Zeros of derivatives of solutions to singular $(p, n-p)$ conjugate BVPs *

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Abstract: Positive solutions of the singular $(p, n-p)$ conjugate BVP are studied. The set of all zeros of their derivatives up to order $n-1$ is described. By means of this, estimates from below of the solutions and the absolute values of their derivatives up to order $n-1$ on the considered interval are reached. Such estimates are necessary for the application of the general existence principle to the BVP under consideration.

Key words: Singular conjugate BVP, positive solutions, zeros of derivatives, estimates from below.

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1 Introduction

Let $n, p \in \mathbb{N}$, $n > 2$, $p \leq n-1$, and T be a positive number. In [?] (for $p = 1$) and [?], the authors have considered the singular $(p, n-p)$ conjugate boundary value problem (BVP)

$$(-1)^p x^{(n)}(t) = f(t, x(t), \ldots, x^{(n-1)}(t)), \tag{1.1}$$

$$x^{(i)}(0) = 0, \quad x^{(j)}(T) = 0 \quad 0 \leq i \leq n-p-1, \ 0 \leq j \leq p-1, \tag{1.2}$$

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where f satisfies the local Carathéodory conditions on the set $\mathcal{D} = [0,T] \times ((0,\infty) \times \mathbb{R}_0^{n-1})$ with $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$ and f is singular at the value 0 of each its phase variable. They have given conditions on f guaranteeing the existence of a positive (on $(0,T)$) solution to BVP (??), (??). The singularities of the function f in (??) 'appear' in any positive solution of BVP (??), (??) and some its derivatives at the fixed points $t = 0$, $t = T$, and all its derivatives up to order $n-1$ 'pass through' singularities of f also at inner points of the interval $(0,T)$ which are not fixed. Therefore for proving the solvability of BVP (??), (??) in the class of positive functions on $(0,T)$ it is very important to give a decomposition analysis of zeros of derivatives up to order $n-1$ of positive solutions to BVP (??), (??). This analysis have been presented for $p = 1$ in [?] and for $p = 2$ in [?] under the assumption that $f \geq c$ on $\mathcal{D}$ with a positive constant c. The aim of this paper is to complete this analysis for all values of p. We note that the singular differential equation

$$(-1)^p x^{(n)}(t) = \phi(t) g(t, x(t)) \tag{1.3}$$

together with the boundary conditions (??) have been discussed for $\phi(t)g(t,x) : (0,1) \times (0,\infty) \to (0,\infty)$ continuous in [?], [?], [?] and [?] (in [?] and [?] with $\phi = 1$). But for BVP (??), (??) singularities of g 'appear' in its positive solutions only at the fixed points $t = 0$ and $t = 1$.

2 Decomposition analysis of zeros to solutions of BVP (??), (??)

Let c be a positive constant and let f in (??) satisfy $f \geq c$ on $\mathcal{D}$. Then the decomposition analysis of zeros to solutions of BVP (??), (??) and their derivatives up to order $n-1$ can be studied by the decomposition analysis of zeros to solutions of the differential inequality

$$(-1)^p x^{(n)}(t) \geq c \tag{2.1}$$

satisfying the boundary conditions (??). By a *solution of problem* (??), (??) we understand a function $x \in AC^{n-1}([0,T])$ (functions having absolutely continuous $(n-1)^{\text{st}}$ derivative on $[0,T]$) satisfying (??) for a.e. $t \in [0,T]$ and fulfilling (??).

Having a solution x of problem (??), (??) we are interested in zeros of $x^{(k)}$, $0 \leq k \leq n-1$, belonging to $(0,T)$. Without loss of generality we can suppose

$$p - 1 \leq n - p - 1 \tag{2.2}$$

that is $p \leq n/2$, because by replacing t by $T-t$ we can transform the case $n/2 < p$ to (??).

For $p = 1, 2$ we have already studied zeros of $x^{(k)}$ and we have proved the following results:

Lemma 2.1. *Let x be a solution of problem (**??**), (**??**) for $p = 1$. Then $x > 0$ on $(0, T)$ and $x^{(k)}$ has just one zero in $(0, T)$, $1 \le k \le n - 1$.*
Proof. Lemma follows from [**?**], Lemmas 2.7 and 2.9. □

Lemma 2.2. *Let x be a solution of problem (**??**), (**??**) for $p = 2$. Then*

(i) $x > 0$ *on* $(0, T)$,

(ii) $x^{(k)}$ *has just one zero in* $(0, T)$ *for* $k = 1$ *and* $k = n - 1$,

(iii) $x^{(k)}$ *has just two zeros in* $(0, T)$ *for* $2 \le k \le n - 2$.

