Upper and lower solutions and topological degree

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Abstract

This paper deals with second order nonlinear boundary value problems. We suppose the existence of upper and lower solutions of the problems which are well ordered, i.e. the lower solution is less than the upper one, and we also consider the case of upper and lower solutions having the opposite ordering. We prove the relation between the topological degree and strict upper and lower solutions in both cases and using this we get the existence and multiplicity results for the boundary value problems under consideration.

1 Introduction

When we study boundary value problems for the second order differential equation

$$x'' = f(t, x, x'), \tag{1}$$

with certain linear or nonlinear boundary conditions on the compact interval $J = [a, b] \subset \mathbf{R}$ we often use the properties of lower and upper solutions for (1). Let us remind the definition.

Definition 1.1. *Let f be continuous on $J \times \mathbf{R}^2$ (or let f satisfy the Carathéodory conditions on $J \times \mathbf{R}^2$). The functions $\sigma_1, \sigma_2 \in C^2(J)$ (or $AC^1(J)$) are called lower and upper solutions for (1), if they satisfy*

$$\begin{aligned} \sigma_1''(t) &\geq f(t, \sigma_1(t), \sigma_1'(t)), \\ \sigma_2''(t) &\leq f(t, \sigma_2(t), \sigma_2'(t)), \end{aligned} \tag{2}$$

for all $t \in J$ (for a.e. $t \in J$). If the inequalities in (2) are strict, then σ_1, σ_2 are called strict lower and upper solutions.

We distinguish two basic cases:
1. The functions σ_1, σ_2 are well ordered, i.e.

$$\sigma_1(t) \leq \sigma_2(t) \text{ for all } t \in J. \tag{3}$$

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2. The functions σ_1, σ_2 are not well ordered, i.e. the condition (3) fails.

Most existence results concern the first case, but there are existence results for the second case, as well. We can refer to the papers [5], [2] or [3].

Here, we want to present existence and multiplicity results for (1) (with various boundary conditions) in the first case and also in the second case where σ_1, σ_2 have the opposite order, i.e.

$$\sigma_2(t) \leq \sigma_1(t) \text{ for all } t \in J. \tag{4}$$

Our results are based on the relation between the topological degree of the operator corresponding to the boundary value problem and strict lower and upper solutions fulfilling (3) or (4) (in the strict sense).

For getting the existence and multiplicity results we need a priori estimates of solutions of the original boundary value problem or of solutions of proper auxiliary boundary value problems. Working with σ_1, σ_2, we want to estimate the solutions just by σ_1, σ_2. For the estimation at the endpoints a, b of J we use certain connection between σ_1, σ_2 and the boundary conditions. It is well known that for the classical two-point boundary conditions such connection has the form:

- for the periodic conditions

$$x(a) = x(b), x'(a) = x'(b), \tag{5}$$

  we suppose

$$\sigma_i(a) = \sigma_i(b), \left(\sigma_i'(b) - \sigma_i'(a)\right)(-1)^i \geq 0, i = 1, 2; \tag{6}$$

- for the Neumann conditions

$$x'(a) = 0, x'(b) = 0, \tag{7}$$

  we assume

$$\sigma_i'(a)(-1)^i \leq 0, \sigma_i'(b)(-1)^i \geq 0, i = 1, 2. \tag{8}$$

Similarly,

- for the four-point conditions

$$x(a) = x(c), x(d) = x(b), a < c \leq d < b, \tag{9}$$

  σ_1, σ_2 have to satisfy

$$\begin{gathered}\left(\sigma_i(c) - \sigma_i(a)\right)(-1)^i \leq 0, \\ \left(\sigma_i(b) - \sigma_i(d)\right)(-1)^i \geq 0, i = 1, 2,\end{gathered} \tag{10}$$

- for the nonlinear conditions

$$g_1\left(x(a), x'(a)\right) = 0, g_2\left(x(b), x'(b)\right) = 0, \tag{11}$$

where $g_1, g_2 \in C(\mathbf{R}^2)$ are increasing in the second argument, we can impose on σ_1, σ_2

$$\begin{gathered} g_1\left(\sigma_i(a), \sigma_i'(a)\right)(-1)^i \leq 0, \\ g_2\left(\sigma_i(b), \sigma_i'(b)\right)(-1)^i \geq 0, i = 1, 2. \end{gathered} \tag{12}$$

Let us note that for more general nonlinear boundary conditions the compatibility of the boundary conditions with σ_1, σ_2 was introduced in [11]. For the special cases of the conditions (5), (7) and (11) this notion leads just to the assumptions (6), (8) and (12).

In this paper we will study the boundary value problems (1),(k), and we will assume the existence of lower and upper solutions σ_1, σ_2 of (1) with the property (k+1), k$\in\{5,7,9,11\}$. We will consider the classical case of f continuous on $J \times \mathbf{R}^2$, here. The case of f satisfying the Carathéodory conditions will be considered in the next paper.

