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On linear ODEs with a time singularity of the first kind and unsmooth inhomogeneity

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Dedicated to Professor Ivan Kiguradze for his merits in the mathematical sciences.

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Abstract

In this paper we investigate the analytical properties of systems of linear ordinary differential equations (ODEs) with unsmooth nonintegrable inhomogeneities and a time singularity of the first kind. We are especially interested in specifying the structure of general linear two-point boundary conditions guaranteeing existence and uniqueness of solutions which are continuous on a closed interval including the singular point. Moreover, we study the convergence behavior of collocation schemes applied to solving the problem numerically. Our theoretical results are supported by numerical experiments.

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1 Introduction

Singular boundary value problems (BVPs) arise in numerous applications in natural sciences and engineering and therefore, since many years, they have been in focus of extensive investigations. An important class of linear singular problems takes the form of the following BVP:

$$y'(t) = \frac{M}{t}y(t) + f(t), \quad t \in (0,1], \qquad B_0 y(0) + B_1 y(1) = \beta, \tag{1}$$

where y is a n-dimensional real function, M is a $n \times n$ matrix and f is a n-dimensional function which is at least continuous, $f \in C[0,1]$. We are mainly interested to find under which circumstances the above problem has a solution $y \in C[0,1]$. B_0 and B_1 are constant matrices and it turns out that they are subject to certain restrictions for a problem with a unique continuous solution. We say that BVP (1) has a time *singularity of the first kind* at $t = 0$.

Problems of type (1), where f may depend in addition on the space variable y and may have a space singularity at $y = 0$, have been studied in [1–4]. The analytical properties of (1) have been discussed in [5, 6], where the attention was focused on the existence and uniqueness of solutions and their smoothness. Especially, the structure of the boundary conditions which are necessary and sufficient for (1) to have a unique continuous solution on $[0,1]$ was of special interest. Our aim is to generalize these analytical results to the problem

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t}, \quad t \in (0,1], \qquad B_0 y(0) + B_1 y(1) = \beta, \tag{2}$$

where $f \in C[0,1]$ but $f(t)/t$ may not be integrable on $[0,1]$. While for the BVP (1) and its applications, comprehensive literature is available, this is not the case for problem (2). The BVPs of type (2) arise in the modeling of the avalanche run up [7] and occur when the regular ODE system $u'(x) = Mu(x) + g(x)$, posed on the semi-infinite interval $x \in [0,\infty)$, is transformed by $x = -\ln t$ to a finite domain $t \in (0,1]$. Moreover, we refer to papers [8–13], where the solvability of similar linear singular problems is discussed. Interesting results for linear BVPs with time singularities in weight-spaces have been provided in [14–17]. Although this framework is close to what we are aiming at here, it is not quite complete. So, in a way our results are closing the existing gaps.

Note that the more general equation

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad t \in (0,T], \tag{3}$$

with a variable coefficient matrix $M(t)$ was investigated in [18], where the existence of a unique continuous solution y of (3) has been studied. The main results of [18] are formulated in [18, Theorem 1.1] and [18, Theorem 1.2]. In Theorem 1.1, f and M are assumed to be continuous and all eigenvalues of $M(0)$ to have negative real parts. In Theorem 1.2 smoothness of higher derivatives of y up to order $m \geq 1$ has been specified. It turns out that for $M, f \in C^m$ there exists a unique solution $y \in C^m$ provided that all real parts of the eigenvalues of $M(0)$ are smaller than m and different from natural numbers.

The current paper completes the results of [18] for the constant matrix M. In contrast to [18], where only particular solutions without boundary conditions are considered, in this paper general structure of linear two-point boundary conditions is in focus. Explicit solution representations and the form of necessary boundary conditions are provided in Theorems 5, 8, and 11 for the eigenvalues of M with negative real parts, positive real parts, and the eigenvalues zero, respectively.

To compute the numerical solution of (1) polynomial collocation was proposed in [19, 20]. This was motivated by its advantageous convergence properties for (1), while in the presence of a singularity other high order methods show order reductions and become inefficient [21]. Consequently, for singular BVPs [22, 23], we have implemented two open domain MATLAB codes based on collocation. The code `sbvp` solves explicit first order ODEs [22], while `bvpsuite` can be applied to arbitrary order problems also in implicit formulation and to differential algebraic equations [23]. Over recent years, both codes were applied to simulate singular BVPs important for applications and proved to work dependably and efficiently. This was our motivation to also propose and analyze polynomial collocation for the approximation of the initial value problems (2).

The paper is organized as follows: In Section 2, we collect the preliminary results and introduce the necessary notation. Further notation can be found in Table 10. In Sections 3, 4, and 5, three case studies are carried out, the case of only negative real parts of the eigenvalues of M, positive real parts of the eigenvalues of M, and zero eigenvalues of M, respectively. These results are summarized and compared with the case of smooth inhomogeneity in Section 6. Finally, the three case studies are used to formulate the respective results for the general initial value problems, terminal value problems, and BVPs in Section 7. We show convergence orders for the collocation scheme in the context of general initial value problems in Section 8 and illustrate the theoretical findings by experiments carried out using the MATLAB code `bvpsuite` in Section 9. In Section 10, we recapitulate the most important results of the study.

2 Preliminaries

We are interested in analyzing the BVP

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t}, \quad t \in (0,1], y \in C[0,1], \qquad B_0 y(0) + B_1 y(1) = \beta, \tag{4}$$

where $M \in \mathbb{R}^{n\times n}$, $B_0, B_1 \in \mathbb{R}^{m\times n}$, $\beta \in \mathbb{R}^m$, and $f \in C[0,1]$. Note that in general $m \le n$ because the requirement $y \in C[0,1]$ results in $n-m$ additional conditions solution y has to satisfy [5].

Before discussing BVP (4), we first consider the easier problem consisting of the ODE system

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t}, \quad t \in (0,1], \tag{5}$$

subject to initial/terminal conditions. This means that we deal with the initial value problem (IVP),

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t}, \quad t \in (0,1], y \in C[0,1], \qquad B_0 y(0) = \beta, \tag{6}$$

where $B_0 \in \mathbb{R}^{m\times n}$, $\beta \in \mathbb{R}^m$, and $m \le n$, or with the terminal value problem (TVP),

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t}, \quad t \in (0,1], y \in C[0,1], \qquad B_1 y(1) = \beta, \tag{7}$$

where $B_1 \in \mathbb{R}^{n\times n}$, $\beta \in \mathbb{R}^n$, respectively. Particular attention is paid to the structure of initial/terminal and boundary conditions which are necessary and sufficient for the existence of a unique continuous solution on the closed interval $[0,1]$. It turns out that the form of such conditions depends on the spectral properties of the coefficient matrix M. Therefore, we distinguish between three cases, where all eigenvalues of M have negative real parts, positive real parts, or are zero.

In the first step, we construct the general solution of (5). We denote by $J \in \mathbb{C}^{n\times n}$ the Jordan canonical form of M and let $E \in \mathbb{C}^{n\times n}$ be the associated matrix of the generalized eigenvectors of M. Thus, $M = EJE^{-1}$. Moreover, let us introduce new variables, $v(t) := E^{-1}y(t)$ and $g(t) := E^{-1}f(t)$, then we can decouple the system (5) and obtain

$$v'(t) = \frac{J}{t}v(t) + \frac{g(t)}{t}. \tag{8}$$

By the variation of constant, any general solution of the linear equation (8) is a complex-valued function of the form

$$v(t) = \Phi(t)d + \Phi(t)\int_1^t \Phi^{-1}(s)\frac{g(s)}{s}\,\mathrm{d}s = t^J d + t^J \int_1^t s^{-J-I} g(s)\,\mathrm{d}s, \quad t \in (0,1],$$

where $d \in \mathbb{C}^n$ is an arbitrary vector and

$$\Phi(t) = t^J := \exp\bigl(J\ln(t)\bigr) = \sum_{j=0}^{\infty} \frac{J^j (\ln t)^j}{j!}$$

is the fundamental solution matrix satisfying

$$\Phi'(t) = \frac{J}{t}\Phi(t), \qquad \Phi(1) = I, \quad t \in (0,1];$$

see [24, Chapter IV]. In the case that the matrix J consists of l Jordan boxes, $J_1, J_2, \dots, J_l$, the fundamental solution matrix has the form of the block diagonal matrix, $t^J = \operatorname{diag}(t^{J_1}, t^{J_2}, \dots, t^{J_l})$, where

$$J_k = \begin{pmatrix} \lambda_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix}, \quad k = 1, \dots, l,$$

and

$$t^{J_k} = t^{\lambda_k} \begin{pmatrix} 1 & \ln t & \frac{(\ln t)^2}{2} & \dots & \frac{(\ln t)^{n_k-1}}{(n_k-1)!} \\ 0 & 1 & \ln t & \dots & \frac{(\ln t)^{n_k-2}}{(n_k-2)!} \\ 0 & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ln t \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}, \quad t \in (0,1]. \tag{9}$$

Here $\lambda_k = \sigma_k + i\rho_k \in \mathbb{C}$ is an eigenvalue of M and $\dim J_1 + \dim J_2 + \cdots + \dim J_l = n$. The general solution of (5) is then given by

$$y(t) = t^M c + t^M \int_1^t s^{-M-I} f(s)\,\mathrm{d}s, \quad t \in (0,1],$$

where $c = Ed \in \mathbb{C}^n$ and $t^M = Et^J E^{-1} \in \mathbb{C}^{n\times n}$. Also,

$$\bigl(t^M\bigr)' = Mt^{M-I}, \quad t \in (0,1],$$

and

$$t^{-M} = \left(\frac{1}{t}\right)^M \quad \Rightarrow \quad \bigl(t^{-M}\bigr)' = -Mt^{-M-I}, \quad t \in (0,1]. \tag{10}$$

From the structure of the matrix t^{J_k} in (9), it is obvious that the solution contribution related to the kth Jordan box may become unbounded for $t = 0$. Apparently, the asymptotic behavior of the solution depends on the sign of the real part σ_k of the associated eigenvalue λ_k. Therefore, we have to distinguish between three cases, $\sigma_k < 0$, $\lambda_k = 0$, and $\sigma_k > 0$.

We assume that M has no purely imaginary eigenvalues to exclude solutions of the form $t^{i\rho} = \cos(\rho \ln t) + i \sin(\rho \ln t)$.

We complete the preliminaries by two technical remarks, which will be frequently used in the following analysis.

Since the paper is considerably long, we tried to keep the presentation as condensed as possible and refer the reader to [25] for technical details.

Remark 1 The main focus of our investigations is on correctly posed initial/terminal conditions which guarantee the existence of continuously differentiable solutions of (5), $y \in C^1[0,1]$. Since logarithm terms occur in the matrix (9), the relation

$$\lim_{t\to 0^+} t^\alpha (\ln t)^k = 0, \quad \forall \alpha \in \mathbb{R}^+, \forall k \in \mathbb{N}, \tag{11}$$

is essential when discussing the smoothness of y.

Remark 2 By integrating (10) we obtain

$$M \int_t^1 s^{-M-I}\,\mathrm{d}s = -s^{-M}\big|_t^1 = t^{-M} - I, \quad t \in (0,1]. \tag{12}$$

Moreover, if M has only eigenvalues with negative real parts, then $\lim_{s\to 0^+} s^{-M} = 0$ due to Remark 1, and therefore

$$\int_0^1 s^{-M-I}\,\mathrm{d}s = (-M)^{-1}. \tag{13}$$

3 Eigenvalues of *M* with negative real parts

In this section, we consider system (5), such that all eigenvalues of M have negative real parts. It turns out that in this case, it is necessary to prescribe initial conditions of a certain structure to guarantee that the solution is continuous on $[0,1]$. Moreover, this continuous solution of the associated IVP (6) is shown to be unique and its form is provided in Theorem 5. In the proof of this theorem, we require the following lemmas.

