Rachůnek and Rachůnková *Boundary Value Problems* 2013, **2013**:195
http://www.boundaryvalueproblems.com/content/2013/1/195

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

First-order nonlinear differential equations with state-dependent impulses

Lukáš Rachůnek and Irena Rachůnková*

*Correspondence:
irena.rachunkova@upol.cz
Department of Mathematics,
Faculty of Science, Palacký
University, 17. listopadu 12,
Olomouc, 77146, Czech Republic

Abstract

The paper deals with the state-dependent impulsive problem

$$z'(t) = f(t, z(t)) \quad \text{for a.e. } t \in [a,b],$$
$$z(\tau+) - z(\tau) = \mathcal{J}(\tau, z(\tau)), \qquad \gamma(z(\tau)) = \tau,$$
$$\ell(z) = c_0,$$

where $[a,b] \subset \mathbb{R}$, $c_0 \in \mathbb{R}$, f fulfils the Carathéodory conditions on $[a,b] \times \mathbb{R}$, the impulse function $\mathcal{J}$ is continuous on $[a,b] \times \mathbb{R}$, the barrier function γ has a continuous first derivative on some subset of $\mathbb{R}$ and ℓ is a linear bounded functional which is defined on the Banach space of left-continuous regulated functions on $[a,b]$ equipped with the sup-norm. The functional ℓ is represented by means of the Kurzweil-Stieltjes integral and covers all linear boundary conditions for solutions of first-order differential equations subject to state-dependent impulse conditions. Here, sufficient and effective conditions guaranteeing the solvability of the above problem are presented for the first time.

MSC: 34B37; 34B15

Keywords: first-order ODE; state-dependent impulses; transversality conditions; general linear boundary conditions; existence; Kurzweil-Stieltjes integral

1 Introduction

The investigation of impulsive differential equations has a long history; see, *e.g.*, the monographs [1–3]. Most papers dealing with impulsive differential equations subject to boundary conditions focus their attention on *impulses at fixed moments*. But this is a very particular case of a more complicated case with *state-dependent impulses*. Boundary value problems with state-dependent impulses, where difficulties with an operator representation appear (*cf.* Remark 6.2), are substantially less developed. We refer to the papers [4–6] and [7] which are devoted to periodic problems, and for problems with other boundary conditions, see [8, 9] or [10–12].

Here, in our paper, we present an approach leading to a new existence principle for impulsive boundary value problems. This approach is applicable to each linear boundary condition which is considered with some first-order differential equation subject to state-dependent impulses. The important step is a proof of a transversality (Remark 2.3 and Lemmas 5.1 and 5.2), which makes possible a construction of a continuous operator (Section 6) whose fixed point leads to a solution of our original impulsive problem (Section 7).

Notation

Let $M \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $[a,b] \subset \mathbb{R}$.

- $\mathbb{C}(M)$ is the set of real functions continuous on M.
- $\mathbb{AC}(M)$ is the set of real functions absolutely continuous on M.
- $\mathbb{L}^1[a,b]$ is the set of real functions Lebesgue integrable on $[a,b]$.
- $\mathbb{L}^\infty[a,b]$ is the set of real functions essentially bounded on $[a,b]$.
- $\mathbb{BV}[a,b]$ is the set of real functions with bounded variation on $[a,b]$.
- $\mathbb{G}_L[a,b]$ is the set of real left-continuous regulated functions on $[a,b]$, that is, $z \in \mathbb{G}_L[a,b]$ if and only if $z\colon [a,b] \to \mathbb{R}$, and for each $\tau_1 \in (a,b]$ and each $\tau_2 \in [a,b)$,

$$z(\tau_1) = z(\tau_1-) = \lim_{t\to\tau_1-} z(t), \qquad z(\tau_2+) = \lim_{t\to\tau_2+} z(t) \in \mathbb{R}. \tag{1.1}$$

- $\mathrm{Car}([a,b]\times M)$ is the set of functions $f\colon [a,b]\times M \to \mathbb{R}$ such that
  (i) $f(\cdot,x)\colon [a,b] \to \mathbb{R}$ is measurable for all $x \in M$,
  (ii) $f(t,\cdot)\colon M \to \mathbb{R}$ is continuous for a.e. $t \in [a,b]$,
  (iii) for each compact set $Q \subset M$, there exists $m_Q \in \mathbb{L}^1[a,b]$ satisfying

$$\bigl|f(t,x)\bigr| \le m_Q(t) \quad \text{for a.e. } t \in [a,b] \text{ and each } x \in Q.$$

- The set $\mathbb{L}^\infty[a,b]$ equipped with the norm

$$\|z\|_\infty = \sup\operatorname{ess}\bigl\{\bigl|z(t)\bigr|: t \in [a,b]\bigr\} \quad \text{for } z \in \mathbb{L}^\infty[a,b] \tag{1.2}$$

  is a Banach space.
- Since $\mathbb{C}[a,b] \subset \mathbb{G}_L[a,b] \subset \mathbb{L}^\infty[a,b]$, we equip the sets $\mathbb{C}[a,b]$ and $\mathbb{G}_L[a,b]$ with the norm $\|\cdot\|_\infty$ and get also Banach spaces (*cf.* [13]). Then (1.2) can be written as

$$\|z\|_\infty = \sup\bigl\{\bigl|z(t)\bigr|: t \in [a,b]\bigr\} \quad \text{for } z \in \mathbb{G}_L[a,b] \tag{1.3}$$

  and

$$\|z\|_\infty = \max\bigl\{\bigl|z(t)\bigr|: t \in [a,b]\bigr\} \quad \text{for } z \in \mathbb{C}[a,b]. \tag{1.4}$$

- $\mathbb{W}^{1,\infty}[a,b]$ is the Banach space of functions $z\colon [a,b] \to \mathbb{R}$ such that $z \in \mathbb{AC}[a,b]$ and $z' \in \mathbb{L}^\infty[a,b]$, where the norm $\|\cdot\|_{1,\infty}$ is given by

$$\|z\|_{1,\infty} = \|z\|_\infty + \bigl\|z'\bigr\|_\infty \quad \text{for } z \in \mathbb{W}^{1,\infty}[a,b]. \tag{1.5}$$

- χ_A is the characteristic function of a set A, where $A \subset \mathbb{R}$.

2 Formulation of problem

We investigate the solvability of the nonlinear differential equation

$$z'(t) = f\bigl(t, z(t)\bigr) \tag{2.1}$$

subject to the state-dependent impulse condition

$$z(\tau+) - z(\tau) = \mathcal{J}\bigl(\tau, z(\tau)\bigr), \qquad \gamma\bigl(z(\tau)\bigr) = \tau, \tag{2.2}$$

and the general linear boundary condition

$$\ell(z) = c_0. \tag{2.3}$$

Here we assume that

$$\begin{cases} f \in \operatorname{Car}([a,b] \times \mathbb{R}), \quad \mathcal{J} \in \mathbb{C}([a,b] \times \mathbb{R}), \quad [a,b] \subset \mathbb{R}, \\ K \in (0,\infty), \quad \gamma \in \mathbb{C}^1[-K,K], \quad c_0 \in \mathbb{R}, \end{cases} \tag{2.4}$$

and $\ell\colon \mathbb{G}_L[a,b] \to \mathbb{R}$ is a linear bounded functional.

