Asymptotic properties of Kneser solutions to nonlinear second order ODEs with regularly varying coefficients

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1. Introduction

In this paper, we utilize the framework of regular variation in order to analyze the existence and asymptotic behaviour of the Kneser solutions to the nonlinear second order ODE equation

$$(p(t)u'(t))' + q(t)f(u(t)) = 0, \tag{1}$$

where $t \in [a, \infty)$, $a \geq 0$. This type of equation is closely related to the extensively studied Emden-Fowler equation

$$(p(t)\Phi_\alpha(u'))' + q(t)\Phi_\gamma(u) = 0,$$

where $\Phi_\alpha(u) =: |u|^\alpha \operatorname{sgn} u$, $\alpha \geq 1$. The Emden-Fowler equation is called sub-half-linear, half-linear or super-half-linear if $\alpha > \gamma$, $\alpha = \gamma$ or $\alpha < \gamma$, respectively. The sub-half-linear case was studied in [16, 19], the half-linear case in [2, 11, 17] and the super-half-linear case in [3, 21], where a different sign condition was posed on the nonlinear term comparing to the present paper (cf. (22) and (29)). According to this terminology, equation (1) can be studied in a neighbourhood of the origin as a super-linear equation, since in our case $\alpha = 1$ and $\gamma = r > 1 = \alpha$, see (41).

We assume that the data functions p and q are regularly varying and $a \geq 0$ and study solutions of (1) satisfying

$$u(a) = u_0 \in (0, L),\ 0 \leq u(t) \leq L \text{ for } t \in [a, \infty), \tag{2}$$

or

$$u(a) = u_0 \in (L_0, 0),\ L_0 \leq u(t) \leq 0 \text{ for } t \in [a, \infty), \tag{3}$$

where the interval $[L_0, L]$ is determined by function f in the following way:

$$L_0 < 0 < L, \quad f(L_0) = f(0) = f(L) = 0.$$

Note that for $a > 0$ equation (1) is regular, while for $a = 0$ there is a time singularity at $t = 0$ due to $p(0) = 0$, cf. (30).

Definition 1.1. A function u is called *a solution of equation* (1) *on* $[a, \infty)$ if $u \in C^1[a, \infty)$, $pu' \in C^1[a, \infty)$ and u satisfies equation (1) for all $t \in [a, \infty)$. The solution u of equation (1) on $[a, \infty)$ is called *a solution of problem* (1), (2) or problem (1), (3) if u additionally satisfies condition (2) or (3), respectively.

Definition 1.2. A solution u of equation (1) on $[a, \infty)$ is called a *Kneser solution* if there exists $t_0 > a$ such that

$$u(t)u'(t) < 0 \text{ for } t \in [t_0, \infty). \tag{4}$$

The aim of the paper is twofold. First of all, we investigate the existence of the Kneser solutions to problems (1), (2) and (1), (3). Moreover, we shall describe the asymptotic properties of the Kneser solutions of (1) in the framework of regularly varying functions. Asymptotic formulas which we provide here are generalizations of those discussed in [30], where the case $p \equiv q$ was investigated. The existence of various types of solutions to (1) with $p \equiv q$ has been also studied in [24–26].
Other asymptotic results for related equations or systems which are characterized by regularly varying functions can be found in [5, 9, 10, 15–18, 22, 27, 28]. We also refer to [4, 12], where Kneser solutions for two-dimensional systems of ODEs were studied.

2. Regularly varying functions

In this section, we introduce regularly varying functions and show some of their basic properties which are necessary for the further analysis. See for example [20].

Definition 2.1. A function p, which is positive and measurable on $(0, \infty)$ is called *regularly varying of index* $\rho \in \mathbb{R}$ if for each $\lambda > 0$

$$\lim_{t\to\infty} \frac{p(\lambda t)}{p(t)} = \lambda^\rho.$$

The set of all regularly varying functions of index ρ is denoted by $RV(\rho)$.

Remark 2.2. A regularly varying function of index $\rho = 0$ is called a *slowly varying* function and the set of those functions is denoted by $\mathcal{SV}$. A slowly varying function may or may not be bounded, but as $t \to \infty$ it can neither grow too fast to infinity, nor decay too fast to zero in the sense, that it satisfies for any $\varepsilon > 0$

$$\lim_{t\to\infty} t^\varepsilon L(t) = \infty, \ \lim_{t\to\infty} t^{-\varepsilon} L(t) = 0.$$

Remark 2.3. Note that Definition 2.1 implies that a regularly varying function p of index ρ can be represented as

$$p(t) = t^\rho L(t), \ t \in [0, \infty), \tag{5}$$

where L is some slowly varying function.

Theorem 2.4. (Karamata integration theorem)
Let $L(t) \in \mathcal{SV}$, $c > 0$.

(i) *If $\alpha > -1$, then*

$$\int_c^t s^\alpha L(s) ds \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t) \ \text{as } t \to \infty,$$

(ii) *if $\alpha < -1$, then*

$$\int_t^\infty s^\alpha L(s) ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t) \ \text{as } t \to \infty,$$

(iii) *if $\alpha = -1$, then*

$$l(t) = \int_c^t \frac{L(s)}{s} ds \in \mathcal{SV} \ \text{and} \ \lim_{t\to\infty} \frac{L(t)}{l(t)} = 0,$$

where the symbol $\sim$ is used to denote the asymptotic equivalence,

$$f(t) \sim g(t) \ \text{as } t \to \infty \Leftrightarrow \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1.$$

In order to be able to investigate the asymptotic behaviour of non-oscillatory solutions of problem (1), (2) and problem (1), (3), we first need to provide auxiliary lemmas for regularly varying functions.

Lemma 2.5. *Let $\rho > 0$ and $p \in RV(\rho)$. Then*

$$\lim_{t\to\infty} p(t) \int_t^\infty \frac{\mathrm{d}s}{p(s)} = \infty. \tag{6}$$

Proof: According to Remark 2.3, the function p can be represented by $p(t) = t^\rho L(t), \ t \in [0, \infty)$, where $L \in \mathcal{SV}$. For $\rho > 1$, property (6) is a simple consequence of Theorem 2.4 (ii), where $-\rho < -1$. For $t \to \infty$, we have

$$\int_t^\infty \frac{\mathrm{d}s}{p(s)} = \int_t^\infty s^{-\rho} L^{-1}(s)\, \mathrm{d}s \sim \frac{1}{\rho-1} t^{-\rho+1} L^{-1}(t)$$

and therefore, the function

$$p(t) \int_t^\infty \frac{\mathrm{d}s}{p(s)} = t^\rho L(t) \int_t^\infty s^{-\rho} L^{-1}(s)\, \mathrm{d}s$$

is asymptotically equivalent to

$$\frac{1}{\rho-1} t L(t) L^{-1}(t) = \frac{t}{\rho-1}.$$

Thus, for $\rho > 1$, property (6) follows.

Let us consider $\rho \in (0,1]$, then

$$\int_t^\infty \frac{\mathrm{d}s}{p(s)} = \int_t^\infty s^{-\rho} L^{-1}(s)\,\mathrm{d}s = \int_t^\infty s^{-\rho-1} s L^{-1}(s)\,\mathrm{d}s \geq t \int_t^\infty s^{-\rho-1} L^{-1}(s)\,\mathrm{d}s,\ t \in [0,\infty).$$

According to Theorem 2.4 (ii), for $-\rho - 1 < -1$, this is asymptotically equivalent to

$$\frac{t^{1-\rho} L^{-1}(t)}{\rho}.$$

Therefore, as $t \to \infty$

$$p(t) \int_t^\infty \frac{\mathrm{d}s}{p(s)} = t^\rho L(t) \int_t^\infty s^{-\rho} L^{-1}(s)\,\mathrm{d}s \sim \frac{t}{\rho},$$

and (6) follows for any $\rho \in (0,1]$. □

Lemma 2.6. *Let us assume that functions p and q satisfy $p \in RV(\rho_p)$ and $q \in RV(\rho_q)$, where $\rho_p > 0$, $\rho_q > 0$, $\rho_q - \rho_p > -1$, and $c > 0$. Then*

$$\lim_{t\to\infty} \frac{1}{p(t)} \int_c^t q(s)\,\mathrm{d}s = \infty.$$

Proof: According to Remark 2.3, the functions p and q can be represented by

$$p(t) = t^{\rho_p} L_p(t), \quad q(t) = t^{\rho_q} L_q(t),\ t \in [0,\infty),$$

where $L_p, L_q \in SV$. Therefore,

$$\frac{1}{p(t)} \int_c^t q(s)\,\mathrm{d}s = t^{-\rho_p} L_p^{-1}(t) \int_c^t s^{\rho_q} L_q(s)\,\mathrm{d}s. \tag{7}$$

Due to Theorem 2.4 (i), the function given by (7) is asymptotically equivalent to the function

$$\frac{1}{\rho_q+1} t^{-\rho_p} L_p^{-1}(t) t^{\rho_q+1} L_q(t) = \frac{1}{\rho_q+1} t^{\rho_q-\rho_p+1} L(t),$$

where $L(t) = L_p^{-1}(t) L_q(t)$. Now, Remark 2.2 and the assumption $\rho_q - \rho_p > -1$ imply

$$\lim_{t\to\infty} \frac{1}{p(t)} \int_c^t q(s)\,\mathrm{d}s = \infty.$$

□

3. Existence of Kneser solutions to regular equation (1)

In this section, we investigate the existence of the Kneser solutions to regular problems (1), (2) and (1), (3) with $a > 0$.
We study problems (1), (2) and (1), (3) under the following assumptions:

$$L_0 < 0 < L,\ f \in C[L_0, L],\ f(L_0) = f(0) = f(L) = 0, \tag{8}$$

$$p \in C[a,\infty), p > 0 \text{ on } [a,\infty), \tag{9}$$

$$q \in C[a,\infty),\ q > 0 \text{ on } (a,\infty). \tag{10}$$

The existence result is shown using the Diagonalization lemma.