Proof. See [**?**], Lemmas 2.2. □

Decomposition analysis of zeros to solutions of BVP (**??**), (**??**) with $p \ge 3$ is described in the next theorem.

Theorem 2.3. *Let x be a solution of problem (**??**), (**??**) for $p \ge 3$ and let (**??**) hold. Then*

(i) $x > 0$ *on* $(0, T)$,

(ii) $x^{(k)}$ *has just j zeros in* $(0, T)$ *for* $k = j$ *and* $k = n - j$ *where* $j = 1, 2, \ldots, p - 1$,

(iii) $x^{(k)}$ *has just p zeros in* $(0, T)$ *for* $p \le k \le n - p$.

Proof. The proof is divided into three parts.

I. *Lower bounds for zeros.* By (**??**) we see that x' has at least one zero $t_1^1 \in (0, T)$. Hence $x'(0) = x'(t_1^1) = x'(T) = 0$, which implies that x'' has at least two zeros $t_1^2, t_2^2 \in (0, T)$. So, we have $x''(0) = x''(t_1^2) = x''(t_2^2) = x''(T) = 0$. By induction we conclude that $x^{(j)}$, $j = 3, \ldots, p - 1$, has at least j zeros $t_1^j, \ldots, t_j^j \in (0, T)$ and, due to (**??**) and (**??**) $x^{(j)}(0) = x^{(j)}(t_1^j) = \ldots = x^{(j)}(t_j^j) = x^{(j)}(T) = 0$, $j = 3, \ldots, p - 1$. Therefore $x^{(p)}$ hat at least p zeros in $(0, T)$. Now we will distinguish two cases: $p < n/2$ and $p = n/2$.

1. Let $p < n/2$. Then $p \le n - p - 1$ and, by (**??**),

$$x^{(j)}(0) = 0, \quad j = p, \ldots, n - p - 1.$$

Thus $x^{(k)}$ has at least p zeros in $(0, T)$ for $k = p + 1, \ldots, n - p$.

2. Let $p = n/2$ (clearly n is even in this case). Then $p = n - p$ and $x^{(n-p)}$ has at least p zeros in $(0, T)$.

Hence we have shown that $x^{(n-p)}$ has at least p zeros in $(0, T)$ in the both cases. Since for $x^{(n-j)}$, $1 \le j \le p - 1$, we cannot already use (**??**), we deduce that $x^{(n-j)}$ has at least j zeros in $(0,T)$ for $j=1,\dots,p-1$. Particularly $x^{(n-1)}$ has at least one zero in $(0,T)$.

II. *Exact number of zeros.* By (**??**), $x^{(n-1)}$ is strictly monotonous and hence it has just one zero in $(0,T)$. Therefore, by I, we deduce that $x^{(n-k)}$ has just k zeros in $(0,T)$ for $2 \le k \le p-1$ and $x^{(k)}$ has just p zeros in $(0,T)$ for $p \le k \le n-p$. Similarly, $x^{(k)}$ has just k zeros in $(0,T)$ for $1 \le k \le p-1$ and x has no zero in $(0,T)$.

III. *Positivity of* x. Denote by t_1^k the first zero of $x^{(k)}$ in $(0,T)$, $1 \le k \le n-1$. Inequality (**??**) implies that $(-1)^p x^{(n-1)} < 0$ on $[0, t_1^{n-1})$ and $(-1)^p x^{(n-2)} > 0$ on $[0, t_1^{n-2})$. Therefore $(-1)^{p+j} x^{(n-j)} > 0$ on $(0, t_1^{n-j})$ for $j = 3, \dots, p$. Particularly we have $x^{(n-p)} > 0$ on $(0, t_1^p)$, wherefrom, by virtue of (**??**), we obtain $x^{(k)} > 0$ on $(0, t_1^k)$, $1 \le k \le n-p-1$, and consequently $x > 0$ on $(0,T)$. □

Our next theorem provides estimates from below of solutions to problem (**??**), (**??**) and of the absolute value of their derivatives up to order $n-1$ on the interval $[0,T]$. These estimations are necessary to apply the general existence principle of [?] to problem (**??**), (**??**) with f in (**??**) satisfying the inequality $f \ge c$ on $\mathcal{D}$.

