Second order BVPs with state dependent impulses via lower and upper functions

IRENA RACHŮNKOVÁ, JAN TOMEČEK

Department of Mathematical Analysis
and Applications of Mathematics,
Faculty of Science, Palacký University,
17. listopadu 12, Olomouc 771 46, Czech Republic
e-mail: irena.rachunkova@upol.cz, jan.tomecek@upol.cz

Abstract

The paper deals with the second order Dirichlet boundary value problem with p state–dependent impulses ($p \in \mathbb{N}$)

$$z''(t) = f(t, z(t)) \quad \text{for a.e. } t \in [0, T],$$
$$z(0) = 0, \quad z(T) = 0,$$
$$z'(\tau_i+) - z'(\tau_i-) = I_i(\tau_i, z(\tau_i)), \quad \tau_i = \gamma_i(z(\tau_i)), \quad i = 1, \dots, p.$$

The solvability of this problem is proved under the assumption that there exists a well-ordered couple of lower and upper functions to the corresponding Dirichlet problem without impulses.

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1 Introduction

We investigate the solvability of the second order Dirichlet boundary value problem on the interval $[0, T]$, $T > 0$, subject to p state–dependent impulses

$$z''(t) = f(t, z(t)), \tag{1}$$

$$z(0) = 0, \quad z(T) = 0, \tag{2}$$

$$z'(\tau_i+) - z'(\tau_i-) = I_i(\tau_i, z(\tau_i)), \quad \tau_i = \gamma_i(z(\tau_i)), \quad i = 1, \dots, p, \tag{3}$$

where we assume

$$p \in \mathbb{N},\ f \in \mathrm{Car}([0,T] \times \mathbb{R}),\ I_i \in C([0,T] \times \mathbb{R}),\ \gamma_i \in C^1(\mathbb{R}),\ i = 1, \dots, p. \tag{4}$$

Almost whole literature on problems with state–dependent impulses is devoted to initial value problems, where existence, stability and other asymptotic properties of solutions have been studied, see e.g. [1], [2], [3], [9], [10], [12], [13], [16], [17]. There are also papers dealing with state–dependent impulsive periodic problems for first order differential equations [4], [14], [20], [23], [25] or for second order differential equations [6], [7]. Other types of boundary value problems with state–dependent impulses have been studied very rarely. We have found the paper [15] by M. Frigon and D. O'Regan, where the authors investigated second order Sturm–Liouville boundary value problems through initial value problems for multivalued maps. Their existence result, which is proved by means of the fixed point theory for composition of acyclic maps, is not applicable to our problem (1)–(3). We refer also to the paper by M. Benchohra, J. R. Graef, S. K. Ntouyas and A. Ouahab [8] dealing with first order differential inclusions subject to nonlinear boundary conditions. To prove the existence of solutions, the authors used a nonlinear alternative of the Leray–Schauder type combined with lower and upper functions (solutions) method. The lower and upper functions method has been also successfully applied to the study of the existence of solutions of the first order state–dependent impulsive problems for differential equations, see e.g. [11], [19], [24]. Important monographs in the area are [5], [18], [22]. Here we present the application of the lower and upper functions method on the second order state–dependent impulsive problem (1)–(3).

In our previous paper [21] we investigated the solvability of problem (1)–(3) with $p = 1$ and the main existence result there (Theorem 7) has been reached by means of the transformation of the studied problem to a fixed point problem for a proper operator in the space $C^1([0,T]) \times C^1([0,T])$. Here, in our present paper, we extend this approach to more state-dependent impulses (see (3)) and we have proved a new existence result for problem (1)–(3) under the assumption that there exists a well-ordered couple of lower and upper functions to the corresponding Dirichlet problem (1), (2) without impulses.

Definition 1 A function $z \in C([0,T])$ is a solution of problem (1)–(3), if for each $i \in \{1, \dots, p\}$ there exists a unique $\tau_i \in (0,T)$ such that $\gamma_i(z(\tau_i)) = \tau_i$, $0 = \tau_0 < \tau_1 < \ldots < \tau_p < \tau_{p+1} = T$, the restrictions $z|_{[\tau_i, \tau_{i+1}]}$, $i = 0, 1, \dots, p$, have absolutely continuous derivatives, z satisfies (1) for a.e. $t \in [0,T]$ and fulfils conditions (2) and (3).

Definition 2 A function $\sigma \in C([0,T])$ is called a lower function of problem (1),(2), if there exists a finite set $S \subset (0,T)$ such that $\sigma \in AC^1_{loc}([0,T] \setminus S)$, $\sigma'(s+), \sigma'(s-) \in \mathbb{R}$ for each $s \in S$ and

$$\sigma''(t) \geq f(t, \sigma(t)) \quad \text{for a.e. } t \in [0,T], \tag{5}$$

$$\sigma(0) \leq 0, \quad \sigma(T) \leq 0, \quad \sigma'(s-) < \sigma'(s+) \quad \text{for } s \in S. \tag{6}$$

If the inequalities in (5) and (6) are reversed, then σ is called an upper function of problem (1),(2).

We will study problem (1)–(3) under the basic assumptions

$$\left\{ \begin{array}{l} \text{there exist lower and upper functions } \alpha \text{ and } \beta \text{ to problem (1),(2)} \\ \text{with } \alpha(t) \leq \beta(t) \text{ for } t \in [0,T], \end{array} \right. \tag{7}$$

$$I_i(t, \alpha(t)) \leq 0, \quad I_i(t, \beta(t)) \geq 0, \quad t \in [0,T], \quad i = 1, \dots, p. \tag{8}$$

Denote

$$m(t) = \sup\{|f(t,x)| : \alpha(t) \leq x \leq \beta(t)\}, \quad K_0 = \int_0^T m(t)\,\mathrm{d}t \tag{9}$$

and

$$\left\{ \begin{array}{l} K_i = \max\{|I_i(t,x)| : t \in [0,T],\ \alpha(t) \leq x \leq \beta(t)\}, \quad i = 1, \dots, p, \\ \tilde{K} = K_0 + \sum_{i=1}^{p} K_i,\ K_0 \text{ is from (9).} \end{array} \right. \tag{10}$$

Further we will work with the assumption

$$\left\{ \begin{array}{l} \exists K > \tilde{K} : |\gamma_i'(x)| < 1/K, \quad i = 1, \dots, p, \\ 0 < \gamma_1(x) < \gamma_2(x) < \dots < \gamma_p(x) < T, \quad \text{for } |x| \leq TK/4, \\ \tilde{K} \text{ is from (10).} \end{array} \right. \tag{11}$$

Under assumptions (4), (7)–(11), we prove the solvability of problem (1)–(3). Our main existence result (Theorem 10), which is based on assumption (7) and which deals with $p \in \mathbb{N}$, can be applied on problems which are not covered by Theorem 7 in [21] even in the case $p = 1$. See Examples 11, 12 and 13.