Lemma 3.1. (Diagonalization lemma)
Let $u_n \in C^1[a,n]$, $n \in \mathbb{N}$, $n > a$ be such that for each $b > a$ there exists $\rho_b > 0$ satisfying

$$|u_n^{(j)}(t)| \le \rho_b \quad \textit{for } t \in [a,b], n \ge b, \; j = 0,1,$$

and

$$\{u_n'\}_{n \ge b} \textit{ is equicontinuous on } [a,b].$$

Then, there exists a subsequence $\{u_{k_n}\} \subset \{u_n\}$ and $u \in C^1[a,\infty)$ such that

$$\lim_{n\to\infty} u_{k_n}^{(j)}(t) = u^{(j)}(t) \textit{ locally uniformly on } [a,\infty), j = 0,1.$$

Proof: Let $\{b_n\} \in \mathbb{N}$ be increasing, $b_n > a$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = \infty$. Then

(i) for $b_1 \in \mathbb{N}$ we have $|u_n^{(j)}(t)| < \rho_{b_1}$ for $t \in [a,b_1], n \ge b_1, j = 0,1$, and, in addition, $\{u_n'\}_{n\ge b_1}$ is equicontinuous on $[a,b_1]$. Hence, by the Arzelà-Ascoli theorem, there is a subsequence $\{u_{k_{1,n}}\} \subset \{u_n\}_{n\ge b_1}$ for which $\{u_{k_{1,n}}^{(j)}(t)\}$ is uniformly convergent on $[a,b_1]$ for $j = 0,1$.

(ii) Next, there exists a subsequence $\{u_{k_{2,n}}\} \subset \{u_{k_{1,n}}\}$ such that $\{u_{k_{2,n}}^{(j)}\}$ is uniformly convergent on $[a,b_2]$ for $j = 0,1$.

(n) We can proceed inductively to obtain a subsequence $\{u_{k_{i,n}}\} \subset \{u_{k_{i-1,n}}\}$ such that $\{u_{k_{i,n}}^{(j)}\}$ is uniformly convergent on $[a,b_i]$ for $j = 0,1$.

Let $k_n := k_{n,n}$ for $n \in \mathbb{N}$ and consider the diagonal sequence $\{u_{k_n}\}$. Choose $\beta > a$. Then $[a,\beta] \subset [a,b_m]$ for some $m \in \mathbb{N}$. Since $\{u_{k_n}\}_{n\ge m}$ is taken from $\{u_{k_{m,n}}\}$ and we know that $\{u_{k_{m,n}}^{(j)}\}$ is uniformly convergent on $[a,b_m]$ for $j = 0,1$, we can see that $\{u_{k_n}^{(j)}\}$ is uniformly convergent on $[a,\beta]$ for $j = 0,1$. Consequently, $\{u_{k_n}^{(j)}\}_{n\ge m}$ is locally uniformly convergent on $[a,\infty)$. Let $\lim_{n\to\infty} u_{k_n}(t) = u(t)$ and $\lim_{n\to\infty} u_{k_n}'(t) = v(t)$ for $t \in [a,\infty)$. Then $u, v \in C[a,\infty)$ and letting $n \to \infty$ in

$$u_{k_n}(t) = u_{k_n}(a) + \int_a^t u_{k_n}'(s)ds, \; t \in [a,n], n \in \mathbb{N},$$

yields

$$u(t) = u(a) + \int_a^t v(s)ds, \; t \in [a,\infty).$$

Hence $u \in C^1[a,\infty)$ and $v = u'$ on $[a,\infty)$ and the result follows.

$\square$

For the existence result stated in Theorem 3.2, the positivity of the initial point a is crucial, since for $a > 0$, equation (1) is regular on $[a,\infty)$, and the solvability of (1), (2) and (1), (3) can be shown using the following standard arguments.

We first define an auxiliary function f^* by

$$f^*(x) = \begin{cases} \frac{L-x}{x-L+1}, & x > L, \\ f(x), & x \in [0,L], \\ \frac{x}{x-1}, & x < 0, \end{cases} \tag{11}$$

and consider the auxiliary equation

$$(p(t)u'(t))' + q(t)f^*(u(t)) = 0. \tag{12}$$

Theorem 3.2. *Let assumptions* (8)–(10) *be satisfied and $a > 0$. Then problem* (1), (2) *(and problem* (1), (3)*) has at least one solution.*

Proof:

Step 1. *Showing solvability of problem* (12), (13):

Let $n > a$, $u_0 \in (0, L)$ and let us assume the Dirichlet boundary conditions

$$u(a) = u_0,\ u(n) = 0 \tag{13}$$

to hold. We first prove the existence of a solution to problem (12), (13). The linear homogeneous problem

$$(p(t)u'(t))' = 0,\ u(a) = 0, u(n) = 0, \tag{14}$$

has only the trivial solution $u \equiv 0$. If we assume the existence of a nontrivial solution, we obtain the following contradiction. Let u be a nontrivial solution of (14). Then there exists $\theta \in (a, n)$ such that $u(\theta) \neq 0, u'(\theta) = 0$. Integrating (14) from θ to $t \in [a, n]$, we obtain

$$p(t)u'(t) = p(\theta)u'(\theta) = 0,\ t \in [a, n].$$

Since p is positive on $[a, \infty)$, $u' = 0$ on $[a, n]$ follows and hence, u has to be a constant function on $[a, n]$. Therefore, since $u(a) = 0$, $u \equiv 0$ is the only solution to (14). Consequently, there exists the unique Green's function $G(t, s)$ to problem (14) of the form

$$G(t, s) = \begin{cases} \left(1 - \dfrac{P(s)}{P(n)}\right) P(t) & \text{for} \quad a \le t \le s \le n, \\ \left(1 - \dfrac{P(t)}{P(n)}\right) P(s) & \text{for} \quad a \le s \le t \le n, \end{cases} \tag{15}$$

where $P(t) = \int_a^t \frac{\mathrm{d}\tau}{p(\tau)}, t \in [a, n]$. The Green's function (15) is bounded by

$$|G(t, s)| \le P(n) \text{ for } t, s \in [a, n].$$

Furthermore, the partial derivative of (15) has the form

$$\frac{\partial G(t, s)}{\partial t} = \begin{cases} \left(1 - \dfrac{P(s)}{P(n)}\right) \dfrac{1}{p(t)} & \text{for} \quad a \le t < s \le n, \\ -\dfrac{P(s)}{P(n)} \dfrac{1}{p(t)} & \text{for} \quad a \le s < t \le n, \end{cases}$$

and it is also bounded,

$$\left| \frac{\partial G(t, s)}{\partial t} \right| \le \frac{1}{p_{min}},\ t \in [a, s) \cup (s, n], s \in [a, n],$$

where $p_{min} = \min\{p(t) : t \in [a, n]\} > 0$.

Then, the unique solution of the nonhomogeneous linear problem

$$(p(t)u'(t))' = y(t),\ u(a) = u_0, u(n) = 0,$$

has the form

$$u(t) = \frac{u_0}{P(n)} \int_t^n \frac{\mathrm{d}\tau}{p(\tau)} - \int_a^n G(t, s)y(s)\,\mathrm{d}s,\ t \in [a, n].$$

Let us define the operator $T : C[a, n] \to C[a, n]$,

$$(Tu)(t) = \frac{u_0}{P(n)} \int_t^n \frac{\mathrm{d}\tau}{p(\tau)} + \int_a^n G(t, s)q(s)f^*(u(s))\,\mathrm{d}s,\ t \in [a, n].$$

Let u be a fixed point of T. Then

$$
\begin{aligned}
u(t) &= \frac{u_0}{P(n)}\int_t^n \frac{\mathrm{d}\tau}{p(\tau)} + \int_a^t \left(1-\frac{P(t)}{P(n)}\right) P(s)q(s)f^*(u(s))\,\mathrm{d}s \\
&\quad + \int_t^n \left(1-\frac{P(s)}{P(n)}\right) P(t)q(s)f^*(u(s))\,\mathrm{d}s, \\
u'(t) &= -\frac{u_0}{P(n)p(t)} - \frac{1}{p(t)}\int_a^t \frac{1}{P(n)} P(s)q(s)f^*(u(s))\,\mathrm{d}s \\
&\quad + \frac{1}{p(t)}\int_t^n \left(1-\frac{P(s)}{P(n)}\right) q(s)f^*(u(s))\,\mathrm{d}s, \\
(p(t)u'(t))' &= -\frac{1}{P(n)} P(t)q(t)f^*(u(t)) - \left(1-\frac{P(t)}{P(n)}\right) q(t)f^*(u(t)) = -q(t)f^*(u(t)),\ t\in[a,n].
\end{aligned}
$$

Therefore, $u\in C^1[a,n]$, $pu'\in C^1[a,n]$ and u is a solution of equation (12). Moreover, since $P(a)=0$, we conclude

$$
\begin{aligned}
u(a) &= \frac{u_0}{P(n)}P(n) + \int_a^n \left(1-\frac{P(s)}{P(n)}\right) P(a)q(s)f^*(u(s))\,\mathrm{d}s = u_0, \\
u(n) &= \int_a^n \left(1-\frac{P(n)}{P(n)}\right) P(s)q(s)f^*(u(s))\,\mathrm{d}s = 0,
\end{aligned}
$$

and so, conditions (13) are satisfied.
In order to show the existence of a fixed point of the operator T, we use the Schauder fixed point theorem. Let $\Omega\subset C[a,b]$,

$$
\Omega = \left\{x\in C[a,n] : \|x\|_{C[a,n]} \le \rho\right\}, \tag{16}
$$

where

$$
\begin{aligned}
\rho &= |u_0| + P(n)MQ, \\
M &= \sup\{|f^*(x)| : x\in\mathbb{R}\}, \\
Q &= \int_a^n q(s)\,\mathrm{d}s.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\|Tu\|_{C[a,n]} &= \max_{t\in[a,n]}\left\{\frac{u_0}{P(n)}\int_t^n \frac{\mathrm{d}\tau}{p(\tau)} + \int_a^n G(t,s)q(s)f^*(u(s))\,\mathrm{d}s\right\} \\
&\le \frac{|u_0|}{P(n)}P(n) + P(n)M\int_a^n q(s)\,\mathrm{d}s = \rho.
\end{aligned}
$$

