\frac{T}{\eta}\right)^\gamma \int_0^{\frac{\eta t_0}{T}} z^\gamma \omega_0(z)\,\mathrm{d}z + \frac{T^{\gamma+1}}{\eta}\left(\frac{T}{\eta}\right)^\delta \int_{T-\frac{\eta(T-t_0)}{T}}^T (T-z)^\delta \omega_0(T-z)\,\mathrm{d}z$$

$$\le \frac{T^{\gamma+\delta+1}}{\eta^2}\int_0^T \left(z^\gamma + z^\delta\right)\omega_0(z)\,\mathrm{d}z = A < \infty \ ,$$

where the constant A is independent on the function u.

Integrating $c \le -u''$ on $[t_0, t]$ we get

$$c(t_0 - t) \le u'(t) = |u'(t)| \text{ for } t \in \ [0, t_0] \ , \tag{3.9}$$

$$c(t - t_0) \le -u'(t) = |u'(t)| \text{ for } t \in [t_0, T] \ . \tag{3.10}$$

Since ω_1 is nonincreasing, we have

$$\int_0^T \omega_1(|u'(t)|)\,\mathrm{d}t = \int_0^{t_0} \omega_1(|u'(t)|)\,\mathrm{d}t + \int_{t_0}^T \omega_1(|u'(t)|)\,\mathrm{d}t$$

$$\le \int_0^{t_0} \omega_1\left(c(t_0-t)\right)\,\mathrm{d}t + \int_{t_0}^T \omega_1\left(c(t-t_0)\right)\,\mathrm{d}t \ .$$

Without loss of generality we can assume $c < 1$ and, by (3.2), substitute $z = c(t_0 - t)$ in the first integral and $z = c(t - t_0)$ in second the integral we get

$$\int_0^T \omega_1(|u'(t)|)\,\mathrm{d}t \le \int_0^{t_0} \omega_1\left(c(t_0-t)\right)\,\mathrm{d}t + \int_{t_0}^T \omega_1\left(c(t-t_0)\right)\,\mathrm{d}t$$

$$= -\frac{1}{c}\int_{ct_0}^0 \omega_1(z)\,\mathrm{d}z + \frac{1}{c}\int_0^{c(T-t_0)} \omega_1(z)\,\mathrm{d}z = B < \infty \ ,$$

where B is independent on u, as well.

We set $\max\{|u'(t)|;\ t \in [0,T]\} = \max\{|u'(0)|; |u'(T)|\} = |u'(\tau_0)| = \rho$. Then

$$-\rho T \le u(t) \le \rho T \text{ for } t \in [0, T] \ .$$

Let $C = \omega_0(1)\int_0^T t^\gamma (T-t)^\delta\,\mathrm{d}t + \omega_1(1)T$. Integrating (3.6) from τ_0 to t_0 we get

$$|u'(\tau_0)| = \rho \le C + \left|\int_{\tau_0}^{t_0} t^\gamma (T-t)^\delta \omega_0(u(s))\,\mathrm{d}s\right| + \left|\int_{\tau_0}^{t_0} \omega_1(|u'(s)|)\,\mathrm{d}s\right|$$

$$+\left|\int_{\tau_0}^{t_0} h_0(s)+h_1(s)(|u(s)|^\alpha+1)+h_2(s)(|u'(s)|^\beta+1)\,\mathrm{d}s\right|$$

and

$$\rho \le A+B+C+\left|\int_{\tau_0}^{t_0} h_0(s)+h_1(s)(|u(s)|^\alpha+1)+h_2(s)(|u'(s)|^\beta+1)\,\mathrm{d}s\right| .$$

Hence, since $|u(s)|^\alpha \le |\rho T|^\alpha$, $|u'(s)|^\beta \le \rho^\beta$ we have

$$\rho \le A+B+C+\left|\int_{\tau_0}^{t_0} h_0(s)+h_1(s)((\rho T)^\alpha+1)+h_2(s)(\rho^\beta+1)\,\mathrm{d}s\right|$$

and consequently

$$\rho \le A+B+C+||h_0||_L+((\rho T)^\alpha+1)||h_1||_L+(\rho^\beta+1)||h_2||_L . \tag{3.11}$$

By contradiction we show, that there exists constant $r^*>0$ (independent on u), such that $\rho<r^*$ for every u. Assume that there is a sequence $\{u_n\}$ satisfying (1.2), (3.5) and (3.6) and that the corresponding sequence $\{\rho_n\}$ is not bounded.

