Upper and lower solutions and multiplicity results

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Abstract

We consider the second order nonlinear differential equation with an unbounded right-hand side and two-point or nonlinear boundary conditions on a compact interval. Using the method of a priori estimates and the relation between the topological degree of the operators associated to the above boundary value problems and strict lower and upper solutions, we get the mutiplicity results for the problems.

1 Introduction

In [11] we have studied the boundary value problems for the second order differential equation

$$x'' = f(t, x, x'), \tag{1}$$

with f continuous and bounded on $J \times \mathbf{R}^2$, $J = [a, b] \subset \mathbf{R}$, i.e.

$$\exists M \in (0, \infty) : |f(t, x, y)| < M \text{ for all } (t, x, y) \in J \times \mathbf{R}^2. \tag{2}$$

We have considered the periodic conditions

$$x(a) = x(b), x'(a) = x'(b), \tag{3}$$

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the Neumann conditions

$$x'(a) = 0, x'(b) = 0, \tag{4}$$

and the nonlinear conditions

$$g_1\left(x(a), x'(a)\right) = 0, g_2\left(x(b), x'(b)\right) = 0, \tag{5}$$

where $g_1, g_2 \in C(\mathbf{R}^2)$ are increasing in the second argument and g_1 is nonincreasing and g_2 nondecreasing in the first argument.

For the problems (1),(k), k∈{3,4}, we have defined the associated operators L and N :
$L : domL \to C(J)$, $x \longmapsto x''$, $domL = \{x \in C^2(J) : x$ fulfills (k)$\}$,
$N : C^1(J) \to C(J), x \longmapsto -f(\cdot, x(\cdot), x'(\cdot))$
and for the problem (1), (5) we have defined
$L : C^2(J) \to C(J) \times \mathbf{R}^2, x \longmapsto (x'', 0, 0)$,
$N : C^1(J) \to C(J) \times \mathbf{R}^2$,
$x \longmapsto (-f(\cdot, x(\cdot), x'(\cdot)), g_1(x(a), x'(a)), g_2(x(b), x'(b)))$.
The problems (1),(k), k∈{3,4,5} can be written in the form of the operator equation

$$(L + N)\, x = 0. \tag{6}$$

For more details see [4], [8], [10]. In [11] we have proven for the degree d_L of the operator $L + N$ the following results:

Theorem 1.1 *Suppose k∈{3,4,5}. Let (2) be fulfilled, (6) be the operator equation corresponding to the problem (1),(k) and let σ_1, σ_2 be strict lower and upper solutions of (1),(k) with*

$$\sigma_1(t) < \sigma_2(t) \text{ for all } t \in J.$$

Then

$$d_L(L + N, \Omega_1) = 1,$$

where

$$\Omega_1 = \{x \in C^1(J) : \sigma_1(t) < x < \sigma_2(t),\ |x'(t)| < c \text{ for all } t \in J\}$$

with $c \geq (2M + r + 1)(b - a)$ for k∈ {3,4}
and $c \geq (2M+r+1)(b-a)+2(r+1)/(b-a)$ for k=5, $r = \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max}$.

Theorem 1.1 concerns the case of well ordered σ_1, σ_2. The case of σ_1, σ_2 ordered by the opposite way is described in Theorem 1.2.

Theorem 1.2 *Suppose k∈{3,4,5}. Let (2) be fulfilled, (6) be the operator equation corresponding to the problem (1),(k) and let σ_1, σ_2 be strict lower and upper solutions of (1),(k) satisfying*

$$\sigma_2(t) < \sigma_1(t) \text{ for all } t \in J.$$

Then

$$d_L(L+N, \Omega_2) = -1,$$

where

$$\Omega_2 = \{x \in C^1(J) : \|x\|_{\max} < A,\ \|x'\|_{\max} < B,$$

$$\exists t_x \in J : \sigma_2(t_x) < x(t_x) < \sigma_1(t_x)\},$$

with $B \geq 2(b-a)M$, $A \geq \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max} + 2(b-a)^2 M$ *for k∈{3,4}, and* $B \geq 2(b-a)M \ + \|\sigma_2'\|_{\max}$, $A \geq \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max} + (b-a)B$ *for k=5.*

Here, we extend the results of Theorems 1.1 and 1.2 onto differential equations with an unbounded right-hand side f. In the whole paper we suppose that k∈ {3, 4, 5}, that (6) is the operator equation corresponding to the problem (1),(k) and that σ_1, σ_2 are strict lower and upper solutions of (1),(k).