Lemma 3 *Let $\gamma \geq 0$ and let the $n \times n$ matrix J be of the form*

$$J = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}, \quad \lambda = \sigma + i\rho, \tag{14}$$

where $\sigma \leq 0$. For $\sigma = 0$, we assume $\lambda = 0$ and $\gamma > 0$. Then, for $t \in (0,1]$,

$$\int_0^t \left|s^{-J}\right| s^{\gamma-1}\,\mathrm{d}s = \sum_{j=0}^{n-1} \sum_{k=0}^{j} \frac{t^{\gamma-\sigma}(-\ln t)^k}{k!(\gamma-\sigma)^{j+1-k}}, \tag{15}$$

and, in particular,

$$\int_0^1 |s^{-J}| s^{\gamma-1}\,\mathrm{d}s = \sum_{j=0}^{n-1} \frac{1}{(\gamma-\sigma)^{j+1}}.$$

Proof Due to the form of J, the norm of s^{-J} for $s \in (0,1]$ is

$$|s^{-J}| = |s^{-\lambda}| \sum_{j=0}^{n-1} \frac{|\ln s|^j}{j!} = s^{-\sigma} \sum_{j=0}^{n-1} \frac{(-\ln s)^j}{j!}. \tag{16}$$

By repeated integration by parts, we obtain

$$\begin{aligned}\int \frac{(-\ln s)^j}{j!} s^{\gamma-\sigma-1}\,\mathrm{d}s &= \frac{s^{\gamma-\sigma}}{\gamma-\sigma}\frac{(-\ln s)^j}{j!} + \int \frac{s^{\gamma-\sigma-1}}{\gamma-\sigma}\frac{(-\ln s)^{j-1}}{(j-1)!}\,\mathrm{d}s \\ &= \sum_{k=0}^{j} \frac{s^{\gamma-\sigma}(-\ln s)^k}{k!(\gamma-\sigma)^{j+1-k}}.\end{aligned}$$

Therefore, due to (11),

$$\int_0^t |s^{-J}| s^{\gamma-1}\,\mathrm{d}s = \int_0^t \sum_{j=0}^{n-1} \frac{(-\ln s)^j}{j!} s^{\gamma-\sigma-1}\,\mathrm{d}s = \sum_{j=0}^{n-1}\sum_{k=0}^{j} \frac{t^{\gamma-\sigma}(-\ln t)^k}{k!(\gamma-\sigma)^{j+1-k}}.$$

Clearly, for $t = 1$,

$$\int_0^1 |s^{-J}| s^{\gamma-1}\,\mathrm{d}s = \sum_{j=0}^{n-1} \frac{1}{(\gamma-\sigma)^{j+1}},$$

which completes the proof. □

Lemma 4 *Assume that all eigenvalues of the matrix M have negative real parts. Then*

$$\lim_{t\to 0^+} \int_0^t |s^{-M-I}|\,\mathrm{d}s = 0. \tag{17}$$

Proof Let $\lambda_k = \sigma_k + i\rho_k$, $k = 1,\dots,l$, be eigenvalues of the matrix M and J_k, $k = 1,\dots,l$, the associated Jordan boxes of M. Then $s^{-M} = Es^{-J}E^{-1}$, where $s^{-J} = \operatorname{diag}(s^{-J_1}, s^{-J_2},\dots,s^{-J_l})$. Therefore,

$$\lim_{t\to 0^+} \int_0^t |s^{-M-I}|\,\mathrm{d}s \le |E||E^{-1}| \lim_{t\to 0^+} \int_0^t |s^{-J}| s^{-1}\,\mathrm{d}s.$$

The result follows from (11) and (15) with $\gamma = 0$. □

Theorem 5 *Let us assume that all eigenvalues of M have negative real parts. Then for every $f \in C[0,1]$ system* (5) *has a unique solution $y \in C[0,1]$. This solution has the form*

$$y(t) = \int_0^1 s^{-M-I} f(ts)\,\mathrm{d}s, \quad t \in [0,1],$$

and satisfies the initial condition $My(0) = -f(0)$. *This condition is necessary and sufficient for* y *to be continuous on* $[0,1]$. *Moreover, if* $f \in C^r[0,1]$, $r \geq 0$, *then* $y \in C^r[0,1]$ *satisfies*

$$\left|y^{(k)}(t)\right| \leq const.\left\|f^{(k)}\right\|, \quad t \in [0,1], \qquad (kI - M)y^{(k)}(0) = f^{(k)}(0), \quad k = 0,\dots,r.$$

Proof The general solution of system (5) can be split into two parts

$$\begin{aligned} y(t) &= t^M c + t^M \int_1^t s^{-M-I} f(s)\,\mathrm{d}s \\ &= t^M \left(c - \int_0^1 s^{-M-I} f(s)\,\mathrm{d}s\right) + t^M \int_0^t s^{-M-I} f(s)\,\mathrm{d}s \\ &=: y_h(t) + y_p(t), \quad t \in (0,1]. \end{aligned} \tag{18}$$

First, we show that $y_p \in C[0,1]$. Change of variable, $u = s/t$, yields

$$y_p(t) = \int_0^1 u^{-M-I} f(ut)\,\mathrm{d}u, \quad t \in (0,1].$$

Let us now introduce the functions,

$$z_m(t) := \int_{\frac{1}{m}}^1 s^{-M-I} f(st)\,\mathrm{d}s, \quad m \in \mathbb{N}, \tag{19}$$

$$z_\infty(t) := \int_0^1 s^{-M-I} f(st)\,\mathrm{d}s. \tag{20}$$

Then, by (17),

$$\lim_{m\to\infty}\left|z_\infty(t) - z_m(t)\right| = \lim_{m\to\infty}\left|\int_0^{\frac{1}{m}} s^{-M-I} f(st)\,\mathrm{d}s\right| \leq \|f\| \lim_{m\to\infty}\int_0^{\frac{1}{m}} \left|s^{-M-I}\right|\mathrm{d}s = 0.$$

Clearly $z_m(t) \in C[0,1]$, for $m \in \mathbb{N}$, and hence z_∞ is continuous as the uniform limit of continuous functions. Consequently, $y_p(t) \in C[0,1]$.

Since all real parts of eigenvalues are negative, y_h is not continuous at $t = 0$ and it is obvious that $y \in C[0,1]$ if and only if

$$\tilde{c} := c - \int_0^1 s^{-M-I} f(s)\,\mathrm{d}s = 0.$$

Thus the unique continuous solution satisfying (5) has the form

$$y(t) = \int_0^1 s^{-M-I} f(st)\,\mathrm{d}s, \quad t \in [0,1], \tag{21}$$

and the estimate

$$\left|y(t)\right| \leq const.\|f\|, \quad t \in [0,1],$$

holds due to Lemma 4. This solution is uniquely determined by $\tilde{c} = 0$ and there are no additional conditions to be imposed. Note that $\tilde{c} = 0$ is equivalent to the condition $My(0) = -f(0)$ which follows from (13) and (21).

We now examine the smoothness of y. Let $f \in C^1[0,1]$. For the first derivative y', we have from (21)

$$y'(t) = \int_0^1 s^{-M} f'(ts)\,\mathrm{d}s, \qquad \left|y'(t)\right| \le const.\left\|f'\right\|, t \in [0,1],$$

due to Lemma 3. Clearly, if $f \in C^r[0,1]$, then

$$y^{(r)}(t) = \int_0^1 s^{(r-1)I-M} f^{(r)}(ts)\,\mathrm{d}s, \qquad \left|y^{(r)}(t)\right| \le const.\left\|f^{(r)}\right\|, t \in [0,1]$$

and by (13) the results follow. □

Theorem 5 shows that if all eigenvalues of M have negative real parts, then there exists a unique continuous solution y of IVP (6) for $B_0 = M$, $\beta = -f(0)$, and $m = n$. Clearly, B_0 has to be nonsingular. Note that for this spectrum of M a terminal problem (7) cannot be set up in a reasonable way.

4 Eigenvalues of *M* with positive real parts

In this section we deal with system (5) whose matrix M has eigenvalues with positive real parts. It turns out that in this case there exists a unique continuous solution of problem (7). Its smoothness depends not only on the smoothness of f but also on the size of real parts of the eigenvalues of M. Before stating the main result of this section formulated in Theorem 8, we show the following two lemmas.

Lemma 6 *Let $\gamma \ge 0$ and let the $n \times n$ matrix J be of the form* (14), *where $\sigma > 0$. Then for $t \in [0,1]$ the function*

$$u(t) = \int_t^1 \left|\left(\frac{t}{s}\right)^J\right| s^{\gamma-1}\,\mathrm{d}s,$$

satisfies the following inequalities:

$$\text{(i)} \quad u(t) \le const.t^\gamma, \qquad \gamma < \sigma, \tag{22}$$

$$\text{(ii)} \quad u(t) \le const.t^\sigma \sum_{j=0}^{n-1} \frac{(-\ln t)^{j+1}}{j!}, \qquad \gamma = \sigma, \tag{23}$$

$$\text{(iii)} \quad u(t) \le const.t^\sigma \sum_{j=0}^{n-1} \frac{(-\ln t)^{j}}{j!}, \qquad \gamma > \sigma. \tag{24}$$

Proof We discuss separately the cases $\gamma < \sigma$, $\gamma = \sigma$, and $\gamma > \sigma$. Note that according to (11) and (16)

$$\int_t^1 \left|\left(\frac{t}{s}\right)^J\right| s^{\gamma-1}\,\mathrm{d}s = \int_t^1 \left(\frac{t}{s}\right)^\sigma \sum_{j=0}^{n-1} \frac{(-\ln(\frac{t}{s}))^j}{j!} s^{\gamma-1}\,\mathrm{d}s$$

holds.

(i) First, let $\gamma < \sigma$. Then there exists a constant $\varepsilon > 0$ such that $\sigma = \gamma + 2\varepsilon$. The term

$$\left(\frac{t}{s}\right)^{\varepsilon} \sum_{j=0}^{n-1} \frac{(-\ln(\frac{t}{s}))^j}{j!}$$

is bounded on $[0,1]$ due to (11) and hence

$$\int_t^1 \left|\left(\frac{t}{s}\right)^J\right| s^{\gamma-1}\,\mathrm{d}s \le const.t^{\gamma+\varepsilon} \int_t^1 s^{-\varepsilon-1}\,\mathrm{d}s = const.t^{\gamma}.$$

(ii) For $\gamma = \sigma$ the function u can be estimated by

$$\int_t^1 \left|\left(\frac{t}{s}\right)^J\right| s^{\gamma-1}\,\mathrm{d}s \le t^{\sigma} \sum_{j=0}^{n-1} \frac{(-\ln t)^j}{j!} \int_t^1 s^{-1}\,\mathrm{d}s \le const.t^{\sigma} \sum_{j=0}^{n-1} \frac{(-\ln t)^{j+1}}{j!}.$$

(iii) Finally, for $\gamma > \sigma$, we have

$$\int_t^1 \left|\left(\frac{t}{s}\right)^J\right| s^{\gamma-1}\,\mathrm{d}s \le t^{\sigma} \sum_{j=0}^{n-1} \frac{(-\ln t)^j}{j!} \int_t^1 s^{-\sigma+\gamma-1}\,\mathrm{d}s \le const.t^{\sigma} \sum_{j=0}^{n-1} \frac{(-\ln t)^j}{j!}.$$

□

Lemma 7 *Let $\gamma \ge 0$ and let all eigenvalues of M have positive real parts. Then the function*

$$u(t) = \int_t^1 \left|\left(\frac{t}{s}\right)^M\right| s^{\gamma-1}\,\mathrm{d}s, \quad t \in [0,1],$$

is bounded on $[0,1]$ and $\lim_{t\to 0^+} u(t) = 0$ for $\gamma > 0$.

Proof Let all eigenvalues of M have positive real parts. Then

$$u(t) = \int_t^1 \left|\left(\frac{t}{s}\right)^M\right| s^{\gamma-1}\,\mathrm{d}s \le |E|\left|E^{-1}\right| \int_t^1 \left|\left(\frac{t}{s}\right)^J\right| s^{\gamma-1}\,\mathrm{d}s.$$

Estimates (22) to (24) and property (11) imply $u(t) \le const.t^{\sigma_0}$ for $t \in [0,1]$, where $\sigma_0 = \min\{\gamma, \frac{\sigma}{2}\} \ge 0$. This means that u is bounded in $[0,1]$. If $\gamma > 0$, then $\sigma_0 > 0$ and the result follows. □

Theorem 8 *Let us assume that all eigenvalues of M have positive real parts. Then for every $f \in C^1[0,1]$ and every constant vector c, there exists a unique solution $y \in C[0,1]$ of (5). This solution has the form*

$$y(t) = \begin{cases} t^M c + t^M \int_1^t s^{-M-I} f(s)\,\mathrm{d}s, & t \in (0,1], \\ -M^{-1} f(0), & t = 0. \end{cases} \tag{25}$$

If the matrix $B_1 \in \mathbb{R}^{n\times n}$ in (7) is nonsingular, then for any $\beta \in \mathbb{R}^n$ there exists a unique solution of TVP (7). This solution is given by (25) with $c = B_1^{-1}\beta$.

Let $f \in C^{r+2}[0,1]$. Then the following statements hold:

(i) *$y \in C^r[0,1] \cap C^{r+3}(0,1]$ for $0 \le r < \sigma_+ \le r+1$,*

(ii) $y \in C^{r+1}[0,1] \cap C^{r+3}(0,1]$ *for* $\sigma_+ > r+1$.

Moreover, higher derivatives of y satisfy for $t \in [0,1]$

(i) $|y^{(k)}(t)| \le const.(t^{\sigma_+ - k}(1 + |\ln(t)|^{n_{\max}-1}) + \|f^{(k)}\|)$ *for* $k = 0,1,\dots,r$,

(ii) $|y^{(k)}(t)| \le const.(t^{\sigma_+ - k}(1 + |\ln(t)|^{n_{\max}-1}) + \|f^{(k)}\|)$ *for* $k = 0,1,\dots,r+1$,

where σ_+ *is the smallest positive real part of the eigenvalues of* M *and* $n_{\max}$ *is the dimension of the largest Jordan box in* J.

Proof The general solution of (5) can be written in the following form:

$$y(t) = t^M c + t^M \int_1^t s^{-M-I} f(s)\,ds = t^M c + \int_1^t \left(\frac{t}{s}\right)^M \frac{f(s)}{s}\,ds =: y_h(t) + y_p(t). \tag{26}$$

Since all eigenvalues have positive real parts, it follows from (11) that $y_h(t) = t^M c$ is continuous on $[0,1]$.