Definition 2.1 A function $z\colon [a,b] \to \mathbb{R}$ is a *solution* of problem (2.1), (2.2) if

- there exists a unique $\tau \in (a,b)$ such that $\gamma(z(\tau)) = \tau$;
- the restrictions $z|_{[a,\tau]}$ and $z|_{(\tau,b]}$ are absolutely continuous;
- $z(\tau+) = z(\tau) + \mathcal{J}(\tau, z(\tau))$;
- z satisfies equation (2.1) for a.e. $t \in [a,b]$.

Definition 2.2 A graph of a function $\gamma\colon [-K,K] \to \mathbb{R}$ is called a *barrier* γ.

Remark 2.3 Let $\mathcal{S}$ be the set of all solutions of problem (2.1), (2.2). According to Definition 2.1, each function $z \in \mathcal{S}$ satisfies a *transversality property*, which means that the graph of z crosses a barrier γ at a unique point $\tau \in (a,b)$, where the impulse $\mathcal{J}$ acts on z. After that (for $t \in (\tau,b]$) the graph of z lies on the right of the barrier γ. This transversality property follows from *transversality conditions* (*cf.* (4.5), (4.6)) and it is proved in Section 5.

Assume that $z_1, z_2 \in \mathcal{S}$ and $z_1 \neq z_2$. Then there exists a unique $\tau_i \in (a,b)$ such that $\gamma(z_i(\tau_i)) = \tau_i$ for $i = 1,2$ and $\tau_1 \neq \tau_2$ can occur. Therefore different functions from $\mathcal{S}$ can have their discontinuities at different points from (a,b). Our aim in this paper is to prove the existence of a solution of problem (2.1), (2.2) satisfying the general linear boundary condition (2.3). To do this, we need a suitable linear space containing $\mathcal{S}$. Due to state-dependent impulses, the Banach space of piece-wise continuous functions on $[a,b]$ with the sup-norm cannot be used here. Therefore we choose the Banach space $\mathbb{G}_L[a,b]$. Clearly, by (1.1), $\mathcal{S} \subset \mathbb{G}_L[a,b]$. The operator ℓ in the general linear boundary condition (2.3) can be written uniquely in the form

$$\ell(z) = kz(a) + {}_{(\mathrm{KS})}\int_a^b v(t)\,\mathrm{d}\big[z(t)\big], \tag{2.5}$$

where $k \in \mathbb{R}$, $v \in \mathbb{BV}[a,b]$ and ${}_{(\mathrm{KS})}\int_a^b$ is the Kurzweil-Stieltjes integral (*cf.* [14], Theorem 3.8). Representation (2.5) is correct on $\mathcal{S}$, because for each $z \in \mathbb{G}_L[a,b]$ the integral ${}_{(\mathrm{KS})}\int_a^b v(t)\,\mathrm{d}[z(t)]$ exists. Its definition and properties can be found in [15] (see Perron-Stieltjes integral based on the work of Kurzweil).

Definition 2.4 A function $z\colon [a,b] \to \mathbb{R}$ is a *solution* of problem (2.1)-(2.3) if z is a solution of problem (2.1), (2.2) and fulfils (2.3).

3 Green's function

For further investigation, we will need a linear homogeneous problem corresponding to problem (2.1)-(2.3). Such problem has the form

$$z'(t) = 0, \tag{3.1}$$

$$\ell(z) = 0, \tag{3.2}$$

because the impulse in (2.2) disappears if $\mathcal{J} \equiv 0$. We will also work with the non-homogeneous equation

$$z'(t) = q(t), \tag{3.3}$$

where $q \in \mathbb{L}^1[a,b]$.

Definition 3.1 A *solution* of problem (3.3), (3.2) is a function $z \in \mathbb{AC}[a,b]$ satisfying equation (3.3) for a.e. $t \in [a,b]$ and fulfilling condition (3.2).

Remark 3.2 If x is a solution of problem (3.3), (3.2), then x belongs to $\mathbb{AC}[a,b]$, and consequently condition (3.2) can be written in the form (*cf.* (2.5))

$$\ell(x) = kx(a) + \int_a^b v(t)x'(t)\,\mathrm{d}t = 0, \tag{3.4}$$

where $k \in \mathbb{R}$, $v \in \mathbb{BV}$ and the Lebesgue integral $\int_a^b v(t)x'(t)\,\mathrm{d}t$ is used.

Definition 3.3 A function $G\colon [a,b] \times [a,b] \to \mathbb{R}$ is the *Green's function* of problem (3.1), (3.2) if

(i) for any $s \in (a,b)$, the restrictions $G(\cdot,s)|_{[a,s)}$, $G(\cdot,s)|_{(s,b]}$ are solutions of equation (3.1) and $G(s+,s) - G(s,s) = 1$, where $G(s,s) = G(s-,s)$;

(ii) $G(t,\cdot) \in \mathbb{BV}[a,b]$ for any $t \in [a,b]$;

(iii) for any $q \in \mathbb{L}^1[a,b]$, the function

$$x(t) = \int_a^b G(t,s)q(s)\,\mathrm{d}s \tag{3.5}$$

fulfils condition (3.4).

Lemma 3.4 *Let ℓ be from* (2.5) *with $k \in \mathbb{R}$ and $v \in \mathbb{BV}[a,b]$.*

(i) *$k \neq 0$ if and only if there exists the Green's function G of problem* (3.1), (3.2) *which has the form*

$$G(t,s) = \begin{cases} -\frac{v(s)}{k} & \text{for } a \le t \le s \le b, \\ 1 - \frac{v(s)}{k} & \text{for } a \le s < t \le b. \end{cases} \tag{3.6}$$

(ii) *$k \neq 0$ if and only if there exists a unique solution x of problem* (3.3), (3.4), *which has a form of* (3.5) *with G from* (3.6).

Proof Clearly, G given by (3.6) fulfils (i) and (ii) of Definition 3.3 if and only if $k \neq 0$. A general solution of equation (3.3) is $x(t) = c + \int_a^t q(s)\,\mathrm{d}s$, where $c \in \mathbb{R}$. By (3.4),

$$\ell(x) = kc + \int_a^b v(t)q(t)\,\mathrm{d}t = 0.$$

The equation

$$kc = -\int_a^b v(t)q(t)\,\mathrm{d}t$$

has a unique solution c if and only if $k \neq 0$. Then a unique solution x of problem (3.3), (3.4) is written as

$$\begin{aligned} x(t) &= -\frac{1}{k}\int_a^b v(s)q(s)\,\mathrm{d}s + \int_a^t q(s)\,\mathrm{d}s \\ &= \int_a^t \left(1 - \frac{v(s)}{k}\right) q(s)\,\mathrm{d}s + \int_t^b \left(-\frac{v(s)}{k}\right) q(s)\,\mathrm{d}s, \quad t \in [a,b]. \end{aligned}$$ □