- Let $\alpha, \beta < 1$, then from (3.11) we get

  $$1 \le \frac{A+B+C+||h_0||_L+||h_1||_L+||h_2||_L}{\rho_n}+\frac{T^\alpha||h_1||_L}{\rho_n^{(1-\alpha)}}+\frac{||h_2||_L}{\rho_n^{(1-\beta)}}$$

  and for $n\to\infty$ we get

  $$1 \le 0 ,$$

  which is a contradiction.

- Let $\alpha=1$, $\beta<1$, then for $n\to\infty$ from (3.11) we get

  $$1 \le \frac{A+B+C+||h_0||_L+||h_1||_L+||h_2||_L}{\rho_n}+T||h_1||_L+\frac{||h_2||_L}{\rho_n^{(1-\beta)}} ,$$

  $$1 \le T||h_1||_L .$$

  By (3.3) we have $T||h_1||_L<1$ and we have a contradiction.

- Let $\alpha<1$, $\beta=1$, then for $n\to\infty$ from (3.11) we get

  $$1 \le \frac{A+B+C+||h_0||_L+||h_1||_L+||h_2||_L}{\rho_n}+\frac{T^\alpha||h_1||_L}{\rho_n^{(1-\alpha)}}+||h_2||_L ,$$

  $$1 \le ||h_2||_L .$$

  By (3.3) we have $||h_2||_L<1$ and we have a contradiction too.

- Let $\alpha = \beta = 1$ then

$$\rho_n \le A + B + C + ||h_0||_L + ||h_1||_L + ||h_2||_L + \rho_n T ||h_1||_L + \rho_n ||h_2||_L \ ,$$

$$1 \le \frac{A + B + C + ||h_0||_L + ||h_1||_L + ||h_2||_L}{\rho_n} + T||h_1||_L + ||h_2||_L \ .$$

By (3.3) we have $T||h_1||_L + ||h_2||_L < 1$ and thus

$$1 \le T||h_1||_L + ||h_2||_L < 1 \ .$$

we have a contradiction again.

Hence there exist $r^* > 0$ such that $\rho < r^*$ for each u satisfying (1.2), (3.5) and (3.6). Since $||u||_{C^1} \le \rho T + \rho$, we set $r = r^* T + r^* + 1$. $\square$

Proof of Theorem 3.1.

Step1. Construction of an auxiliary singular problem.

Let $r \in (1, \infty)$ be given by Lemma 3.3. For a.e. $t \in [0, T]$ and for all $x, y, z \in \mathbb{R}$ define auxiliary functions

$$\sigma(z) = \begin{cases} z & \text{for } |z| \le r \\ r \operatorname{sign} z & \text{for } |z| > r \end{cases}$$

and

$$g(t, x, y) = f(t, |\sigma(x)|, \sigma(y)) \ .$$

We will apply Theorems 2.1 and 2.2 to the auxiliary singular problem

$$-u'' = g(t, u, u') \ , \quad u(0) = 0 \ , \quad u(T) = 0 \tag{3.12}$$

and we will prove that problem (3.12) has a solution u such that

$$0 < u(t) \le r \text{ for } t \in (0, T) \text{ and } ||u'||_C \le r \ . \tag{3.13}$$

Then u will be also a solution of problem (1.1), (1.2).

Step 2. Construction of approximate regular problems.