The problem (1),(k), k$\in\{5,7,9,11\}$, can be written in the form of the operator equation

$$(L + N)\, x = 0, \tag{13}$$

where $L : domL \to Y$ is a linear operator and it is a Fredholm map of index 0, and $N : C^1(J) \to Y$ is, in general, nonlinear and it is $L-$compact on any open bounded set $\Omega \subset C^1(J)$. The form of L and N and the choice of the spaces $domL$ and Y depend on the type of boundary value problems. Here we put for k$\in\{5,7,9\}$ $domL = \{x \in C^2(J) : x$ satisfies (k)$\}$, $Y = C(J)$, $L : x \longmapsto x''$, $N : x \longmapsto -f(\cdot, x(\cdot), x'(\cdot))$; for the boundary condition (11) we put $domL = C^2(J)$, $Y = C(J) \times \mathbf{R}^2$, $L : x \longmapsto (x'', 0, 0)$, $N : x \longmapsto (-f(\cdot, x(\cdot), x'(\cdot)), g_1(x(a), x'(a)), g_2(x(b), x'(b)))$. For more details see [1], [6], [7].

If the equation (13) has no solution on the boundary of Ω then there exists the degree of the map $L + N$ in Ω with respect to L

$$d_L(L + N, \Omega).$$

In [4], the relation between the degree and strict lower and upper solutions satisfying (3) (in the strict sense) is shown. In the following section we will formulate and prove this relation for the above boundary value problems.

2 Well ordered lower and upper solutions

For the simplicity we will suppose that f is bounded:

$$\exists M \in (0, \infty) : |f(t, x, y)| < M \text{ for } \forall (t, x, y) \in J \times \mathbf{R}^2. \tag{14}$$

For f unbounded we can use the method of a priori estimates and replace the condition (14) by conditions of the growth or sign types. For such results see the papers [8], [9] and [10].

Theorem 2.1. *Suppose k∈{5,7,9,11}. Let (14) be fulfilled, (13) be the operator equation corresponding to the problem(1),(k) and let σ_1, σ_2 be strict lower and upper solutions of (1),(k) with*

$$\sigma_1(t) < \sigma_2(t) \text{ for all } t \in J.$$

Then

$$d_L(L + N, \Omega_1) = 1, \tag{15}$$

with

$$\begin{aligned} \Omega_1 &= \{x \in C^1(J) : \sigma_1(t) < x(t) < \sigma_2(t), |x'(t)| < c \\ \text{for all } t &\in J\}, \\ \text{where } c &\geq (2M + r + 1)(b - a) \text{ for } k \in \{5, 7, 9\} \\ \text{and } c &\geq (2M + r + 1)(b - a) + 2(r + 1)/(b - a) \text{ for } k{=}11, \\ r &= \parallel \sigma_1 \parallel_{\max} + \parallel \sigma_2 \parallel_{\max} . \end{aligned}$$

In the proof we will need the following two lemmas which concern the case of constant lower and upper solutions $-r, r$ for the problems (1),(k), k∈{5,7,9,11}.

Lemma 2.2. *Consider the problem (1),(k), k∈{5,7,9} and the corresponding equation (13). Let (14) be fulfilled and let there exists $r \in (0, \infty)$ such that*

$$f(t, -r, 0) < 0,\ f(t, r, 0) > 0 \text{ for all } t \in J. \tag{16}$$

Then

$$d_L(L + N, \Omega_2) = 1,$$

where

$$\Omega_2 = \{x \in C^1(J) : |x(t)| < r,\ |x'(t)| < c,\ \text{for all } t \in J\},$$
$$\text{with } c \geq (M + r)(b - a).$$

Proof. Let us put

$$\tilde{f}(t, x, y, \lambda) = \lambda f(t, x, y) + (1 - \lambda)x$$

for $\lambda \in [0, 1]$. Consider the parameter system of equations

$$x'' = \tilde{f}(t, x, x', \lambda),\ \lambda \in [0, 1],$$

with the boundary conditions (k), k∈{5,7,9}, and the corresponding operator equations

$$Lx + \tilde{N}(x, \lambda) = 0, \tag{17}$$

where

$$\begin{aligned} domL &= \{x \in C^2(J) : x \text{ fulfils (k)}\}, \\ L &: domL \to C(J),\ x \longmapsto x'', \\ \tilde{N}(\cdot, \lambda) &: C^1(J) \to C(J), x \longmapsto -\tilde{f}(\cdot, x(\cdot), x'(\cdot), \lambda). \end{aligned}$$