Consequently, $T(\Omega)$ is bounded in $C[a,n]$. Due to (16), $T(\Omega)\subset\Omega$. Since f^* is a continuous function, the inequality

$$
\begin{aligned}
\|Tu_m - Tu\|_{C[a,n]} &\le \max_{t\in[a,b]}\{\int_a^n |G(t,s)|q(s)|f^*(u_m(s)) - f^*(u(s))|\,\mathrm{d}s\} \\
&\le P(n)\|f^*(u_m) - f^*(u)\|Q \le P(n)Q\varepsilon, \{u_m\}\subset\Omega, u\in\Omega,
\end{aligned}
$$

yields the continuity of T on Ω. Moreover, for $u \in \Omega$, $t_1, t_2 \in [a,n]$, there exists ξ between t_1 and t_2 such that

$$\begin{aligned}
|(Tu)(t_1) - (Tu)(t_2)| &\leq \left| \frac{u_0}{P(n)} \int_{t_1}^{t_2} \frac{\mathrm{d}\tau}{p(\tau)} \right| + \int_a^n |G(t_1,s) - G(t_2,s)| q(s) |f^*(u(s))| \,\mathrm{d}s \\
&\leq \frac{|u_0|}{P(n)} \frac{|t_1 - t_2|}{p_{min}} + \int_a^n \frac{|\partial G(\xi,s)|}{\partial t} |t_1 - t_2| q(s) |f^*(u(s))| \,\mathrm{d}s \\
&\leq |t_1 - t_2| \left(\frac{|u_0|}{P(n)} \frac{1}{p_{min}} + M \int_a^n \frac{q(s)}{p_{min}} \,\mathrm{d}s \right) \\
&\leq |t_1 - t_2| \left(\frac{|u_0|}{P(n)} \frac{1}{p_{min}} + \frac{MQ}{p_{min}} \right).
\end{aligned}$$

This implies the compactness of T on Ω, due to the Arzelà-Ascoli theorem. Since the operator T is continuous and compact on Ω and $T(\Omega) \subset \Omega$, there exists a fixed point $u = Tu$ according to the Schauder fixed point theorem.

Step 2. *Showing solvability of problem* (1), (13):

Let u be a solution of problem (12), (13). We will prove that

$$0 \leq u(t) \leq L \text{ for } t \in [a,n]. \tag{17}$$

On a contrary, let us assume that

$$u(t_0) = \max\{u(t) : t \in [a,n]\} > L. \tag{18}$$

Since $u(a) = u_0 \in (0, L)$ and $u(n) = 0$, it follows that $t_0 \in (a,n)$ and $u'(t_0) = 0$. Therefore, we can find $\delta > 0$ such that $u(t) > L$ on $(t_0, t_0 + \delta) \subset (a,n)$ and, by (11),

$$(p(t)u'(t))' = -q(t) \frac{L - u(t)}{u(t) - L + 1} > 0, \; t \in (t_0, t_0 + \delta) \tag{19}$$

follows. Integrating (19) over $(t_0, t), t \in (t_0, t_0 + \delta)$, we obtain

$$0 < -\int_{t_0}^t q(s) \frac{L - u(s)}{u(s) - L + 1} \,\mathrm{d}s = p(t)u'(t).$$

Thus, $u' > 0$ on $(t_0, t_0 + \delta)$, which contradicts (18).

Analogously, the contradiction follows when we assume

$$\min\{u(t) : t \in [a,n]\} < 0$$

to holds. Finally, it follows from (11) and (17) that u is a solution of equation (1) on $[a,n]$.

Step 3. *Showing solvability of problem* (1), (2):

It follows from Step 2 that for each $n \in \mathbb{N}$, $n \geq a$ we have a solution u_n of equation (1) on $[a,n]$. This solution satisfies

$$u_n(a) = u_0, \; 0 \leq u_n(t) \leq L \text{ for } t \in [a,n]. \tag{20}$$

We now show that there exists a subsequence $\{u_\nu\} \subset \{u_n\}$ which locally uniformly converges on $[a,\infty)$ to a solution u of problem (1), (2). To this aim we consider an arbitrary compact interval $[a,b] \subset [a,\infty)$. Then, the following holds

$$0 \leq u_n(t) \leq L, \; t \in [a,b], \; n > b.$$

Consequently, there exists $\tau_n \in [a,b]$ such that $|u_n'(\tau_n)| \leq \frac{L}{b-a}$.

Let us estimate the first derivative of the solution u_n on $[a,b]$. Integrating equation (1) from $t \in [a,b]$ to τ_n we obtain

$$\begin{aligned} p(t)u_n'(t) &= p(\tau_n)u_n'(\tau_n) + \int_t^{\tau_n} q(s)f(u_n(s))\,\mathrm{d}s, \\ u_n'(t) &= \frac{p(\tau_n)}{p(t)}u_n'(\tau_n) + \frac{1}{p(t)}\int_t^{\tau_n} q(s)f(u_n(s))\,\mathrm{d}s, \\ |u_n'(t)| &\le \frac{p_{max}}{p_{min}}\frac{L}{b-a} + \frac{1}{p_{min}}q_{max}f_{max}(b-a) =: \rho_b,\ t \in [a,b], \end{aligned} \tag{21}$$

where

$$\begin{aligned} p_{max} &= \max\{p(t) : t \in [a,b]\}, \\ p_{min} &= \min\{p(t) : t \in [a,b]\}, \\ q_{max} &= \max\{q(t) : t \in [a,b]\}, \\ f_{max} &= \max\{|f(x)| : 0 \le x \le L\}. \end{aligned}$$

According to (21), the sequence $\{u_n\}$ is equicontinuous on $[a,b]$. Equation (1) yields

$$|(p(t)u_n'(t))'| \quad \le \quad |q(t)f(u_n(t))| \le q_{max}f_{max},\ t \in [a,b],$$

and hence the sequence $\{pu_n'\}$ is equicontinuous on $[a,b]$. Since $p_{min} > 0$, we have by (21) for $t_1, t_2 \in [a,b]$

$$|u_n'(t_1) - u_n'(t_2)| \le \frac{1}{p_{min}}\left(|p(t_1)u_n'(t_1) - p(t_2)u_n'(t_2)| + \rho_b|p(t_2) - p(t_1)|\right).$$

This implies that the sequence $\{u_n'\}$ is also equicontinuous on $[a,b]$. From the Arzelà-Ascoli theorem and the Diagonalization lemma 3.1 it follows, that there exists a subsequence $u_m \rightrightarrows^{loc} u$, $u_m' \rightrightarrows^{loc} u'$ on $[a,\infty)$ and u is a solution of equation (1) on $[a,\infty)$. By (20), u satisfies (2).

For problem (1), (3) we consider $u_0 \in (L_0, 0)$ and use the dual argument.

□

Imposing some additional assumptions on f, p, and q, enables to derive two different limits of solutions to problem (1), (2) and (1), (3).

Theorem 3.3. *Let* (8)–(10) *hold and* $a > 0$. *Moreover, we assume that*

$$f \in \mathrm{Lip}_{loc}(0,L], \quad f(x) > 0 \text{ for } x \in (0,L). \tag{22}$$

Then problem (1), (2) *has a solution* u, *such that*

$$0 < u(t) < L \text{ for } t \in [a,\infty). \tag{23}$$

If in addition

$$\lim_{t\to\infty} \frac{1}{p(t)}\int_a^t q(s)\,\mathrm{d}s = \infty, \tag{24}$$

with

$$\liminf_{t\to\infty} p(t) > 0, \tag{25}$$

then either

$$u'(t) > 0 \text{ for } t \ge a \text{ and } \lim_{t\to\infty} u(t) = L, \tag{26}$$

or

$$u \text{ is a Kneser solution.} \tag{27}$$

Proof: According to Theorem 3.2, problem (1), (2) has a solution u. Let us assume that $u(b) = L$ for some $b > a$. Due to (2), $u'(b) = 0$. By virtue of the first condition in (22), $u \equiv L$ is the only solution satisfying $u(b) = L, u'(b) = 0$. Therefore, $u(t) < L$ for $t \in [a, \infty)$. Assume that $u(c) = 0$ for some $c > a$. Due to (2), $u'(c) = 0$. Integrating equation (1) over $(c, t), t \in [a, \infty)$, and using the second condition in (22), we conclude $u'(t) \le 0$ for $t > c$ and $u'(t) \ge 0$ for $t < c$. This yields $u \equiv 0$ which contradicts $u(a) = u_0 > 0$. Therefore (23) holds. By (1), (2), (10), and (22)

$$(pu')'(t) = -q(t)f(u(t)) < 0, \ t \ge a, \tag{28}$$

and thus, pu' is decreasing on $[a, \infty)$.

(i) Assume that $u' > 0$ on $[a, \infty)$. Then there exists $\lim_{t\to\infty} u(t) =: l_0 \in (u_0, L]$. Let $l_0 \in (u_0, L)$ and let us denote $m_0 := \min\{f(x) : x \in [u_0, l_0]\} > 0$. Integration of (1) over $[a, t]$ yields

$$p(t)u'(t) - p(a)u'(a) \le -m_0 \int_a^t q(s)\,\mathrm{d}s, \ t \in [a, \infty),$$

$$0 < u'(t) \le \frac{1}{p(t)}\left(p(a)u'(a)\right) - m_0 \frac{1}{p(t)} \int_a^t q(s)\,\mathrm{d}s, \ t \in [a, \infty).$$

Letting $t \to \infty$ and using (24), (25), we arrive at $0 \le \liminf_{t\to\infty} u'(t) \le -\infty$, which is a contradiction. Therefore $l_0 = L$ and (26) holds.

(ii) Assume that there exists $b \ge a$ such that $u'(b) \le 0$. Then $(pu')(b) \le 0$ and having in mind that pu' is decreasing, we can find $t_0 > b$ such that $pu' < 0$ on $[t_0, \infty)$. By (9), $u' < 0$ on $[t_0, \infty)$ and (27) follows.

□

The dual theorem formulated below holds for problem (1), (3) and can be shown using arguments from the proof of Theorem 3.3.