Since f has not time singularities at $t = 0$ and $t = T$, we can put $\Delta_n = [0, T]$ for $n \in \mathbb{N}$. Now, choose an arbitrary $n \in \mathbb{N}$ and for a.e. $t \in [0, T]$ and all $x, y \in \mathbb{R}$

$$g_n(t, x, y) = \begin{cases} g(t, |x|, y) & \text{if } |x| \ge \frac{1}{n} \\ \\ g\left(t, \frac{1}{n}, y\right) & \text{if } |x| < \frac{1}{n} \end{cases}$$

and

$$f_n(t, x, y) = \begin{cases} g_n(t, x, y) & \text{if } |y| \ge \frac{1}{n} \\ \\ \frac{n}{2}\left[g_n\left(t, x, \frac{1}{n}\right)\left(y + \frac{1}{n}\right) - g_n\left(t, x, -\frac{1}{n}\right)\left(y - \frac{1}{n}\right)\right] & \text{if } |y| < \frac{1}{n} \end{cases}$$

We see that $f_n \in Car([0,T] \times \mathbb{R}^2$ fulfils

$$\begin{array}{c} f_n(t,x,y) = g(t,x,y) \text{ for a.e. } t \in [0,T] \\ \text{and all } x \in \left[\frac{1}{n}, \infty\right) \;, \quad |y| \in \left[\frac{1}{n}, \infty\right) \;. \end{array} \tag{3.14}$$

Further we have

$$c \le f_n(t,x,y) \le t^\gamma (T-t)^\delta \omega_0 \left(\frac{1}{n}\right) + \omega_1 \left(\frac{1}{n}\right) + h_0(t) + h_1(t) r^\alpha + h_2(t) r^\beta = m_n(t)$$

for a.e. $t \in [0,T]$.

Since $m_n \in L[0,T]$, Theorem 1.5 yields a solution u_n of problem

$$-u'' = f_n(t,u,u') \;, \quad u(0) = 0 \;, \quad u(T) = 0 \tag{3.15}$$

for each $n \in \mathbb{N}$.

Step 3. Convergence of sequence $\{u_n\}$ *of approximate solutions.*

By (3.4) and (3.14) we get

$$c \le -u_n''(t) \le t^\gamma (T-t)^\delta \left(\omega_0(u_n(t)) + \omega_0(1)\right) + \omega_1(1) + \omega_1\left(|u_n'(t)|\right)$$
$$+ h_0(t) + h_1(t)\left(u_n(t)^\alpha + 1\right) + h_2(t)\left(|u_n'(t)|^\beta + 1\right)$$

for a.e. $t \in [0,T]$ and all $n \in \mathbb{N}$.

Therefore, due to Lemma 3.3,

$$||u_n||_{C^1} \le r \text{ for each } n \in \mathbb{N} \;. \tag{3.16}$$

Define the set

$$\Omega = \left\{x \in C^1[0,T] : \; ||x||_{C^1} \le r\right\} \;.$$

By Theorem 2.1 there exists $u \in C[0,T]$ and a subsequence $\{u_k\} \subset \{u_n\}$ such that (2.3) holds.

Further we have

$$u_n(0) = 0 \;, \quad u_n(T) = 0 \text{ and } u_n''(t) < 0 \text{ for a.e. } t \in [0,T] \;.$$

Therefore $u_n > 0$ on $(0,T)$ and u_n has a unique maximum point $t_n \in (0,T)$. By Lemma 3.2, there is $\eta \in \left(0, \frac{rT}{2}\right)$ such that

$$u_n(t_n) > \eta \;, \quad u_n(t) \ge \begin{cases} \frac{\eta t}{T} & \text{for } t \in [0,t_n] \\ \frac{\eta(T-t)}{T} & \text{for } t \in [t_n,T] \;, \end{cases}$$

$$u_n'(t_n) = 0 \;, \quad \begin{array}{ll} c\,(t_n - t) \le u_n'(t) & \text{for } t \in [0,t_n] \\ c\,(t - t_n) \le -u_n'(t) & \text{for } t \in [t_n,T] \;, \quad n \in \mathbb{N} \end{array} \tag{3.17}$$