Let us show that no solution of (17) for $\lambda \in [0,1]$ and k$\in\{5,7,9\}$ lies on $\partial\Omega_2$. Suppose on the contrary that for some $\lambda \in [0,1]$ and for some solution $u \in \bar{\Omega}_2$ of (17) there exists $t_1 \in J$ such that $\max\{u(t) : t \in J\} = u(t_1) = r$. Then $u'(t_1) = 0$, $u''(t_1) \leq 0$ and, simultaneously $u''(t_1) = \lambda f(t_1, r, 0) + (1-\lambda)r > 0$, a contradiction. Supposing $\min\{u(t) : t \in J\} = -r$, we can argue similarly. Moreover, from (k) and (14) it follows

$$|u'(t)| < (M+r)(b-a) \text{ for all } t \in J.$$

Therefore the degree $d_L(L+\tilde{N}(\cdot,\lambda),\Omega_2)$ is well defined for all $\lambda \in [0,1]$. By the invariance of the degree under a homotopy we get

$$d_L(L+\tilde{N}(\cdot,0),\Omega_2) = d_L(L+\tilde{N}(\cdot,1),\Omega_2).$$

Since $Lx + \tilde{N}(x,0) = x'' - x$, we get

$$d_L(L+\tilde{N}(\cdot,0),\Omega_2) = 1.$$

From the equality $\tilde{N}(\cdot,1) = N$ the assertion of Lemma 2.2 follows. $\diamondsuit$

Lemma 2.3. *Consider the problem (1),(11) and the corresponding equation (13). Let f satisfy the assumptions of Lemma 2.2 and moreover (according to (12))*

$$\left.\begin{array}{l} g_1(-r,0) \geq 0,\ g_1(r,0) \leq 0, \\ g_2(-r,0) \leq 0,\ g_2(r,0) \geq 0. \end{array}\right\} \qquad (18)$$

Then

$$d_L(L+N,\Omega_3) = 1,$$

where

$$\Omega_3 = \{x \in C^1(J) : |x(t)| < r,\ |x'(t)| < c \textit{ for all } t \in J\}$$
$$\textit{with } c \geq (M+r)(b-a) + 2r/(b-a).$$

Proof. We can follow the proof of Lemma 2.2 with this small modification: we put

$$\begin{array}{rcl} \tilde{g}_i(x,y,\lambda) &=& \lambda g_i(x,y) + (1-\lambda)x(-1)^i,\ i=1,2, \\ domL &=& C^2(J), \\ L &:& domL \to C(J)\times\mathbf{R}^2,\ x \longmapsto (x'',0,0), \end{array}$$

$$\begin{array}{rcl} \tilde{N}(\cdot,\lambda) &:& C^1(J) \to C(J)\times\mathbf{R}^2, \\ x &\longmapsto& \left(-\tilde{f}(\cdot,x(\cdot),x'(\cdot),\lambda),\ \tilde{g}_1(x(a),x'(a),\lambda),\ \tilde{g}_2(x(b),x'(b),\lambda)\right), \end{array}$$

and prove that solutions of (17) for $\lambda \in [0,1]$ do not belong to $\partial\Omega_3$. Supposing for a $\lambda \in [0,1]$ and for a solution $u \in \bar{\Omega}_3$ of (17) that

$$\max\{u(t) : t \in J\} = u(t_1) = r,$$

we get for $u'(t_1) = 0$ the contradiction like in the proof of Lemma 2.2. For $t_1 = a$ and $u'(a) < 0$, we get by (18),

$$\tilde{g}_1\left(u(a), u'(a), \lambda\right) = \lambda g_1\left(u(a), u'(a)\right) - (1-\lambda)r < 0,$$

a contradiction. If $t_1 = b$ and $u'(b) > 0$ we use (18) for g_2. Similarly we get that $\min\{u(t) : t \in J\} \neq -r$. Thus, supposing $u \in \bar{\Omega}_3$, we have $|u(t)| < r$ on J. From the latter inequality we get a point $\xi \in J$ such that $|u'(\xi)| < 2r/(b-a)$. Since u fulfils (17), we have on J

$$u'' = \lambda f(t, u, u') + (1-\lambda)u.$$

Integrating this equation we obtain

$$|u'(t)| < (b-a)(M+r) + 2r/(b-a),$$

which implies that $u \notin \partial\Omega_3$. Therefore the degree $d_L(L + \tilde{N}(\cdot, \lambda), \Omega_3)$ is well defined for all $\lambda \in [0, 1]$ and we can finish our proof like in the proof of Lemma 2.2. $\diamondsuit$

Proof of Theorem 2.1. Put

$$h(t,x,y) = \begin{cases} f(t,\sigma_2(t),y) & \text{for} \quad x > \sigma_2(t) \\ f(t,x,y) & \text{for} \quad \sigma_1(t) \le x \le \sigma_2(t), \\ f(t,\sigma_1(t),y) & \text{for} \quad x < \sigma_1(t) \end{cases}$$

$$f^*(t,x,y) = \begin{cases} h(t,x,y) + M & \text{for} \quad x \ge r+1 \\ h(t,x,y) + (x-r)M & \text{for} \quad r < x < r+1 \\ h(t,x,y) & \text{for} \quad -r \le x \le r \\ h(t,x,y) + (x+r)M & \text{for} \quad -r-1 < x < -r \\ h(t,x,y) - M & \text{for} \quad x \le -r-1 \end{cases}$$

and

$$\begin{aligned} \Omega_2^* &= \{x \in C^1(J) : |x(t)| < r+1, \; |x'(t)| < (2M+r+1)(b-a) \\ \text{for all } t &\in J\}. \end{aligned}$$

We can see that f^* satisfies the assumptions of Lemma 2.2 with $2M$ and $r+1$.