Here, we denote by $C(J)$ the set of all continuous functions on the interval J, $C^1(J)$ the set of all functions having continuous derivatives on the interval J and $L^1(J)$ the set of all Lebesgue integrable functions on J. For a compact interval J we consider the linear spaces $C(J)$ and $C^1(J)$ equipped with the norms

$$\|x\|_\infty = \max_{t \in J} |x(t)| \quad \text{and} \quad \|x\|_1 = \|x\|_\infty + \|x'\|_\infty,$$

respectively. In the paper we work with the linear space

$$X = \left(C^1([0,T])\right)^{p+1}, \tag{12}$$

equipped with the norm

$$\|(u_1, \ldots, u_{p+1})\| = \sum_{i=1}^{p+1} \|u_i\|_1 \quad \text{for } (u_1, \ldots, u_{p+1}) \in X.$$

It is well–known that the mentioned normed spaces are Banach spaces. Recall that for $\mathcal{A} \subset \mathbb{R}$, a function $f : [a,b] \times \mathcal{A} \to \mathbb{R}$ satisfies the Carathéodory conditions on $[a,b] \times \mathcal{A}$ (we write $f \in \mathrm{Car}([a,b] \times \mathcal{A})$) if

- $f(\cdot, x) : [a,b] \to \mathbb{R}$ is measurable for all $x \in \mathcal{A}$,
- $f(t, \cdot) : \mathcal{A} \to \mathbb{R}$ is continuous for a.e. $t \in [a,b]$,
- for each compact set $K \subset \mathcal{A}$ there exists a function $m_K \in L^1([a,b])$ such that $|f(t,x)| \le m_K(t)$ for a.e. $t \in [a,b]$ and each $x \in K$.

2 Operators and auxiliary problem

In this section we assume that (4), (7)–(11) are fulfilled. We construct an auxiliary problem (21)–(23) and transform it to a fixed point problem for a proper operator in the space X introduced in (12), Section 1. To this end we use the approach of [21], where such transformation has been done for $p = 1$. Let us consider K of (11) and define a set

$$B = \{u \in C^1([0,T]) : \|u\|_\infty < TK/4, \ \|u'\|_\infty < K\}. \tag{13}$$

The following three lemmas and their proofs are simple generalizations of those in the paper [21]. For the sake of independence of this paper, we state them with their full proofs.

Lemma 3 *Let $u \in \overline{B}$, $i \in \{1, \ldots, p\}$ and let $\gamma_i \in C^1(\mathbb{R})$ satisfy* (11). *Then there exists a unique $\tau_i \in (0,T)$ such that*

$$\gamma_i(u(\tau_i)) = \tau_i. \tag{14}$$

Proof. Let us take an arbitrary $u \in \overline{B}$ and $i \in \{1, \ldots, p\}$. Obviously, the constant τ_i is a solution of the equation

$$\gamma_i(u(t)) = t,$$

i.e. τ_i is a root of the function

$$\sigma(t) = \gamma_i(u(t)) - t, \quad t \in [0,T].$$

According to (11) and (13), we get $\sigma(0) = \gamma_i(u(0)) > 0$, $\sigma(T) = \gamma_i(u(T)) - T < 0$ and

$$\sigma'(t) = \gamma_i'(u(t))u'(t) - 1 \le |\gamma_i'(u(t))||u'(t)| - 1 < \frac{1}{K}K - 1 = 0, \quad t \in (0,T). \tag{15}$$

Therefore σ is strictly decreasing on $[0, T]$ and hence it has exactly one root in $(0, T)$. $\square$

Due to Lemma 3 each function $u \in \overline{B}$ crosses each barrier curve $x = \gamma_i(t)$, $i = 1, \dots, p$, at exactly one point $\tau_i \in (0, T)$. Therefore we can define functionals $\mathcal{P}_i : \overline{B} \to (0, T)$ by

$$\mathcal{P}_i u = \tau_i, \quad i = 1, \dots, p, \tag{16}$$

where τ_i fulfils (14).

In order to construct a proper operator fixed point problem, the following lemma is crucial.

Lemma 4 *Let $i \in \{1, \dots, p\}$ and let γ_i satisfy* (11)*. Then the functional $\mathcal{P}_i$ is continuous on $\overline{B}$.*

Proof. Let us consider u_n, $u \in \overline{B}$ for $n \in \mathbb{N}$ such that $u_n \to u$ in $C^1([0, T])$. Choose $i \in \{1, \dots, p\}$ and denote

$$\sigma_n(t) = \gamma_i(u_n(t)) - t, \quad \sigma(t) = \gamma_i(u(t)) - t, \quad \text{for } t \in [0, T].$$

By Lemma 3, $\sigma_n(\tau_i^n) = 0$ and $\sigma(\tau_i) = 0$, where $\tau_i^n = \mathcal{P}_i u_n$ and $\tau_i = \mathcal{P}_i u$, respectively. According to (4) we get $\sigma_n, \sigma \in C^1([0, T])$ for $n \in \mathbb{N}$ and

$$\sigma_n \to \sigma \quad \text{in } C([0, T]). \tag{17}$$

We will prove that $\lim_{n\to\infty} \tau_i^n = \tau_i$. Let us take an arbitrary $\epsilon > 0$. Since $\sigma(\tau_i) = 0$ and $\sigma'(\tau_i) < 0$ (cf. (15)), we can find $\xi \in (\tau_i - \epsilon, \tau_i)$ and $\eta \in (\tau_i, \tau_i + \epsilon)$ such that

$$\sigma(\xi) > 0 \quad \text{and} \quad \sigma(\eta) < 0.$$

From (17) it follows the existence of $n_0 \in \mathbb{N}$ such that

$$\sigma_n(\xi) > 0 \quad \text{and} \quad \sigma_n(\eta) < 0$$

for each $n \geq n_0$. By Lemma 3 and the continuity of σ_n there follows that $\tau_i^n \in (\xi, \eta) \subset (\tau_i - \epsilon, \tau_i + \epsilon)$ for $n \geq n_0$. $\square$