Now, we show that $\lim_{t\to 0} y_p(t)$ exists and therefore $y \in C[0,1]$. Using the integration formula (12) we obtain

$$\int_1^t \left(\frac{t}{s}\right)^M \frac{f(0)}{s}\,ds = M^{-1}\big(t^M - I\big)f(0),$$

and hence

$$-M^{-1}f(0) = \int_1^t \left(\frac{t}{s}\right)^M \frac{f(0)}{s}\,ds - M^{-1}t^M f(0).$$

Therefore

$$\int_1^t \left(\frac{t}{s}\right)^M \frac{f(s)}{s}\,ds - (-M)^{-1}f(0) = \int_1^t \left(\frac{t}{s}\right)^M \frac{f(s)-f(0)}{s}\,ds + M^{-1}t^M f(0). \tag{27}$$

Since $f \in C^1[0,1]$, there exists $M_0 \in (0,\infty)$ such that

$$\left|\frac{f(s)-f(0)}{s}\right| \le M_0, \quad s \in [0,1]. \tag{28}$$

Equation (27) together with (28) yield

$$\left|\int_1^t \left(\frac{t}{s}\right)^M \frac{f(s)}{s}\,ds - (-M)^{-1}f(0)\right| \le M_0 \int_t^1 \left|\left(\frac{t}{s}\right)^M\right| ds + \big|(-M)^{-1}t^M f(0)\big|.$$

Since all eigenvalues of M have positive real parts, (11) implies

$$\lim_{t\to 0^+} \big|(-M)^{-1}t^M f(0)\big| = 0.$$

Moreover, by Lemma 7 with $\gamma = 1$

$$\lim_{t\to 0^+} \left|\int_1^t \left(\frac{t}{s}\right)^M \frac{f(s)}{s}\,ds - (-M)^{-1}f(0)\right| = 0$$

follows. Thus, $\lim_{t\to 0^+} y_p(t) = (-M)^{-1}f(0)$ and $y \in C[0,1]$.

It is clear from (26) that the solution y of (5) becomes unique if we specify the constant vector $c \in \mathbb{R}^n$. Note that at $t = 0$, $y(0)$ satisfies n linearly independent conditions $My(0) = -f(0)$ for any $c \in \mathbb{R}^n$. Therefore, we have to specify c via the terminal conditions given in (7). Let $\beta \in \mathbb{R}^n$ and let $B_1 \in \mathbb{R}^{n\times n}$ be nonsingular, then it follows from $B_1 y(1) = B_1 c = \beta$ that the unique solution of TVP (7) is given by (26), where $c = B_1^{-1}\beta$.

We now provide the estimate for y. To this aim, we utilize Lemma 7 with $\gamma = 0$ and the inequality

$$\left|t^M\right| = \left|Et^J E^{-1}\right| \le const.\left|t^J\right| \le const.t^{\sigma_+}\left(1 + \left|\ln(t)\right|^{n_{\max}-1}\right). \tag{29}$$

Hence, according to (26)

$$\begin{aligned}\left|y(t)\right| &\le \left|t^M B_1^{-1}\beta\right| + \left|\int_1^t \left(\frac{t}{s}\right)^M s^{-1} f(s)\,\mathrm{d}s\right| \\ &\le const.t^{\sigma_+}\left(1 + \left|\ln(t)\right|^{n_{\max}-1}\right)\left|B_1^{-1}\beta\right| + const.\|f\|.\end{aligned}$$

In order to discuss the smoothness of y, we first study the general solution of the homogeneous problem y_h. Since σ_+ is positive, there always exists a constant $l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ such that $0 \le l < \sigma_+ \le l + 1$. Then we have

$$\begin{aligned}&y_h'(t) = \left(t^M c\right)' = Mt^{M-I}c, \\ &y_h^{(k)}(t) = \left(t^M c\right)^{(k)} = M(M-I)\cdots\bigl(M-(k-1)I\bigr)t^{M-kI}c, \quad k = 1,\dots,l,\end{aligned}$$

and it is easily seen that $y_h \in C^l[0,1] \cap C^\infty(0,1]$. The estimates for higher derivatives of y_h follow from (29).

We now turn to the smoothness of the particular solution of the inhomogeneous problem y_p. We integrate by parts

$$\begin{aligned}y_p(t) &= t^M \int_1^t s^{-M-I} f(s)\,\mathrm{d}s \\ &= t^M\left((-M)^{-1}t^{-M}f(t) - (-M)^{-1}If(1) - (-M)^{-1}\int_1^t s^{-M}f'(s)\,\mathrm{d}s\right) \\ &= (M)^{-1}\left(t^M f(1) - f(t) + t^M \int_1^t s^{-M}f'(s)\,\mathrm{d}s\right).\end{aligned}$$

Note that t^M and M^{-1} are commutative if t^M and M are commutative, since $t^M M^{-1} = (Mt^{-M})^{-1}$. The latter property will be shown in Lemma 18.

We differentiate the above equation and obtain

$$\begin{aligned}y_p'(t) &= (M)^{-1}\left(Mt^{M-I}f(1) - f'(t) + Mt^{M-I}\int_1^t s^{-M}f'(s)\,\mathrm{d}s + t^M t^{-M}f'(t)\right) \\ &= t^{M-I}f(1) + t^{M-I}\int_1^t s^{-M}f'(s)\,\mathrm{d}s.\end{aligned}$$

Let $f \in C^2[0,1]$ and $\sigma_+ > 1$, then we argue as at the beginning of the proof (in context of y and $\sigma_+ > 0$) and conclude that $y_p \in C^1[0,1]$. Moreover, the following estimate holds:

$$\begin{aligned}\left|y_p'(t)\right| &\le \left|f(1)\right| const.t^{\sigma_+-1}\left(1 + \left|\ln(t)\right|^{n_{\max}-1}\right) + \left\|f'\right\| t^{-1}\left|\int_1^t \left(\frac{t}{s}\right)^M \mathrm{d}s\right| \\ &\le \left|f(1)\right| const.t^{\sigma_+-1}\left(1 + \left|\ln(t)\right|^{n_{\max}-1}\right) + \left\|f'\right\| t^{-1} const.t \\ &\le const.\left(t^{\sigma_+-1}\left(1 + \left|\ln(t)\right|^{n_{\max}-1}\right) + \left\|f'\right\|\right), \quad t \in [0,1].\end{aligned}$$

Similarly, if $f \in C^{r+2}[0,1]$ and $\sigma_+ > r+1$, then $y_p \in C^{r+1}[0,1]$ and the following estimate holds:

$$\left|y_p^{(r+1)}(t)\right| \le const.\left(t^{\sigma_+-r-1}\left(1 + \left|\ln(t)\right|^{n_{\max}-1}\right) + \left\|f^{(r+1)}\right\|\right), \quad t \in [0,1].$$

It follows from (5) that if $f \in C^{r+2}[0,1]$, then $y_p \in C^{r+3}(0,1]$. Consequently, we have $y_p \in C^r[0,1] \cap C^{r+3}(0,1]$ for $r < \sigma_+ \le r+1$ and $y_p \in C^{r+1}[0,1] \cap C^{r+3}(0,1]$ for $\sigma_+ > r+1$. The above smoothness results and estimates for y_p and y_h complete the proof. □

We recapitulate the case when all eigenvalues of M have positive real parts: For any $f \in C^1[0,1]$ and any vector $\beta \in \mathbb{R}^n$ there exists a unique continuous solution y of TVP (7) if and only if the matrix $B_1 \in \mathbb{R}^{n\times n}$ is nonsingular. Each continuous solution y of (5) satisfies the initial condition $My(0) = -f(0)$ independently on $c \in \mathbb{R}^n$ from (26). Consequently, in this case there exists no IVP with a unique solution.

Remark 9 A continuous solution to (5) exists also in the case when f is not continuously differentiable in $[0,1]$. However, in this case, we need some more structure in f close to the singularity. Let us assume that $f(t) = O(t^\alpha h(t))$ as $t \to 0$, for some constant $\alpha > 0$ and a function $h \in C[0,\delta_1]$, $\delta_1 > 0$. Then the solution of (5) is still continuous on $[0,1]$. For the proof see [25].

5 Eigenvalues λ = 0

Finally, we consider the case when all eigenvalues of the matrix M are zero. We begin with the scalar equation (5) which for $M = \lambda = 0$ immediately reduces to

$$y'(t) = \frac{f(t)}{t}, \tag{30}$$

and show that additional structure in the function f is necessary to guarantee that the solution y is continuous on $[0,1]$. To see this, assume that f is a constant function, $f(t) \equiv 1$. Then any solution y of (30) has the following form:

$$y(t) = y(1) + \int_1^t \frac{1}{s}\,\mathrm{d}s = y(1) + \ln t, \quad t \in (0,1]$$

and, clearly, y is not continuous at $t = 0$. Motivated by the scalar case, we require the inhomogeneity f to satisfy additional conditions providing the continuity of the associated solution. Before formulating the main result of this section we show the following lemma.

Lemma 10 *Let us assume that all eigenvalues of the matrix M are zero. Then for $\alpha > 0$*

$$\lim_{t\to 0^+} \int_0^t \left|s^{-M}\right| s^{\alpha-1}\, ds = 0. \tag{31}$$

Proof Let J_k, $k = 1,\dots,l$, be the Jordan boxes of M. Then we can write $s^{-M} = Es^{-J}E^{-1}$, $s^{-J} = \operatorname{diag}(s^{-J_1},\dots,s^{-J_l})$, and thus

$$\lim_{t\to 0^+} \int_0^t \left|s^{-M}\right| s^{\alpha-1}\, ds \le |E|\left|E^{-1}\right| \int_0^t \left|s^{-J}\right| s^{\alpha-1}\, ds.$$

Applying (15) and (11) we obtain (31). □

To obtain results for zero eigenvalues a projection matrix R onto the eigenspace of M and the matrix $\tilde{R}$ consisting of the linearly independent columns of R are required. For respective notation, see Table 10.

Theorem 11 *Let all eigenvalues of the matrix M be zero and $m = \dim X_0^{(e)}$. Moreover, let us assume that there exist a constant $\alpha > 0$ and a function $h \in C[0,\delta]$, $\delta > 0$ such that*

$$f(t) = O\big(t^\alpha h(t)\big) \quad \textit{for } t \to 0. \tag{32}$$

Then for any $B_0 \in \mathbb{R}^{m\times n}$ such that the matrix $B_0\tilde{R} \in \mathbb{R}^{m\times m}$ is nonsingular and for any $f \in C[0,1]$ and $\beta \in \mathbb{R}^m$, there exists a unique solution $y \in C[0,1]$ of IVP (6). *This solution has the form*

$$y(t) = \tilde{R}(B_0\tilde{R})^{-1}\beta + \int_0^1 s^{-M} s^{-1} f(st)\, ds, \quad t \in (0,1],$$

and satisfies the initial condition $My(0) = 0$, which is necessary and sufficient for $y \in C[0,1]$. Moreover,

$$\left|y(t)\right| \le \left|\tilde{R}(B_0\tilde{R})^{-1}\beta\right| + const.\big(\|f\| + t^\alpha \|h\|_\delta\big), \quad t \in [0,1],$$

and if $\alpha \ge r+1$, $f \in C^r[0,1]$, and $h \in C^r[0,\delta]$, then $y \in C^{r+1}[0,1]$ and the following estimates hold for any $k = 0,\dots,r+1$:

$$\begin{aligned} \left|y^{(k)}\right| &\le const. \sum_{j=0}^{k-1} \big(t^{\alpha-1}\big)^{(k-1-j)} \left\|h^{(j)}\right\|_\delta, \quad t \in [0,\delta), \\ \left|y^{(k)}\right| &\le const. \sum_{j=0}^{k-1} \Big(\big(t^{\alpha-1}\big)^{(k-1-j)} \left\|h^{(j)}\right\|_\delta + \big(t^{-1}\big)^{(k-1-j)} \left\|f^{(j)}\right\|\Big), \quad t \in [\delta,1]. \end{aligned} \tag{33}$$

Proof We split the general solution of (5) into two parts $y(t) = y_h(t) + y_p(t)$ as defined in (18). To prove that $y_p \in C[0,1]$, we again use the functions z_m with $m \in \mathbb{N}$ and z_∞ specified in (19) and (20). Due to (15), (31), and (32), we obtain

$$\lim_{m\to\infty} \left|z_\infty(t) - z_m(t)\right| \le \|h\|_\delta t^\alpha \lim_{m\to\infty} \int_0^{\frac{1}{m}} \left|s^{-M}\right| s^{\alpha-1}\, ds = 0. \tag{34}$$

Therefore, $y_p = z_\infty \in C[0,1]$ and $y_p(0) = 0$ since $f(0) = 0$ due to (32).

We now examine the continuity of

$$y_h(t) = t^M \left(c + \int_1^0 s^{-M} s^{-1} f(s)\,\mathrm{d}s \right) =: t^M \eta,$$

cf. (18). The fundamental solution matrix is given by $t^M = Et^J E^{-1}$, where t^J has the form $t^J = \operatorname{diag}(t^{J_1}, \ldots, t^{J_l})$ and

$$E = \left(v_1, h_1^{(1)}, h_1^{(2)}, \ldots, h_1^{(n_1-1)}, v_2, h_2^{(1)}, \ldots, h_2^{(n_2-1)}, \ldots, v_l, h_l^{(1)}, \ldots, h_l^{(n_l-1)}\right),$$

where for $k = 1, \ldots, l$, v_k are the eigenvectors of M, $h_k^{(1)}, \ldots, h_k^{(n_k-1)}$ are the associated principal vectors, and n_k are the dimensions of the Jordan boxes J_k. Clearly, because of the logarithmic terms occurring in t^J, see (9), y_h is not continuous at $t = 0$ in general. Only when the contributions including the logarithmic terms vanish, y_h becomes continuous on $[0,1]$. It is clear from (9) that the only bounded contributions to y_h are linear combinations of the eigenvectors of M. Consequently, any linear combination of principal vectors has to vanish. This is the case when $\eta_i = 0$, $\forall i \neq 1, n_1 + 1, n_1 + n_2 + 1, \ldots, \sum_{k=1}^{l} n_k + 1$ and arbitrary η_i for all $i = 1, n_1 + 1, n_1 + n_2 + 1, \ldots, \sum_{k=1}^{l} n_k + 1$. Thus, y_h is continuous on $[0,1]$ if and only if it is a constant linear combination of the eigenvectors of M. In other words, by setting $y_h(t) := \eta$, we have

$$y(t) \in C[0,1] \quad \Leftrightarrow \quad My(0) = M\eta = 0 \quad \Leftrightarrow \quad \eta \in \operatorname{Ker} M.$$

Consequently, $My(0) = 0$ is necessary and sufficient for the solution

$$y(t) = \eta + \int_0^1 s^{-M-I} f(ts)\,\mathrm{d}s, \quad t \in [0,1] \tag{35}$$

to be continuous on $[0,1]$.