By (3.16) and the Mean Value Theorem for u_n on intervals $[0,t_n]$ and $[t_n,T]$ we get

$$0 < \frac{\eta}{r} \le t_n \le T - \frac{\eta}{r} < T \;, \quad n \in \mathbb{N}$$

and we can choose the subsequence $\{u_k\}$ in such a way that it satisfies (2.3) and $\lim_{k\to\infty} t_k = t_0 \in (0,T)$. Then

$$u(t) \geq \begin{cases} \frac{\eta t}{T} & \text{for } t \in [0,t_0] \\ \frac{\eta(T-t)}{T} & \text{for } t \in [t_0,T] \ . \end{cases} \tag{3.18}$$

Put $\Sigma = \{t_0\}$ and choose an arbitrary interval $[a,b] \subset (0,T) \setminus \Sigma$. For example let $[a,b] \subset (0,t_0)$. Then there exists $k_0 \in \mathbb{N}$ such that for $k \geq k_0$ we have

$$|t_k - t_0| \leq \frac{1}{2}(t_0 - b) \ , \quad [a,b] \subset \left(\frac{1}{k}, t_k\right) \ ,$$

$$u_k(t) \geq \frac{\eta a}{T} = m_0 \ , \quad u_k'(t) \geq c(t_k - t) \geq c(t_k - b) \geq \frac{1}{2}(t_0 - b) = m_1$$

for $t \in [a,b]$. If we choose $[a,b] \subset (t_0,T)$, we argue similarly.

Thus, for a.e. $t \in [a,b]$

$$|f_k(t, u_k(t), u_k'(t))| \leq \psi(t) \in L[a,b] \ ,$$

where

$$\psi(t) = \sup\{|f(t,x,y) : \quad m_0 \leq x \leq r \ , \quad m_1 \leq |y| \leq r\} \ .$$

We have proved that on each $[a,b] \subset (0,T) \setminus \Sigma$ there exists $\psi \in L[a,b]$ such that

$$|u_k''(t)| \leq \psi(t) \text{ for a.e. } t \in [a,b] \text{ and all } k \in \mathbb{N} \ , \quad k \geq k_0 \ .$$

By virtue of the absolute continuity of the Lebesgue integral we see that the sequence $\{u_k'\}$ satisfies (2.4) and, by Theorem 2.1, $u \in C^1((0,T) \setminus \Sigma)$ and (2.5) is valid.

Step 4. The function u is a solution of problem (1.1), (1.2).

Conditions (2.5) and (3.17) imply

$$\begin{array}{ll} c\,(t_0 - t) \leq u'(t) & \text{for } t \in (0,t_0) \\ c\,(t - t_0) \leq -u'(t) & \text{for } t \in (t_0,T) \ . \end{array} \tag{3.19}$$

Since u_k' is decreasing on [0,T] for each $k \geq k_0$, u' is noincreasing on $(0,t_0)$ and on (t_0,T). Therefore there exist the limits

$$\lim_{t\to t_0-} u'(t) \ , \quad \lim_{t\to t_0+} u'(t)$$

and

$$\lim_{t\to 0+} u'(t) \geq ct_0 > 0 \ , \quad \lim_{t\to T-} u'(t) \leq -c(T-t_0) < 0 \ .$$

We summarize that $u'(t) > 0$ on $[0,t_0)$ and $u'(t) < 0$ on $(t_0,T]$. So, t_0 is the unique point where $u'(t_0) = 0$ or $u'(t_0)$ does not exist. By (3.18), u is positive on $(0,T)$. Hence $\Sigma = \{t_0\}$ satisfies (2.8) and, by Theorem 2.2, $u \in AC^1_{loc}((0,T) \setminus \Sigma)$

is a w-solution of problem (3.12). Finally by (3.4), we have $f_k(t, u_k(t), u_k'(t)) \geq 0$ for a.e. $t \in [0, T]$ and all $k \in \mathbb{N}$. Hence (2.9) holds and, by Theorem 2.2, u is a solution of problem (3.12). Having in mind that $u > 0$ on $(0, T)$ and $||u_k||_{C^1} \leq r$ hold, we get by (2.3) and (2.5) that estimate (3.13) is satisfied and consequently u is a solution of problem (1.1), (1.2). $\square$