Having the lower function α and upper function β due to (7), we define for a.e. $t \in [0, T]$ and all $x \in \mathbb{R}$,

$$h(t, x) = \begin{cases} f(t, \beta(t)) + \dfrac{x - \beta(t)}{x - \beta(t) + 1}\epsilon_0 & \text{for } x > \beta(t), \\ f(t, x) & \text{for } \alpha(t) \leq x \leq \beta(t), \\ f(t, \alpha(t)) - \dfrac{\alpha(t) - x}{\alpha(t) - x + 1}\epsilon_0 & \text{for } x < \alpha(t). \end{cases} \tag{18}$$

Here $\epsilon_0 > 0$ is such that

$$\tilde{K} + (T + p)\epsilon_0 < K, \tag{19}$$

where K and $\tilde{K}$ are from (11) and (10), respectively. Further, we define on $[0,T]\times\mathbb{R}$ for $i=1,\dots,p$,

$$\tilde{I}_i(t,x)=\begin{cases} I_i(t,\beta(t))+\dfrac{x-\beta(t)}{x-\beta(t)+1}\epsilon_0 & \text{if } x>\beta(t)\\ I_i(t,x) & \text{if } \alpha(t)\le x\le\beta(t),\\ I_i(t,\alpha(t))-\dfrac{\alpha(t)-x}{\alpha(t)-x+1}\epsilon_0 & \text{if } x<\alpha(t).\end{cases} \tag{20}$$

Let us consider an auxiliary problem

$$z''(t)=h(t,z(t)), \tag{21}$$

$$z(0)=0, \quad z(T)=0, \tag{22}$$

$$z'(\tau_i+)-z'(\tau_i-)=\tilde{I}_i(\tau_i,z(\tau_i)), \quad \tau_i=\gamma_i(z(\tau_i)), \quad i=1,\dots,p. \tag{23}$$

Definition 5 A function $z\in C([0,T])$ is a solution of problem (21)–(23), if for each $i\in\{1,\dots,p\}$ there exists a unique $\tau_i\in(0,T)$ such that $\gamma_i(z(\tau_i))=\tau_i$, $0=\tau_0<\tau_1<\ldots<\tau_p<\tau_{p+1}=T$, the restrictions $z|_{[\tau_i,\tau_{i+1}]}$, $i=0,1,\dots,p$, have absolutely continuous derivatives, z satisfies (21) for a.e. $t\in[0,T]$ and fulfils conditions (22) and (23).

We will define an operator representation of problem (21)–(23). For this purpose we define a set Ω by

$$\Omega=B^{p+1}\subset X. \tag{24}$$

Then we put

$$\tilde{f}_u(t)=\begin{cases} h(t,u_1(t)) & \text{for a.e. } t\in[0,\tau_1],\\ \dots & \dots\\ h(t,u_{p+1}(t)) & \text{for a.e. } t\in[\tau_p,T]\end{cases} \tag{25}$$

for every $u=(u_1,\dots,u_{p+1})\in\overline{\Omega}$ and define an operator $\mathcal{F}:\overline{\Omega}\to X$ by $\mathcal{F}(u_1,\dots,u_{p+1})=(x_1,\dots,x_{p+1})$, where

$$\begin{aligned} x_j(t)=&\int_0^T G(t,s)\tilde{f}_u(s)\,\mathrm{d}s+\sum_{j\le i\le p} g_1(t,\tau_i)\tilde{I}_i(\tau_i,u_i(\tau_i))\\ &+\sum_{1\le i<j} g_2(t,\tau_i)\tilde{I}_i(\tau_i,u_i(\tau_i)),\end{aligned} \tag{26}$$

for $t\in[0,T]$, $\tau_i=\mathcal{P}_iu_i$, $j=1,\dots,p+1$. Here

$$g_1(t,s)=\frac{t(s-T)}{T}, \quad g_2(t,s)=\frac{s(t-T)}{T}, \quad s,t\in[0,T],$$

and G is the Green function of the problem $u''=0$, $u(0)=u(T)=0$, that is

$$G(t,s)=\begin{cases} g_1(t,s) & \text{for } 0\le t\le s\le T,\\ g_2(t,s) & \text{for } 0\le s\le t\le T.\end{cases}$$

Lemma 6 *Assume that Ω and $\mathcal{F}$ are given by* (24) *and* (26)*, respectively. The operator $\mathcal{F}$ is compact on $\overline{\Omega}$.*

Proof. First, we will prove the continuity of the operator $\mathcal{F}$. Let us take a sequence $\{u^{[n]}\}_{n=1}^{\infty} = \{(u_1^{[n]}, \dots, u_{p+1}^{[n]})\}_{n=1}^{\infty} \subset X$ and $u = (u_1, \dots, u_{p+1}) \in X$ such that

$$u^{[n]} \to u \quad \text{in } X. \tag{27}$$

Let us denote for each $n \in \mathbb{N}$, $j = 1, \dots, p$,

$$\tau_0^{[n]} = \tau_0 = 0, \quad \tau_{p+1}^{[n]} = \tau_{p+1} = T, \quad \tau_j^{[n]} = \mathcal{P}_j u_j^{[n]}, \quad \tau_j = \mathcal{P}_j u_j,$$
$$x = (x_1, \dots, x_{p+1}) = \mathcal{F}u, \quad x^{[n]} = (x_1^{[n]}, \dots, x_{p+1}^{[n]}) = \mathcal{F}u^{[n]}.$$