Note that the regularity requirement $My(0) = 0$ contains $n - l$ linearly independent conditions and can be equivalently expressed by $Hy(0) = 0$, $y(0) = Ry(0)$ or $y(0) \in \operatorname{Ker} M$. The remaining l free constants have to be uniquely specified by appropriately prescribed initial conditions. Let us consider the initial conditions specified in (6), where $B_0 \in \mathbb{R}^{m\times n}$ and $\beta \in \mathbb{R}^m$. Since $y_p(0) = 0$ and $y_h(0) = \eta$, the initial condition $B_0 y(0) = \beta$ is equivalent to $B_0 \eta = \beta$. Due to the fact that $\eta \in \operatorname{Im} R$, there exists a unique l-dimensional vector d, $l = \dim X_0^{(e)}$, such that $\eta = \tilde{R}d$, where $\tilde{R}$ is the $n \times l$ matrix containing the linearly independent columns of R. Clearly, the problem is uniquely solvable if and only if $m = l = \dim X_0^{(e)}$ and the $m \times m$ matrix $B_0\tilde{R}$ is nonsingular. Hence,

$$B_0\eta = \beta \quad \Leftrightarrow \quad B_0\tilde{R}d = \beta \quad \Rightarrow \quad d = (B_0\tilde{R})^{-1}\beta \quad \Rightarrow \quad \eta = \tilde{R}(B_0\tilde{R})^{-1}\beta,$$

and the solution y has the form

$$y(t) = \tilde{R}(B_0\tilde{R})^{-1}\beta + \int_0^1 s^{-M} s^{-1} f(st)\,\mathrm{d}s, \quad t \in [0,1].$$

This solution is bounded by

$$\left|y(t)\right| \leq \left|\tilde{R}(B_0\tilde{R})^{-1}\beta\right| + const.\left(t^\alpha \|h\|_\delta + \|f\|\right), \quad t \in [0,1],$$

due to (15) and

$$\begin{aligned}\left|y_p(t)\right| &= \left|\int_\delta^1 s^{-M}s^{-1}f(ts)\,\mathrm{d}s\right| + \left|\int_0^\delta s^{-M}s^{-1}f(ts)\,\mathrm{d}s\right| \\ &\leq const.\|f\| + const.t^\alpha \|h\|_\delta \int_0^\delta \left|s^{-M}\right|s^{\alpha-1}\,\mathrm{d}s, \quad t \in [0,1] \\ &\leq const.\left(\|f\| + t^\alpha \|h\|_\delta\right), \quad t \in [0,1].\end{aligned}$$

In order to derive the following form of the first derivative, we substitute the solution given by (35) into (5) and use the property $M\eta = 0$, then

$$y'(t) = \frac{M}{t}\int_0^1 s^{-M-I}f(st)\,\mathrm{d}s + \frac{f(t)}{t}, \quad t \in (0,1].$$

If $\alpha \geq 1$, then the first derivative is bounded by

$$\begin{aligned}\left|y'(t)\right| &\leq const.\frac{|M|}{t}t^\alpha\left|\int_0^1 s^{-M}s^{\alpha-1}h(st)\,\mathrm{d}s\right| + const.\frac{t^\alpha|h(t)|}{t} \\ &\leq const.t^{\alpha-1}\|h\|_\delta, \quad t \in [0,\delta), \\ \left|y'(t)\right| &\leq const.\frac{|M|}{t}t^\alpha\left|\int_0^\delta s^{-M}s^{\alpha-1}h(st)\,\mathrm{d}s\right| + \frac{|M|}{t}\left|\int_\delta^1 s^{-M}s^{-1}f(st)\,\mathrm{d}s\right| + \frac{|f(t)|}{t} \\ &\leq const.t^{\alpha-1}\|h\|_\delta + const.t^{-1}\|f\|, \quad t \in [\delta,1].\end{aligned}$$

Analogously, for $f \in C^r[0,1]$, $h \in C^r[0,\delta]$, $\alpha \geq r+1$, we have the following bounds for the higher derivatives:

$$\left|y^{(r+1)}\right| \leq const.\sum_{k=0}^{r}\left(t^{\alpha-1}\right)^{(r-k)}\left\|h^{(k)}\right\|_\delta, \quad t \in [0,\delta),$$

$$\left|y^{(r+1)}\right| \leq const.\sum_{k=0}^{r}\left(\left(t^{\alpha-1}\right)^{(r-k)}\left\|h^{(k)}\right\|_\delta + \left(t^{-1}\right)^{(r-k)}\left\|f^{(k)}\right\|\right), \quad t \in [\delta,1].$$

The above estimates imply $y \in C^{r+1}[0,1]$. □

Remark 12 Note that a purely polynomial inhomogeneity of the form

$$f(t) = \left(t^{\alpha_1},\ldots,t^{\alpha_n}\right)^\top,$$

where $\alpha_i \in \mathbb{N}$, for $i = 1,\ldots,n$, yields $y \in C^\infty[0,1]$. For the proof see [25].

In Theorem 11, we described the unique solvability of IVP (6) in case when all eigenvalues of M are zero. The dimension of the corresponding eigenspace $X_0^{(e)}$ was $m < n$ and it turned out that the following regularity requirement $My(0) = 0$ has to be satisfied. If

$m = n$, then $M = 0$ and the regularity condition holds. In this case we can also investigate the unique solvability of TVP (7). We address this question in the next lemma.

Lemma 13 *Consider system* (5) *with the matrix* $M = 0$. *Let* $f \in C[0,1]$ *and assume that* (32) *is satisfied. Then, for any vector* $\beta \in \mathbb{R}^n$ *and a nonsingular matrix* $B_1 \in \mathbb{R}^{n\times n}$, *there exists a unique solution of* (7),

$$y(t) = B_1^{-1}\beta + \int_1^t \frac{f(s)}{s}\,\mathrm{d}s,$$

bounded by

$$\left|y(t)\right| \le \left|B_1^{-1}\beta\right| + const.\left(\|f\| + t^{\alpha}\|h\|_{\delta}\right).$$

Moreover, if $f \in C^r[0,1]$, $h \in C^r[0,\delta]$, *and* $\alpha \ge r+1$, *then* $y \in C^{r+1}[0,1]$ *and the estimates* (33) *hold*.

Proof For $M = 0$ the system (5) reduces to $y'(t) = f(t)/t$, and its solution is $y(t) = y(1) + \int_1^t f(s)/s\,\mathrm{d}s$. To show that $y \in C[0,1]$, we follow the arguments given in the proof of Theorem 11. The terminal condition $B_1 y(1) = \beta$ yields $y(1) = B_1^{-1}\beta$. Moreover,

$$\begin{aligned}\left|y(t)\right| &\le \left|B_1^{-1}\beta\right| + \int_1^{\delta}\frac{f(s)}{s}\,\mathrm{d}s + \int_{\delta}^{t}\frac{f(s)}{s}\,\mathrm{d}s \\ &\le \left|B_1^{-1}\beta\right| + \|f\|\left|\ln(\delta)\right| + const.\|h\|_{\delta}\int_{\delta}^{t} s^{\alpha-1}\,\mathrm{d}s \\ &\le \left|B_1^{-1}\beta\right| + const.\left(\|f\| + \|h\|_{\delta}\right).\end{aligned}$$

Estimates for the higher derivatives of y follow in an analogous manner. □

6 Differences between linear systems with smooth and unsmooth inhomogeneity

Before discussing the case of an arbitrary spectrum of M which enables to consider more general IVPs, TVPs, and BVPs, we summarize here the results from the previous sections and point out the differences when compared to the framework given in [5, 26], where linear systems with smooth inhomogeneity,

$$y'(t) = \frac{M}{t}y(t) + f(t), \quad t \in (0,1], \tag{36}$$

were studied.

6.1 Eigenvalues with negative real parts

Let us consider the ODE system (36) and assume that all eigenvalues of M have negative real parts. Then, according to [5, 26], $y \in C[0,1]$ if and only if $y(0) = 0$. Therefore, the following IVP has a unique solution:

$$y'(t) = \frac{M}{t}y(t) + f(t), \qquad y(0) = 0.$$

Moreover, $y \in C^{r+1}[0,1]$ if $f \in C^r[0,1]$, $r \geq 0$.

According to Theorem 5, ODE system (5) has a solution $y \in C[0,1]$ if and only if $My(0) = -f(0)$. Consequently, the IVP specified below has a unique solution,

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t}, \qquad My(0) = -f(0).$$

Here $y \in C^r[0,1]$ if $f \in C^r[0,1]$, $r \geq 0$.

The conditions $y(0) = 0$ and $My(0) = -f(0)$ are necessary and sufficient for the solution y to be continuous, $y \in C[0,1]$, in case of the system (36) and (5), respectively.

6.2 Eigenvalues with positive real parts

For this spectrum of M both, for system (36) and for (5), we have to specify the boundary conditions at $t = 1$ and solve a TVP. In particular, the TVP

$$y'(t) = \frac{M}{t}y(t) + f(t), \qquad B_1 y(1) = \beta, \tag{37}$$

where $B_1 \in \mathbb{R}^{n\times n}$ is nonsingular and $\beta \in \mathbb{R}^n$, has a unique solution $y \in C[0,1]$. This solution satisfies $y(0) = 0$. If $f \in C^r[0,1]$ and $\sigma_+ > r+1$, then $y \in C^{r+1}[0,1]$, *cf.* [5]. In contrast to system (36), we need extra smoothness of the function f to obtain a unique continuous solution of the TVP

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t}, \qquad B_1 y(1) = \beta,$$

where $B_1 \in \mathbb{R}^{n\times n}$ is nonsingular and $\beta \in \mathbb{R}^n$. Theorem 8 states that $y \in C[0,1]$ if $f \in C^1[0,1]$. Additionally, if $f \in C^{r+2}[0,1]$ and $\sigma_+ > r+1$, then $y \in C^{r+1}[0,1]$, $r \geq 0$.

6.3 Eigenvalues λ = 0

Let all eigenvalues of M be zero. Consider the IVP associated with (36) which takes the form

$$y'(t) = \frac{M}{t}y(t) + f(t), \qquad My(0) = 0, \qquad B_0 y(0) = \beta,$$

where the $m \times m$ matrix $B_0\tilde{R}$ is nonsingular, $\beta \in \mathbb{R}^m$, and $m = \dim X_0^{(e)}$. The initial condition $My(0) = 0$ is necessary and sufficient for the solution to by continuous. The remaining m conditions necessary for its uniqueness are specified by $B_0 y(0) = \beta$. For $f \in C^r[0,1]$, $r \geq 0$, $y \in C^{r+1}[0,1]$; see [5, 26].

In case of the unsmooth inhomogeneity in (5), f has to satisfy an additional requirement,

$$f(t) = O\bigl(t^\alpha h(t)\bigr) \quad \text{as } t \to 0, \alpha > 0, h \in C[0,\delta], \delta > 0,$$

to enable a continuous solution of the following IVP:

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t}, \qquad My(0) = 0, \qquad B_0 y(0) = \beta,$$

where the $m \times m$ matrix $B_0\tilde{R}$ is nonsingular, $\beta \in \mathbb{R}^m$, and $m = \dim X_0^{(e)}$.

Finally, if $f \in C^r[0,1]$, $h \in C^r[0,\delta]$, and $\alpha \geq r+1$, then $y \in C^{r+1}[0,1]$.

7 General IVPs, TVPs, and BVPs

In this section we study general IVPs, TVPs, and BVPs. For notation see Table 10. All projections were constructed using the eigenbasis of M.

First, we discuss general IVPs (6) and TVPs (7), where all conditions which are necessary and sufficient to specify a unique solutions $y \in C[0,1]$ are posed at only one point, either at $t = 0$ or at $t = 1$. According to the results derived above, restrictions on the spectrum of M need to be made.

A.1 For IVP (6) we assume that the matrix M has only eigenvalues with nonpositive real parts and if $\sigma = 0$, then $\lambda = 0$.

A.2 For TVP (7) we assume that the matrix M has only eigenvalues with nonnegative real parts and if $\sigma = 0$, then $\lambda = 0$. Additionally, if zero is an eigenvalue of M, then the associated invariant subspace is assumed to be the eigenspace of M.

Results formulated below without proofs are simple consequences of Theorems 5, 8, 11, and Lemma 13.

Lemma 14 *Let us assume that* $f \in C[0,1]$, $Sf \in C^1[0,1]$, *and* Zf *satisfies condition* (32).