Theorem 2.4. *Let x be a solution of problem (**??**), (**??**). Then for any $i \in \{1, \dots, n-1\}$ there are $p_i + 1$ disjoint intervals (a_k, a_{k+1}), $0 \le k \le p_i$, $p_i < (n-1)p$, such that*

$$\bigcup_{k=0}^{p_i} [a_k, a_{k+1}] = [0,T] \tag{2.3}$$

and for each $k \in \{0, \dots, p_i\}$ one of the inequalities

$$|x^{(n-i)}(t)| \ge \frac{c}{i!}(t - a_k)^i \quad \text{for } t \in [a_k, a_{k+1}] \tag{2.4}$$

or

$$|x^{(n-i)}(t)| \ge \frac{c}{i!}(a_{k+1} - t)^i \quad \text{for } t \in [a_k, a_{k+1}] \tag{2.5}$$

is satisfied.

Proof. Let x be a solution of problem (**??**), (**??**) and let $t_i^j \in (0,T)$ be zeros of $x^{(j)}$ described in Lemmas **??**, **??** and Theorem **??**. Integrating (**??**) we get

$$\left.\begin{aligned} (-1)^{p+1} x^{(n-1)}(t) &\ge c(t_1^{n-1} - t) \quad && \text{for } t \in [0, t_1^{n-1}] \\ (-1)^{p} x^{(n-1)}(t) &\ge c(t - t_1^{n-1}) && \text{for } t \in [t_1^{n-1}, T]. \end{aligned}\right\} \tag{2.6}$$

Now, integrate the first inequality in (**??**) from $t \in [0, t_1^{n-2})$ to t_1^{n-2}, we have

$$(-1)^p x^{n-2}(t) \ge \frac{c}{2}\Big(- (t_1^{n-1} - t_1^{n-2})^2 + (t_1^{n-1} - t)^2 \Big) \ge \frac{c}{2!}(t_1^{n-2} - t)^2.$$

Hence,we get in such a way

$$\left.\begin{array}{ll}(-1)^p x^{(n-2)}(t) \geq \frac{c}{2!}(t_1^{n-2}-t)^2 & \text{for } t \in [0, t_1^{n-2}] \\ (-1)^{p+1} x^{(n-2)}(t) \geq \frac{c}{2!}(t-t_1^{n-1})^2 & \text{for } t \in [t_1^{n-2}, t_1^{n-1}] \\ (-1)^{p+1} x^{(n-2)}(t) \geq \frac{c}{2!}(t_2^{n-2}-t)^2 & \text{for } t \in [t_1^{n-1}, t_2^{n-2}] \\ (-1)^p x^{(n-2)}(t) \geq \frac{c}{2!}(t-t_2^{n-2})^2 & \text{for } t \in [t_2^{n-2}, T]. \end{array}\right\} \tag{2.7}$$

Choose $i \in \{1, \ldots, n-1\}$ and take all different zeros of functions $x^{(n-1)}, \ldots, x^{(n-i)}$, which are in $(0, T)$. By Lemmas **??**, **??** and Theorem **??**, there is a finite number $p_i < (n-1)p$ of these zeros. Let us put them in order and denote by $a_1, \ldots, a_{p_i}$. Set $a_0 = 0$, $a_{p_i+1} = T$. In this way we get $p_i + 1$ disjoint intervals (a_k, a_{k+1}), $0 \leq k \leq p_i$, satisfying (**??**).

If $i = 1$, then for $a_1 = t_1^{n-1}$, $a_2 = T$, we get by (**??**) that $|x^{(n-1)}(t)| \geq c(a_1 - t)$ for $t \in [a_0, a_1]$ and $|x^{(n-1)}(t)| \geq c(t - a_1)$ for $t \in [a_1, a_2]$.

If $i = 2$, we put $t_1^{n-1} = a_1$, $t_1^{n-2} = a_2$, $t_2^{n-2} = a_3$, $T = a_4$, and then (**??**) gives (**??**) or (**??**).

If $i > 2$ and we integrate the inequalities in (**??**) $(i-2)$-times, we get that on each $[a_k, a_{k+1}]$, $k \in \{0, \ldots, p_i\}$ either (**??**) or (**??**) has to be fulfilled. □

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