We will prove that $x^{[n]} \to x$ in X, i.e. $x_j^{[n]} \to x_j$ in $C^1([0,T])$ for each $j = 1, \dots, p+1$. Let us take $j \in \{1, \dots, p+1\}$. For each $t \in [0,T]$ we get by (25) and (26)

$$\begin{aligned} x_j^{[n]}(t) - x_j(t) = \sum_{i=0}^{p} \Bigg(&\int_{\tau_i}^{\tau_{i+1}} G(t,s)[h(s, u_{i+1}^{[n]}(s)) - h(s, u_{i+1}(s))] \, \mathrm{d}s \\ &+ \int_{\tau_i^{[n]}}^{\tau_i} G(t,s) h(s, u_{i+1}^{[n]}(s)) \, \mathrm{d}s + \int_{\tau_{i+1}}^{\tau_{i+1}^{[n]}} G(t,s) h(s, u_{i+1}^{[n]}(s)) \, \mathrm{d}s \Bigg) \\ &+ \sum_{j \le i \le p} \big(g_1(t, \tau_i^{[n]}) \tilde{I}_i(\tau_i^{[n]}, u_i^{[n]}(\tau_i^{[n]})) - g_1(t, \tau_i) \tilde{I}_i(\tau_i, u_i(\tau_i)) \big) \\ &+ \sum_{1 \le i < j} \big(g_2(t, \tau_i^{[n]}) \tilde{I}_i(\tau_i^{[n]}, u_i^{[n]}(\tau_i^{[n]})) - g_2(t, \tau_i) \tilde{I}_i(\tau_i, u_i(\tau_i)) \big) \end{aligned}$$

and

$$\begin{aligned} (x_j^{[n]})'(t) - (x_j)'(t) = \sum_{i=0}^{p} \Bigg(&\int_{\tau_i}^{\tau_{i+1}} \frac{\partial G}{\partial t}(t,s)[h(s, u_{i+1}^{[n]}(s)) - h(s, u_{i+1}(s))] \, \mathrm{d}s \\ &+ \int_{\tau_i^{[n]}}^{\tau_i} \frac{\partial G}{\partial t}(t,s) h(s, u_{i+1}^{[n]}(s)) \, \mathrm{d}s + \int_{\tau_{i+1}}^{\tau_{i+1}^{[n]}} \frac{\partial G}{\partial t}(t,s) h(s, u_{i+1}^{[n]}(s)) \, \mathrm{d}s \Bigg) \\ &+ \sum_{j \le i \le p} \big(\frac{\partial g_1}{\partial t}(t, \tau_i^{[n]}) \tilde{I}_i(\tau_i^{[n]}, u_i^{[n]}(\tau_i^{[n]})) - \frac{\partial g_1}{\partial t}(t, \tau_i) \tilde{I}_i(\tau_i, u_i(\tau_i)) \big) \\ &+ \sum_{1 \le i < j} \big(\frac{\partial g_2}{\partial t}(t, \tau_i^{[n]}) \tilde{I}_i(\tau_i^{[n]}, u_i^{[n]}(\tau_i^{[n]})) - \frac{\partial g_2}{\partial t}(t, \tau_i) \tilde{I}_i(\tau_i, u_i(\tau_i)) \big). \end{aligned}$$

Since

$$|G(t,s)| \le \frac{T}{4}, \quad \left| \frac{\partial G}{\partial t}(t,s) \right| \le 1 \quad \text{for } t, s \in [0,T], t \ne s, \tag{28}$$

we get

$$
\begin{aligned}
\|x_j^{[n]} - x_j\|_1 \leq \left(\frac{T}{4}+1\right) \sum_{i=0}^{p} \Bigg(& \int_0^T |h(s, u_{i+1}^{[n]}(s)) - h(s, u_{i+1}(s))| \, \mathrm{d}s \\
& + \left| \int_{\tau_i^{[n]}}^{\tau_i} |h(s, u_{i+1}^{[n]}(s))| \, \mathrm{d}s \right| + \left| \int_{\tau_{i+1}}^{\tau_{i+1}^{[n]}} |h(s, u_{i+1}^{[n]}(s))| \, \mathrm{d}s \right| \Bigg) \\
& + \sum_{j \leq i \leq p} \max_{t \in [0,T]} |g_1(t, \tau_i^{[n]}) \tilde{I}_i(\tau_i^{[n]}, u_i^{[n]}(\tau_i^{[n]})) - g_1(t, \tau_i) \tilde{I}_i(\tau_i, u_i(\tau_i))| \\
& + \sum_{j \leq i \leq p} \max_{t \in [0,T]} \left| \frac{\partial g_1}{\partial t}(t, \tau_i^{[n]}) \tilde{I}_i(\tau_i^{[n]}, u_i^{[n]}(\tau_i^{[n]})) - \frac{\partial g_1}{\partial t}(t, \tau_i) \tilde{I}_i(\tau_i, u_i(\tau_i)) \right| \\
& + \sum_{1 \leq i < j} \max_{t \in [0,T]} |g_2(t, \tau_i^{[n]}) \tilde{I}_i(\tau_i^{[n]}, u_i^{[n]}(\tau_i^{[n]})) - g_2(t, \tau_i) \tilde{I}_i(\tau_i, u_i(\tau_i))| \\
& + \sum_{1 \leq i < j} \max_{t \in [0,T]} \left| \frac{\partial g_2}{\partial t}(t, \tau_i^{[n]}) \tilde{I}_i(\tau_i^{[n]}, u_i^{[n]}(\tau_i^{[n]})) - \frac{\partial g_2}{\partial t}(t, \tau_i) \tilde{I}_i(\tau_i, u_i(\tau_i)) \right|.
\end{aligned}
$$

By (27), there exists a compact set $K \subset \mathbb{R}$ such that $u_i^{[n]}(t) \in K$ for each $t \in [0,T]$, $n \in \mathbb{N}$ and $i = 1, \ldots, p+1$. Consequently, by (4) and (18), there exists $m_K \in L^1([0,T])$ such that

$$|h(t, u_i^{[n]}(t))| \leq m_K(t)$$

for a.e. $t \in [0,T]$, all $n \in \mathbb{N}$, $i = 1, \ldots, p+1$. Since

$$\lim_{n \to \infty} h(t, u_i^{[n]}(t)) = h(t, u_i(t))$$

for a.e. $t \in [0,T]$ and each $i = 1, \ldots, p+1$, then due to the Lebesgue dominated convergence theorem it follows that

$$\int_0^T |h(s, u_i^{[n]}(s)) - h(s, u_i(s))| \, \mathrm{d}s \to 0$$

as $n \to \infty$ for $i = 1, \ldots, p+1$. Lemma 4 and (27) give $\lim_{n\to\infty} \tau_i^{[n]} = \tau_i$ for $i = 0, \ldots, p$, and hence the absolute continuity of the Lebesgue integral yields for each $i = 0, \ldots, p$,

$$
\begin{aligned}
& \lim_{n \to \infty} \left(\left| \int_{\tau_i^{[n]}}^{\tau_i} |h(s, u_{i+1}^{[n]}(s))| \, \mathrm{d}s \right| + \left| \int_{\tau_{i+1}}^{\tau_{i+1}^{[n]}} |h(s, u_{i+1}^{[n]}(s))| \, \mathrm{d}s \right| \right) \\
& \qquad \leq \lim_{n \to \infty} \left(\left| \int_{\tau_i^{[n]}}^{\tau_i} m_K(s) \, \mathrm{d}s \right| + \left| \int_{\tau_{i+1}}^{\tau_{i+1}^{[n]}} m_K(s) \, \mathrm{d}s \right| \right) = 0.
\end{aligned}
$$