Thus, for k∈{5,7,9}, we get

$$d_L(L + N^*, \Omega_2^*) = 1, \tag{19}$$

where L is from Lemma 2.2 and

$$N^* : C^1(J) \to C(J), \; x \longmapsto -f^*(\cdot, x(\cdot), x'(\cdot)).$$

For k=11 we put

$$\varphi_i(x,y) = \begin{cases} g_i(\sigma_2(t),y) & \text{for} \quad x > \sigma_2(t) \\ g_i(x,y) & \text{for} \quad \sigma_1(t) \le x \le \sigma_2(t) \\ g_i(\sigma_1(t),y) & \text{for} \quad x < \sigma_1(t) \end{cases} \;, \; i = 1, 2,$$

and

$$g_i^*(x,y) = \begin{cases} \varphi_i(x,y) + m(-1)^i & \text{for} \quad x \geq r+1 \\ \varphi_i(x,y) + (x-r)m(-1)^i & \text{for} \quad r < x < r+1 \\ \varphi_i(x,y) & \text{for} \quad -r \leq x \leq r \\ \varphi_i(x,y) + (x+r)m(-1)^i & \text{for} \quad -r-1 < x < -r \\ \varphi_i(x,y) - m(-1)^i & \text{for} \quad x \leq -r-1 \end{cases},$$

where

$$m = \max\left\{ \sum_{i,,j=1}^{2} |g_i(\sigma_j(t),0)| : t \in J \right\}.$$

Let L be from Lemma 2.3 and

$$\begin{aligned} H^* \quad &: \quad C^1(J) \to C(J) \times \mathbf{R}^2, \\ x \quad &\longmapsto \quad (-f^*(\cdot, x(\cdot), x'(\cdot)), g_1^*(x(a), x'(a)), g_2^*(x(b), x'(b))) \,. \end{aligned}$$

Put

$$\begin{aligned} \Omega_3^* \quad &= \quad \{x \in C^1(J) : |x(t)| < r+1, \, |x'(t)| < \\ &\quad\quad (2M+r+1)(b-a) + 2(r+1)/(b-a) \\ \text{for all } t \quad &\in \quad J\}. \end{aligned}$$

Since g_1^*, g_2^* satisfy (18) with $r+1$, we get from Lemma 2.3

$$d_L(L + H^*, \Omega_3^*) = 1. \tag{20}$$

Now, let us show that:

(i) for k$\in$$\{5,7,9\}$, each solution u of the equation $(L+N^*)x = 0$ satisfies $u \in \Omega_2^* \Rightarrow u \in \Omega_1$;

(ii) for k=11, each solution u of the equation $(L+H^*)x = 0$ satisfies $u \in \Omega_3^* \Rightarrow u \in \Omega_1$.

Suppose the contrary and put

$$v_2(t) = u(t) - \sigma_2(t) \ , \ v_1(t) = \sigma_1(t) - u(t).$$

Then for an $i \in \{1,2\}, \max\{v_i(t) : t \in J\} = v_i(t_0) \geq 0$.

(i) By (k), k$\in$$\{5,7,9\}$, $v_i'(t_0) = 0$, $v_i''(t_0) \leq 0$ for $t_0 \in (a,b)$ as well as for $t_0 = a$, $t_0 = b$. On the other hand if $i = 2$, $v_2''(t_0) = u''(t_0) - \sigma_2''(t_0) \geq f(t_0, \sigma_2(t_0), \sigma_2'(t_0)) - \sigma_2''(t_0) > 0$ and if $i = 1$, $v_1''(t_0) = \sigma_1''(t_0) - u''(t_0) > 0$. We get the contradiction in the both cases.

(ii) If k=11, then either $v_i'(t_0) = 0$ and $v_i''(t_0) \le 0$ and we get the same contradiction like in (i), or t_0 is one of the endpoints of J and $v_i'(t_0) \neq 0$. If $t_0 = a$, then $v_i'(a) < 0$ and provided $i = 2$ we have

$$g_1^*(u(a), u'(a)) \le g_1(\sigma_2(a), u'(a)) < g_1(\sigma_2(a), \sigma_2'(a)) \le 0,$$

and provided $i = 1$ we have

$$g_1^*(u(a), u'(a)) \ge g_1(\sigma_1(a), u'(a)) > g_1(\sigma_1(a), \sigma_1'(a)) \ge 0,$$

a contradiction. For $t_0 = b$ we can use the similar arguments.