Theorem 3.4. *Let (8)–(10) hold and $a > 0$. Moreover, we assume that*

$$f \in \mathrm{Lip}_{loc}[L_0, 0), \quad f(x) < 0 \text{ for } x \in (L_0, 0). \tag{29}$$

Then problem (1), (3) has a solution u, such that

$$L_0 < u(t) < 0 \text{ for } t \in [a, \infty).$$

If in addition (24) and (25) hold, then either

$$u'(t) < 0 \text{ for } t \ge a \text{ and } \lim_{t\to\infty} u(t) = L_0,$$

or

$$u \text{ is a Kneser solution.}$$

4. Existence of Kneser solutions to singular equation (1)

In this section we discuss the existence of the Kneser solutions to problems (1), (2) and (1), (3) for the singular case $a = 0$ under the assumptions (8) and

$$p \in C[0, \infty), p > 0 \text{ on } (0, \infty), \ p(0) = 0, \tag{30}$$

$$q \in C[0, \infty), \ q > 0 \text{ on } (0, \infty). \tag{31}$$

Let us stress that for $a = 0$, equation (1) is singular, because $p(0) = 0$, cf. (30), and the results obtained in the previous section cannot be easily extended to the singular case. Note also, that for the case $p(t) = q(t) = t^\alpha$, $\alpha \in (0, 1]$, $\int_1^\infty \frac{\mathrm{d}s}{p(s)} = \infty$ follows, and thus, problems (1), (2) and (1), (3) have no Kneser solutions, see [29], Theorem 4.6. Therefore, α should be chosen greater than 1. However, then $\int_0^1 \frac{\mathrm{d}s}{p(s)} = \infty$ and the functions P and G in (15) are not defined at $t = a = 0$. Hence, the approach leading for $a > 0$ to the existence results in Theorems 3.2–3.4 cannot be used for $a = 0$.

The special case of problems (1), (2) and (1), (3) with $a = 0$ and $p \equiv q$, was studied in [30]. The existence of the Kneser solutions can be found in [30] in Theorems 3.4 and 3.5, which are proved by arguments different from those used here in the proofs of Theorems 3.2–3.4. The two following results are corollaries to Theorems 3.4 and 3.5 from [30].

Theorem 4.1. *Let us assume that* (8)*,* (22)*,* (30) *and the following assumptions:*

$$p \equiv q \tag{32}$$

$$p \in C^1(0,\infty), p' > 0 \text{ on } (0,\infty), \ \lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0. \tag{33}$$

$$\frac{p'(t)P(t)}{p^2(t)} \geq c, \ t \in (0,\infty), \tag{34}$$

$$\frac{xf(x)}{F(x)} \geq \frac{2}{2c-1}, \ x \in (0, A_0], \tag{35}$$

hold for some $c > \frac{1}{2}$ *and* $A_0 \in (0, L)$*, where* $P(t) = \int_0^t p(s)\,\mathrm{d}s$*,* $F(x) = \int_0^x f(z)\,\mathrm{d}z$*. Then for each* $u_0 \in (0, A_0]$ *there exists a unique Kneser solution* u *to problem* (1)*,* (2) *with* $a = 0$*. This solution has the following properties:*

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0, \quad u'(0) = 0, \quad u'(t) < 0, \quad t \in (0,\infty).$$

A dual assertion for an initial condition u_0 from a negative neighbourhood of zero is given in the following theorem.

Theorem 4.2. *Assume* (8)*,* (29)*,* (30)*,* (32)*, and* (33) *to hold. Let condition* (34) *hold with a constant* $c > \frac{1}{2}$ *and assume that there exists* $B_0 \in (L_0, 0)$ *such that the inequality*

$$\frac{xf(x)}{F(x)} \geq \frac{2}{2c-1}, \ x \in [B_0, 0) \tag{36}$$

is satisfied. Then for each $u_0 \in [B_0, 0)$*, there exists a unique Kneser solution* u *to problem* (1)*,* (3) *with* $a = 0$*. This solution has the following properties:*

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0, \quad u'(0) = 0, \quad u'(t) > 0, \quad t \in (0,\infty).$$

To our knowledge, the existence of Kneser solutions for the case $a = 0$ and $p \neq q$ under assumptions (8), (30), and (31) remains an open problem.

Remark 4.3. Let us note, that the condition $u'(0) = 0$ is necessary for the smoothness of the solution in the case where $p \equiv q$. To see this, let us consider a solution u of (1). Since $u \in C^1[0,\infty)$, the assumption $p(0) = 0$ yields $p(0)u'(0) = 0$. Therefore, there exist $M > 0$ and $\delta > 0$ such that $|f(u(t))| \leq M$ for $r \in (0,\delta)$. We now integrate (1) and use (33) to obtain

$$|u'(t)| = \left| \frac{1}{p(t)} \int_0^t p(s) f(u(s))\,\mathrm{d}s \right| \leq \frac{M}{p(t)} \int_0^t p(s)\,\mathrm{d}s \leq Mt, \quad t \in (0,\delta).$$

Consequently, $u'(0) = 0$ holds.

Remark 4.4. In Theorems 3.4 [30] additional restrictions for f were made. Those assumptions are related to the interval $\mathbb{R} \setminus (0, L)$. Since the Kneser solution obtained in Theorem 4.1 satisfies $u(t) \in (0, L)$ for $t \in [0,\infty)$, we can omit those additional assumptions and still refer to Theorem 3.4 [30]. Similar remark can be made on context of Theorem 4.2.

5. Asymptotic properties of Kneser solutions

In this section we focus on properties of Kneser solutions of problems (1), (2) and (1), (3) in the neighbourhood of infinity. We derive asymptotic formulas for such solutions and for their first derivatives. These results about asymptotic properites can be applied to Kneser solutions of the regular differential equation (1) with $a > 0$, as well as to Kneser solutions of singular equation (1) with $a = 0$, provided that $u(a) \in [L_0, L]$.

In the whole section we assume that function f satisfies condition (8) and the second conditions in (22) and (29). Let us recapitulate these assumptions,

$$L_0 < 0 < L,\ f \in C[L_0, L],\ f(L_0) = f(0) = f(L) = 0,\ xf(x) > 0 \text{ for } x \in (L_0, 0) \cup (0, L). \tag{37}$$

First asymptotic properties of the Kneser solutions to problem (1) are formulated in the following lemma.

Theorem 5.1. *Assume that* (37) *holds and $a \geq 0$. Moreover, assume that $p \in RV(\rho_p) \cap C[a,\infty)$, $q \in RV(\rho_q) \cap C[a,\infty)$, $\rho_p > 0$, $\rho_q > 0$, $\rho_q - \rho_p > -1$. Let u be a Kneser solution of equation* (1)*. Then*

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0. \tag{38}$$

Proof: Let u be a Kneser solution of problem (1), (2). By (4), there exists $t_0 > a$ such that

$$0 < u(t) < L,\ u'(t) < 0,\ t \in [t_0, \infty). \tag{39}$$

Hence, there exists

$$\lim_{t\to\infty} u(t) =: l_1 \in [0, u(t_0)).$$

Assume that $l_1 \in (0, u(t_0))$ and let us denote $m_1 := \min\{f(x) : x \in [l_1, u(t_0)]\} > 0$. The integration of (1) over $[t_0, t]$ gives

$$p(t)u'(t) - p(t_0)u'(t_0) \leq -m_1 \int_{t_0}^{t} q(s)\,\mathrm{d}s,\ t \in [t_0, \infty),$$

$$0 < u'(t) \leq \frac{p(t_0)u'(t_0)}{p(t)} - m_1 \frac{t_0}{p(t)} \int_{t_0}^{t} q(s)\,\mathrm{d}s,\ t \in [t_0, \infty).$$

Since p satisfies (5) with $\rho = \rho_p > 0$, we see that $\liminf_{t\to\infty} p(t) > 0$. Consequently, it follows from Lemma 2.6 $0 \leq \lim_{t\to\infty} u'(t) \leq -\infty$, which is a contradiction. Therefore $l_1 = 0$, i.e.,

$$\lim_{t\to\infty} u(t) = 0. \tag{40}$$

The second condition in (39) implies $\liminf_{t\to\infty} u'(t) \leq 0$. Assume that $\liminf_{t\to\infty} u'(t) < 0$. Then there exist a sequence $\{t_n\} \subset (a, \infty)$ and $\varepsilon > 0$ such that

$$\lim_{n\to\infty} t_n = \infty,\ \lim_{n\to\infty} u'(t_n) = -\varepsilon.$$

By (1), (2), (10), and (37) function $(pu')'(t) < 0$ for $t \geq a$. Therefore, the inequalities

$$u'(t_n) \leq -\frac{\varepsilon}{2},\ p(t)u'(t) < p(t_n)u'(t_n),\ t > t_n,$$

hold for each sufficiently large $n \geq n_0$. Then $u'(t) < -\frac{\varepsilon}{2} p(t_n) \frac{1}{p(t)},\ t > t_n, n \geq n_0$. Integrating this inequality we obtain

$$u(t) - u(t_n) \leq -\frac{\varepsilon}{2} p(t_n) \int_{t_n}^{t} \frac{1}{p(s)}\,\mathrm{d}s,\ t > t_n, n \geq n_0.$$

Let $t \to \infty$. Then, according to (40),

$$-u(t_n) \leq -\frac{\varepsilon}{2} p(t_n) \int_{t_n}^{\infty} \frac{1}{p(s)}\,\mathrm{d}s,\ n \geq n_0.$$

Now, letting $n \to \infty$ and using Lemma 2.5 we obtain

$$0 = -\lim_{n\to\infty} u(t_n) \leq -\frac{\varepsilon}{2} \lim_{n\to\infty} p(t_n) \int_{t_n}^{\infty} \frac{\mathrm{d}s}{p(s)} = -\frac{\varepsilon}{2} \lim_{t\to\infty} p(t) \int_{t}^{\infty} \frac{\mathrm{d}s}{p(s)} = -\infty.$$

This is again a contradiction. Therefore, $\liminf_{t\to\infty} u'(t) = 0$. By virtue of (39), $\limsup_{t\to\infty} u'(t) \leq 0$, and consequently,

$$\lim_{t\to\infty} u'(t) = 0.$$

The proof for a Kneser solution of problem (1), (3) is analogous.

□

We are now in the position to describe the asymptotic behaviour of the Kneser solutions in a more precise way.