The continuity of g_1, $\frac{\partial g_1}{\partial t}$ g_2, $\frac{\partial g_2}{\partial t}$ and $\tilde{I}_i$ implies that

$$g_1(t, \tau_i^{[n]}) \tilde{I}_i(\tau_i^{[n]}, u_i^{[n]}(\tau_i^{[n]})) \to g_1(t, \tau_i) \tilde{I}_i(\tau_i, u_i(\tau_i)),$$

$$\frac{\partial g_1}{\partial t}(t,\tau_i^{[n]})\tilde{I}_i(\tau_i^{[n]},u_i^{[n]}(\tau_i^{[n]})) \to \frac{\partial g_1}{\partial t}(t,\tau_i)\tilde{I}_i(\tau_i,u_i(\tau_i)),$$
$$g_2(t,\tau_i^{[n]})\tilde{I}_i(\tau_i^{[n]},u_i^{[n]}(\tau_i^{[n]})) \to g_2(t,\tau_i)\tilde{I}_i(\tau_i,u_i(\tau_i)),$$
$$\frac{\partial g_2}{\partial t}(t,\tau_i^{[n]})\tilde{I}_i(\tau_i^{[n]},u_i^{[n]}(\tau_i^{[n]})) \to \frac{\partial g_2}{\partial t}(t,\tau_i)\tilde{I}_i(\tau_i,u_i(\tau_i)),$$

as $n \to \infty$ uniformly w.r.t. $t \in [0,T]$. Therefore $x_j^{[n]}$ converges to x_j in $C^1([0,T])$ for each $j = 1,\dots,p+1$.
Now we will prove that $\mathcal{F}(\overline{\Omega})$ is relatively compact. Choose an arbitrary $u = (u_1,\dots,u_{p+1}) \in \overline{\Omega}$. By (7), (9), (10), (18), (20) and (25), it holds

$$|\tilde{I}_i(t,u_i(t))| \le K_i + \varepsilon_0, \quad t \in [0,T],\ i = 1,\dots,p, \tag{29}$$

$$|\tilde{f}_u(t)| \le m(t) + \varepsilon_0 \quad \text{for a.e. } t \in [0,T]. \tag{30}$$

Denote $(x_1,\dots,x_{p+1}) = \mathcal{F}(u_1,\dots,u_{p+1})$. Then, by (9), (10), (26), (28), (29) and (30), we get for $j = 1,\dots,p+1$

$$|x_j(t)| \le \frac{T}{4}\left(\int_0^T |\tilde{f}_u(s)|\,\mathrm{d}s + p\varepsilon_0 + \sum_{i=1}^p K_i\right)$$
$$\le \frac{T}{4}\left(K_0 + T\varepsilon_0 + p\varepsilon_0 + \sum_{i=1}^p K_i\right) = \frac{T}{4}\left((T+p)\varepsilon_0 + \tilde{K}\right),$$

and similarly
$$|x_j'(t)| \le (T+p)\varepsilon_0 + \tilde{K}.$$

We have proved that the set $\mathcal{F}(\overline{\Omega})$ is bounded in X. In addition, since $K > (T+p)\varepsilon_0 + \tilde{K}$ (see (19)), we get

$$|x_j(t)| < \frac{T}{4}K, \quad |x_j'(t)| < K, \quad t \in [0,T],\ j = 1,\dots,p+1.$$

Consequently, by virtue of (13) and (24), we see that $(x_1,\dots,x_{p+1}) \in \Omega$ which implies
$$\mathcal{F}(\overline{\Omega}) \subset \overline{\Omega}. \tag{31}$$

Now, we show that the set $\{(x_1',\dots,x_{p+1}') : (x_1,\dots,x_{p+1}) \in \mathcal{F}(\overline{\Omega})\}$ is equicontinuous on $[0,T]$. For a.e. $t \in [0,T]$ and all $(x_1,\dots,x_{p+1}) \in \mathcal{F}(\overline{\Omega})$ we have by (26), (30) and from the properties of Green function G that

$$|x_j''(t)| \le m(t) + \varepsilon_0 \quad \text{for a.e. } t \in [0,T] \text{ and all } j = 1,\dots,p+1.$$

As a result, for each $\epsilon > 0$ there exists $\delta > 0$ such that for each t_1, $t_2 \in [0,T]$ satisfying $|t_1 - t_2| < \delta$ the inequality

$$\sum_{j=1}^{p+1} |x_j'(t_1) - x_j'(t_2)| \le (p+1)\left|\int_{t_2}^{t_1} (m(t) + \varepsilon_0)\,\mathrm{d}t\right| < \epsilon$$

holds for all $(x_1, \dots, x_{p+1}) \in \mathcal{F}(\overline{\Omega})$. Consequently, $\mathcal{F}(\overline{\Omega})$ is relatively compact in X by the Arzelà–Ascoli theorem.

$\square$

Theorem 7 *Assume that* Ω *and* $\mathcal{F}$ *are given by* (24) *and* (26), *respectively. The operator* $\mathcal{F}$ *has a fixed point in* $\overline{\Omega}$.

Proof. By Lemma 6, $\mathcal{F}$ is compact on $\overline{\Omega}$. Therefore, by (31), the Schauder fixed point theorem yields a fixed point of $\mathcal{F}$ in $\overline{\Omega}$. $\square$

Lemma 8 *Let* $(u_1, \dots, u_{p+1}) \in \overline{\Omega}$ *be a fixed point of* $\mathcal{F}$. *Consider* $\mathcal{P}_i$, $i = 1, \dots, p$, *from* (16). *Then the function*

$$z(t) = \begin{cases} u_1(t), & t \in [0, \mathcal{P}_1 u_1], \\ u_2(t), & t \in (\mathcal{P}_1 u_1, \mathcal{P}_2 u_2], \\ \dots, & \dots, \\ u_{p+1}(t) & t \in (\mathcal{P}_p u_p, T], \end{cases} \tag{32}$$

is a solution of problem (21)–(23).