Lemma 3.5 *Let G be the Green's function of problem* (3.1), (3.2), *where ℓ is from* (2.5) *and $k \neq 0$. Then, for each $s \in [a,b)$, the function $G(\cdot,s)$ belongs to $\mathbb{G}_L[a,b]$ and*

$$\ell\bigl(G(\cdot,s)\bigr) = 0, \quad s \in [a,b). \tag{3.7}$$

Proof Choose $s \in [a,b)$. By (3.6),

$$G(t,s) = \chi_{(s,b]}(t) - \frac{v(s)}{k} \quad \text{for } t \in [a,b].$$

Consequently, the function $G(\cdot,s)$ belongs to $\mathbb{G}_L[a,b]$. This yields that the integral ${}_{(\mathrm{KS})}\int_a^b v(t)\,\mathrm{d}[G(t,s)]$ exists for each $v \in \mathbb{BV}[a,b]$. Note that since $G(\cdot,s)$ is not continuous on $[a,b]$, formula (3.4) cannot be used for $G(\cdot,s)$ in place of x. Instead, we use the properties of the Kurzweil-Stieltjes integral which justify the following computation

$$\begin{aligned} {}_{(\mathrm{KS})}\int_a^b v(t)\,\mathrm{d}\bigl[G(t,s)\bigr] &= {}_{(\mathrm{KS})}\int_a^b v(t)\,\mathrm{d}\left[\chi_{(s,b]}(t) - \frac{v(s)}{k}\right] \\ &= {}_{(\mathrm{KS})}\int_a^b v(t)\,\mathrm{d}\bigl[\chi_{(s,b]}(t)\bigr] - {}_{(\mathrm{KS})}\int_a^b v(t)\,\mathrm{d}\left[\frac{v(s)}{k}\right] = v(s). \end{aligned}$$

Hence, by (2.5), we get

$$\ell\bigl(G(\cdot,s)\bigr) = kG(a,s) + {}_{(\mathrm{KS})}\int_a^b v(t)\,\mathrm{d}\bigl[G(t,s)\bigr] = k\left(\frac{-v(s)}{k}\right) + v(s) = 0.$$ □

Example 3.6 Consider a solution x of problem (3.3), (3.2), where ℓ has a form of the two-point boundary condition

$$\ell(x) = \alpha x(a) + \beta x(b) = 0, \quad \alpha, \beta \in \mathbb{R}. \tag{3.8}$$

We will show that ℓ can be expressed in a form of (3.4). If $\alpha + \beta \neq 0$, then k and ν can be found from the equality

$$\alpha x(a) + \beta x(b) = kx(a) + \int_a^b \nu(t)x'(t)\,\mathrm{d}t.$$

Assuming that $\nu(t) \equiv \nu_0 \in \mathbb{R}$, we get

$$\alpha x(a) + \beta x(b) = kx(a) + \nu_0\bigl(x(b) - x(a)\bigr),$$

and hence $k = \alpha + \beta$, $\nu_0 = \beta$. In addition, if $\alpha + \beta \neq 0$, then (*cf.* (3.6))

$$G(t,s) = \begin{cases} -\frac{\beta}{\alpha+\beta} & \text{for } a \le t \le s \le b, \\ 1 - \frac{\beta}{\alpha+\beta} & \text{for } a \le s < t \le b. \end{cases}$$

Example 3.7 Consider a solution x of problem (3.3), (3.2), where ℓ has a form of the multi-point boundary condition

$$\ell(x) = \sum_{i=0}^{n} \alpha_i x(t_i), \quad \alpha_i \in \mathbb{R}, i = 0, 1, \dots, n, n \in \mathbb{N}. \tag{3.9}$$

Here $a = t_0 < t_1 < \cdots < t_n = b$. If $\sum_{i=0}^{n} \alpha_i \neq 0$, then k and ν of (3.4) can be found from the equality

$$\sum_{i=0}^{n} \alpha_i x(t_i) = kx(a) + \int_a^b \nu(t)x'(t)\,\mathrm{d}t. \tag{3.10}$$

Assume that ν is a piece-wise constant right-continuous function on $[a,b]$, that is,

$$\nu(s) = \nu_i \quad \text{for } s \in [t_i, t_{i+1}), i = 0, \dots, n-2,$$
$$\nu(s) = \nu_{n-1} \quad \text{for } s \in [t_{n-1}, b],$$

where $\nu_i \in \mathbb{R}$, $i = 0, \dots, n-1$. By (3.10), we get

$$\begin{aligned} \sum_{i=0}^{n} \alpha_i x(t_i) &= kx(a) + \sum_{i=0}^{n-1} \nu_i \int_{t_i}^{t_{i+1}} x'(t)\,\mathrm{d}t \\ &= kx(a) + \nu_0\bigl(x(t_1) - x(a)\bigr) + \nu_1\bigl(x(t_2) - x(t_1)\bigr) + \cdots + \nu_{n-1}\bigl(x(b) - x(t_{n-1})\bigr). \end{aligned}$$

Consequently,

$$\nu_i = \sum_{j=i+1}^{n} \alpha_j, \quad i = 0, \dots, n-1, \qquad k = \sum_{j=0}^{n} \alpha_j.$$

To summarize, if $\sum_{j=0}^{n} \alpha_j \neq 0$, then

$$\nu(s) = \sum_{j=i+1}^{n} \alpha_j \quad \text{for } s \in [t_i, t_{i+1}), i = 0, \dots, n-2,$$

$$\nu(s) = \alpha_n \quad \text{for } s \in [t_{n-1}, b],$$

and further (*cf.* (3.6))

$$G(t,s)=\begin{cases}-\frac{\nu(s)}{\sum_{j=0}^{n}\alpha_j} & \text{for } a\le t\le s\le b,\\ 1-\frac{\nu(s)}{\sum_{j=0}^{n}\alpha_j} & \text{for } a\le s<t\le b.\end{cases}$$

Example 3.8 Consider a solution x of problem (3.3), (3.2), where ℓ has a form of the integral condition

$$\ell(x)=x(b)-\int_a^b h(\xi)x(\xi)\,\mathrm{d}\xi,$$

where $h\in\mathbb{L}^1[a,b]$. If $\int_a^b h(\xi)\,\mathrm{d}\xi\neq 1$, then k and ν of (3.4) can be found from the equality

$$x(b)-\int_a^b h(\xi)x(\xi)\,\mathrm{d}\xi=kx(a)+\int_a^b \nu(t)x'(t)\,\mathrm{d}t. \tag{3.11}$$

Let us put

$$\nu(s)=\int_a^s h(\xi)\,\mathrm{d}\xi+\nu(a).$$

Then

$$\int_a^b \nu(\xi)x'(\xi)\,\mathrm{d}\xi=-\int_a^b h(\xi)x(\xi)\,\mathrm{d}\xi+\nu(b)x(b)-\nu(a)x(a)$$

and (3.11) gives $\nu(a)=k$, $\int_a^b h(\xi)\,\mathrm{d}\xi+k=1$. Consequently,

$$k=1-\int_a^b h(\xi)\,\mathrm{d}\xi,\qquad \nu(s)=1-\int_s^b h(\xi)\,\mathrm{d}\xi,\quad s\in[a,b].$$

Similarly, if

$$\ell(x)=x(a)-\int_a^b h(\xi)x(\xi)\,\mathrm{d}\xi,$$

and $\int_a^b h(\xi)\,\mathrm{d}\xi\neq 1$, we derive

$$k=1-\int_a^b h(\xi)\,\mathrm{d}\xi,\qquad \nu(s)=-\int_s^b h(\xi)\,\mathrm{d}\xi,\quad s\in[a,b].$$

In both cases, G is written as

$$G(t,s)=\begin{cases}-\frac{\nu(s)}{1-\int_a^b h(\xi)\,\mathrm{d}\xi} & \text{for } a\le t\le s\le b,\\ 1-\frac{\nu(s)}{1-\int_a^b h(\xi)\,\mathrm{d}\xi} & \text{for } a\le s<t\le b.\end{cases}$$

4 Assumptions

An existence result for problem (2.1)-(2.3) will be proved in the next sections under the basic assumption (2.4) and the following additional assumptions imposed on f, ℓ, $\mathcal{J}$ and γ.