In the mathematical literature there are many existence results via lower and upper solutions method. It should me mentioned the papers by H. B. Thompson [14, 15], where the existence of solutions to the equation (1) with fully nonlinear two point boundary conditions has been established and many references and further information can be found. The boundary conditions (3), (4) and (5) in our paper are a specification of those in [14, 15]. However, thanks to their special properties we need only one-sided growth restrictions for f instead of two-sided ones which are required in [14, 15]. Moreover, the crucial assumption in [14, 15] is the existence of a lower solution σ_1 and an upper solution σ_2 which are well ordered, i.e. $\sigma_1 \leq \sigma_2$. But we consider also the opposite order $\sigma_2 < \sigma_1$. Our theorems on multiplicity results from Section 4 combine the both types of ordering of lower and upper solutions and so they cannot be obtained by the approach presented in [14, 15]. For other results concerning non-ordered or reverse ordered lower and upper solutions we refer to the papers [2, 5, 6, 9]. In the literature we can find multiplicity results reached via proper modifications of the Leggett-Williams fixed point theorem. In this way the existence of three positive solutions of autonomous differential equation with the Dirichlet boundary conditons has been proved in [1]. It should be interesting to apply this method on the nonautonomous differential equation (1) with the boundary conditions (3), (4) or (5) and to compare obtained results with ours in Section 4.

2 Nagumo-Knobloch-Schmitt conditions

Using the method of a priori estimates we can replace the conditon (2) in Theorem 1.1 by the Nagumo-Knobloch-Schmitt condition with bounding functions φ_1, φ_2 :

$$\exists \varphi_1, \varphi_2 \in C^1(K) : \varphi_1(t, \sigma_i(t)) \leq \sigma_i'(t), \varphi_2(t, \sigma_i(t)) \geq \sigma_i'(t),$$

$$f(t, x, \varphi_1(t, x) < \tfrac{\partial \varphi_1(t,x)}{\partial t} + \tfrac{\partial \varphi_1(t,x)}{\partial x} \varphi_1(t, x),$$

$$f(t, x, \varphi_2(t, x) > \tfrac{\partial \varphi_2(t,x)}{\partial t} + \tfrac{\partial \varphi_2(t,x)}{\partial x} \varphi_2(t, x), \tag{7}$$

for $i \in \{1, 2\}$ and for all $(t, x) \in K = J \times [\sigma_1(t), \sigma_2(t)]$.

Theorem 2.1 *Let (7) be fulfilled and let*

$$\sigma_1(t) < \sigma_2(t) \text{ for all } t \in J.$$

Further suppose that for k=3

$$(\varphi_i(b, x) - \varphi_i(a, x))(-1)^i \geq 0, \tag{8}$$

for k=4

$$(\varphi_i(b, x) - \sigma_i'(b))(-1)^i > 0, \tag{9}$$

and for k=5

$$g_2(x, \varphi_i(b, x))(-1)^i > 0, \tag{10}$$

with $i = 1, 2$, $x \in [\sigma_1(t), \sigma_2(t)]$.
Then

$$d_L(L + N, \Omega_3) = 1,$$

where

$$\begin{aligned} \Omega_3 &= \{x \in C^1(J) : \sigma_1(t) < x(t) < \sigma_2(t), \\ \varphi_1(t,x) &< x'(t) < \varphi_2(t,x) \ \ on\ K\}. \end{aligned}$$