Proof. Let $(u_1, \dots, u_{p+1}) \in \overline{\Omega}$ be such that $(u_1, \dots, u_{p+1}) = \mathcal{F}(u_1, \dots, u_{p+1})$, that is (see (26))

$$\begin{aligned} u_j(t) = \int_0^T G(t,s) \tilde{f}_u(s) \, \mathrm{d}s + \sum_{j \le i \le p} g_1(t, \tau_i) \tilde{I}_i(\tau_i, u_i(\tau_i)) \\ + \sum_{1 \le i < j} g_2(t, \tau_i) \tilde{I}_i(\tau_i, u_i(\tau_i)), \end{aligned} \tag{33}$$

for $t \in [0,T]$, $j = 1, \dots, p+1$, where $\tau_i = \mathcal{P}_i u_i$ for $i = 1, \dots, p$. Let us assume the function z defined in (32). Hence, $z(0) = u_1(0) = 0$, $z(T) = u_{p+1}(T) = 0$, and since $g_1(\tau_j, \tau_j) = g_2(\tau_j, \tau_j)$, we get

$$z(\tau_j) = u_j(\tau_j) = u_{j+1}(\tau_j) = z(\tau_j +)$$

$j = 1, \dots, p$. By Lemma 3,

$$\gamma_j(z(\tau_j)) = \tau_j, \quad j = 1, \dots, p, \tag{34}$$

and τ_j is a unique point in $(0,T)$ satisfying (34). In addition, (11) yields $0 < \tau_1 < \ldots < \tau_p < T$. Denote $\tau_0 = 0$, $\tau_{p+1} = T$. We get

$$\begin{aligned} u_j'(t) = \int_0^T \frac{\partial G}{\partial t}(t,s) \tilde{f}_u(s) \, \mathrm{d}s + \sum_{j \le i \le p} \frac{\partial g_1}{\partial t}(t, \tau_i) \tilde{I}_i(\tau_i, u_i(\tau_i)) \\ + \sum_{1 \le i < j} \frac{\partial g_2}{\partial t}(t, \tau_i) \tilde{I}_i(\tau_i, u_i(\tau_i)), \end{aligned}$$

$t \in [0, T]$, and

$$u_j''(t) = \tilde{f}_u(t) = h(t, u_j(t)) \quad \text{for a.e. } t \in [\tau_{j-1}, \tau_j],$$

$j = 1, \dots, p+1$. Therefore, by (32),

$$z''(t) = h(t, z(t)) \quad \text{for a.e. } t \in [0, T],$$

and the restrictions $z|_{[\tau_i, \tau_{i+1}]}$, $i = 0, \dots, p$, have absolutely continuous derivatives. Finally,

$$\begin{aligned} z'(\tau_j+) - z'(\tau_j-) &= u_{j+1}'(\tau_j) - u_j'(\tau_j) \\ &= \left(\frac{\partial g_2}{\partial t}(\tau_j, \tau_j) - \frac{\partial g_1}{\partial t}(\tau_j, \tau_j)\right) \tilde{I}_j(\tau_j, u_j(\tau_j)) = \tilde{I}_j(\tau_j, u_j(\tau_j)), \end{aligned}$$

for $j = 1, \dots, p$. Due to Definition 5 this completes the proof. □

Lemma 9 *Each solution z of problem* (21)–(23) *is a solution of problem* (1)–(3) *and satisfies the inequalities*

$$\alpha(t) \le z(t) \le \beta(t), \quad t \in [0, T], \tag{35}$$

where α and β are from (7).

Proof. Let z be a solution of problem (21)–(23). First, we will prove by contradiction that z fulfils (35). Let us define

$$w(t) = z(t) - \beta(t), \quad t \in [0, T]$$

and assume that

$$\max\{w(t) : t \in [0, T]\} = w(t_0) > 0. \tag{36}$$

Due to (22) and Definition 2 of an upper function β we can see that

$$w(0) \le 0 \quad \text{and} \quad w(T) \le 0,$$

and therefore $t_0 \in (0, T)$. According to Definition 5, for each $i \in \{1, \dots, p\}$ there exists a unique $\tau_i \in (0, T)$ such that $\gamma_i(z(\tau_i)) = \tau_i$, $0 = \tau_0 < \tau_1 < \ldots < \tau_p < \tau_{p+1} = T$ and the restrictions $z|_{[\tau_i, \tau_{i+1}]}$, $i = 0, 1, \dots, p$, have absolutely continuous derivatives. There are two possibilites:

Case A. Let $t_0 \in (\tau_i, \tau_{i+1})$ for some $i \in \{0, \dots, p\}$. If $t_0 \in S$, i.e. $\beta'(t_0-) > \beta'(t_0+)$, then

$$w'(t_0-) = z'(t_0) - \beta'(t_0-) < z'(t_0) - \beta'(t_0+) = w'(t_0+),$$

which contradicts (36). Therefore $t_0 \notin S$ and hence there exists $w'(t_0)$ and $w'(t_0) = 0$ holds. Having in mind (36) and the finiteness of the set S, there exists $\delta > 0$ such that

$$w(t) > 0 \quad \text{and} \quad w'(t-) = w'(t+) \quad \text{for } t \in [t_0, t_0 + \delta). \tag{37}$$