(i) *Assume* A.1 *to hold. Let* y *be a continuous solution of IVP* (6). *Then*

$$MNy(0) = -Nf(0), \qquad Hy(0) = 0.$$

(ii) *Assume* A.2 *to hold. Let* y *be a continuous solution of TVP* (7). *Then*

$$MSy(0) = -Sf(0).$$

In both cases

$$My(0) = M(S+N)y(0) = -f(0).$$

The statement of Lemma 14 means that the conditions which are necessary for the solution of IVP (6) to be continuous are equivalent to

$$\operatorname{rank} M = \operatorname{rank} H + \operatorname{rank} N = \operatorname{rank} Q = n - \operatorname{rank} R$$

initial conditions, which the solution y has to satisfy. In case of TVP (7), where A.2 holds, any solution of (5) is continuous on $[0,1]$ and no regularity conditions have to be prescribed.

From Theorems 5 and 11 we obtain the following result for a general IVP (6).

Theorem 15 *Let us assume that* A.1 *holds, the* $m \times m$ *matrix* $B_0\tilde{R}$ *is nonsingular, and* $\beta \in \mathbb{R}^m$. *Then, for every* $f \in C[0,1]$ *such that* Zf *satisfies* (32), *there exists a unique solution* $y \in C[0,1]$ *of IVP* (6),

$$y(t) = \tilde{R}(B_0\tilde{R})^{-1}\beta + \int_0^1 s^{-M}s^{-1}f(st)\,\mathrm{d}s.$$

This solution is bounded by

$$|y(t)| \leq const.\bigl(t^{\alpha}\|h\|_{\delta} + \|f\|\bigr) + \bigl|\tilde{R}(B_0\tilde{R})^{-1}\beta\bigr|.$$

Let $Nf \in C^{r+1}[0,1]$ *and* $Zf \in C^{r}[0,1]$ *satisfy condition* (32) *with* $\alpha \geq r+1$. *Then* $y \in C^{r+1}[0,1]$.

The analogous result for a general TVP (7) follows from Theorems 8 and 11.

Theorem 16 *Let us assume that* A.2 *holds*, $B_1 \in \mathbb{R}^{n\times n}$ *is nonsingular, and* $\beta \in \mathbb{R}^n$. *Then, for every* $f \in C[0,1]$ *such that* Rf *satisfies* (32) *and* $Sf \in C^1[0,1]$, *there exists a unique solution* $y \in C[0,1]$ *of TVP* (7),

$$y(t) = t^M B_1^{-1}\beta + t^M \int_1^t s^{-M}s^{-1}f(s)\,\mathrm{d}s, \quad t \in (0,1].$$

This solution satisfies $My(0) = -f(0)$ *and is bounded by*

$$\bigl|y(t)\bigr| \leq const.\bigl(1 + t^{\sigma_+}\bigl(1 + \bigl|\ln(t)\bigr|^{n_{\max}-1}\bigr)\bigr)\bigl|B_1^{-1}\beta\bigr| + const.\bigl(\|f\| + t^{\alpha}\|h\|_{\delta}\bigr).$$

Let $r < \sigma_+ \leq r+1$, $Sf \in C^{r+2}[0,1]$, *and* $Zf \in C^{r+1}[0,1]$ *satisfy condition* (32) *with* $\alpha \geq r$, *then* $y \in C^r[0,1]$. *For* $\sigma_+ > r+1$, $Sf \in C^{r+2}[0,1]$, *and* $Zf \in C^r[0,1]$ *satisfying condition* (32) *with* $\alpha \geq r+1$, *we have* $y \in C^{r+1}[0,1]$. *Here* σ_+ *denotes the smallest positive real part of the eigenvalues of* M *and* $n_{\max}$ *is the dimension of the largest Jordan box of* M.

Next, we consider the linear BVP of the form

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t}, \quad t \in (0,1], y \in C[0,1], \qquad B_0 y(0) + B_1 y(1) = \beta, \tag{38}$$

where the matrix M may have an arbitrary spectrum, $B_0, B_1 \in \mathbb{R}^{m\times n}$, $m \leq n$, $\beta \in \mathbb{R}^m$, and $f \in C[0,1]$. It is clear from the previous considerations that the form of the boundary conditions which guarantee the existence of a unique continuous solution of (38) will depend on the spectral properties of the coefficient matrix M.

Before proceeding with the analysis, we show the following two auxiliary results. For proofs see [25].

Lemma 17 *Let* R *be a projection onto the eigenspace associated with eigenvalues* $\lambda = 0$. *Then*

$$t^M R = R, \quad t \in [0,1].$$

Lemma 18 *The projection matrices* S, Z, *and* N *commute with the matrices* t^M *and* M.

To specify the boundary conditions which guarantee the unique solvability of BVP (38) the following lemma is required.

Lemma 19 *Consider the following BVP*:

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t}, \quad t \in (0,1],$$

$$Hy(0) = 0, \qquad MNy(0) = -Nf(0), \qquad Sy(1) = S\gamma, \qquad Ry(0) = R\gamma.$$

Then, for every $f \in C[0,1]$, such that Zf satisfies (32) *and $Sf \in C^1[0,1]$, and for any constant vector γ, there exists a unique continuous solution of the form*

$$y(t) = t^M P\gamma + t^M S \int_1^t s^{-M-I} f(s)\,\mathrm{d}s + (Q+R)\int_0^1 s^{-M-I} f(st)\,\mathrm{d}s.$$

Proof According to the previous results, the contributions to the solution y depend on the signs of the eigenvalues of M. For the eigenvalues with negative real parts the contribution has the form

$$y_-(t) = N \int_0^1 s^{-M-I} f(st)\,\mathrm{d}s, \quad t \in [0,1].$$

For the eigenvalues with positive real parts the contribution is given by

$$y_+(t) = t^M S\gamma + t^M S \int_1^t s^{-M-I} f(s)\,\mathrm{d}s, \quad t \in (0,1],$$

and can be continuously extended to $t = 0$. Finally, for the eigenvalues $\lambda = 0$, we have

$$y_0(t) = R\gamma + Z \int_0^1 s^{-M-I} f(st)\,\mathrm{d}s, \quad t \in [0,1].$$

The solution y is the sum of all contributions, $y(t) = (N + S + Z)y(t)$. Therefore, we obtain

$$\begin{aligned} y(t) &= R\gamma + t^M S\gamma + (N+R+H)\int_0^1 s^{-M-I} f(st)\,\mathrm{d}s + t^M S \int_1^t s^{-M-I} f(s)\,\mathrm{d}s \\ &= t^M P\gamma + t^M S \int_1^t s^{-M-I} f(s)\,\mathrm{d}s + (Q+R)\int_0^1 s^{-M-I} f(st)\,\mathrm{d}s. \end{aligned}$$

We now evaluate y at the boundaries to show that the above boundary conditions are satisfied. According to (32), $Zf(0) = 0$ holds. This yields

$$(R+H)y(0) = Zy(0) = R\gamma + Z\int_0^1 s^{-M-I} f(0)\,\mathrm{d}s = R\gamma + \int_0^1 s^{-M-I}\,\mathrm{d}s Zf(0) = R\gamma.$$

Therefore, $Hy(0) = 0$ and $Ry(0) = R\gamma$. Moreover,

$$Sy(t) = t^M S\gamma + t^M S \int_1^t s^{-M-I} f(s)\,\mathrm{d}s \quad \Rightarrow \quad Sy(1) = S\gamma.$$

Finally, we show that $MNy(0) = -Nf(0)$. First note that

$$MNy(0) = NM\int_0^1 s^{-M-I}\,\mathrm{d}s f(0).$$

According to (12),

$$M\int_t^1 s^{-M-I}\,\mathrm{d}s = t^{-M} - I$$

for $t \in (0,1]$. Taking into account (13) and letting $t \to 0^+$, we obtain

$$NM \int_0^1 s^{-M-I}\,\mathrm{d}s = \lim_{t\to 0^+} Nt^{-M} - N = -N,$$

since the matrix Nt^{-M} consists only of Jordan boxes corresponding to eigenvalues with negative real parts. Therefore, $MNy(0) = -Nf(0)$. □

We now turn to the general boundary conditions specified in (38). For the investigation of these general boundary conditions, we have to rewrite the representation of the solution y, especially the term y_0,

$$\begin{aligned} y_0(t) &= R\gamma + Z\int_0^1 s^{-M-I} f(st)\,\mathrm{d}s = R\gamma + Zt^M \int_0^t s^{-M-I} f(s)\,\mathrm{d}s \\ &= R\gamma + t^M R \int_0^t s^{-M-I} f(s)\,\mathrm{d}s + t^M H \int_0^t s^{-M-I} f(s)\,\mathrm{d}s \\ &= R\tilde{\gamma} + t^M R \int_1^t s^{-M-I} f(s)\,\mathrm{d}s + t^M H \int_0^t s^{-M-I} f(s)\,\mathrm{d}s, \end{aligned}$$

where $\tilde{\gamma} := \gamma + \int_0^1 s^{-M-I} f(s)\,\mathrm{d}s$.

Remark 20 Note that the function

$$t^M H \int_0^t s^{-M-I} f(s)\,\mathrm{d}s = t^M H t^{-M} \int_0^1 s^{-M-I} f(st)\,\mathrm{d}s,$$

is continuous on $[0,1]$. In order to see this, we again use functions z_m and z_∞ given by (19) and (20). Due to (11), (31), (32), and (34), we have

$$\lim_{m\to\infty} \left|(\ln t)^k z_\infty(t) - (\ln t)^k z_m(t)\right| \le \|h\|_\delta \left\|t^\alpha (\ln t)^k\right\| \lim_{m\to\infty} \int_0^{\frac{1}{m}} \left|s^{-M}\right| s^{\alpha-1}\,\mathrm{d}s = 0$$

for $k \in \mathbb{N}_0$. Since each entry of the matrix $t^M H t^{-M}$ is a sum of terms $const.(\ln t)^k$, $k \in \mathbb{N}_0$, the function

$$t^M H t^{-M} \int_0^1 s^{-M-I} f(st)\,\mathrm{d}s = t^M H t^{-M} z_\infty$$

is continuous on $[0,1]$.

Consequently, the general continuous solution of the ODE system given in (38) can be represented as

$$y(t) = t^M P\gamma + t^M P \int_1^t s^{-M-I} f(s)\,\mathrm{d}s + t^M Q \int_0^t s^{-M-I} f(s)\,\mathrm{d}s, \tag{39}$$

and it satisfies the following boundary conditions:

$$Hy(0) = 0, \qquad MNy(0) = -Nf(0), \qquad Py(1) = P\gamma.$$

In the following lemma, we use the superposition principle to rewrite the solution (39) of (38) in a way convenient to discus the boundary conditions specified in (38).

Lemma 21 *Let us assume that the inhomogeneity $f \in C[0,1]$ is given in such a way that Zf satisfies* (32) *and $Sf \in C^1[0,1]$. Let the $n \times m$ matrix $\tilde{P}$ be a matrix consisting of the linearly independent columns of P. Then the general continuous solution of* (38) *has the form*

$$y(t) = \tilde{y}(t) + Y(t)\alpha, \quad t \in [0,1], \tag{40}$$

where α is a constant m-dimensional vector, $\tilde{y}$ is the unique solution of

$$\tilde{y}'(y) = \frac{M}{t}\tilde{y}(t) + \frac{f(t)}{t}, \quad t \in [0,1],$$
$$H\tilde{y}(0) = 0, \qquad MN\tilde{y}(0) = -Nf(0), \qquad P\tilde{y}(1) = 0,$$

and $Y(t)$ is the unique continuous fundamental solution matrix satisfying

$$Y'(t) = \frac{M}{t}Y(t), \quad t \in [0,1], \qquad Y(1) = \tilde{P}.$$

The case of the general boundary conditions (38) is covered by the following lemma.

Lemma 22 *Let $f \in C[0,1]$ be given in such a way that Zf satisfies* (32) *and $Sf \in C^1[0,1]$. Then there exists a unique solution $y \in C[0,1]$ of BVP* (38) *if and only if the $m \times m$ matrix*

$$B_0\tilde{R} + B_1\tilde{P}$$

is nonsingular. Here $B_0, B_1 \in \mathbb{R}^{m\times n}$, $\beta \in \mathbb{R}^m$, and $m = \operatorname{rank} P$.

Proof We use (39) and (40) to calculate $y(0)$ and $y(1)$. Since $Hy(0) = 0$ and $\lim_{t\to 0} t^M S = 0$, we first deduce

$$\begin{aligned} y(0) &= (H + P + N)y(0) = (P + N)\big(\tilde{y}(0) + Y(0)\alpha\big) = (P + N)\tilde{y}(0) + PY(0)\alpha \\ &= (P + N)\tilde{y}(0) + (R + S)Y(0)\alpha = (P + N)\tilde{y}(0) + \tilde{R}\alpha. \end{aligned}$$

Moreover, from $\tilde{P}y(1) = 0$ and $1^M S = S$, we have

$$\begin{aligned} y(1) &= (Q + P)y(1) = (Q + P)\big(\tilde{y}(1) + Y(1)\alpha\big) = Q\tilde{y}(1) + \big(QY(1) + \tilde{P}\big)\alpha \\ &= Q\tilde{y}(1) + \tilde{P}\alpha. \end{aligned}$$

Finally, we substitute $y(0)$ and $y(1)$ into the boundary condition and obtain

$$B_0y(0) + B_1y(1) = B_0\big((P + N)\tilde{y}(0) + \tilde{R}\alpha\big) + B_1\big(Q\tilde{y}(1) + \tilde{P}\alpha\big) = \beta.$$

Thus,

$$(B_0\tilde{R} + B_1\tilde{P})\alpha = \beta - B_0\big(P\tilde{y}(0) + N\tilde{y}(0)\big) - B_1Q\tilde{y}(1),$$

and the unknown vector α can be uniquely determined if the $m \times m$ matrix

$$B_0\tilde{R} + B_1\tilde{P}$$

is nonsingular. This completes the proof. □

The following theorem, stated without proof, is a consequence of the above results.