Proof. Put

$$\sigma(t,x) = \begin{cases} \sigma_2(t) & \text{for} \quad x > \sigma_2(t) \\ x & \text{for} \quad \sigma_1(t) \le x \le \sigma_2(t) \\ \sigma_1(t) & \text{for} \quad x < \sigma_1(t) \end{cases},$$

$$\varphi(t,x,y) = \begin{cases} \varphi_2(t,x) & \text{for} \quad y > \varphi_2(t,x) \\ y & \text{for} \quad \varphi_1(t,x) \le y \le \varphi_2(t,x) \\ \varphi_1(t,x) & \text{for} \quad y < \varphi_1(t,x) \end{cases},$$

$$f^*(t,x,y) = f(t,\sigma(t,x),\varphi(t,x,y)),$$

and consider the auxiliary equation

$$x'' = f^*(t,x,x'). \tag{11}$$

We can see that f^* satisfies (2) with $M > \max\{|f(t,x,y)| : (t,x,y) \in J \times [\sigma_1(t),\sigma_2(t)] \times [\varphi_1(t,x),\varphi_2(t,x)]\}$ and σ_1, σ_2 are strict lower and upper solutions for the problem (11),(k), k$\in$ {3, 4, 5}. Let us define the set

$$\Omega = \{x \in C^1(J) : \sigma_1(t) < x < \sigma_2(t), |x'(t)| < c \text{ for all } t \in J\},$$

where
$c \ge (2M + r + 1)(b - a)$ for k$\in \{3, 4\}$ and
$c \ge (2M + r + 1)(b - a) + 2(r + 1)/(b - a)$ for k=5,
$r = \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max}$.
Further, for k$\in\{3, 4\}$ we put
$N^* : C^1(J) \to C(J), x \longmapsto -f^*(\cdot, x(\cdot), x'(\cdot))$,
for k=5 we put
$N^* : C^1(J) \to C(J) \times \mathbf{R}^2$,
$x \longmapsto (-f^*(\cdot, x(\cdot), x'(\cdot)), g_1(x(a), x'(a)), g_2(x(b), x'(b)))$.
Then, by Theorem 1.1,

$$d_L(L + N^*, \Omega) = 1. \tag{12}$$

Let us prove that for any solution u of the problem (11),(k), k$\in$ {3, 4, 5} the implication $u \in \Omega \Longrightarrow u \in \Omega_3$ holds. Put $z_2(t) = u'(t) - \varphi_2(t,u)$ and

$z_1(t) = \varphi_1(t, u) - u'(t)$ and suppose that there exists $i \in \{1, 2\}$ and $t_0 \in J$ such that

$$\max\{z_i(t) : t \in J\} = z_i(t_0) \geq 0.$$

Then $z_i'(t_0) \leq 0$ for $t_0 \in [a, b)$. On the other hand, by (7), $z_i'(t_0) > 0$, a contradiction. Now, suppose that $t_0 = b$. For k=3, we have $u'(b) = u'(a)$ and, by (8), $(\varphi_i(b, u(b)) - \varphi_i(a, u(a)))(-1)^i \geq 0$, thus $z_2(b) = u'(b) - \varphi_2(b, u(b)) \leq u'(a) - \varphi_2(a, u(a)) = z_2(a)$, and $z_1(b) = \varphi_1(b, u(b)) - u'(b) \leq \varphi_1(a, u(a)) - u'(a) \leq z_1(a)$. So $z_i(b) = z_i(a)$ and we can put $t_0 = a$. For k=4 we have $u'(a) = u'(b) = 0$ and $\sigma_1'(a) \geq 0$, $\sigma_1'(b) \leq 0$, $\sigma_2'(a) \leq 0$, $\sigma_2'(b) \geq 0$, thus, by (9), $\varphi_1(b, u(b)) < \sigma_1'(b) \leq 0$, $\varphi_2(b, u(b)) > \sigma_2'(b) \geq 0$ and $z_2(b) < 0$, $z_1(b) < 0$, which is a contradiction. For k=5, according to (10), we have $g_2(u(b), u'(b)) \geq g_2(u(b), \varphi_2(b, u(b)) > 0$ if $i = 2$, and $g_2(u(b), u'(b)) \leq g_2(u(b), \varphi_1(b, u(b)) < 0$ if $i = 1$. In the both cases we get a contradiction. Therefore