Further, by Definition 2, (18) and (21),

$$w''(t) = z''(t) - \beta''(t) \geq h(t, z(t)) - f(t, \beta(t)) = \frac{z(t) - \beta(t)}{z(t) - \beta(t) + 1}\epsilon_0 > 0$$

for a.e. $t \in (t_0, t_0 + \delta)$. Therefore

$$0 < \int_{t_0}^{t} w''(s)\,\mathrm{d}s = w'(t) - w'(t_0) = w'(t) \quad \text{for } t \in (t_0, t_0 + \delta),$$

which contradicts (36).
Case B. Let $t_0 = \tau_i$ for some $i \in \{1, \ldots, p\}$. Since $\beta'(t_0-) \geq \beta'(t_0+)$ it follows from (8), (20) and (23) that

$$\begin{aligned} w'(t_0-) &= z'(t_0-) - \beta'(t_0-) \\ &\leq z'(t_0+) - \tilde{I}_i(t_0, z(t_0)) - \beta'(t_0+) \\ &= w'(t_0+) - I_i(t_0, \beta(t_0)) - \frac{z(t_0) - \beta(t_0)}{z(t_0) - \beta(t_0) + 1}\epsilon_0 < w'(t_0+), \end{aligned}$$

which contradicts (36).
We have proved the inequatity $z(t) \leq \beta(t)$ for $t \in [0, T]$. The inequality $z(t) \geq \alpha(t)$ for $t \in [0, T]$ can be obtained in a similar way. These facts together with (18) implies that z satisfies (1) for a.e. $t \in [0, T]$. The boundary conditions (2) and (22) are the same. According to (20) and (23) we get

$$z'(\tau_i+) - z'(\tau_i-) = \tilde{I}_i(\tau_i, z(\tau_i)) = I_i(\tau_i, z(\tau_i))$$

for each $i = 1, \ldots, p$. Due to Definition 1 this completes the proof. □

3 Main result

Now, we are ready to present the main result of this paper.

Theorem 10 *Let assumptions* (4), (7)–(11) *be fulfilled. Then there exists a solution z of the problem* (1)–(3) *satisfying* (35).

Proof. Assume that the operator $\mathcal{F}$ and the set Ω are given by (26) and (24), respectively. According to Theorem 7, the operator $\mathcal{F}$ has a fixed point $u = (u_1, \ldots, u_{p+1})$ in the set $\overline{\Omega}$. From Lemma 8 it follows that the function z constructed from u in (32) is a solution of the auxiliary problem (21)–(23). Lemma 9 implies that z is a solution of problem (1)–(3) and satisfies the inequalities (35). □

4 Examples

In this section we show the applicability of the obtained results. The examples are chosen such that the existence results from the paper [21] cannot be applied.

Example 11 *(Sublinear problem)* Let us consider problem (1)–(3) with

$$p = T = 1,\ f(t,x) = t^2 + |x|^a \operatorname{sgn} x,\ I_1(t,x) = |x|^b \operatorname{sgn} x,\ a \in (0,1), b > \frac{1}{a}.$$

We see that f is sublinear in x and that f, I_1 fulfil (4). The functions

$$\alpha(t) = -1, \quad \beta(t) = 0, \quad t \in [0,1],$$

satisfy Definition 2 and they form the well–ordered couple of lower and upper functions to problem (1),(2). In addition, $I_1(t,\alpha(t)) = I_1(t,-1) = -1 < 0$, $I_1(t,\beta(t)) = I_1(t,0) = 0$. Therefore (7) and (8) are valid. If we put

$$m(t) = t^2 + 1, \quad K_0 = \int_0^1 (t^2+1)\,\mathrm{d}t = \frac{4}{3}, \quad K_1 = 1, \quad \widetilde{K} = \frac{7}{3},$$

then (9) and (10) hold. Summarizing assumptions for γ_1 contained in (4) and (11), we get that γ_1 should fulfil for some $K > \widetilde{K}$

$$\gamma_1 \in C^1(\mathbb{R}), \quad 0 < \gamma_1(x) < 1, \quad |\gamma_1'(x)| < \frac{1}{K} \quad \text{for } |x| < \frac{K}{4}. \tag{38}$$

Hence, consider an arbitrary $K > 7/3$. If we choose $c \in (0, 2/K^2)$ and put

$$\gamma_1(x) = cx^2 + \frac{1}{2}, \quad x \in \mathbb{R}, \tag{39}$$

or if we choose $c \in (0, 1/2)$, $n > Kc$ and put

$$\gamma_1(x) = c\sin\frac{x}{n} + \frac{1}{2}, \quad x \in \mathbb{R}, \tag{40}$$

we can check that (38) is fulfilled in both cases. Therefore, by Theorem 10, the corresponding problem (1)–(3) has at least one solution.
Let us show that Theorem 7 in [21] cannot be applied in this case. The basic assumption needed in Theorem 7 has the form

$$\exists K > 0: \quad \frac{1}{K}\left[\int_0^T h(s, K + TJ(K))\,\mathrm{d}s + J(K)\right] < \min\left\{1, \frac{1}{T}\right\}, \tag{41}$$

where h and J are majorants for f and I_1, respectively. Here we have

$$h(t,x) = t^2 + x^a, \quad J(x) = x^b, \quad x \in (0,\infty),$$

and (41) can be written as

$$\exists x > 0: \quad \frac{1}{x}\left[\int_0^1 (s^2 + (x + x^b)^a)\,\mathrm{d}s + x^b\right] < 1. \tag{42}$$

Let us put
$$\Phi(x) = \frac{1}{3} + (x + x^b)^a + x^b - x, \quad x \in (0, \infty).$$
Since $b > 1$, we have $x^b - x \geq 0$ for $x \geq 1$ and hence $\Phi(x) > 1/3$ for $x \geq 1$. Since $a \in (0,1)$, we have $(x+x^b)^a > x^a > x$ for $x \in (0,1)$ and hence $\Phi(x) > 1/3$ for $x \in (0,1)$. Consequently $\Phi(x) > 1/3 > 0$ for $x > 0$ and (42) fails.

Example 12 *(Linear problem)* Let $p = 2$ and let us consider problem (1)–(3) with f, I_1, I_2 having linear behaviour in x. In particular, we put for $t \in [0,T]$, $x \in \mathbb{R}$
$$f(t,x) = t^2 + x, \quad I_1(t,x) = I_2(t,x) = x.$$
As a lower and upper functions to problem (1),(2) we can take for instance
$$\alpha(t) = -T^2, \quad \beta(t) = 0, \quad t \in [0,T].$$
Then f, I_1, I_2 fulfil (4), (7) and (8). If we put
$$m(t) = t^2 + 1, \quad K_0 = \frac{4}{3}, \quad K_1 = K_2 = 1, \quad \widetilde{K} = \frac{10}{3},$$
then (9) and (10) hold. Choose an arbitrary $K > 10/3$ and take γ_1 defined by (39) and γ_2 defined by (40). Then by Theorem 10, the corresponding problem (1)–(3) is solvable.