Theorem 5.2. *Assume that* (37) *holds and* $a \geq 0$. *Moreover, let us assume that* $p \in RV(\rho_p) \cap C[a,\infty)$, $\rho_p > 0$, $q \in RV(\rho_q) \cap C[a,\infty)$, $\rho_q > 0$, $\rho_q - \rho_p > -1$, *and*

$$\exists r > 1 : \liminf_{x\to 0} \frac{|f(x)|}{|x|^r} > 0, \ \limsup_{x\to 0} \frac{|f(x)|}{|x|^r} < \infty. \tag{41}$$

Let u *be a Kneser solution of problem* (1), (2) *or* (1), (3). *Then, for any* $\varepsilon > 0$

$$\lim_{t\to\infty} t^{\frac{\rho_q-\rho_p+2}{r-1} - \varepsilon}|u(t)| = 0. \tag{42}$$

Proof: Let u be a Kneser solution of problem (1), (2) or (1), (3). Consider $t_0 > a$ from (4). According to (41) and (38), there exist $\alpha, \beta, \delta > 0$ and $t_1 \geq t_0$ such that

$$\alpha < \frac{|f(x)|}{|x|^r} < \beta, \ x \in (0,\delta) \ \text{ and } 0 < |u(t)| < \delta, \ t \geq t_1.$$

Hence, we have

$$\alpha|u(t)|^r < |f(u(t))| < \beta|u(t)|^r, \ t \geq t_1. \tag{43}$$

We now integrate equation (1) from t_1 to $t \geq t_1$ and obtain

$$p(t)u'(t) - p(t_1)u'(t_1) + \int_{t_1}^t q(s)f(u(s))\,\mathrm{d}s = 0.$$

Since $u(t)u'(t) < 0$ and $u(t)$ is monotone for $t > t_1$,

$$p(t)|u'(t)| > \int_{t_1}^t q(s)|f(u(s))|\,\mathrm{d}s > \alpha|u(t)|^r \int_{t_1}^t q(s)\,\mathrm{d}s$$

follows. Therefore,

$$\frac{|u'(t)|}{\alpha|u(t)|^r} > \frac{1}{p(t)}\int_{t_1}^t q(s)\,\mathrm{d}s, \ t > t_1.$$

Let L_p and L_q be slowly varying functions such that $p(t) = t^{\rho_p}L_p(t)$ and $q(t) = t^{\rho_q}L_q(t)$, respectively. Functions L_p, L_q always exist due to Remark 2.3. According to Theorem 2.4 (i), there exists a sufficiently large $b \geq t_1$ such that

$$\frac{1}{p(t)}\int_{t_1}^t q(s)\,\mathrm{d}s = \frac{\int_{t_1}^t s^{\rho_q}L_q(s)\,\mathrm{d}s}{t^{\rho_p}L_p(t)} \geq \frac{1}{2(\rho_q+1)}t^{\rho_q-\rho_p+1}\frac{L_q(t)}{L_p(t)}, \ t > b.$$

Therefore,

$$\frac{|u'(t)|}{\alpha|u(t)|} > c_1 t^{\rho_q-\rho_p+1}L(t), \ t > b,$$

where $c_1 = \frac{1}{2(\rho_q+1)}$ and $L(t) = \frac{L_q(t)}{L_p(t)}$. By Theorem 2.4 (i), there exists a sufficiently large $T \geq b$ such that

$$\begin{aligned} \frac{1}{\alpha(r-1)}\left(\frac{1}{|u(t)|^{r-1}} - \frac{1}{|u(T)|^{r-1}}\right) \quad &> \quad c_1\int_T^t s^{\rho_q-\rho_p+1}L(s)\,\mathrm{d}s \\ &\geq \quad \frac{c_1}{2(\rho_q-\rho_p+2)}t^{\rho_q-\rho_p+2}L(t) \end{aligned}$$

holds for $t > T$. Consequently,

$$0 < |u(t)| < \left(\alpha(r-1)c_2 t^{\rho_q-\rho_p+2}L(t)\right)^{-\frac{1}{r-1}}, \ t > T,$$

where $c_2 = \frac{c_1}{2(\rho_q-\rho_p+2)}$. Let us denote $L_2(t) := (L(t))^{-\frac{1}{r-1}}$ and $c_3 := \alpha(r-1)c_2$, then

$$0 < t^{\frac{\rho_q-\rho_p+2}{r-1}}|u(t)| < c_3 L_2(t), \ t > T. \tag{44}$$

Finally, we choose an $\varepsilon > 0$ and multiply inequality (44) by $t^{-\varepsilon}$. Remark 2.2 yields

$$\lim_{t\to\infty} t^{\frac{\rho_q-\rho_p+2}{r-1} - \varepsilon}|u(t)| = 0.$$

$\square$

Once the asymptotic formula for the Kneser solutions is derived, we focus our attention on their first derivatives.

Theorem 5.3. *Let all assumptions of Theorem 5.2 be satisfied. Then for any $\varepsilon > 0$*

I. *If $\rho_q > r\rho_p - r - 1$, then*

$$\lim_{t\to\infty} t^{\rho_p-\varepsilon}|u'(t)| = 0. \tag{45}$$

II. *If $\rho_q \le r\rho_p - r - 1$, then*

$$\lim_{t\to\infty} t^{\frac{\rho_q-\rho_p+r+1}{r-1}-\varepsilon}|u'(t)| = 0. \tag{46}$$

Proof: Let u be a Kneser solution of problem (1), (2) or (1), (3) and let $a \le t_0 \le t_1$ be the points from the proof of Theorem 5.2. Then $uu' < 0$ on $[t_1, \infty)$ and (43) holds. Let us choose $\varepsilon_1 > 0$. Due to (42), for each $c > 0$ there exists $T_1 \ge t_1$ such that

$$0 < t^{\frac{\rho_q-\rho_p+2}{r-1}-\varepsilon_1}|u(t)| < c,\ t > T_1. \tag{47}$$

We first integrate equation (1) over (T_1, t) and set $A_1 := p(T_1)|u'(T_1)|$. Then, by (41),

$$0 < p(t)|u'(t)| \;=\; A_1 + \int_{T_1}^{t} q(s)|f(u(s))|\,\mathrm{d}s < A_1 + \beta\int_{T_1}^{t} q(s)|u(s)|^r\,\mathrm{d}s,\ t > T_1.$$

Let L_p and L_q be slowly varying functions such that $p(t) = t^{\rho_p}L_p(t)$ and $q(t) = t^{\rho_q}L_q(t)$, respectively. This implies

$$0 < t^{\rho_p}L_p(t)|u'(t)| < A_1 + \beta\int_{T_1}^{t} s^{\rho_q}L_q(s)|u(s)|^r\,\mathrm{d}s,\ t > T_1.$$

Due to (47),

$$\begin{aligned} 0 < t^{\rho_p}L_p(t)|u'(t)| &< A_1 + \beta\int_{T_1}^{t} s^{\rho_q-r\mu+r\varepsilon_1}L_q(s)|s^{\mu-\varepsilon_1}u(s)|^r\,\mathrm{d}s \\ &< A_1 + \beta c^r\int_{T_1}^{t} s^{\rho_q-r\mu+\varepsilon_2}L_q(s)\,\mathrm{d}s, \end{aligned}$$

where $\mu = \frac{\rho_q-\rho_p+2}{r-1}, \varepsilon_2 = r\varepsilon_1 > 0$.

I. Let $\rho_q > r\rho_p - r - 1$. Then

$$\begin{aligned} \rho_q - r\mu &= \frac{(r\rho_p - r - 1) + 1 - \rho_q - r}{r-1} \\ &< \frac{\rho_q - \rho_q + 1 - r}{r-1} = -1. \end{aligned}$$

We can choose ε_1 and consequently ε_2 to be sufficiently small for $\rho_q - r\mu + \varepsilon_2 < -1$. The Karamata integration Theorem 2.4 (ii) now provides the existence of a sufficiently large $T > T_1$, such that

$$\begin{aligned} 0 < t^{\rho_p}L_p(t)|u'(t)| &< A_1 + \beta c^r\int_{T_1}^{\infty} s^{\rho_q-r\mu+\varepsilon_2}L_q(s)\,\mathrm{d}s \\ &= A_1 + \beta c^r\int_{T_1}^{T} s^{\rho_q-r\mu+\varepsilon_2}L_q(s)\,\mathrm{d}s + \beta c^r\int_{T}^{\infty} s^{\rho_q-r\mu+\varepsilon_2}L_q(s)\,\mathrm{d}s \\ &< A_1 + A_2 + \frac{2\beta c^r}{|\rho_q - r\mu + \varepsilon_2 + 1|}T^{\rho_q-r\mu+\varepsilon_2+1}L_q(T)\,\mathrm{d}s := A_3, \end{aligned}$$

holds for $t > T$. Here, $A_2 = \beta c^r\int_{T_1}^{T} s^{\rho_q-r\mu+\varepsilon_2}L_q(s)\,\mathrm{d}s$. We now choose an arbitrary $\varepsilon > 0$ and multiply the above inequality by $t^{-\varepsilon}L_p^{-1}$. Thus, we obtain

$$0 < t^{\rho_p-\varepsilon}|u'(t)| < A_3t^{-\varepsilon}L_p^{-1}(t),$$

for $t > T$. Due to Remark 2.2, asymptotic formula (45) follows.

II. Let $\rho_q \leq r\rho_p - r - 1$. Then for arbitrary $\varepsilon_2 > 0$,

$$\begin{aligned}\rho_q - r\mu + \varepsilon_2 &= \frac{r\rho_q - \rho_q}{r-1} - \frac{r}{r-1}(\rho_q - \rho_p + 2) + \varepsilon_2 \\ &= \frac{(r\rho_p - r - 1) + 1 - \rho_q - r}{r-1} + \varepsilon_2 \\ &\geq \frac{\rho_q - \rho_q + 1 - r}{r-1} + \varepsilon_2 \geq -1 + \varepsilon_2 > -1.\end{aligned}$$

The Karamata integration Theorem 2.4 (i) provides the existence of a sufficiently large $T > T_1$, such that

$$0 < t^{\rho_p} L_p(t)|u'(t)| < A_1 + \frac{2\beta c^r}{\rho_q - r\mu + \varepsilon_2 + 1} t^{\rho_q - r\mu + \varepsilon_2 + 1} L_q(t) = A_1 + A_2 t^{\omega} L_q(t)$$

holds for $t > T$, where $A_2 = \frac{2\beta c^r}{\rho_q - r\mu + \varepsilon_2 + 1}$ and $\omega = \rho_q - r\mu + \varepsilon_2 + 1 > 0$. Therefore,

$$t^{\rho_p - \omega}|u'(t)| \quad < \quad A_1 t^{-\omega} L_p^{-1}(t) + A_2 L(t),$$

where $t > T, L(t) = \frac{L_q(t)}{L_p(t)}$. Finally, we choose an arbitrary $\varepsilon_3 > 0$ and multiply the above inequality by $t^{-\varepsilon_3}$. Consequently, we obtain

$$0 < t^{\frac{\rho_q - \rho_p + r + 1}{r-1} - \varepsilon}|u'(t)| < A_1 t^{-\omega - \varepsilon_3} L_p^{-1}(t) + A_2 t^{-\varepsilon_3} L(t),$$

where $\varepsilon = \varepsilon_2 + \varepsilon_3$. By Remark 2.2 the asymptotic formula (46) holds and this completes the proof.