So, by the excision property of the degree, using (19) and (20), we get

$$d_L(L + N^*, \Omega_1) = 1$$

for k$\in\{5,7,9\}$, and

$$d_L(L + H^*, \Omega_1) = 1$$

for k=11.

Since $N^* = N$ for k$\in\{5,7,9\}$ ($H^* = N$ for k=11) on $\bar{\Omega}_1$, Theorem 2.1 is proved. $\diamondsuit$

3 Upper and lower solutions with opposite order

Theorem 3.1. *Suppose k$\in\{5,7,9\}$. Let (14) be fulfilled, (13) be the operator equation corresponding to the problem (1),(k) and let σ_1, σ_2 be strict lower and upper solutions of (1),(k) satisfying*

$$\sigma_2(t) < \sigma_1(t) \text{ for all } t \in J.$$

Then

$$d_L(L + N, \Omega_4) = -1, \tag{21}$$

where

$$\begin{aligned} \Omega_4 &= \{x \in C^1(J) : \|x\|_{\max} < A,\ \|x'\|_{\max} < B, \\ \exists t_x &\in J : \sigma_2(t_x) < x(t_x) < \sigma_1(t_x)\}, \end{aligned}$$

with $B \ge 2(b-a)M$, $A \ge \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max} + 2(b-a)^2 M$.

Proof. Put

$$f^*(t,x,y) = \begin{cases} f(t,x,y) + M & \text{for} \quad x \ge A+1 \\ f(t,x,y) + (x-A)M & \text{for} \quad A < x < A+1 \\ f(t,x,y) & \text{for} \quad -A \le x \le A \\ f(t,x,y) + (A+x)M & \text{for} \quad -A-1 < x < -A \\ f(t,x,y) - M & \text{for} \quad x \le -A-1 \end{cases},$$

$$\Omega = \{x \in C^1(J) : \|x\|_{\max} < A+1,\ \|x'\|_{\max} < B + (A+2)(b-a)\}.$$

We can see that f^* satisfies (14) with $2M$ and (16) with $A+1$. Thus, by Lemma 2.2

$$d_L(L + F^*, \Omega) = 1, \tag{22}$$

where $F^* : C^1(J) \to C(J)$, $x \longmapsto -f^*(\cdot, x(\cdot), x'(\cdot))$.

Now, consider the pairs $-A-1, \sigma_2(t)$ and $\sigma_1(t), A+1$. They are well ordered strict lower and upper solutions for the problem

$$x'' = f^*(t, x, x'), (k). \tag{23}$$

So, we can define the sets

$$\Delta_1 = \{x \in \Omega : \sigma_1(t) < x(t) \text{ for all } t \in J\}$$

and

$$\Delta_2 = \{x \in \Omega : x(t) < \sigma_2(t) \text{ for all } t \in J\}.$$

By Theorem 2.1, we get

$$d_L(L + F^*, \Delta_1) = 1 \tag{24}$$

and

$$d_L(L + F^*, \Delta_2) = 1. \tag{25}$$

Now, consider the set

$$\Delta = \Omega \backslash (\overline{\Delta_1 \cup \Delta_2}).$$

We can see that

$$\Delta = \{x \in \Omega : \exists t_x \in J \text{ with } \sigma_2(t_x) < x(t_x) < \sigma_1(t_x)\}.$$

Let us show that if $u \in \bar{\Delta}$ is a solution of (23), than $u \notin \partial\Delta$. Clearly $u \notin \partial\Omega$ because

$$\|u'\|_{\max} < 2(b-a)M,\ \|u\|_{\max} < A. \tag{26}$$

Put $v_2(t) = u(t) - \sigma_2(t)$ and $v_1(t) = \sigma_1(t) - u(t)$ and suppose $u \in \partial\Delta$. Then for an $i \in \{1, 2\}$ we have $\max\{v_i(t) : t \in J\} = v_i(t_0) = 0$. Now we can get a contradiction like in (i) in the proof of Theorem 2.1. Thus $u \notin \partial\Delta$, and by the additivity property of the degree

$$\begin{aligned} d_L(L + F^*, \Omega) \quad &= \quad d_L(L + F^*, \Delta_2) + \\ &\qquad d_L(L + F^*, \Delta_1) + d_L(L + F^*, \Delta). \end{aligned}$$

From (22), (24) and (25) it follows that

$$d_L(L + F^*, \Delta) = -1.$$

With respect to (26) and the excision property of the degree we get

$$d_L(L + F^*, \Omega_4) = -1.$$

Since $F^* = N$ on Ω_4, Theorem 3.1 is proved. $\diamondsuit$

Theorem 3.2. *Suppose k=11. Let all other assumptions of Theorem 3.1 be fulfilled. Moreover suppose that g_1 is nonincreasing and g_2 nondecreasing in the first argument. Then the assertion of Theorem 3.1 is valid with $B \geq 2(b-a)M + \|\sigma_2'\|_{\max}$, $A \geq \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max} + (b-a)B$.*