(i) Boundedness of f

$$\begin{cases} \text{There exists } h \in \mathbb{L}^\infty[a,b] \text{ such that} \\ |f(t,x)| \le h(t) \quad \text{for a.e. } t \in [a,b] \text{ and all } x \in \mathbb{R}. \end{cases} \tag{4.1}$$

(ii) Boundedness of $\mathcal{J}$

$$\begin{cases} \text{There exists } J_0 \in (0,\infty) \text{ such that} \\ |\mathcal{J}(t,x)| \le J_0 \quad \text{for } t \in [a,b], x \in \mathbb{R}. \end{cases} \tag{4.2}$$

(iii) Boundedness of γ

$$\begin{cases} \text{There exist } a_1, b_1 \in (a,b) \text{ such that} \\ a_1 \le \gamma(x) \le b_1 \quad \text{for } x \in [-K,K]. \end{cases} \tag{4.3}$$

(iv) Properties of ℓ

$$\ell \text{ fulfils (2.5)}, \quad \text{where } k \in \mathbb{R}, k \neq 0, v \in \mathbb{BV}[a,b] \cap \mathbb{C}[a_1,b_1]. \tag{4.4}$$

(v) Transversality conditions

$$\left|\gamma'(x)\right| < \frac{1}{\|h\|_\infty} \quad \text{for } x \in [-K,K], \tag{4.5}$$

$$\begin{cases} \text{either} & \mathcal{J}(t,x) \ge 0, \quad \gamma'(x) \le 0 \quad \text{for } t \in [a_1,b_1], x \in [-K,K], \\ \text{or} & \mathcal{J}(t,x) \le 0, \quad \gamma'(x) \ge 0 \quad \text{for } t \in [a_1,b_1], x \in [-K,K], \end{cases} \tag{4.6}$$

where h is from (4.1) and a_1, b_1 are from (4.3).

(vi) $\mathbb{L}^\infty$-continuity of f

$$\begin{cases} \text{For any } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that} \\ |x-y| < \delta \Rightarrow \|f(\cdot,x) - f(\cdot,y)\|_\infty < \varepsilon, \quad x,y \in [-K,K]. \end{cases} \tag{4.7}$$

Remark 4.1

(a) Boundedness of f and $\mathcal{J}$ can be replaced by more general conditions, for example, growth or sign ones, if the method of *a priori* estimates is used. See, *e.g.*, [16, 17].

(b) Continuity of v on $[a_1,b_1]$ is necessary for the construction of a continuous operator in Section 6. Note that then we need $t_1,\dots,t_{n-1} \notin [a_1,b_1]$ in Example 3.7.

(c) Clearly, if f is continuous on $[a,b] \times [-K,K]$, then f fulfils (4.7).

(d) Let there exist $p \in \mathbb{N}$, $\psi \in \mathbb{L}^\infty[a,b]$ and $g_i \in \mathbb{C}(\mathbb{R})$, $i = 1,\dots,p$, such that

$$\left|f(t,x) - f(t,y)\right| \le \psi(t) \sum_{i=1}^{p} \left|g_i(x) - g_i(y)\right|$$

for a.e. $t \in [a,b]$ and all $x, y \in [-K,K]$. Then f fulfils (4.7). An example of such a function f is

$$f(t,x) = \sum_{i=1}^{p} f_i(t) g_i(x) + f_0(t),$$

where $f_j \in \mathbb{L}^\infty[a,b]$, $j = 0,1,\dots,p$, $g_i \in \mathbb{C}[-K,K]$, $i = 1,\dots,p$.

5 Transversality

Consider $K \in (0,\infty)$, $h \in \mathbb{L}^\infty[a,b]$ and define a set $\mathcal{B}$ by

$$\mathcal{B} = \left\{ u \in \mathbb{W}^{1,\infty}[a,b] \colon \|u\|_\infty < K, \|u'\|_\infty < \|h\|_\infty \right\}. \tag{5.1}$$

The following two lemmas for functions from $\mathcal{B}$ are the modifications of lemmas in [10] and provide the transversality (*cf.* Remark 2.3) which will be essential for operator constructions in Section 6.

Lemma 5.1 *Let γ satisfy* (2.4), (4.3) *and* (4.5). *Then, for each $u \in \overline{\mathcal{B}}$, there exists a unique $\tau \in (a,b)$ such that*

$$\tau = \gamma\big(u(\tau)\big). \tag{5.2}$$

In addition $\tau \in [a_1, b_1]$.

Proof Let us take an arbitrary $u \in \overline{\mathcal{B}}$ and denote

$$\sigma(t) = \gamma\big(u(t)\big) - t, \quad t \in [a,b].$$

Then, by (2.4) and (5.1), we see that $\sigma \in \mathbb{AC}[a,b]$ and

$$\sigma'(t) = \gamma'\big(u(t)\big)u'(t) - 1 \quad \text{for a.e. } t \in [a,b].$$

Since $u(a), u(b) \in [-K,K]$, condition (4.3) gives

$$\sigma(a) = \gamma\big(u(a)\big) - a \ge a_1 - a > 0,$$
$$\sigma(b) = \gamma\big(u(b)\big) - b \le b_1 - b < 0.$$

Consequently, there exists at least one zero of σ in (a,b). Let $\tau \in (a,b)$ be a zero of σ. By virtue of (4.5) and (5.1), we get, for $t \in [a,b]$, $t \ne \tau$,

$$\begin{aligned}
\operatorname{sign}(t-\tau)\sigma(t) &= \operatorname{sign}(t-\tau)\int_\tau^t \sigma'(s)\,\mathrm{d}s = \operatorname{sign}(t-\tau)\int_\tau^t \big(\gamma'\big(u(s)\big)u'(s) - 1\big)\,\mathrm{d}s \\
&\le \operatorname{sign}(t-\tau)\int_\tau^t \big(\big|\gamma'\big(u(s)\big)\big| \cdot \|u'\|_\infty - 1\big)\,\mathrm{d}s \\
&< \operatorname{sign}(t-\tau)\int_\tau^t \left(\frac{1}{\|h\|_\infty}\|h\|_\infty - 1\right)\mathrm{d}s = 0.
\end{aligned}$$

That is,

$$\sigma > 0 \quad \text{on } [a,\tau), \qquad \sigma < 0 \quad \text{on } (\tau,b]. \tag{5.3}$$

Hence τ is a unique zero of σ, and (4.3) yields $\tau \in [a_1,b_1]$. □

Due to Lemma 5.1, we can define a functional $\mathcal{P}\colon \overline{\mathcal{B}} \to [a_1,b_1]$ by

$$\mathcal{P}u = \tau, \tag{5.4}$$

where τ fulfils (5.2).