□

6. Numerical simulations

In this section we use the open domain MATLAB Code *bvpsuite* to numerically simulate three model problems in order to illustrate theoretical statements made above. The aim is to give a numerical evidence for the existence of Kneser solutions. We focus on the singular problems (1), (2) and (1), (3) with $a = 0$ and simulate Kneser solutions on the interval $[0, \infty)$ which contains the singular point $t = 0$. Moreover, asymptotic properties of such solutions are investigated and compared with the analytically derived asymptotic formulas (42) and (46).

The MATLAB™ software package `bvpsuite` [14] is designed to solve BVPs in ODEs and differential algebraic equations. The solver routine is based on a class of collocation methods whose orders may vary from 2 to 8. Collocation has been investigated in context of singular differential equations of first and second order in [7, 31], respectively. This method could be shown to be robust with respect to *singularities in time* and retains its high convergence order in case that the analytical solution is appropriately smooth. The code also provides an asymptotically correct estimate for the global error of the numerical approximation. To enhance the efficiency of the method, a mesh adaptation strategy is implemented, which attempts to choose grids related to the solution behavior, in such a way that the tolerance is satisfied with the least possible effort. Error estimate procedure and the mesh adaptation work dependably provided that the solution of the problem and its global error are appropriately smooth[1]. The code and the manual can be downloaded from `http://www.math.tuwien.ac.at/~ewa`. For further information see [14]. This software proved useful for the approximation of numerous singular BVPs important for applications, see e.g. [1, 6, 13, 23].

Since we intend to solve a scalar second order differential equation, we have to specify two boundary/initial conditions which are correctly posed to guarantee the (at least local) uniqueness of the solution. More

[1]The required smoothness of higher derivatives is related to the order of the used collocation method.

precisely, we try to solve the problems (1), (2) and (1), (3) with $a = 0$ but we do not know the values of $u(0)$. Therefore, we solve the differential equation (1),

$$(p(t)u'(t))' + q(t)f(u(t)) = 0, \quad t \in (0, \infty),$$

subject to the following boundary conditions:

$$u'(0) = 0, \quad u(\infty) := \lim_{t\to\infty} u(t) = 0. \tag{48}$$

The first condition in (48) is motivated by the results obtained for $p \equiv q$, where *this condition is necessary for any solution to be continuous*, cf. Remark 4.3. The second condition in (48) has to be satisfied by any Kneser solution under the assumption of Theorem 5.1. It turns out that from the numerical point of view, the problem is very involved and the numerical treatment is by no means straightforward.

For the first tests, we choose the simplest regularly varying functions p and q,

$$p(t) = t^{\alpha}, \ \ \alpha > 1, \quad q(t) = t^{\beta}, \ \ \beta \geq \alpha, \quad t \in [0, \infty). \tag{49}$$

The restriction on $\alpha > 1$ is motivated by results given in [29], where for $\alpha \in (0, 1]$ all solutions of (1) subject to the initial conditions $u(0) \in (L_0, L) \setminus \{0\}$, $u'(0) = 0$ are oscillatory. In order to recover the solution asymptotics specified in (42), the parameter β has to satisfy $\beta > \alpha - 1$. Here, we restrict our attention to the case[2] $\beta \geq \alpha$.
For p, q from (49), we rewrite equation (1) and obtain

$$u''(t) + \frac{\alpha}{t}u'(t) + t^{\beta-\alpha}f(u(t)) = 0, \quad t \in (0, \infty), \quad u'(0) = 0, \ \ u(\infty) = 0. \tag{50}$$

To solve this boundary value problem we reduce the differential equation to the finite interval $[0, 1]$. To this aim, we rewrite the problem as follows:

$$v_1''(t) + \frac{\alpha}{t}v_1'(t) + t^{\beta-\alpha}f(v_1(t)) = 0, \ \ t \in (0, 1], \quad v_2''(t) + \frac{\alpha}{t}v_2'(t) + t^{\beta-\alpha}f(v_2(t)) = 0, \ \ t \in [1, \infty), \tag{51}$$

and use the transformation $\tau = \frac{1}{t}$ in the equation for v_2. Then the problem is solved on (0,1] subject to boundary conditions

$$v_1'(0) = 0, \ \ v_2(0) = 0, \ \ v_1(1) = v_2(1), \ \ v_1'(1) = -v_2'(1).$$

Example 1. The following model is used to illustrate the existence of positive and negative Kneser solutions of equation (50). The problem data reads:

$$p(t) = t^5, \quad q(t) = t^7, \quad t \in [0, \infty),$$

and

$$f(x) = \begin{cases} -12 - 2x & \text{for} \quad x < -2, \\ x^3 & \text{for} \quad x \in [-2, 1], \\ 2 - x & \text{for} \quad x > 1, \end{cases} \tag{52}$$

and $L_0 = -6$, $L = 2$, $r = 3$. As shown in Figure 1, we have found two different Kneser solutions u_1, u_2, lying in the regions indicated in (2) and (3), respectively. The solutions satisfy $\lim_{t\to\infty} u_i'(t) = 0$, $i = 1, 2$ in correspondence to the theory. According to Theorem 5.2, the asymptotic behaviour of any Kneser solution u of (1), (2) or (1), (3) is specified by (42):

$$\lim_{t\to\infty} t^{\frac{\rho_q - \rho_p + 2}{r-1} - \varepsilon}|u(t)| = 0.$$

Thus, for $\alpha = 5$, $\beta = 7$, and $r = 3$ the formula becomes

$$\lim_{t\to\infty} t^{2-\varepsilon}|u_i(t)| = 0, \quad i = 1, 2.$$

The first derivative of the Kneser solution behaves asymptotically as specified in (46), which here means

$$\lim_{t\to\infty} t^{\frac{\rho_q - \rho_p + r + 1}{r-1} - \varepsilon}|u_i'(t)| = \lim_{t\to\infty} t^{3-\varepsilon}|u_i'(t)| = 0, \quad i = 1, 2.$$

[2] For $\beta \in (\alpha - 1, \alpha)$ no Kneser solutions were found.

We illustrate the asymptotic behaviour of the Kneser solutions using graphs with double logarithmic scales, and therefore the power k in the relation $y = ax^k$ corresponds to the slope of the line. Figure 1 clearly indicates that not only solutions u_1, u_2 tend to zero for $t \to \infty$, but also the expressions $t^2 u_i(t)$, $i = 1, 2$ do. Similar observations can be made for the first derivatives of both solutions u_1', u_2', where $t^3 u_i'(t)$, $i = 1, 2$ tends to zero for $t \to \infty$.

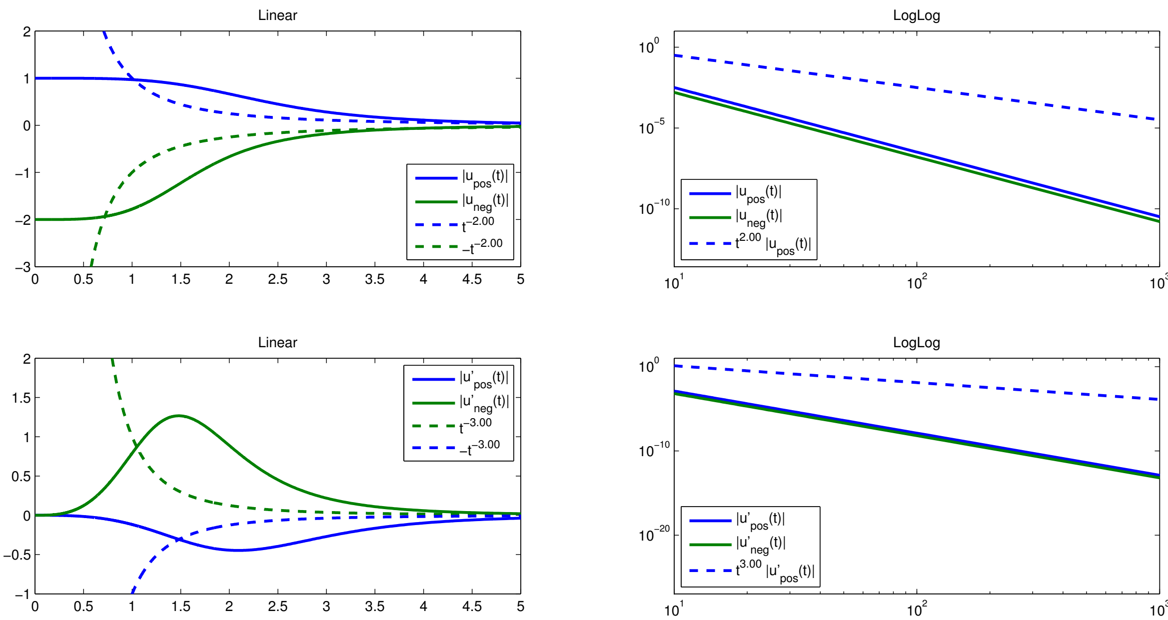

FIGURE 1. Example 1: Solutions u_i, $i = 1, 2$, of (50), (48) corresponding to $\alpha = 5$ and $\beta = 7$ plotted using linear scales, upper graph (left), and double logarithmic scales, upper graph (right). First derivatives of the solutions u_i', $i = 1, 2$ are shown in lower graphs.

Numerical simulations for problem (50), (48) with parameters $(\alpha, \beta) \in \{(4,4), (4,5), (5,6)\}$ and function f satisfying condition (41) with $r = 2, 3, 4$ show similar behaviour, and therefore, they are not discussed here. For details see [8].