Proof. Put

$$g_i^*(x,y) = \begin{cases} g_i(x,y) + m(-1)^i & \text{for} \quad x \geq A+1 \\ g_i(x,y) + (x-A)m(-1)^i & \text{for} \quad A < x < A+1 \\ g_i(x,y) & \text{for} \quad -A \leq x \leq A \\ g_i(x,y) + (x+A)m(-1)^i & \text{for} \quad -A-1 < x < -A \\ g_i(x,y) - m(-1)^i & \text{for} \quad x \leq -A-1 \end{cases},$$

where $m = \max\left\{\sum_{i,j=1}^{2} |g_i((A+1)(-1)^j, 0)| : t \in J\right\}$ and consider f^* from the proof of Theorem 3.1. Define the set

$$\Omega = \left\{ \begin{array}{c} x \in C^1(J) : \|x\|_{\max} < A+1,\ \|x'\|_{\max} < B + (A+2)(b-a) + \\ 2(A+2)/(b-a) + \|\sigma_2'\|_{\max} \end{array} \right\}.$$

We can see that f^*, g_1^* and g_2^* satisfy (14) with $2M$ instead of M and (16), (18) with $A+1$ instead of r. Therefore if we put

$$H^* : C^1(J) \to C(J) \times \mathbf{R}^2$$

$$x \longmapsto (-f^*(\cdot, x(\cdot), x'(\cdot)), g_1^*(x(a), x'(a)), g_2^*(x(b), x'(b)))$$

and

$$L : C^2(J) \to C(J) \times \mathbf{R}^2,\ x \longmapsto (x'', 0, 0),$$

we get by Lemma 2.3

$$d_L(L + H^*, \Omega) = 1.$$

By the same way like in the proof of Theorem 3.1, we define the sets $\Delta_1, \Delta_2, \Delta$ and get

$$d_L(L + H^*, \Delta_1) = 1$$

and

$$d_L(L + H^*, \Delta_2) = 1.$$

Now, we need to prove that for any solution u of the problem

$$x'' = f^*(t, x, x'),\ (11) \tag{27}$$

the implication $u \in \bar{\Delta} \Longrightarrow u \notin \partial\Delta$ holds. Let us put $v(t) = u(t) - \sigma_2(t)$. Since $u \in \bar{\Delta}$, there exists a $t_u \in J$ with $v(t_u) \geq 0$. Suppose $v'(t) > 0$ for all $t \in J$. Then $v(b) \geq 0$ and $g_2(u(b), u'(b)) > g_2(\sigma_2(b), \sigma_2'(b)) \geq 0$, a contradiction. If $v'(t) < 0$ on J, we get $v(a) \geq 0$ and the contradiction $g_1(u(a), u'(a)) > 0$. Therefore $v'(t_0) = 0$ for a $t_0 \in J$, i.e.

$$u'(t_0) = \sigma_2'(t_0).$$

(Similarly we can prove $u'(t_1) = \sigma_1'(t_1)$ for a $t_1 \in J$.) Integrating the equation in (27) we get $|u'(t)| < B$ on J which implies $|u(t)| < A$ on J. Thus $u \notin \partial\Omega$. Suppose $u \in \partial\Delta$ and put

$$v_i(t) = (u(t) - \sigma_i(t))\,(-1)^i,\ i \in \{1,2\}.$$

Then we can find an $i \in \{1,2\}$ and a $t_0 \in J$ such that

$$\max\{v_i(t) : t \in J\} = v_i(t_0) = 0.$$

We can argue like in (ii) in the proof of Theorem 2.1 and get a contradiction. Since we have proven $u \in \bar{\Delta} \Longrightarrow u \in \Omega_4$,we finish this proof by the same way like the proof of Theorem 3.1 working with H^* instead of F^*. ♢

Corollary 3.3. *Suppose k∈{5,7,9,11}. If σ_1, σ_2 in Theorem 2.1 (3.1, 3.2) are not strict, then either the problem (1),(k) has a solution on $\partial\Omega_1$($\partial\Omega_4$) or the condition (15) ((21)) is valid.*

Proof. Suppose that all assumptions of Theorem 2.1 are fulfilled but σ_1, σ_2 are not strict. Let us choose $\mu_0 \in (0,\infty)$ such that

$$|f(t,x,y) + \mu_0| < M \ \text{ for all } (t,x,y) \in J \times \mathbf{R}^2,$$

and for $\mu \in [0, \mu_0]$ put

$$\epsilon(t,\mu,x) = \begin{cases} \mu & \text{for} \quad x \geq \sigma_2(t) \\ \mu\frac{2x-\sigma_2(t)-\sigma_1(t)}{\sigma_2(t)-\sigma_1(t)} & \text{for} \quad \sigma_1(t) < x < \sigma_2(t) \\ -\mu & \text{for} \quad x \leq \sigma_1(t) \end{cases},$$