Lemma 5.2 *Let γ satisfy* (2.4), (4.3) *and* (4.5). *Then the functional $\mathcal{P}$ is continuous.*

Proof Let us choose a sequence $\{u_n\}_{n=1}^{\infty} \subset \overline{\mathcal{B}}$ which is convergent in $\mathbb{W}^{1,\infty}[a,b]$. Then

$$u_n \in \mathbb{W}^{1,\infty}[a,b], \qquad \|u_n\|_\infty \le K, \qquad \|u_n'\|_\infty \le \|h\|_\infty, \quad n \in \mathbb{N}, \tag{5.5}$$

and there exists $u \in \mathbb{W}^{1,\infty}[a,b]$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{1,\infty} = 0. \tag{5.6}$$

So, by virtue of (1.5) and (5.5),

$$\|u\|_\infty \le \lim_{n\to\infty} \|u - u_n\|_\infty + \lim_{n\to\infty} \|u_n\|_\infty \le K,$$
$$\|u'\|_\infty \le \lim_{n\to\infty} \|u' - u_n'\|_\infty + \lim_{n\to\infty} \|u_n'\|_\infty \le \|h\|_\infty.$$

We see that $u \in \overline{\mathcal{B}}$. For $n \in \mathbb{N}$, define

$$\sigma_n(t) = \gamma\big(u_n(t)\big) - t, \qquad \sigma(t) = \gamma\big(u(t)\big) - t, \quad t \in [a,b].$$

By Lemma 5.1,

$$\sigma_n(\tau_n) = 0, \qquad \sigma(\tau) = 0, \quad \text{where } \tau_n = \mathcal{P}u_n, \tau = \mathcal{P}u, n \in \mathbb{N}. \tag{5.7}$$

We need to prove that

$$\lim_{n\to\infty} \tau_n = \tau. \tag{5.8}$$

Conditions (2.4), (1.5) and (5.6) yield

$$\lim_{n\to\infty} \sigma_n = \sigma \quad \text{in } \mathbb{C}[a,b]. \tag{5.9}$$

Let us take an arbitrary $\varepsilon > 0$. By (5.3) and (5.9) we can find $\xi \in (\tau - \varepsilon, \tau)$, $\eta \in (\tau, \tau + \varepsilon)$ and $n_0 \in \mathbb{N}$ such that $\sigma_n(\xi) > 0$, $\sigma_n(\eta) < 0$ for each $n \ge n_0$. By Lemma 5.1 and the continuity of σ_n, we see that $\tau_n \in (\xi,\eta) \subset (\tau - \varepsilon, \tau + \varepsilon)$ for $n \ge n_0$, and (5.8) follows. □

6 Fixed point problem

In this section we assume that

$$\text{conditions (2.4), (4.1)-(4.7) are fulfilled,} \tag{6.1}$$

and we construct a fixed point problem whose solvability leads to a solution of problem (2.1)-(2.3). To this aim, having the set $\mathcal{B}$ from (5.1), we define a set Ω by

$$\Omega = \mathcal{B} \times \mathcal{B} \subset \mathbb{W}^{1,\infty}[a,b] \times \mathbb{W}^{1,\infty}[a,b], \tag{6.2}$$

and for $u = (u_1, u_2) \in \Omega$, we define a function $f_u \colon [a,b] \to \mathbb{R}$ as follows. We set, for a.e. $t \in [a,b]$,

$$f_u(t) = \begin{cases} f(t, u_1(t)) & \text{if } t \in [a, \mathcal{P}u_1], \\ f(t, u_2(t)) & \text{if } t \in (\mathcal{P}u_1, b], \end{cases} \tag{6.3}$$

where $\mathcal{P}$ is defined by (5.4) and the point $\mathcal{P}u_1 \in [a_1, b_1]$ is uniquely determined due to Lemma 5.1. By (4.1)

$$f_u \in \mathbb{L}^\infty[a,b], \qquad \|f_u\|_\infty \le \|h\|_\infty. \tag{6.4}$$

Now, we can define an operator $\mathcal{F} \colon \overline{\Omega} \to \mathbb{W}^{1,\infty}[a,b] \times \mathbb{W}^{1,\infty}[a,b]$ by $\mathcal{F}(u_1, u_2) = (x_1, x_2)$, where

$$x_1(t) = \begin{cases} \int_a^b G(t,s) f_u(s)\,\mathrm{d}s + \frac{c_0}{k} \\ \quad - \frac{\nu(\mathcal{P}u_1)}{k} \mathcal{J}(\mathcal{P}u_1, u_1(\mathcal{P}u_1)) & \text{if } t \le \mathcal{P}_{u_1}, \\ \int_a^b G(t,s) f(s, u_1(s))\,\mathrm{d}s + \frac{c_0}{k} \\ \quad - \frac{\nu(\mathcal{P}u_1)}{k} \mathcal{J}(\mathcal{P}u_1, u_1(\mathcal{P}u_1)) + \mathcal{A}_1 u & \text{if } t > \mathcal{P}_{u_1}, \end{cases} \tag{6.5}$$

$$x_2(t) = \begin{cases} \int_a^b G(t,s) f(s, u_2(s))\,\mathrm{d}s + \frac{c_0}{k} \\ \quad + (1 - \frac{\nu(\mathcal{P}u_1)}{k}) \mathcal{J}(\mathcal{P}u_1, u_1(\mathcal{P}u_1)) + \mathcal{A}_2 u & \text{if } t \le \mathcal{P}_{u_1}, \\ \int_a^b G(t,s) f_u(s)\,\mathrm{d}s + \frac{c_0}{k} \\ \quad + (1 - \frac{\nu(\mathcal{P}u_1)}{k}) \mathcal{J}(\mathcal{P}u_1, u_1(\mathcal{P}u_1)) & \text{if } t > \mathcal{P}_{u_1}. \end{cases} \tag{6.6}$$

Here the functionals $\mathcal{A}_1 \colon \overline{\Omega} \to \mathbb{R}$ and $\mathcal{A}_2 \colon \overline{\Omega} \to \mathbb{R}$ are defined such that the functions x_1 and x_2 are continuous at the point $\mathcal{P}u_1$. Therefore

$$\begin{cases} \mathcal{A}_1 u = \int_a^b G(\mathcal{P}u_1, s) f_u(s)\,\mathrm{d}s - \int_a^b G(\mathcal{P}u_1, s) f(s, u_1(s))\,\mathrm{d}s, \\ \mathcal{A}_2 u = \int_a^b G(\mathcal{P}u_1, s) f_u(s)\,\mathrm{d}s - \int_a^b G(\mathcal{P}u_1, s) f(s, u_2(s))\,\mathrm{d}s. \end{cases} \tag{6.7}$$

Differentiating (6.5) and using (3.6) and (6.3), we get

$$x_i'(t) = f\big(t, u_i(t)\big) \quad \text{for a.e. } t \in [a,b], i = 1, 2. \tag{6.8}$$

This together with (4.1) yields

$$\|x_i'\|_\infty \le \|h\|_\infty, \quad i = 1, 2. \tag{6.9}$$

Since $v \in \mathbb{BV}[a,b]$ (*cf.* (4.4)), we see that (6.4)-(6.6), (3.6), (4.1) and (4.2) give

$$\begin{aligned} \|x_i\|_\infty \le{} & 3\left(1 + \frac{\|v\|_\infty}{|k|}\right)(b-a)\|h\|_\infty + \frac{|c_0|}{|k|} \\ & + \left(1 + \frac{\|v\|_\infty}{|k|}\right)J_0, \quad i = 1, 2. \end{aligned} \tag{6.10}$$

Due to (6.8)-(6.10), we see that $x_i \in \mathbb{W}^{1,\infty}[a,b]$, $i = 1, 2$, and the operator $\mathcal{F}$ is defined well.