Example 2. Using this example, we illustrate how the difference $\beta - \alpha$ effects the asymptotic behaviour of the Kneser solutions. According to (42), if $\beta - \alpha$ grows we expect that the solution decay towards zero becomes faster. To see this, we consider problem (50), (48) with f specified in (52) and

$$(\alpha, \beta) \in \{(3,3), (4,5), (5,7)\}.$$

We can observe in Figure 2 that indeed, larger difference $\beta - \alpha$ results in a steeper decline of the solution towards zero.

Example 3. Here we study Kneser solutions of problem (1), (48) with a function f given by (52) and

$$p(t) = t^\alpha \in RV(\alpha), \quad q(t) = t^\beta (1 + \exp(-t)) \in RV(\beta), \quad t \in [0, \infty).$$

Now, equation (1) takes the form

$$u''(t) + \frac{\alpha}{t} u'(t) + t^{\beta - \alpha}(1 + \exp(-t)) f(u(t)) = 0.$$

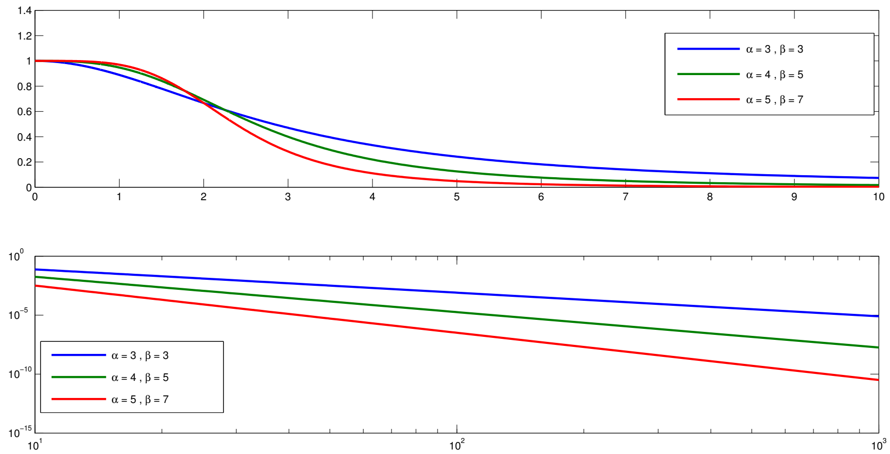

FIGURE 2. Example 2: Comparison of Kneser solutions depending on the difference $\beta - \alpha$. Graph of the Kneser solutions using linear scales (above), double logarithmic scales (below).

Two Kneser solutions and their first derivatives, for the parameters $(\alpha, \beta) = (4, 5)$, $(\alpha, \beta) = (5, 7)$, respectively, can be found in Figure 3.

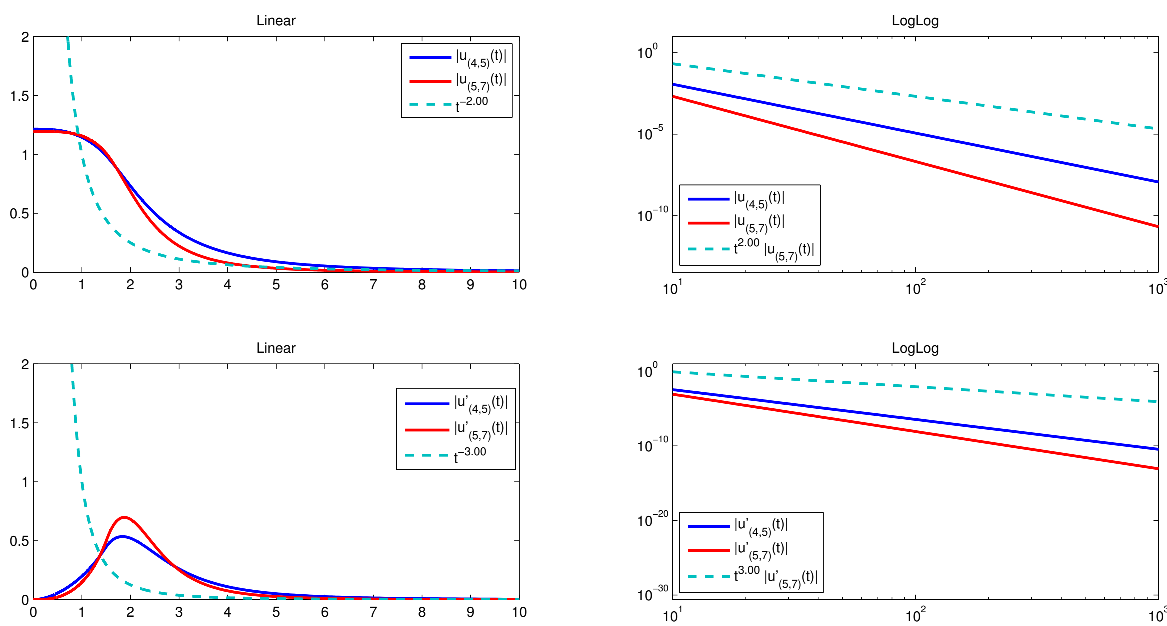

FIGURE 3. Example 3: Solutions corresponding to $\alpha = 4$, $\beta = 5$ and $\alpha = 5$, $\beta = 7$, respectively, plotted using linear scales, upper graph (left), and double logarithmic scales, upper graph (right). First derivatives of the solutions are shown in lower graphs.

Example 4. We designed this example to illustrate the influence of parameters α and β. To this aim, we choose

$$p(t) = t^{\alpha}\log(1+t) \in RV(\alpha), \quad q(t) = t^{\beta} \in RV(\beta), f(x) = \mathrm{sgn}(x)x^4(2+x)(1-x), \quad x \in \mathbb{R}.$$

This yields

$$u''(t) + \left(\frac{\alpha}{t} + \frac{1}{(1+t)\log(1+t)}\right)u'(t) + \frac{t^{\beta-\alpha}}{\log(1+t)}\,\mathrm{sgn}(u(t))u^4(t)(2+u(t))(1-u(t)) = 0. \tag{53}$$

First, we fix $\alpha = 4$ and vary $\beta \in \{4,5,6,7\}$. Numerical results are shown in Figure 4. All solutions seem to have a similar asymptotic behaviour. Moreover, we observe that the function $t^{5/3}u(t)$ for u corresponding to $\alpha = 4$, $\beta = 7$ tends to zero for large values of t. The same holds for all other solutions although formula (42) provides somewhat weaker estimates for the asymptotic behaviour of the Kneser solutions.

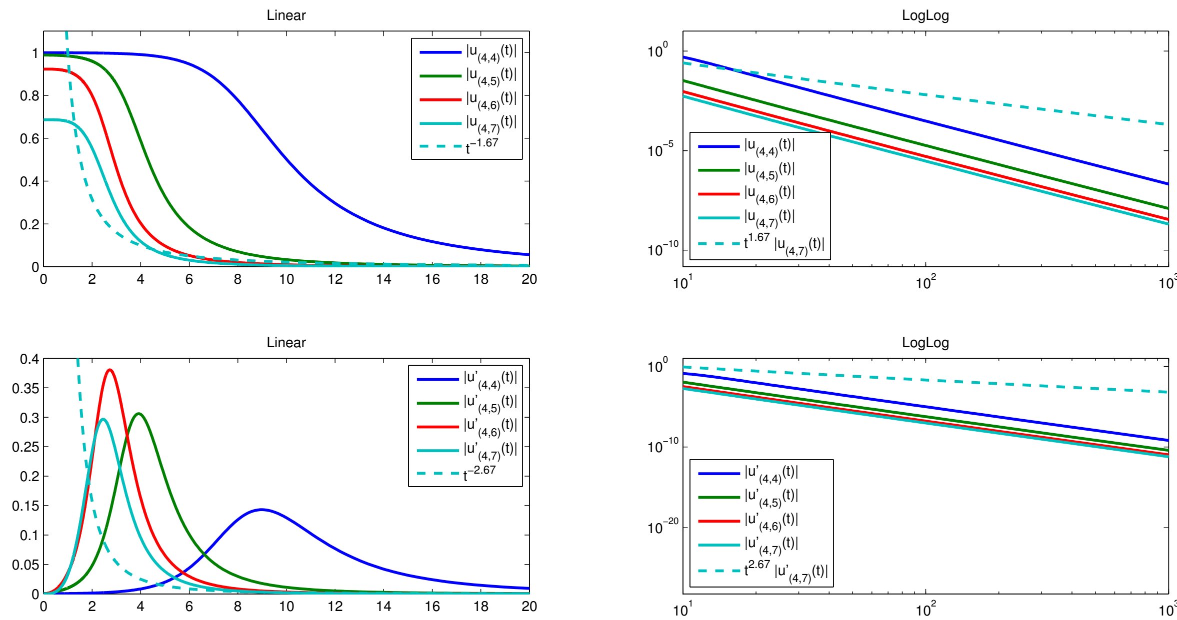

FIGURE 4. Example 4: Comparison of Kneser solution with fixed α plotted in a graph with linear scales (left above) and double logarithmic scales (right above). First derivatives of solutions are shown in the lower graphs.

We now fix $\beta = 10$ and vary $\alpha \in \{5,6,7,8\}$. Figure 5 shows the related Kneser solutions and their first derivatives. We look closer on the solution u with the slowest tendency to converge to zero and see that the multiple $t^{7/3}u$ goes to zero for $t \to \infty$. Other solutions show faster convergence towards zero, although we (42) suggests a slower decay.