Theorem 23 *Consider BVP* (38), *where the inhomogeneity* f *is given in such a way such that* $f \in C[0,1]$, Zf *satisfies* (32), *and* $Sf \in C^1[0,1]$. *Moreover, let* $B_0, B_1 \in \mathbb{R}^{m\times n}$, $\beta \in \mathbb{R}^m$, *and* $m = \operatorname{rank} P$. *Let us assume that the* $m \times m$ *matrix* $B_0\tilde{R} + B_1\tilde{P}$ *is nonsingular. Then BVP* (38) *has a unique continuous solution* $y \in C[0,1]$. *This solution satisfies two initial conditions*,

$$Hy(0) = 0, \qquad MNy(0) = -Nf(0),$$

which are necessary and sufficient for $y \in C[0,1]$.

8 Collocation method

In this section we propose and analyze the polynomial collocation [19], for the numerical treatment of IVP, *cf.* (6),

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t}, \quad t \in (0,1], y \in C[0,1], \qquad B_0y(0) = \beta,$$

which we assume to be uniquely solvable. Here the matrix M has only eigenvalues with nonpositive real parts, and if $\sigma = 0$, then $\lambda = 0$. Moreover, $B_0 \in \mathbb{R}^{m\times n}$, $\beta \in \mathbb{R}^m$, where $\operatorname{rank} R = m \le n$. For the numerical treatment, we have to augment the m initial conditions specified by $B_0y(0) = \beta$ by the $n - m$ linearly independent initial conditions singled out from the set

$$Hy(0) = 0, \qquad MNy(0) = -Nf(0).$$

Consequently, we have to solve the initial value problem,

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t}, \qquad B_0y(0) = \beta, \qquad Hy(0) = 0, \qquad MNy(0) = -Nf(0). \tag{41}$$

We first discretize the analytical problem (41). The interval of integration $[0,1]$ is partitioned by an equidistant mesh Δ,

$$\Delta := \{0 = t_0 < t_1 < \cdots < t_{I-1} < t_I = 1, t_j = jh, j = 0, \dots, I = 1/h\},$$

and in each subinterval $[t_j, t_{j+1}]$, we introduce k collocation nodes $t_{jl} := t_j + u_l h$, $j = 0, \dots, I-1$, $l = 1, \dots, k$, where $0 < u_1 < u_2 < \cdots < u_k \le 1$. The computational grid including the mesh points and the collocation points is shown in Figure 1.

By $\mathcal{P}_{k,h}$ we denote the class of piecewise polynomial function of degree less or equal to k on each subinterval $[t_j, t_{j+1}]$. We approximate the analytical solution y by a piecewise

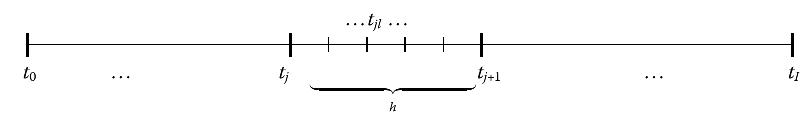

Figure 1 The computational grid.

polynomial function $p \in \mathcal{P}_{k,h} \cap C[0,1]$, $p(t) := p_j(t)$, $t \in [t_j, t_{j+1}]$, $j = 0, \ldots, I-1$, such that p satisfies ODE system (5) at the collocation points,

$$p'(t_{jl}) - \frac{M}{t_{jl}} p(t_{jl}) = \frac{f(t_{jl})}{t_{jl}}, \quad l = 1, \ldots, k, j = 0, \ldots, I-1, \tag{42}$$

together with the continuity relations,

$$p_{j-1}(t_j) = p_j(t_j), \quad j = 1, \ldots, I-1, \tag{43}$$

and p_0 satisfies the initial conditions

$$B_0 p(0) = \gamma, \qquad Hp(0) = 0, \qquad MNp(0) = -Nf(0). \tag{44}$$

Note that $\operatorname{rank} B_0 + \operatorname{rank} H + \operatorname{rank} N = n$. Since in each subinterval $[t_j, t_{j+1}]$ $p(t) = p_j(t)$ is a polynomial of degree smaller or equal to k, the total number of unknowns, the coefficients in the ansatz function p, is $(k+1)In$. On the other hand, the system (42) consists of kIn equations, (43) provides $(I-1)n$, and (44) n conditions, which together add up to $(k+1)In$. This means that the collocation scheme (42), (43), and (44) is closed.

The collocation applied to solve (36) was studied in [19], where in particular, unique solvability of the collocation scheme and the convergence properties have been shown. For the reader's convenience, we recapitulate in the next theorem an important auxiliary result from [19] required in the subsequent investigations. Note that since the analytical problem (41) has a unique solution, its value $y(0)$ is known. Therefore, in Theorem 4.1 [19], a slightly simpler problem is considered, where instead of the initial conditions the correct value of $y(0) := \delta$ is prescribed.

Theorem 24 (Theorem 4.1 in [19]) *Let us consider the collocation scheme*,

$$p'(t_{jl}) - \frac{M}{t_{jl}} p(t_{jl}) = M^{\mu} \frac{c_{jl}}{t_{jl}^{\beta}}, \quad l = 1, \ldots, k, j = 0, \ldots I-1, \qquad p(0) = \delta, \tag{45}$$

where $\mu, \beta = 0, 1$, *and* $p \in \mathcal{P}_{k,h} \cap C[0,1]$. *Then problem* (45) *has a unique solution, provided that h is sufficiently small. This solution satisfies*

$$\left|p(t)\right| \le const.\left(|\delta| + \left|\ln(h)\right|^{d} |M\delta| + \left|\ln(h)\right|^{(\beta(d-\mu))_+} C\right), \quad t \in [0,1],$$

where $C = \max_{0 \le j \le I-1} \max_{1 \le l \le k} |c_{jl}|$, *d is the dimension of the largest Jordan box of M associated to the eigenvalue* $\lambda = 0$ *and*

$$(x)_+ = \begin{cases} x, & x \ge 0, \\ 0, & x < 0. \end{cases}$$

We are now in the position to formulate the convergence result for the collocation method.

Theorem 25 *Let us consider the initial value problem*

$$y'(t) - \frac{M}{t}y(t) = \frac{f(t)}{t}, \qquad y(0) = \delta,$$

where $H\delta = 0$ *and* $MN\delta = -Nf(0)$. *Let us assume that the function* f *satisfies* $Nf \in C^{k+1}[0,1]$, $Zf = O(t^\alpha z(t))$, *with* $\alpha \geq k+1$, $Zf \in C^k[0,1]$, *and* $z \in C^k[0,1]$. *Let the function* $p \in \mathcal{P}_{k,h} \cap C[0,1]$ *satisfy the collocation scheme*

$$p'(t_{jl}) - \frac{M}{t_{jl}}p(t_{jl}) = \frac{f(t_{jl})}{t_{jl}}, \quad l = 1, \dots k, j = 0, \dots, I-1, \qquad p(0) = \delta.$$

Then

$$\bigl|p(t) - y(t)\bigr| \leq const.h^k, \quad t \in [0,1].$$

Proof The idea of the proof is to introduce an *error function* $e \in \mathcal{P}_{k,h} \cap C[0,1]$ and investigate how it is related to the global error $p - y$ of the scheme. Let e be defined as follows:

$$e'(t_{jl}) := y'(t_{jl}) - p'(t_{jl}), \quad l = 1, \dots, k, j = 0, \dots, I-1, \qquad e(0) := 0.$$

Since on each subinterval $[t_j, t_{j+1}]$ the function $e'(t)$ is a polynomial of degree less or equal to $k-1$ it is uniquely determined by its values at k distinct points in this interval,

$$e'(t) = \sum_{i=1}^{k} l_i\left(\frac{t-t_j}{h}\right)y'(t_{ji}) - p'(t), \quad t \in [t_j, t_{j+1}],$$

where $l_i(t) = w(t)/((t-u_i)w'(u_i))$, $i = 1, \dots, k$, $w(t) = (t-u_1)(t-u_2)\cdots(t-u_k)$. Since $y \in C^{k+1}[0,1]$ the interpolation error is $O(h^k)$ and, hence,

$$e'(t) = y'(t) - p'(t) + O\bigl(h^k\bigr).$$

By integration in $[0,t]$, we obtain $e(t) = y(t) - p(t) + O(h^k t)$, which means that e differs from $y - p$ by $O(h^k)$ terms. Moreover, we see that e satisfies the following collocation scheme:

$$\begin{aligned} e'(t_{jl}) - \frac{M}{t_{jl}}e(t_{jl}) &= y'(t_{jl}) - \frac{M}{t_{jl}}y(t_{jl}) - \left(p'(t_{jl}) - \frac{M}{t_{jl}}p(t_{jl})\right) - \frac{M}{t_{jl}}O\bigl(t_{jl}h^k\bigr) \\ &= \frac{f(t_{jl})}{t_{jl}} - \frac{f(t_{jl})}{t_{jl}} - \frac{M}{t_{jl}}O\bigl(t_{jl}h^k\bigr) = O\bigl(h^k\bigr), \qquad e(0) = 0. \end{aligned}$$

According to Theorem 24, we conclude that $e(t) = O(h^k)$ which together with $e(t) = y(t) - p(t) + O(h^k)$ yields $|p(t) - y(t)| \leq const.h^k$. □

The especially attractive property of the collocation is the so-called superconvergence. For regular ODEs and certain choices of the collocation points (Gaussian, Lobatto, Radau), the convergence order in the *mesh points* can be considerably higher than k, provided that the solution y is sufficiently smooth. For the Gaussian points the superconvergence order is $O(h^{2k})$. Since already for problem (36) counterexamples show that the superconvergence does not hold [19], we do not expect it for the problem at hand either. However, the so-called small superconvergence uniform in t can be shown; see the next theorem. The main prerequisite for the proof is the property

$$\int_0^1 w(s)\,\mathrm{d}s = 0, \tag{46}$$

which holds for an appropriate choice of the collocation points.

Theorem 26 *Let $Nf \in C^{k+2}[0,1]$, $Zf \in C^{k+1}[0,1]$, and $Zf = O(t^\alpha z(t))$, where $\alpha \geq k+2$ and $z \in C^{k+1}[0,1]$. If* (46) *holds, then the estimate for the global error given in Theorem* 25 *can be replaced by*

$$\left|p(t) - y(t)\right| \leq const.h^{k+1}\left|\ln(h)\right|^{(d-1)_+}.$$

Proof Consider again the error function e defined in Theorem 25. Due to the smoothness assumptions made for the problem data $y \in C^{k+2}[0,1]$ follows. Therefore,

$$\begin{aligned} e'(t) &= \sum_{i=1}^{k} l_i\left(\frac{t-t_j}{h}\right)y'(t_{ji}) - p'(t) \\ &= y'(t) - p'(t) + \frac{h^k}{k!}w\left(\frac{t-t_j}{h}\right)y^{(k+1)}(t_j) + O\left(h^{k+1}\right), \quad t \in [t_j, t_{j+1}]. \end{aligned}$$

We integrate e' on $[0,1]$ and use (46) to obtain

$$\begin{aligned} e(t) &= y(t) - p(t) + \sum_{i=0}^{j-1}\frac{h^k}{k!}y^{(k+1)}(t_i)\int_{t_i}^{t_{i+1}} w\left(\frac{s-t_i}{h}\right)\mathrm{d}s \\ &\quad + \frac{h^k}{k!}y^{(k+1)}(t_j)\int_{t_j}^{t} w\left(\frac{s-t_j}{h}\right)\mathrm{d}s + O\left(th^{k+1}\right) = y(t) - p(t) + O\left(h^{k+1}\right). \end{aligned}$$

This implies

$$\begin{aligned} e'(t_{jl}) - \frac{M}{t_{jl}}e(t_{jl}) &= y'(t_{jl}) - \frac{M}{t_{jl}}y(t_{jl}) - \left(p'(t_{jl}) - \frac{M}{t_{jl}}p(t_{jl})\right) - \frac{M}{t_{jl}}O\left(h^{k+1}\right) \\ &= -\frac{M}{t_{jl}}O\left(h^{k+1}\right), \qquad e(0) = 0. \end{aligned}$$

According to Theorem 24 we have $|e(t)| \leq const.(|\ln(h)|^{(d-1)_+}h^{k+1})$, and finally $|p(t) - y(t)| \leq const.(|\ln(h)|^{(d-1)_+}h^{k+1})$. □

9 Numerical experiments

In order to illustrate the theoretical results derived in the previous section, we have constructed model problems and run the collocation code `bvpsuite` on coherently refined meshes to compare the empirically estimated convergence orders of the scheme with the theoretically predicted ones.