Now, assume that $p = 1$ and check assumption (41) of Theorem 7 in [21], which can be written here as
$$\exists x > 0: \quad \frac{1}{x}\left[\int_0^T (s^2 + x + Tx)\,\mathrm{d}s + x\right] < 1. \tag{43}$$
Since $\int_0^T (s^2 + x + Tx)\,\mathrm{d}s > 0$ for $x > 0$, (43) fails.

Example 13 *(Superlinear problem)* Let us consider problem (1)–(3) with $p = T = 1$, $f(t,x) = t^3 + 2x^3$, $I_1(t,x) = 2x$. We see that f is superlinear in x and that f and I_1 fulfil (4). As lower and upper functions to problem (1),(2) we can take for instance
$$\alpha(t) = -\frac{1}{\sqrt[3]{2}}, \quad \beta(t) = 0, \quad t \in [0,1].$$
Then f and I_1 fulfil (4), (7) and (8). If we put
$$m(t) = t^3 + 1, \quad K_0 = \int_0^1 (t^3 + 1)\,\mathrm{d}t = \frac{5}{4}, \quad K_1 = \frac{2}{\sqrt[3]{2}} = \sqrt[3]{4}, \quad \widetilde{K} = \frac{5}{4} + \sqrt[3]{4},$$
then (9) and (10) hold. Choose an arbitrary $K > 5/4 + \sqrt[3]{4}$. Then problem (1)–(3) has a solution for γ_1 given by (39) or for γ_1 given by (40).
Finally, let us show that Theorem 7 in [21] cannot be applied because assumption (41) fails here. In this case assumption (41) can be written in the form
$$\exists x > 0: \quad \frac{1}{x}\left[\int_0^1 (s^3 + 2(x + 2x)^3)\,\mathrm{d}s + 2x\right] < 1. \tag{44}$$
Since $\int_0^1 (s^3 + 2(x + 2x)^3)\,\mathrm{d}s > 0$ for $x > 0$, (44) fails.

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References

[1] S. M. Afonso, E. M. Bonotto, M. Federson, Š. Schwabik, Discontinuous local semiflows for Kurzweil equations leading to LaSalle's invariance principle for differential systems with impulses at variable times, *J. Differential Equations* **250** (2011), 2969–3001.

[2] M.U. Akhmet, On the general problem of stability for impulsive differential equations, *J. Math. Anal. Appl.* **288** (2003), 182–196.

[3] M.U. Akhmetov, A. Zafer, Stability of the zero solution of impulsive differential equations by the Lyapunov second method, *J. Math. Anal. Appl.* **248** (2000), 69–82.

[4] I. Bajo and E. Liz, Periodic boundary value problem for first order differential equations with impulses at variable times, *J. Math. Anal. Appl.* **204** (1996), 65–73.

[5] D. Bainov, P. Simeonov, Impulsive Differential Equations: Periodic Solutions and Applications, Pitman Monographs and Surveys in Pure and Applied Mathematics 66, Longman Scientific and Technical, Essex 1993.

[6] J. Belley, M. Virgilio, Periodic Duffing delay equations with state dependent impulses, *J. Math. Anal. Appl.* **306** (2005), 646–662.

[7] J. Belley, M. Virgilio, Periodic Liénard–type delay equations with state–dependent impulses, *Nonlinear Anal.* **64** (2006), 568–589.

[8] M. Benchohra, J. R. Graef, S. K. Ntouyas and A. Ouahab, Upper and lower solutions method for impulsive differential inclusions with nonlinear boundary conditions and variable times, *Dynamics of Continuous, Discreate and Impulsive Systems, Series A: Mathematical Analysis* **12** (2005), 383–396.

[9] M. Benchohra, J. Henderson, S. K. Ntouyas and A. Ouahab, Impulsive functional differential equations with variable times, *Comp. and Math. with Applications* **47** (2004), 1659–1665.

[10] F. Cordova–Lepe, M. Pinto, E. Gonzalez–Olivares, A new class of differential equations with impulses at instants dependent on preceding pulses. Applications to management of renewable resources *Nonlinear Anal. RWA*, **13** (2012), 2313–2322.

[11] J. V. Devi, A. S. Vatsala, Generalized quasilinearization for an impulsive differential equation with variable moments of impulse, *Dynamic Systems and Applications*, **12** (2003), 369–382.

[12] A. Domoshnitsky, M. Drakhlin, E. Litsyn, Nonoscillation and positivity of solutions to first order state–dependent differential equations with impulses in variable moments, *J. Differential Equations* **228** (2006), 39–48.

[13] M. Frigon and D. O'Regan, Impulsive differential equations with variable times, *Nonlinear Anal.* **26** (1996), 1913–1922.

[14] M. Frigon and D. O'Regan, First order impulsive initial and periodic problems with variable moments, *J. Math. Anal. Appl.* **233** (1999), 730–739.

[15] M. Frigon and D. O'Regan, Second order Sturm–Liouville BVP's with impulses at variable times, Dynamics of Continuous, *D*iscrete and Impulsive Systems, Series A: Mathematical Analysis **8** (2001), 149–159.

[16] G. Gabor, The existence of viable trajectories in state-dependent impulsive systems, *Nonlinear Anal. TMA*, **72** (2010), 3828–3836.

[17] S. Kaul, V. Lakshmikantham and S. Leela, Extremal solutions, comparison principle and stability criteria for impulsive differential equations with variable times, *Nonlinear Anal.* **22** (1994), 1263–1270.

[18] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[19] V. Lakshmikantham, N.S. Papageorgiou, J. Vasundhara Devi, The method of upper and lower solutions and monotone technique for impulsive differential equations with variable moments, *Applicable Analysis* **51** (1993), 41–58.

[20] L. Liu, J. Sun, Existence of periodic solution of a harvested system with impulses at variable times, *Physics Letters A* **360** (2006), 105–108.

[21] I. Rachůnková, J. Tomeček, A new approach to BVPs with state–dependent impulses, submitted.

[22] A. M. Samoilenko and N. A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.

[23] J. Qi, X. Fu, Existence of limit cycles of impulsive differential equations with impulses at variable times, *Nonlinear Anal.* **44** (2001), 345–353.

[24] A.S. Vatsala, J. Vasundhara Devi, Generalized monotone technique for an impulsive differential equation with variable moments of impulse, *Nonlinear Studies* **9** (2002), 319–330.

[25] L. Yong, C. Fuzhong and L. Zhanghua, Boundary value problems for impulsive differential equations, *Nonlinear Anal. TMA* **29** 11 (1997), 1253–1264.