The above observations mean that the asymptotic formula (42) can be applied to all numerical Kneser solutions of problem (53), (48), but it seem not to optimally recover the speed of their decay.
Asymptotic behaviour of the numerically computed solutions shows that the second term in equation (50)

$$\frac{\alpha}{t}u'(t), \quad t \in [0,\infty)$$

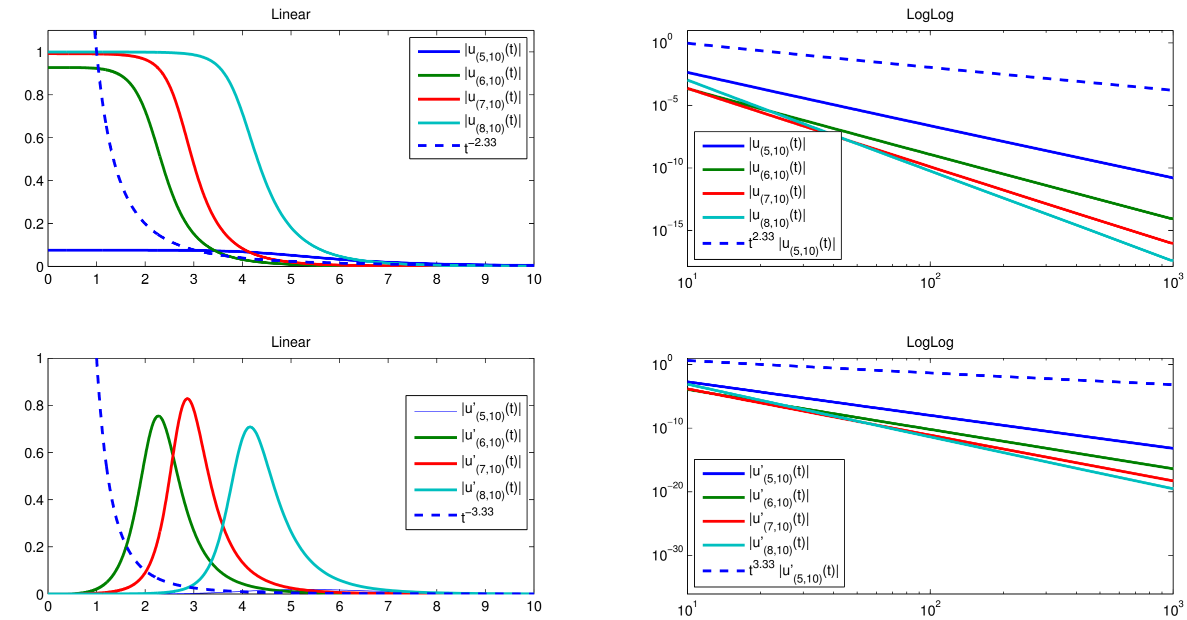

FIGURE 5. Example 4: Comparison of Kneser solution with fixed β plotted in a graph with linear scales (left above) and double logarithmic scales (right above). First derivatives of the solutions are shown in the lower graphs.

becomes dominant as $t \to \infty$, and therefore, properties of the solutions seem to be mainly controlled by the parameter α.

7. Conclusions

In this paper, we investigated the existence and asymptotic properties of the Kneser solutions to the second order ODE (1),

$$(p(t)u'(t))' + q(t)f((u(t)) = 0, \;\; t \in [a, \infty).$$

More precisely, we discussed two classes of initial value problems,

$$(p(t)u'(t))' + q(t)f((u(t)) = 0, \;\; t \in [a, \infty), \quad u(a) = u_0 \in (0, L), \; 0 \le u(t) \le L \text{ for } t \in [a, \infty), \tag{54}$$

and

$$(p(t)u'(t))' + q(t)f((u(t)) = 0, \;\; t \in [a, \infty), \quad u(a) = u_0 \in (L_0, 0), \; L_0 \le u(t) \le 0 \text{ for } t \in [a, \infty). \tag{55}$$

It turns out that in the regular case ($a > 0$), there exists a non-oscillatory solution of problem (54) (and problem (55)), which is either a Kneser solution or a monotonically increasing solution whose limit is L for t tending to infinity (and a monotonically decreasing solution whose limit is L_0 for t tending to infinity).

For the singular case where $a = 0$, the existence of the Kneser solutions can be shown for $p \equiv q$ [30], while for $p \neq q$ the existence results is still an open question.

Moreover, we have provided asymptotic formulas for the Kneser solutions and their first derivatives to the ODE (1) with regularly varying coefficients, which was here considered to be super-linear in the neighbourhood of the origin.

The aim of further investigations is to prove the existence of Kneser solutions to equation (1) with a time singularity at $a = 0$ and $p \neq q$, and to more precisely describe the speed of decay towards zero for the Kneser solutions to equation (1).

Acknowledgements

The authors Jana Burkotová and Irena Rachůnková gratefully acknowledge support received from the grant Mathematical models, IGA_PrF_2015_013 and from the grant No. 14-06958S of the Grant Agency of the Czech Republic.

References

[1] Ch. Budd, O. Koch, E.B. Weinmüller, From nonlinear PDEs to singular ODEs, Appl. Numer. Math. 56 (2006), 413–422
[2] M. Cecchi, Z. Došlá, M. Marini, On intermediate solutions and the Wronskian for half-linear differential equations, J. Math. Anal. Appl. 336 (2007), 905–918.
[3] M. Cecchi, Z. Došlá, M. Marini, I. Vrkoč, Integral conditions for nonoscillation of second order nonlinear differential equations, Nonlinear Anal. 64 (2006), 1278–1289.
[4] A. Domoshnitsky, R. Koplatadze, On the Kneser-type solutions for two-dimensional linear differential systems with deviating arguments, J. Inequal. Appl. 2007 (2007), Article number 52304.
[5] Z. Došlá, M. Marini, S. Matucci, A boundary value problem on a half-line for differential equations with indefinite weight, Commun. Appl. Anal. 15 (2011), 341–352.
[6] R. Hammerling, O. Koch, C. Simon, and E.B. Weinmüller, Numerical Solution of Singular ODE Eigenvalue Problems in Electronic Structure Computations, Comput. Phys. Comm. 181 (2010), 1557–1561.
[7] F. de Hoog, R. Weiss, Collocation methods for singular boundary value problems, SIAM J. Numer. Anal. 15 (1978), 198–217.
[8] M. Hubner, E.B. Weinmüller, Numerical simulations of Kneser solutions to a singular boundary value problem of second order, Technical Report, Vienna University of Technology, Vienna, Austria, in preparation.
[9] J. Jaroš, T. Kusano, Slowly varying solutions of a class of first order systems of nonlinear differential equations, Acta Math. Univ. Comenian. 2 (2013), 265–284.
[10] J. Jaroš, T. Kusano, J.V. Manojlović, Asymptotic analysis of positive solutions of generalized Emden-Fowler differential equations in the framework of regular variation, Cent. Eur. J. Math. 11(12) (2013), 2215–2233.
[11] J. Jaroš, T. Kusano, T. Tanigawa, Nonoscillatory half-linear differential equations and generalized Karamata functions, Nonlinear Anal. 64 (2006), 762–787.
[12] I. Kiguradze, N. Partsvania, On the Kneser problem for two-dimensional differential systems with advanced arguments, J. Inequal. Appl. 7 (2002), 453–477.
[13] G. Kitzhofer, O. Koch, P. Lima, and E.B. Weinmüller, Efficient numerical solution of the density profile equation in hydrodynamics, J. Sci. Comput. 32 (3) (2007), 411–424.
[14] G. Kitzhofer, G. Pulverer, C. Simon, O. Koch, E.B. Weinmüller, The New Matlab Solver `BVPSUITE` for the Solution of Singular Implicit BVPs, JNAIAM, J. Numer. Anal. Ind. Appl. Math. 5 (2010), 113–134.
[15] T. Kusano, J.V. Manojlović, Asymptotic analysis of Emden-Fowler differential equations in the framework of regular variation, Ann. Mat. Pura Appl. 190 (2011), 619–644.
[16] T. Kusano, J.V. Manojlović, J. Milošević, Intermediate solutions of second order quasilinear ordinary differential equations on the framework of regular variation, Appl. Math. Comput. 219 (2013), 8178–8191.
[17] T. Kusano, J.V. Manojlović, T. Tanigawa, Existence of regularly varying solutions with nonzero indices of half-linear differential equations with retarded arguments, Comput. Math. Appl. 59 (2010), 411–425.
[18] T. Kusano, V. Marić, T. Tanigawa, Regularly varying solutions of generalized Thomas-Fermi equations, Bull. Cl. Sci. Mat. Nat. Sci. Math. 34 (2009), 43–73.
[19] T. Kusano, V. Marić, T. Tanigawa, An asymptotic analysis of positive solutions of generalized Thomas-Fermi differential equations - The sub-half-linear case, Nonlinear Anal. 75 (2012), 2474–2485.
[20] V. Marić, Regular Variation and Differential Equation, Lecture Notes in Mathematics 1726, Springer–Verlag, Berlin–Heidelberg–New York, 2000.
[21] M. Marini, S. Matucci, A boundary value problem on the half-line for superlinear differential equations with changing sign weight, Rend. Istit. Mat. Univ. Trieste 44 (2012), 117–132.
[22] S. Matucci, P. Řehák, Asymptotics of decreasing solutions of coupled p-Laplacian systems in the framework of regular variation, Ann. Mat. Pura Appl. 193 (2014), 837–858.
[23] I. Rachůnková, G. Pulverer, and E.B. Weinmüller, A unified approach to singular problems arising in the membrane theory, Appl. Math. 55 (2010), 47–75.
[24] I. Rachůnková, J. Tomeček, Bubble-type solutions of nonlinear singular problem, Math. Comput. Modelling 51 (2010), 658–669.
[25] I. Rachůnková, L. Rachůnek, J. Tomeček, Existence of oscillatory solutions of singular nonlinear differential equations, Abstr. Appl. Anal. 2011 (2011), Article number 408525, 20 pages.

[26] I. Rachůnková, J. Tomeček, Homoclinic solutions of singular nonautonomous second order differential equations, Bound. Value Probl. 2009 (2009), Article number 959636, 21 pages.

[27] P. Řehák, Asymptotic behaviour of increasing solutions to a system of n nonlinear differential equations, Nonlinear Anal. 77 (2013), 45–58.

[28] P. Řehák, De Haan type increasing solutions of half-linear differential equations, J. Math. Anal. Appl. 412 (2014), 236–243.

[29] M. Rohleder, On the existence of oscillatory solutions of the second order nonlinear ODE, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. 51, 2 (2012) 107–127.

[30] J. Vampolová, On existence and asymptotic properties of Kneser solutions to singular second order ODE, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. 52, 1 (2013) 135–152.

[31] E.B. Weinmüller, Collocation for singular boundary value problems of second order, SIAM J. Numer. Anal. 23 (1986), 1062–1095.