Lemma 6.1 *Assume that* (6.1) *holds and that* Ω *and* $\mathcal{F}$ *are given by* (6.2) *and* (6.5), (6.6), *respectively. Then the operator* $\mathcal{F}$ *is compact on* $\overline{\Omega}$.

Proof

Step 1. We show that $\mathcal{F}$ is continuous on $\overline{\Omega}$. Choose a sequence

$$\{u^{[n]}\}_{n=1}^\infty = \{(u_1^{[n]}, u_2^{[n]})\}_{n=1}^\infty \subset \overline{\Omega}$$

which is convergent in $\mathbb{W}^{1,\infty}[a,b] \times \mathbb{W}^{1,\infty}[a,b]$, that is, (*cf.* (1.5)) there exists $u = (u_1, u_2) \in \overline{\Omega}$ such that

$$\lim_{n\to\infty} \|u_1^{[n]} - u_1\|_{1,\infty} = 0, \qquad \lim_{n\to\infty} \|u_2^{[n]} - u_2\|_{1,\infty} = 0. \tag{6.11}$$

Lemma 5.1 and Lemma 5.2 yield

$$\mathcal{P}u_1, \mathcal{P}u_1^{[n]} \in [a_1, b_1], \quad n \in \mathbb{N}, \qquad \lim_{n\to\infty} \mathcal{P}u_1^{[n]} = \mathcal{P}u_1, \tag{6.12}$$

where $\mathcal{P}$ is defined by (5.4). Denote

$$x = (x_1, x_2) = \mathcal{F}(u_1, u_2), \qquad x^{[n]} = (x_1^{[n]}, x_2^{[n]}) = \mathcal{F}(u_1^{[n]}, u_2^{[n]}), \quad n \in \mathbb{N}. \tag{6.13}$$

We will prove that

$$\lim_{n\to\infty} \|x_1^{[n]} - x_1\|_{1,\infty} = 0, \qquad \lim_{n\to\infty} \|x_2^{[n]} - x_2\|_{1,\infty} = 0. \tag{6.14}$$

By (4.7), (6.8), (6.11) and (6.13),

$$\lim_{n\to\infty} \|(x_i^{[n]})' - x_i'\|_\infty = \lim_{n\to\infty} \|f(\cdot, u_i^{[n]}(\cdot)) - f(\cdot, u_i(\cdot))\|_\infty = 0, \quad i = 1, 2. \tag{6.15}$$

Using (4.1), we get

$$\lim_{n\to\infty} \left| \int_\tau^{\tau_n} |f(s, u_1^{[n]}(s)) - f(s, u_2^{[n]}(s))| \,\mathrm{d}s \right| \le 2 \lim_{n\to\infty} \left| \int_\tau^{\tau_n} h(s)\,\mathrm{d}s \right| = 0. \tag{6.16}$$

Since

$$\begin{aligned}\int_a^b \big(f_{u^{[n]}}(s) - f_u(s)\big)\,\mathrm{d}s = \int_a^\tau \big(f\big(s,u_1^{[n]}(s)\big) - f\big(s,u_1(s)\big)\big)\,\mathrm{d}s \\ + \int_\tau^b \big(f\big(s,u_2^{[n]}(s)\big) - f\big(s,u_2(s)\big)\big)\,\mathrm{d}s \\ + \int_\tau^{\tau_n} \big(f\big(s,u_1^{[n]}(s)\big) - f\big(s,u_2^{[n]}(s)\big)\big)\,\mathrm{d}s,\end{aligned}$$

the Lebesgue dominated convergence theorem and (6.16) give

$$\lim_{n\to\infty} \int_a^b \big|f_{u^{[n]}}(s) - f_u(s)\big|\,\mathrm{d}s = 0. \tag{6.17}$$

Using (6.13) and (6.5), we get

$$\begin{aligned}\big|x_1^{[n]}(a) - x_1(a)\big| \le \int_a^b \big|G(a,s)\big| \cdot \big|f_{u^{[n]}}(s) - f_u(s)\big|\,\mathrm{d}s \\ + \left|\frac{\nu(\mathcal{P}u_1^{[n]})}{k}\mathcal{J}\big(\mathcal{P}u_1^{[n]}, u_1^{[n]}\big(\mathcal{P}u_1^{[n]}\big)\big) - \frac{\nu(\mathcal{P}u_1)}{k}\mathcal{J}\big(\mathcal{P}u_1, u_1(\mathcal{P}u_1)\big)\right|.\end{aligned}$$

The continuity and boundedness of $\mathcal{P}$, $\mathcal{J}$ and ν (*cf.* Lemma 5.2, (2.4), (4.2), (4.4) and (6.12)) imply

$$\begin{aligned}&\lim_{n\to\infty}\left|\frac{\nu(\mathcal{P}u_1^{[n]})}{k}\mathcal{J}\big(\mathcal{P}u_1^{[n]}, u_1^{[n]}\big(\mathcal{P}u_1^{[n]}\big)\big) - \frac{\nu(\mathcal{P}u_1)}{k}\mathcal{J}\big(\mathcal{P}u_1, u_1(\mathcal{P}u_1)\big)\right| \\ &\quad\le \frac{\|\nu\|_\infty}{|k|}\lim_{n\to\infty}\big|\mathcal{J}\big(\mathcal{P}u_1^{[n]}, u_1^{[n]}\big(\mathcal{P}u_1^{[n]}\big)\big) - \mathcal{J}\big(\mathcal{P}u_1, u_1(\mathcal{P}u_1)\big)\big| \\ &\qquad + \frac{J_0}{|k|}\lim_{n\to\infty}\big|\nu\big(\mathcal{P}u_1^{[n]}\big) - \nu(\mathcal{P}u_1)\big| = 0,\end{aligned}$$

wherefrom, by the boundedness of G and (6.17),

$$\lim_{n\to\infty}\big|x_1^{[n]}(a) - x_1(a)\big| = 0. \tag{6.18}$$

Using (6.13) and integrating (6.8), we get

$$x_1(t) = x_1(a) + \int_a^t f\big(s,u_1(s)\big)\,\mathrm{d}s, \qquad x_1^{[n]}(t) = x_1^{[n]}(a) + \int_a^t f\big(s,u_1^{[n]}(s)\big)\,\mathrm{d}s,$$

and, due to (6.15) and (6.18), we arrive at

$$\lim_{n\to\infty}\big\|x_1^{[n]} - x_1\big\|_\infty = 0. \tag{6.19}$$

Similarly, we derive

$$\lim_{n\to\infty}\big|x_2^{[n]}(b) - x_2(b)\big| = 0, \qquad \lim_{n\to\infty}\big\|x_2^{[n]} - x_2\big\|_\infty = 0. \tag{6.20}$$

Properties (6.15), (6.19) and (6.20) yield (6.14).