Example 3.4. Let $h_1, h_2 \in L[0, T]$ be nonnegative. For a.e. $t \in [0, T]$ and all $x, y \in \mathbb{R}$ define the function

$$f(t, x, y) = 1 + \frac{t^{\frac{3}{2}}(T-t)^{\frac{3}{2}}}{x^2} + h_1(t)\sqrt{x} + \frac{1}{\sqrt{|y|}}(1 + h_2(t)|y|) \ .$$

The second term of f has the space singularity at $x = 0$ and the last one the singularity at $y = 0$. We can check that f satisfies the conditions of Theorem 3.1 with $h_0(t) \equiv c = 1$, $\alpha = \beta = \frac{1}{2}$, $\gamma = \delta = \frac{3}{2}$, $\omega_0(x) = \frac{1}{x^2}$ and $\omega_1(|y|) = \frac{1}{\sqrt{|y|}}$.

Example 3.5. Let $T = 1$. For a.e. $t \in [0, 1]$ and all $x, y \in \mathbb{R}$ define the function

$$f(t, x, y) = \sqrt{1-t}\left(1 + \frac{t^2}{x}\right) + \frac{3}{\sqrt[3]{|y|}} + \frac{1}{6\sqrt{t}}(x + |y|) \ .$$

The first term has the space singularity at $x = 0$ and the second one at $y = 0$. We see that f satisfies the conditions of Theorem 3.1 if we put $\alpha = \beta = 1$, $\gamma = 2$, $\delta = \frac{1}{2}$, $\omega_0(x) = \frac{1}{x}$, $\omega_1(|y|) = \frac{3}{\sqrt[3]{|y|}}$, $h_0(t) = \sqrt{1-t}$, $h_1(t) = \frac{1}{6\sqrt{t}}$, $h_2(t) = \frac{1}{6\sqrt{t}}$, and choose $c > 0$ sufficiently small.

Example 3.6. Let $T = 2\pi$. We define the function f for a.e. $t \in [0, 2\pi]$ and all $x, y \in \mathbb{R}$ by

$$f(t, x, y) = t\sqrt[5]{t^3}\left(e + 10\frac{\sqrt[3]{(2\pi - t)^4}}{x^2}\right) + \frac{e}{\sqrt[5]{|y|}} + t^3\sqrt[6]{x} + \frac{5t^4 + 2t}{10000}|y| \ .$$

The function f has the space singularities at $x = 0$ and $y = 0$. We can check that f satisfies the assumptions of Theorem 3.1 for $\alpha = \frac{1}{6}$, $\beta = 1$, $\gamma = \frac{8}{5}$, $\delta = \frac{4}{3}$, $\omega_0(x) = \frac{10}{x^2}$, $\omega_1(|y|) = \frac{e}{\sqrt[5]{|y|}}$, $h_0(t) = et\sqrt[5]{t^3}$, $h_1(t) = t^3$, $h_2(t) = \frac{5t^4+2t}{10000}$ and sufficiently small c.

4 Acknowledgemetns

This research was supported by the Council of Czech Goverment MSM6198959214 and by the grant No. A100190703 of the Grant Agency of the Academy of Sciences of the Czech Republic.

References

[1] R.P. Agarwal, H. Lü and D. O'Regan: *An upper and lower solution method for one-dimensional singular p-Laplacian*, Memoirs on Differential Equations nad Math. Phys. **28**(2003), 13-31.

[2] R.P. Agarwal and D. O'Regan: *Singular boundary value problems for superlinear second order ordinary and delay differential equations*, J. Differential Equations **130**(1996), 333-355.

[3] R.P. Agarwal and D. O'Regan: *Nonlinear superlinear singular and nonsingular second order boundary value problems*, J. Differential Equations **143**(1998), 60-95.