9.1 General IVP with smooth solution

We first deal with a linear system of ODEs,

$$y'(t) = \frac{1}{t}\begin{pmatrix} -4 & 2 & -1 \\ -8 & 4 & -2 \\ -12 & 8 & -4 \end{pmatrix} y(t) + \frac{f(t)}{t}, \quad t \in (0,1], \tag{47}$$

subject to initial conditions

$$B_0 y(0) = \begin{pmatrix} 3 & -2 & 1 \end{pmatrix} y(0) = 1, \qquad \begin{pmatrix} -2 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix} y(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{48}$$

Here

$$f(t) = \begin{pmatrix} t\exp(t) + 2\exp(t) + \sin(t)\cos(t) + t\cos^2(t) - t\sin^2(t) \\ 2t\exp(t) + 4\exp(t) + 2\sin(t)\cos(t) + 2t\cos^2(t) - 2t\sin^2(t) + 2t^2 \\ 2t\exp(t) + 4\exp(t) + t\cos^2(t) - t\sin^2(t) + 4t^2 \end{pmatrix}.$$

The matrix M has a double eigenvalue $\lambda_1 = \lambda_2 = -2$, and a single eigenvalue $\lambda_3 = 0$. The IVP (47), (48) satisfies the assumptions of Theorem 15 and the exact solution y has the form

$$y(t) = \begin{pmatrix} \exp(t) + \sin(t)\cos(t) \\ 2\exp(t) + 2\sin(t)\cos(t) + t^2 \\ 2\exp(t) + \sin(t)\cos(t) + 2t^2 \end{pmatrix}.$$

We see that $y \in C^\infty[0,1]$, *cf.* Remark 12.

In Tables 1 to 4, we illustrate the convergence behavior for the collocation executed with equidistant and Gaussian collocation points. The number of the collocation points k was chosen to vary from 1 to 8. However, in the simulations shown here, we report only on the values 1 to 4 since the results for 5 to 8 are very similar. The maximal global error is computed either in the mesh points,

$$\|Y_h - Y\|_\infty := \max_{0\le j\le l} \left|p(t_j) - y(t_j)\right|,$$

Table 1 IVP (47), (48): Convergence of the collocation scheme, $k = 1$

h	Equidistant points			Gaussian points					
	Uniform			Mesh			Uniform		
	$\|Y_h - Y\|_\infty$	*c*	*p*	$\|Y_h - Y\|_\infty$	*c*	*p*	$\|Y_h - Y\|_\infty$	*c*	*p*
1/2	1.4e–01	–	–	9.2e–02	–	–	1.4e–01	–	–
1/4	3.5e–02	2.1e+00	1.96	2.3e–02	3.7e–01	2.01	3.5e–02	2.1e+00	1.96
1/8	9.0e–03	2.2e+00	1.98	5.8e–03	3.7e–01	2.00	9.0e–03	2.2e+00	1.98
1/16	2.3e–03	2.2e+00	1.99	1.4e–03	3.7e–01	2.00	2.3e–03	2.2e+00	1.99
1/32	5.7e–04	2.3e+00	2.00	3.6e–04	3.7e–01	2.00	5.7e–04	2.3e+00	2.00
1/64	1.4e–04	2.3e+00	2.00	9.0e–05	3.7e–01	2.00	1.4e–04	2.3e+00	2.00
1/128	3.6e–05	2.3e+00	2.00	2.2e–05	3.7e–01	2.00	3.6e–05	2.3e+00	2.00
1/256	8.9e–06	2.3e+00	2.00	5.6e–06	3.7e–01	2.00	8.9e–06	2.3e+00	2.00
1/512	2.2e–06	2.3e+00	2.00	1.4e–06	3.7e–01	2.00	2.2e–06	2.3e+00	2.00

Table 2 IVP (47), (48): Convergence of the collocation scheme, *k* = 2

h	Equidistant points			Gaussian points					
	Uniform			Mesh			Uniform		
	$\|Y_h - Y\|_\infty$	*c*	*p*	$\|Y_h - Y\|_\infty$	*c*	*p*	$\|Y_h - Y\|_\infty$	*c*	*p*
1/2	3.7e–03	–	–	3.7e–03	–	–	7.8e–03	–	–
1/4	5.1e–04	6.4e–01	2.87	1.6e–04	8.4e–02	4.50	1.2e–03	3.6e+00	2.73
1/8	6.6e–05	8.1e–01	2.96	8.1e–06	6.7e–02	4.34	1.5e–04	6.4e+00	2.93
1/16	8.3e–06	8.8e–01	2.99	4.4e–07	5.1e–02	4.21	2.0e–05	7.6e+00	2.98
1/32	1.0e–06	9.0e–01	3.00	2.5e–08	4.0e–02	4.12	2.5e–06	8.1e+00	2.99
1/64	1.3e–07	9.1e–01	3.00	1.5e–09	3.3e–02	4.06	3.1e–07	8.4e+00	3.00
1/128	1.6e–08	9.2e–01	3.00	9.3e–11	2.9e–02	4.03	3.8e–08	8.5e+00	3.00
1/256	2.0e–09	9.2e–01	3.00	5.7e–12	2.7e–02	4.02	4.8e–09	8.5e+00	3.00
1/512	2.6e–10	9.2e–01	3.00	3.5e–13	2.8e–02	4.03	6.0e–10	8.5e+00	3.00

Table 3 IVP (47), (48): Convergence of the collocation scheme, *k* = 3

h	Equidistant points			Gaussian points					
	Uniform			Mesh			Uniform		
	$\|Y_h - Y\|_\infty$	*c*	*p*	$\|Y_h - Y\|_\infty$	*c*	*p*	$\|Y_h - Y\|_\infty$	*c*	*p*
1/2	2.0e–04	–	–	3.6e–05	–	––	3.5e–04	–	–
1/4	7.4e–06	3.8e+00	4.75	5.1e–07	2.6e–03	6.15	1.3e–05	2.8e+02	4.73
1/8	3.0e–07	2.7e+00	4.63	7.6e–09	2.3e–03	6.07	5.4e–07	1.8e+02	4.61
1/16	1.4e–08	1.5e+00	4.46	1.2e–10	2.1e–03	6.03	2.5e–08	9.1e+01	4.44
1/32	6.9e–10	8.0e–01	4.30	1.8e–12	2.1e–03	6.02	1.3e–09	4.2e+01	4.29
1/64	3.8e–11	4.4e–01	4.18	2.0e–14	8.9e–03	6.44	7.1e–11	2.1e+01	4.17
1/128	2.2e–12	2.8e–01	4.10	3.4e–14	1.0e–15	–0.72	4.2e–12	1.3e+01	4.09
1/256	1.3e–13	2.0e–01	4.04	1.1e–14	1.1e–10	1.66	2.5e–13	8.5e+00	4.03
1/512	7.1e–15	8.0e–01	4.24	2.0e–14	5.9e–17	–0.94	1.6e–14	8.7e+00	4.03

Table 4 IVP (47), (48): Convergence of the collocation scheme, *k* = 4

h	Equidistant points			Gaussian points					
	Uniform			Mesh			Uniform		
	$\|Y_h - Y\|_\infty$	*c*	*p*	$\|Y_h - Y\|_\infty$	*c*	*p*	$\|Y_h - Y\|_\infty$	*c*	*p*
1/2	2.7e–05	–	–	6.9e–07	–	–	2.6e–05	–	–
1/4	9.1e–07	2.2e+00	4.91	5.8e–09	8.2e–05	6.90	8.6e–07	3.7e+02	4.90
1/8	2.9e–08	2.7e+00	4.98	4.7e–11	8.7e–05	6.95	2.7e–08	5.0e+02	4.98
1/16	9.0e–10	2.9e+00	5.00	3.7e–13	9.3e–05	6.97	8.6e–10	5.4e+02	5.00
1/32	2.8e–11	2.9e+00	5.00	6.2e–15	4.7e–06	5.90	2.7e–11	5.6e+02	5.00
1/64	8.8e–13	3.1e+00	5.01	1.5e–14	7.4e–17	–1.28	8.4e–13	5.8e+02	5.01
1/128	3.1e–14	1.2e+00	4.84	8.9e–15	3.6e–13	0.77	2.4e–14	1.1e+03	5.10
1/256	2.7e–15	2.4e–04	3.52	7.1e–15	4.2e–14	0.32	3.1e–15	1.3e–04	2.97
1/512	1.3e–15	3.4e–12	1.00	1.3e–14	4.7e–17	–0.91	3.6e–15	6.4e–16	–0.19

or 'uniformly' in t, $\|Y_h - Y\|_\infty := \max_{0 \le i \le 1.000} |p(\tau_i) - y(\tau_i)|$, $\tau_i = ih$, $h = 10^{-3}$. The estimated order of convergence p and the error constant c are estimated using two consecutive meshes with the step sizes h and $h/2$.

Since $\|Y_h - Y\| \approx ch^p$ for $h \to 0$, we have

$$\|Y_h - Y\|_\infty = ch^p, \qquad \|Y_{h/2} - Y\|_\infty = c\left(\frac{h}{2}\right)^p \quad \Rightarrow \quad p = \ln\left(\frac{\|Y_h - Y\|_\infty}{\|Y_{h/2} - Y\|_\infty}\right)\frac{1}{\ln(2)}.$$

Having p, we calculate the error constant from $c = \|Y_{h/2} - Y\|_\infty / (\frac{h}{2})^p$.

According to the experiments, the empirical convergence orders very well reflect the theoretical findings. For Gaussian points, we observe the small superconvergence order

Table 5 IVP (49): Convergence of the collocation scheme, $k = 4$

h	Equidistant points			Gaussian points					
	Uniform			Mesh			Uniform		
	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/2	3.2e–03	–	–	6.2e–04	–	–	1.2e–03	–	–
1/4	1.1e–03	1.0e–01	1.49	2.2e–04	1.7e–03	1.50	4.4e–04	1.9e–01	1.50
1/8	4.1e–04	1.0e–01	1.50	7.7e–05	1.7e–03	1.50	1.6e–04	1.9e–01	1.50
1/16	1.4e–04	1.0e–01	1.50	2.7e–05	1.7e–03	1.50	5.5e–05	1.9e–01	1.50
1/32	5.1e–05	1.0e–01	1.50	9.6e–06	1.7e–03	1.50	1.9e–05	1.9e–01	1.50
1/64	1.8e–05	1.0e–01	1.50	3.4e–06	1.7e–03	1.50	2.4e–04	3.6e–15	–3.65
1/128	8.0e–05	7.4e–11	–2.15	1.2e–06	1.7e–03	1.50	4.9e–03	3.4e–17	–4.34
1/356	7.9e–04	4.0e–14	–3.31	4.3e–07	1.7e–03	1.50	1.0e–01	2.1e–17	–4.40
1/512	2.5e–02	2.5e–19	–4.99	1.5e–07	1.7e–03	1.50	4.8e–01	1.5e–09	–2.20

$k + 1$ uniformly in t. The superconvergence order $2k$ in the mesh points does not hold in general; see the case $k = 4$. For uniformly spaced equidistant collocation points we again observe the order $k + 1$, which for this model is slightly better than we can show theoretically.

9.2 General IVP with 'unsmooth' solution

Next, we discuss an IVP whose solution is less smooth than in the previous model. The problem reads

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t}, \qquad (1 \quad 0 \quad 0)y(0) = \frac{1}{4}, \qquad \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} y(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{49}$$

where

$$M = \begin{pmatrix} -4 & 0 & 0 \\ -2 & -2 & 0 \\ 2 & -2 & 0 \end{pmatrix}, \qquad f(t) = \begin{pmatrix} \exp(t) \\ \exp(t) + t \\ t + \frac{1}{2}t^{\frac{1}{2}}\sin(t) + t^{\frac{3}{2}}\cos(t) \end{pmatrix}.$$

The eigenvalues of M are $\lambda_1 = -4$, $\lambda_2 = -2$, and $\lambda_3 = 0$; and the initial conditions are designed in such a way that IVP (49) satisfies the assumptions of Theorem 15. The analytical solution $y \in C^1[0,1]$ is given by

$$y(t) = \begin{pmatrix} t^{-4}(6 - 6\exp(t) + 6t\exp(t) - 3t^2\exp(t) + t^3\exp(t)) \\ t^{-4}(6 - 6\exp(t) + 6t\exp(t) - 3t^2\exp(t) + t^3\exp(t)) + \frac{t}{3} \\ \frac{t}{3} + \sqrt{t}\sin(t) \end{pmatrix}.$$

The related numerical results are listed for $k = 4$ in Table 5. As expected, we observe an order reduction down to 1.5, not only for $k = 4$, but also for all other values of k.

9.3 General TVP with small positive eigenvalues

The case of the matrix M having eigenvalues with positive real parts has not been investigated yet, since the related theory is particularly tedious and involved, *cf.* [27]. However, some interesting numerical simulations are already available and, therefore, the results of these experiments are briefly discussed here to complete the picture.