$$f_\mu(t,x,y) = f(t,x,y) + \epsilon(t,\mu,x).$$

Then for any $\mu \in (0, \mu_0]$, σ_1 and σ_2 are strict lower and upper solutions to the problem

$$x'' = f_\mu(t,x,x'),\ (k). \tag{28}$$

If we define the operator

$$N_\mu : C^1(J) \to C(J),\ x \longmapsto -f_\mu(\cdot, x(\cdot), x'(\cdot)),$$

for $\mu \in [0, \mu_0]$, then, by Theorem 2.1

$$d_L(L + N_\mu, \Omega_1) = 1$$

for each $\mu \in (0, \mu_0]$. Suppose that no solution of $(L+N)x = 0$ lies on $\partial\Omega_1$. Then, using the invariance of the degree under a homotopy and the fact that $N_0 = N$, we get (15). In the case of Theorems 3.1 and 3.2 we can use the same arguments but σ_1 and σ_2 interchange themselves in the formula for the function $\epsilon(t,\mu,x)$. ♢

4 Existence results

As the direct consequence of the Corollary 3.3, using a limiting process, we obtain the following existence results for the problems (1),(k), k$\in\{5,7,9,11\}$.

Theorem 4.1. *Suppose k$\in\{5,7,9,11\}$. Let (14) be fulfilled and let σ_1, σ_2 be lower and upper solutions of (1),(k) with*

$$\sigma_1(t) \leq \sigma_2(t) \text{ for all } t \in J.$$

Then the problem (1),(k) has at least one solution in $\overline{\Omega}_1$, where Ω_1 is the set from Theorem 2.1.

Remark 4.2. *The existence results of Theorem 4.1 are known and they are presented here for the completeness, only.*

Theorem 4.3. *Suppose k$\in\{5,7,9,11\}$. Let (14) be fulfilled and let σ_1, σ_2 be lower and upper solutions of (1),(k) with*

$$\sigma_2(t) \leq \sigma_1(t) \text{ for all } t \in J.$$

For k=11 suppose g_1 nonincreasing and g_2 nondecreasing in the first argument. Then the problem (1),(k) has at least one solution in $\overline{\Omega}_4$, where for k$\in\{5,7,9,\}$ Ω_4 is the set from Theorem 3.1 and for k=11 it is the set from Theorem 3.2.

Remark 4.4. *For k$\in\{5,7\}$ the similar existence result is proven in [5] or [2].*

5 Multiplicity results

In this section, using Theorems 2.1, 3.1 and 3.2, we get several multiplicity results for (1),(k), both for the linear two-point or multipoint boundary conditions k$\in\{5,7\}$ or k=9, and for the nonlinear boundary condition k=11. In the last case we suppose that g_1 is nonincreasing and g_2 nondecreasing in the first argument.

Theorem 5.1. *Suppose k$\in\{5,7,9,11\}$. Let (14) be fulfilled and let $\sigma_1, \sigma_2, \sigma_3$ be strict lower, upper and lower solutions of (1),(k) with*

$$\sigma_1(t) < \sigma_2(t) < \sigma_3(t) \text{ for all } t \in J. \tag{29}$$

Then (1),(k) has at least two different solutions u, v satisfying

$$\sigma_1(t) < u(t) < \sigma_2(t), \sigma_1(t) < v(t) \text{ for all } t \in J,$$

$$\sigma_2(t_v) < v(t_v) < \sigma_3(t_v) \text{ for a } t_v \in J.$$

Proof. From Theorem 2.1 it follows the existence of a solution u in Ω_1. We define an auxiliary function

$$h(t,x,y) = \begin{cases} f(t,x,y) & \text{for} \quad x \geq \sigma_1(t) \\ f(t,\sigma_1(t),y) & \text{for} \quad x < \sigma_1(t) \end{cases}$$

and from Theorem 3.1 (Theorem 3.2 for k=11) we get a solutions v of the problem

$$x'' = h(t,x,x'), (k).$$

Moreover, v lies in Ω_4 which is defined by the couple σ_2, σ_3 instead of σ_1, σ_2. The inequality $\sigma_1(t) < v(t)$ on J can be proven like in (i) or (ii) in the proof of Theorem 2.1. This inequality implies that v is a solutions of (1),(k), as well. $\diamondsuit$

The dual situation is described in Theorem 5.2.

Theorem 5.2. *Let all assumptions of Theorem 5.1 be fulfilled with the exception that now $\sigma_1, \sigma_2, \sigma_3$ are strict upper, lower and upper solutions. Then (1),(k) has at least two different solutions u, v satisfying*

$$\sigma_2(t) < v(t) < \sigma_3(t), u(t) < \sigma_3(t) \text{ for all } t \in J,$$

$$\sigma_1(t_u) < u(t_u) < \sigma_2(t_u) \text{ for a } t_u \in J.$$

For constant lower and upper solutions we get the multiplicity result of the Ambrosetti-Prodi type.