[4] R.P. Agarwal and D. O'Regan: *A survey of Recent for Initial and Boundary Value Problems Singular in the Depend Variable*, In: Handbook of Differential Equations, Ordinary Differential Equations, Vol. 1, A. Cañada, P. Drábek, A. Fonda, eds., Elsevier, North Holland, Amsterodam (2004), pp. 1-68.

[5] R.P. Agarwal and D. O'Regan, V. Lakshmikantham: *Existence of positive solutions for singular initial and boundary value problems via the classical upper and lower solution approach*, Nonlinear Anal., Theory Methods Appl. **50**(2002), 215-222.

[6] R.G. Bartle: *A Modern Theory of Integration*, AMS Providence, Rhode Island 2001.

[7] J.V. Baxley: *Some singular nonlinear boundary value problems*, SIAM J. Math. Anal. **22**(1991), 463-479.

[8] P. Habets and F. Zanolin: *Upper and lower solutions for a generalized Emded-Fowler equation*, J. Math. Anal. Appl. **181**(1994), 684-700.

[9] J.K. Hunter, B. Nachtergaele: *Applied Analysis*, World Scientific, London 2001.

[10] D.Q. Jiang: *Upper a lower solutions method and a superlinear singular boundary value problems*, Comp. Math. Appl. **44**(2002), 323-337.

[11] I.T. Kiguradze: *On some Singular Boundary Value Problems for Ordinary Differential Equations*, (in Russian), Tbilisi Univ. Press, Tbilisi 1975.

[12] I.T. Kiguradze and B.L. Shekhter: *Singular boundary value problems for second order ordinary differential equations*, (in Russian), Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Noveishie Dostizh. **30**(1987), 105-201, translated in J. Sov. Math. **43**(1988), 2340-2417.

[13] A. Lomtatitdze: *Positive solutions of boundary value problems for second order differential equations with singular points*, (in Russian), Differentsial'nye Uravneniya **23**(1987), 1685-1692, translated in Differential Equations **23**(1987), 1146-1152.

[14] A. Lomtatidze and P. Torres: *On a two-point boundary value problem for second order singular equations*, Czechoslovak Math. J. **53**(2003), 19-43.

[15] D. O'Regan: *Theory of singular boundary value problems*, World Scientific, Singapore 1994.

[16] I. Rachůnková and S. Staněk: *Sign-changing solutions of singular Dirichlet boundary value problems*, Archives of Inequal. Appl. **1**(2003), 11-30.

[17] I. Rachůnková and S. Staněk: *Connections between types of singularities in differential equations and smoothness of solutions of Dirichlet BVPs*, Dyn. Contin. Discrete Impulsive Syst. **10**(2003), 209-222.

[18] I. Rachůnková, S. Staněk: *General existence principle for singular BVPs and its aplication*, Georgian Math. J. **11**(2004), 549-565.

[19] G.E. Shilov, B.L. Gurevich: *Integral, measure and derivative*: a unified approach. Rev. Engl. ed. Transl. by R.A. Silverman, Dover Publ., New York 1977.

[20] S. Staněk: *Positive solutions of singular positone Dirichlet boundary value problems*, Math. Comp. Modelling **33**(2001), 341-351.

[21] S.D. Taliaferro: *A nonlinear singular boundary value problem*, Nonlinear Anal., Theory Methods Appl. **3**(1979), 897-904.

[22] A. Tineo: *Existence theorems for a singular two-point Dirichlet problem*, Nonlinear Anal., Theory Methods Appl. **19**(1992), 323-333.

[23] N.I. Vasiljev, J.A. Klokov: *Foundation of the Theory of Boundary Value Problems for Ordinary Differential Equations*, Zinatne, Riga 1978 (in Russian).

Addresses:

Irena Rachůnková
Department of Mathematics
Palacký University
Tomkova 40
771 46 Olomouc
Czech Republic
e-mail: rachunko@inf.upol.cz

Jakub Stryja
Department of Mathematics and Descriptive Geometry
VŠB - Technical University of Ostrava
17. listopadu 15
708 33 Ostrava-Poruba
Czech Republic
e-mail: jakub.stryja@vsb.cz