Step 2. We show that the set $\mathcal{F}(\overline{\Omega})$ is relatively compact in $\mathbb{W}^{1,\infty}[a,b] \times \mathbb{W}^{1,\infty}[a,b]$. Choose an arbitrary sequence

$$\big\{\big(x_1^{[n]}, x_2^{[n]}\big)\big\}_{n=1}^{\infty} \subset \mathcal{F}(\overline{\Omega}) \subset \mathbb{W}^{1,\infty}[a,b] \times \mathbb{W}^{1,\infty}[a,b].$$

We need to prove that there exists a convergent subsequence. Clearly, there exists $\{(u_1^{[n]}, u_2^{[n]})\}_{n=1}^{\infty} \subset \overline{\Omega}$ such that

$$\mathcal{F}\big(u_1^{[n]}, u_2^{[n]}\big) = \big(x_1^{[n]}, x_2^{[n]}\big), \quad n \in \mathbb{N}.$$

Choose $i \in \{1,2\}$. By (5.1) and (6.2), it holds

$$\big\{u_i^{[n]}\big\}_{n=1}^{\infty} \subset \mathbb{W}^{1,\infty}[a,b], \qquad \big\|u_i^{[n]}\big\|_{\infty} \le K,$$
$$\big|u_i^{[n]}(t_1) - u_i^{[n]}(t_2)\big| = \left|\int_{t_1}^{t_2} \big(u_i^{[n]}\big)'(s)\,\mathrm{d}s\right| \le \|h\|_{\infty}|t_1 - t_2|$$

for $t_1, t_2 \in [a,b]$, $n \in \mathbb{N}$. Therefore, the Arzelà-Ascoli theorem yields that there exists a subsequence

$$\big\{\big(u_1^{[m]}, u_2^{[m]}\big)\big\}_{m=1}^{\infty} \subset \big\{\big(u_1^{[n]}, u_2^{[n]}\big)\big\}_{n=1}^{\infty}$$

which converges in $\mathbb{C}[a,b] \times \mathbb{C}[a,b]$. Consequently, for each $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that for each $m \in \mathbb{N}$,

$$m \ge m_0 \quad \Rightarrow \quad \big\|u_i^{[m_0]} - u_i^{[m]}\big\|_{\infty} < \varepsilon, \quad i = 1,2.$$

Similarly as in Step 1, we prove (*cf.* (6.15), (6.19), (6.20))

$$\big\|\big(x_i^{[m_0]}\big)' - \big(x_i^{[m]}\big)'\big\|_{\infty} < \varepsilon, \qquad \big\|x_i^{[m_0]} - x_i^{[m]}\big\|_{\infty} < \varepsilon, \quad i = 1,2,$$

which gives by (1.5) that $\{(x_1^{[m]}, x_2^{[m]})\}_{m=1}^{\infty}$ is convergent in $\mathbb{W}^{1,\infty}[a,b] \times \mathbb{W}^{1,\infty}[a,b]$. □

Remark 6.2 If there exists $\tau_0 \in [a_1, b_1]$ such that $\gamma(x) = \tau_0$ for $x \in [-K, K]$, then problem (2.1)-(2.3) has an impulse at fixed time τ_0 and a standard operator $\mathcal{F}_0$, acting on the space of piece-wise continuous functions on $[a,b]$ and having the form

$$(\mathcal{F}_0 z)(t) = \int_a^b G(t,s) f\big(s, z(s)\big)\,\mathrm{d}s + \frac{c_0}{k} + G(t,\tau_0)\mathcal{J}\big(\tau_0, z(\tau_0)\big), \quad t \in [a,b], \tag{6.21}$$

can be used instead of the operator $\mathcal{F}$ from (6.5), (6.6). But this is not possible if γ is not constant on $[-K, K]$. The reason is that then an impulse is realized at a state-dependent point $\tau = \gamma(z(\tau))$, and $\mathcal{F}_0$ with τ instead of τ_0 should be investigated on the space $\mathbb{G}_L[a,b]$. But if we write a state-dependent τ instead of a fixed τ_0 in (6.21), $\mathcal{F}_0$ loses its continuity on $\mathbb{G}_L[a,b]$, which we show in the next example.

Example 6.3 Let $a = 0$, $b = 2$ and ℓ be from (2.5) with $k \in \mathbb{R}$, $k \neq 0$ and $v \in \mathbb{C}[0,2]$. Consider the functions

$$u(t) = 1, \qquad u_n(t) = 1 - \frac{1}{n}, \quad t \in [0,2], n \in \mathbb{N}.$$

Clearly, $u_n \to u$ uniformly on $[0,2]$ and hence

$$\lim_{n\to\infty} \|u_n - u\|_\infty = 0.$$

For $n \in \mathbb{N}$, denote $x_n = \mathcal{F}_0 u_n$ and $x = \mathcal{F}_0 u$. Assume that the barrier γ is given by the linear function $\gamma(x) = x$ on $\mathbb{R}$ and the impulse function $\mathcal{J}(t,x) = 1$ for $t \in [0,2]$, $x \in \mathbb{R}$. Then

$$\tau = \gamma\big(u(\tau)\big) = u(\tau) = 1,$$
$$\tau_n = \gamma\big(u_n(\tau_n)\big) = u_n(\tau_n) = 1 - \frac{1}{n}, \quad n \in \mathbb{N},$$

and, according to (6.21), we have for $t \in [0,2]$

$$x_n(t) = \int_0^2 G(t,s) f\left(s, 1 - \frac{1}{n}\right) \mathrm{d}s + \frac{c_0}{k} + G\left(t, 1 - \frac{1}{n}\right), \quad n \in \mathbb{N},$$
$$x(t) = \int_0^2 G(t,s) f(s,1)\, \mathrm{d}s + \frac{c_0}{k} + G(t,1).$$

Consequently,

$$\begin{aligned}\lim_{n\to\infty} \big(x_n(1) - x(1)\big) &= \lim_{n\to\infty} \int_0^2 G(1,s)\left(f\left(s, 1 - \frac{1}{n}\right) - f(s,1)\right) \mathrm{d}s \\ &\quad + \lim_{n\to\infty}\left(G\left(1, 1 - \frac{1}{n}\right) - G(1,1)\right) \\ &= 1 - \frac{v(1)}{k} - \left(-\frac{v(1)}{k}\right) = 1\end{aligned}$$

due to (3.6). Hence $x_n(1) \nrightarrow x(1)$ and we have also $\|x_n - x\|_\infty \nrightarrow 0$, and $\mathcal{F}_0$ is not continuous on $\mathbb{G}_L[0,2]$.

Lemma 6.1 results in the following theorem.

Theorem 6.4 *Assume that* (6.1) *holds and that the set* Ω *is given by* (6.2), *where*

$$K \geq \left(1 + \frac{\|v\|_\infty}{|k|}\right)\big(3(b-a)\|h\|_\infty + J_0\big) + \frac{|c_0|}{|k|}. \tag{6.22}$$

Further, let the operator $\mathcal{F}$ *be given by* (6.5), (6.6). *Then* $\mathcal{F}$ *has a fixed point in* $\overline{\Omega}$.

Proof By Lemma 6.1, $\mathcal{F}$ is compact on $\overline{\Omega}$. Due to (5.1), (6.2), (6.5), (6.6), (6.10) and (6.22),

$$\mathcal{F}(\overline{\Omega}) \subset \overline{\Omega}.$$

Therefore, the Schauder fixed point theorem yields a fixed point of $\mathcal{F}$ in $\overline{\Omega}$. □

7 Main result

The main result, which is contained in Theorem 7.1, guarantees the solvability of problem (2.1)-(2.3) provided the data functions f, $\mathcal{J}$ and γ are bounded (*cf.* (4.1)-(4.3)). As it is mentioned in Remark 4.1, Theorem 7.1 serves as an existence principle which, in combination with the method of *a priori* estimates, can lead to more general existence results for unbounded f and $\mathcal{J}$ and concrete boundary conditions.