Theorem 5.3. *Suppose k∈{5,7,9}. Let (14) be fulfilled and let $n \in \mathbf{N}$, $n \geq 2$, $s_1 \in (-M, M)$, $r_1, ..., r_{n+1} \in \mathbf{R}$ be such that*

$$r_1 < r_2 < ... < r_{n+1}$$

and

$$(s_1 - f(t, r_i, 0))\,(-1)^i > 0 \text{ for all } t \in J,\ i \in \{1, ..., n\}. \tag{30}$$

Then there exist $s_2, s_3 \in (-M, s_1)$, $s_3 \leq s_2$, such that the problem

$$x'' + f(t,x,x') = s,\ (k) \tag{31}$$

has:
(i) at least n different solutions greater than r_1 for $s \in (s_2, s_1]$;
(ii) at least $\frac{n+1}{2}$ ($\frac{n}{2}$) solutions greater than r_1 for $s = s_2$ and n odd (even);
(iii)provided $s_3 < s_2$ at least one solution greater or equal to r_1 for $s \in [s_3, s_2)$;
(iv) no solution greater or equal to r_1 for $s < s_3$.

Proof. Let $j \in \{1, ..., n+1\}$. The condition (30) implies that there exists $s_2 < s_1$ such that for j odd (even) r_j is a strict lower (an upper) solution to (31) for $s \in (s_2, s_1]$. Therefore, using Theorem 5.1 we get (i). For $s = s_2$ at least one of the strict upper solutions r_j of the problem (31) becames nonstrict and so two solutions of this problem can identify. In the case where all the upper solutions became nonstrict for $s = s_2$, all neighbour pairs of solutions of (31) can be identic. Thus (ii) is proved. Suppose that x is a solution of (31). Let k=5, 7. Then, integrating the equation (31) from a to b and using (14), we get $-M < s$. For k=9 we integrate from α to β where $\alpha \in (a, c), \beta \in (d, b)$ are zeros of x' and get $-M < s$ as well. Thus for $s \leq -M$ the problem (31) has no solution. Suppose that for some $s^* \in (-M, s_1)$ the problem (31) has a solution u^*. Then there exists a solution of (31) for all $s \in [s^*, s_1]$, because u^* is an upper solution and r_1 a lower solution of (31) for $s \in [s^*, s_1]$, and $u^*(t) > r_1$ on J. So, we can put $s_3 = \inf\{s : s < s_1,$ (31) has a solution greater than $r_1\}$. Then $s_3 \in (-M, s_2]$. If $s_3 < s_2$, we consider a sequence $\{\sigma_n\} \subset (s_3, s_2)$ converging to s_3 and the corresponding sequence of solutions $\{u_n\}$ of the problems $\{(31),\ s{=}\sigma_n\}$. This sequence is equi-bounded and equi-continuous in $C^1(J)$ and by the Arzelà-Ascoli theorem, we can choose a subsequence converging in the space $C^1(J)$ to a solution of (31) for $s = s_3$. Thus (iii) and (iv) are valid. $\diamondsuit$

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References

[1] R.E. Gaines & J.L. Mawhin: Coincidence Degree and Nonlinear Differential Equations, Lecture Notes in Math. 568, Springer-Verlag, Berlin 1977.

[2] J.P. Gossez & P. Omari: Periodic solutions of a second order ordinary differential equation: a necessary and sufficient condition for nonresonance, J. Differential Equations 94 (1991), 67-82.

[3] P. Habets & P. Omari: Existence and localization of solutions of second order elliptic problems using lower and upper solutions in the reversed order, preprint U.C.L., June 1994, in print on Topological Methods in Nonlinear Analysis.

[4] J. Mawhin: Points fixes, points critiques et problèmes aux limites, Sémin. Math. Sup., No. 92, Presses Univ. Montréal, Montréal 1985.

[5] P. Omari: Non-ordered lower and upper solutions and solvability of the periodic problem for the Liénard and the Rayleigh equations, Rend. Inst. Mat. Univ. Trieste, 20 (1988), 54-64.

[6] I. Rachůnková: Sign conditions in nonlinear boundary value problems, Acta Univ. Palack. Olom., Fac. rer. nat. 114, 33 (1994), 117-124.

[7] I. Rachůnková: On a transmission problem, Acta Univ. Palack. Olom., Fac.rer.nat. 105, 31 (1992), 45-59.

[8] I. Rachůnková: Multiplicity results for four-point boundary value problems, Nonlinear Analysis TMA 18 (1992), 495-505.

[9] I. Rachůnková: On the existence of two solutions of the periodic problem for the ordinary second-order differential equation, Nonlinear Analysis TMA 22(1994), 1315-1322.

[10] I. Rachůnková: Upper and lower solutions and multiplicity results. Preprint 25/1997, Faculty of Science, Palacky University, Olomouc.

[11] H.B. Thompson: Second order ordinary differential equations with fully nonlinear two-point boundary conditions, Pacif. Journal Math., 172 (1996), 255-297.