We first consider the following model problem:

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t}, \qquad B_1 y(1) = \begin{pmatrix} 4 & -1 & 1 \\ 0 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} y(1) = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}, \tag{50}$$

where

$$M = \begin{pmatrix} 3.5 & -1 & 1 \\ -14 & 5 & -4 \\ -24.5 & 8 & -7 \end{pmatrix}, \qquad f(t) = \begin{pmatrix} 1 - 4t^2 \\ 4 + t^2 \ln(t) \\ 1 + t^2 \ln(t) + 14t^2 \end{pmatrix}.$$

The eigenvalues of M are $\lambda_1 = 0.5$, $\lambda_2 = 1$, and $\lambda_3 = 0$; and the solution $y \in C[0,1]$ of (50) is

$$y(t) = \begin{pmatrix} 3\sqrt{t} - 2t^2 - 4 \\ 12\sqrt{t} - 8 + 4t + t^2 \ln(t) - t^2 \\ 3\sqrt{t} + 4t + 5 + 6t^2 + t^2 \ln(t) \end{pmatrix}.$$

In Tables 6 and 7, we again see the order reduction down to 0.5, due to the fact that the first derivative y' is unbounded for $t \to 0$. Moreover, we see that the problem is hard to solve and the convergence is very slow. For $h \approx 2 \cdot 10^{-3}$ the level of the global error is only $\|Y_h - Y\|_\infty \approx 10^{-1}$. In [6] the convergence order of the collocation scheme for TVPs of the

Table 6 TVP (50): Convergence of the collocation scheme, *k* = 2

h	**Equidistant points**			**Gaussian points**					
	Uniform			**Mesh**			**Uniform**		
	$\|Y_h - Y\|_\infty$	***c***	***p***	$\|Y_h - Y\|_\infty$	***c***	***p***	$\|Y_h - Y\|_\infty$	***c***	***p***
1/2	2.2e+00	–	–	1.7e+00	–	–	1.7e+00	–	–
1/4	1.5e+00	5.4e+00	0.51	1.2e+00	2.4e+00	0.51	1.2e+00	5.4e+00	0.51
1/8	1.1e+00	5.3e+00	0.50	8.5e–01	2.4e+00	0.50	8.5e–01	5.3e+00	0.50
1/16	7.5e–01	5.3e+00	0.50	6.0e–01	2.4e+00	0.50	6.0e–01	5.2e+00	0.50
1/32	5.3e–01	5.2e+00	0.50	4.2e–01	2.4e+00	0.50	4.2e–01	5.2e+00	0.50
1/64	3.8e–01	5.2e+00	0.50	3.0e–01	2.4e+00	0.50	3.0e–01	5.2e+00	0.50
1/128	2.7e–01	5.2e+00	0.50	2.1e–01	2.4e+00	0.50	2.1e–01	5.2e+00	0.50
1/256	1.9e–01	5.2e+00	0.50	1.5e–01	2.4e+00	0.50	1.5e–01	5.2e+00	0.50
1/512	1.3e–01	5.2e+00	0.50	1.1e–01	2.4e+00	0.50	1.1e–01	5.2e+00	0.50

Table 7 TVP (50): Convergence of the collocation scheme, *k* = 3

h	**Equidistant points**			**Gaussian points**					
	Uniform			**Mesh**			**Uniform**		
	$\|Y_h - Y\|_\infty$	***c***	***p***	$\|Y_h - Y\|_\infty$	***c***	***p***	$\|Y_h - Y\|_\infty$	***c***	***p***
1/2	1.7e+00	–	–	1.2e+00	–	–	1.2e+00	–	–
1/4	1.2e+00	4.7e+00	0.50	8.6e–01	1.7e+00	0.50	8.6e–01	5.2e+00	0.50
1/8	8.3e–01	4.7e+00	0.50	6.1e–01	1.7e+00	0.50	6.1e–01	5.1e+00	0.50
1/16	5.8e–01	4.7e+00	0.50	4.3e–01	1.7e+00	0.50	4.3e–01	5.1e+00	0.50
1/32	4.1e–01	4.7e+00	0.50	3.0e–01	1.7e+00	0.50	3.0e–01	5.1e+00	0.50
1/64	2.9e–01	4.7e+00	0.50	2.1e–01	1.7e+00	0.50	2.1e–01	5.1e+00	0.50
1/128	2.1e–01	4.7e+00	0.50	1.5e–01	1.7e+00	0.50	1.5e–01	5.1e+00	0.50
1/256	1.5e–01	4.7e+00	0.50	1.1e–01	1.7e+00	0.50	1.1e–01	5.1e+00	0.50
1/512	1.0e–01	4.7e+00	0.50	7.6e–02	1.7e+00	0.50	7.6e–02	5.1e+00	0.50

type (37), where all eigenvalues of M have positive real parts, has been specified. Provided that the Jordan box associated with the eigenvalue whose real part is σ_+ is diagonal, the uniform stage convergence order was shown to be $\min\{\sigma_+, k\}$. This result suggests that also in the case of the model (50) the convergence order will drop down to 0.5, *cf.* Tables 6 and 7.

The remedy for this lack of smoothness due to the small size of the positive eigenvalues of M is to make a change of the independent variable [28], $t = \tau^\mu$, for some $\mu > 1$. Then $\tilde{y}(\tau) := y(\tau^\mu)$ satisfies the transformed ODE system

$$\tilde{y}'(\tau) = \frac{\tilde{M}}{\tau}\tilde{y}(\tau) + \frac{\tilde{f}(\tau)}{\tau}, \quad \tau \in (0,1], \tag{51}$$

where $\tilde{M} = \mu M$ and $\tilde{f}(\tau) = \mu f(\tau^\mu)$. The eigenvalues of the matrix $\tilde{M}$ become $\tilde{\lambda} = \mu\lambda$ and therefore, the solution $\tilde{y}$ of the transformed equation is smoother than the solution y of the original one. For instance, if $f \in C^{r+2}[0,1]$ and $\sigma_+ \leq r+1$, then we can choose μ such that $\mu\sigma_+ > r+1$. Consequently, by Theorem 8(ii), $\tilde{y}(\tau) = y(\tau^\mu) \in C^{r+1}[0,1]$. One can also interpret the above smoothing in terms of the mesh adaptation - solving the ODE system (51) on an equidistant mesh, means solving the original ODE system on a mesh which is adequately refined close to the singularity, where the solution y and its derivatives rapidly change. Consequently, we solve (51) subject to terminal conditions given in (50), where, for $\mu = 8$,

$$\tilde{M} = \begin{pmatrix} 28 & -8 & 8 \\ -112 & 40 & -32 \\ -196 & 64 & -56 \end{pmatrix}, \qquad \tilde{f}(\tau) = \begin{pmatrix} 8 - 32\tau^{16} \\ 32 + 8\tau^{16}\ln(\tau^8) \\ 8 + 8\tau^{16}\ln(\tau^8) + 112\tau^{16} \end{pmatrix}.$$

While the eigenvalues of M are $\lambda_1 = 0.5$, $\lambda_2 = 1$, and $\lambda_3 = 0$, the eigenvalues of $\tilde{M}$ become $\tilde{\lambda}_1 = 4$, $\tilde{\lambda}_2 = 8$, and $\tilde{\lambda}_3 = 0$. The solution of the transformed problem reads

$$\tilde{y}(\tau) = \begin{pmatrix} 3\tau^4 - 2\tau^{16} - 4 \\ -8 + 12\tau^4 + 4\tau^8 + \tau^{16}\ln\tau^8 - \tau^{16} \\ 5 + 6\tau^{16} + 4\tau^8 + \tau^{16}\ln\tau^8 + 3\tau^4 \end{pmatrix}.$$

Tables 8 and 9 show the desired effect. For $k = 2$ and equidistant collocation points, we observe the $O(h^k)$ behavior of the global error uniformly in t, as was the case for a smooth

Table 8 Transformed TVP (50): Convergence of the collocation scheme, $k = 2$

h	Equidistant points			Gaussian points					
	Uniform			Mesh			Uniform		
	$\Vert Y_h - Y\Vert_\infty$	c	p	$\Vert Y_h - Y\Vert_\infty$	c	p	$\Vert Y_h - Y\Vert_\infty$	c	p
1/2	5.4e+00	–	–	2.1e+00	–	–	1.9e+00	–	–
1/4	2.3e+00	5.2e+01	1.26	2.3e–01	1.8e+01	3.17	2.2e–01	2.2e+03	3.14
1/8	6.8e–01	1.7e+02	1.73	1.7e–02	4.2e+01	3.76	1.6e–02	1.3e+04	3.74
1/16	1.8e–01	3.1e+02	1.93	1.1e–03	6.1e+01	3.94	1.1e–03	2.6e+04	3.93
1/32	4.5e–02	3.8e+02	1.98	7.0e–05	6.9e+01	3.98	6.8e–05	3.3e+04	3.98
1/64	1.1e–02	4.1e+02	2.00	4.4e–06	7.2e+01	4.00	4.3e–06	3.5e+04	4.00
1/128	2.8e–03	4.2e+02	2.00	2.7e–07	7.3e+01	4.00	2.7e–07	3.6e+04	4.00
1/256	7.1e–04	4.2e+02	2.00	1.7e–08	7.3e+01	4.00	1.7e–08	3.6e+04	4.00
1/512	1.8e–04	4.2e+02	2.00	1.1e–09	7.4e+01	4.00	1.0e–09	3.6e+04	4.00

Table 9 Transformed TVP (50): Convergence of the collocation scheme, $k = 3$

h	Equidistant points			Gaussian points					
	Uniform			Mesh			Uniform		
	$\Vert Y_h - Y\Vert_\infty$	c	p	$\Vert Y_h - Y\Vert_\infty$	c	p	$\Vert Y_h - Y\Vert_\infty$	c	p
1/2	2.2e+00	–	–	1.5e–01	–	–	1.5e–01	–	–
1/4	2.8e–01	1.1e+03	2.98	3.7e–03	5.7e+00	5.29	4.1e–03	4.5e+05	5.18
1/8	2.1e–02	8.2e+03	3.71	8.3e–05	7.5e+00	5.49	8.3e–05	2.1e+06	5.62
1/16	1.4e–03	1.7e+04	3.93	5.2e–06	3.4e–01	4.00	5.2e–06	2.1e+03	4.00
1/32	8.8e–05	2.2e+04	3.98	3.3e–07	3.4e–01	4.00	3.3e–07	2.1e+03	4.00
1/64	5.5e–06	2.3e+04	4.00	2.0e–08	3.4e–01	4.00	2.0e–08	2.1e+03	4.00
1/128	3.5e–07	2.4e+04	4.00	1.3e–09	3.4e–01	4.00	1.3e–09	2.1e+03	4.00
1/256	2.2e–08	2.4e+04	4.00	8.0e–11	3.3e–01	3.99	8.0e–11	2.0e+03	3.99
1/512	1.4e–09	2.4e+04	4.00	7.0e–12	2.2e–02	3.50	7.0e–12	4.6e+01	3.50

Table 10 Notation

$\mathbb{R}^n$	n-dimensional vector space of real-valued vectors
$\mathbb{C}^n$	n-dimensional vector space of complex-valued vectors
$\vert x\vert := \max_{1\le i\le n}\vert x_i\vert$	maximum norm for a vector $x \in \mathbb{C}^n$
$C[0,1]$	space of continuous real vector-valued functions on $[0,1]$
$C^p[0,1]$	space of p-times continuously differentiable real vector-valued functions on $[0,1]$
$\Vert y\Vert := \max_{t\in[0,1]}\vert y(t)\vert$	maximum norm for a function $y \in C[0,1]$
$\Vert y\Vert_\delta := \max_{t\in[0,\delta]}\vert y(t)\vert$	norm restricted to the interval $[0,\delta]$, $\delta > 0$
$\vert A\vert := \max_{1\le i\le m}\sum_{j=1}^n \vert a_{ij}\vert$	induced operator norm for a matrix $A \in \mathbb{C}^{m\times n}$
X_+	invariant subspace associated with the eigenvalues with positive real parts
$X_0^{(e)}$	space spanned by eigenvectors associated with eigenvalues $\lambda = 0$
X_-	invariant subspace associated with the eigenvalues with negative real parts
$X_0^{(h)}$	space spanned by principal eigenvectors associated with the eigenvalue $\lambda = 0$
S	orthogonal projection onto X_+
R	orthogonal projection onto $X_0^{(e)}$
$P := R + S$	projection onto $X_+ \oplus X_0^{(e)}$
$Q := I - P$	projection onto $X_- \oplus X_0^{(h)}$
$Z := R + H$	orthogonal projection onto $X_0^{(e)} \oplus X_0^{(h)}$
N	orthogonal projection onto X_-
H	orthogonal projection onto $X_0^{(h)}$

IVP. For the Gaussian points we see the superconvergence $O(h^{2k})$, both in the mesh points and uniformly in t, which is better than expected. However, this very fast convergence for the Gaussian points is put into the right perspective by the data for $k = 3$ in Table 9. Here only the expected order $O(h^{k+1})$ uniformly in t can be observed.

The above experiments in context of the TVPs suggest the following working hypothesis: The polynomial collocation shows the same convergence behavior for the TVPs and IVPs, provided that their solutions are appropriately smooth. This hypothesis may become a subject of further studies.

10 Conclusions

In this paper, we investigated the analytical properties of the singular BVP

$$y'(t) = \frac{M}{t}y(t) + \frac{f(t)}{t}, \quad t \in (0,1], y \in C[0,1], \qquad B_0y(0) + B_1y(1) = \beta.$$

It turns out that the structure of the initial/terminal/boundary conditions to guarantee that the problem has a unique solution which is at least continuous on $[0,1]$ depends on the spectral properties of the matrix M.

In context of an IVP with appropriately smooth solution, polynomial collocation method executed with k arbitrary collocation points retains its classical stage order $O(h^k)$ uniformly in t. For Gaussian points the so-called small superconvergence order $O(h^{k+1})$ can also be shown to hold uniformly in t. In general, the superconvergence order $O(h^{2k})$ in the mesh points cannot be expected to hold.

Our next task is to generalize the results discussed in the present article to the linear case with a variable coefficient matrix $M(t)$ and to the nonlinear case, cf. [29].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to the analytical part of the paper. JB and EBW contributed to its numerical part. All authors read and approved the final version of the manuscript.

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