Theorem 7.1 *Assume that* (6.1) *and* (6.22) *hold. Then there exists a solution z of problem* (2.1)-(2.3) *such that*

$$\|z\|_\infty \le K. \tag{7.1}$$

Proof By Theorem 6.4, there exists $u = (u_1, u_2) \in \overline{\Omega}$ which is a fixed point of the operator $\mathcal{F}$ defined in (6.5) and (6.6). This means that

$$u_1(t) = \begin{cases} \int_a^b G(t,s) f_u(s)\,\mathrm{d}s + \frac{c_0}{k} \\ \quad - \frac{\nu(\mathcal{P}u_1)}{k} \mathcal{J}(\mathcal{P}u_1, u_1(\mathcal{P}u_1)) & \text{if } t \le \mathcal{P}_{u_1}, \\ \int_a^b G(t,s) f(s, u_1(s))\,\mathrm{d}s + \frac{c_0}{k} \\ \quad - \frac{\nu(\mathcal{P}u_1)}{k} \mathcal{J}(\mathcal{P}u_1, u_1(\mathcal{P}u_1)) + \mathcal{A}_1 u & \text{if } t > \mathcal{P}_{u_1}, \end{cases} \tag{7.2}$$

$$u_2(t) = \begin{cases} \int_a^b G(t,s) f(s, u_2(s))\,\mathrm{d}s + \frac{c_0}{k} \\ \quad + (1 - \frac{\nu(\mathcal{P}u_1)}{k}) \mathcal{J}(\mathcal{P}u_1, u_1(\mathcal{P}u_1)) + \mathcal{A}_2 u & \text{if } t \le \mathcal{P}_{u_1}, \\ \int_a^b G(t,s) f_u(s)\,\mathrm{d}s + \frac{c_0}{k} \\ \quad + (1 - \frac{\nu(\mathcal{P}u_1)}{k}) \mathcal{J}(\mathcal{P}u_1, u_1(\mathcal{P}u_1)) & \text{if } t > \mathcal{P}_{u_1}, \end{cases} \tag{7.3}$$

where $G, \mathcal{P}, f_u, \mathcal{A}_1, \mathcal{A}_2$ are given by (3.6), (5.4), (6.3), (6.7), respectively. Recall that $\mathcal{P}u_1$ is a unique point in (a,b) satisfying

$$\mathcal{P}u_1 = \tau_1 \in [a_1, b_1], \quad \text{where } \tau_1 = \gamma\bigl(u_1(\tau_1)\bigr). \tag{7.4}$$

For $t \in [a,b]$, define a function z by

$$z(t) = \begin{cases} u_1(t) & \text{if } t \in [a, \tau_1], \\ u_2(t) & \text{if } t \in (\tau_1, b]. \end{cases} \tag{7.5}$$

Differentiating (7.2), (7.3) and using (3.6) and (6.3), we get $u_i'(t) = f(t, u_i(t))$ for a.e. $t \in [a,b]$, $i = 1, 2$, and consequently

$$z'(t) = f\bigl(t, z(t)\bigr) \quad \text{for a.e. } t \in [a,b].$$

By virtue of (7.2)-(7.5), we have

$$z(\tau_1+) - z(\tau_1) = u_2(\tau_1) - u_1(\tau_1) = \mathcal{J}\bigl(\tau_1, u_1(\tau_1)\bigr) = \mathcal{J}\bigl(\tau_1, z(\tau_1)\bigr). \tag{7.6}$$

Let us show that τ_1 is a unique solution of the equation

$$t = \gamma\bigl(z(t)\bigr) \tag{7.7}$$

in $[a,b]$. According to (7.4) and (7.5), it suffices to prove

$$t \neq \gamma\bigl(u_2(t)\bigr), \quad t \in (\tau_1, b]. \tag{7.8}$$

Since $(u_1, u_2) \in \overline{\Omega}$, we have (*cf.* (5.1) and (6.2))

$$\|u_i\|_\infty \le K, \qquad \|u_i'\|_\infty \le \|h\|_\infty, \quad i = 1, 2.$$

Assume that the first condition in (4.6) is fulfilled. Then $\mathcal{J}(\tau_1, x) \ge 0$, $\gamma'(x) \le 0$ for $x \in [-K, K]$. Put

$$\sigma(t) = \gamma\bigl(u_2(t)\bigr) - t, \quad t \in [a,b].$$

By (7.6), $u_2(\tau_1) - u_1(\tau_1) = \mathcal{J}(\tau_1, u_1(\tau_1)) \ge 0$, and since γ is non-increasing, we have

$$\sigma(\tau_1) = \gamma\bigl(u_2(\tau_1)\bigr) - \tau_1 \le \gamma\bigl(u_1(\tau_1)\bigr) - \tau_1 = 0$$

due to (7.4). Using (4.5), we derive for $t \in (\tau_1, b]$

$$\begin{aligned}\sigma(t) &= \int_{\tau_1}^t \bigl(\gamma'\bigl(u_2(s)\bigr)u_2'(s) - 1\bigr)\,\mathrm{d}s \le \int_{\tau_1}^t \bigl(\bigl|\gamma'\bigl(u_2(s)\bigr)\bigr| \cdot \|u_2'\|_\infty - 1\bigr)\,\mathrm{d}s \\ &< \int_{\tau_1}^t \left(\frac{1}{\|h\|_\infty}\|h\|_\infty - 1\right)\mathrm{d}s = 0.\end{aligned}$$

So, (7.8) is valid. If the second condition in (4.6) is fulfilled, we use the dual arguments.

Finally, let us check that $\ell(z) = c_0$. By (7.2)-(7.6) and (3.6), we have

$$z(t) = \int_a^b G(t,s)f\bigl(s, z(s)\bigr)\,\mathrm{d}s + \frac{c_0}{k} + G(t,\tau_1)\mathcal{J}\bigl(\tau_1, z(\tau_1)\bigr). \tag{7.9}$$

Put

$$x(t) = \int_a^b G(t,s)f\bigl(s, z(s)\bigr)\,\mathrm{d}s. \tag{7.10}$$

Then, according to (iii) of Definition 3.3 and Remark 3.2, we get $\ell(x) = 0$. Further, using (3.7) from Lemma 3.5, we arrive at $\ell(G(\cdot,\tau_1)) = 0$. Consequently, due to (2.5), (7.9) and (7.10), $\ell(z)$ results in

$$\begin{aligned}\ell(z) &= \ell(x) + \ell\left(\frac{c_0}{k}\right) + \ell\bigl(G(\cdot,\tau_1)\bigr)\mathcal{J}\bigl(\tau_1, z(\tau_1)\bigr) \\ &= \ell\left(\frac{c_0}{k}\right) = k\frac{c_0}{k} + {}_{(\mathrm{KS})}\int_a^b v(t)\,\mathrm{d}\left[\frac{c_0}{k}\right] = c_0.\end{aligned}$$

□

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the manuscript and read and approved the final manuscript.

Acknowledgements
This research was supported by the grant Matematické modely, PrF_2013_013. The authors thank the referees for suggestions which improved the paper.

Received: 13 May 2013 Accepted: 13 August 2013 Published: 28 August 2013

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doi:10.1186/1687-2770-2013-195
Cite this article as: Rachůnek and Rachůnková: **First-order nonlinear differential equations with state-dependent impulses.** *Boundary Value Problems* 2013 **2013**:195.