$$\varphi_1(t, u(t)) < u'(t) < \varphi_2(t, u(t)) \text{ on } K$$

and thus $u \in \Omega_3$. By the excision property of the degree we get from (12)

$$d_L(L + N^*, \Omega_3) = 1$$

and since $N = N^*$ on Ω_3, Theorem 2.1 is proved.

For the constant functions $\sigma_1, \sigma_2, \varphi_1, \varphi_2$ Theorem 2.1 implies

Corollary 2.2 *Suppose that there exist real numbers $r_1 < r_2$, $c_1 < 0 < c_2$, such that*

$$f(t, r_1, 0) < 0,\ f(t, r_2, 0) > 0, \tag{13}$$

$$f(t, x, c_1) < 0,\ f(t, x, c_2) > 0, \tag{14}$$

for all $(t, x) \in J \times [r_1, r_2]$.
If k=5 we suppose moreover that for $x \in [r_1, r_2]$

$$g_2(x, c_i)(-1)^i > 0,\ i = 1, 2, \tag{15}$$

$$g_1(r_1, 0) \geq 0,\ g_1(r_2, 0) \leq 0, \tag{16}$$

$$g_2(r_1, 0) \leq 0,\ g_2(r_2, 0) \geq 0.$$

Then

$$d_L(L + N, \Omega_4) = 1,$$

where

$$\Omega_4 = \{x \in C^1(J) : r_1 < x(t) < r_2,\ c_1 < x'(t) < c_2, \forall\, t \in J\}.$$

Now, let us consider the special case of bounding functions depending on t only:

$$\exists\beta_1,\beta_2\in C^1(J)\colon\ \beta_1(t)\le\sigma_i'(t),\ \beta_2(t)\ge\sigma_i'(t),$$
$$f(t,x,\beta_1(t))<\beta_1'(t),\ f(t,x,\beta_2(t))>\beta_2'(t), \tag{17}$$

for all $(t,x)\in J\times[s_2,s_1]$, where $s_2=\min\{\sigma_2(t):t\in J\}-\int_a^b\gamma(t)dt$, $s_1=\max\{\sigma_1(t):t\in J\}+\int_a^b\gamma(t)dt$, $\gamma(t)=\max\{|\beta_1(t)|,|\beta_2(t)|\}$.

Theorem 2.3 *Let (17) be fulfilled and let*

$$\sigma_2(t)<\sigma_1(t)\ \textit{for all}\ t\in J.$$

Further suppose that for k=3

$$(\beta_i(b)-\beta_i(a))(-1)^i\ge 0, \tag{18}$$

for k=4

$$(\beta_i(b)-\sigma_i'(b))(-1)^i>0, \tag{19}$$

and for k=5

$$g_2(x,\beta_i(b))(-1)^i>0, \tag{20}$$

with $i\in\{1,2\}$, $x\in[s_2,s_1]$.
Then

$$d_L(L+N,\Omega_5)=-1,$$

where

$$\begin{aligned}\Omega_5 &= \{x\in C^1(J): s_2<x(t)<s_1,\ \beta_1(t)<x'(t)<\beta_2(t)\\ \textit{for all}\ t &\in J,\\ \exists t_x &\in J:\sigma_2(t_x)<x(t_x)<\sigma_1(t_x)\}.\end{aligned}$$

Proof. Put

$$\rho(x)=\begin{cases} s_1 & \text{for}\quad x>s_1\\ x & \text{for}\quad s_2\le x\le s_1\\ s_2 & \text{for}\quad x<s_2\end{cases},$$

$$\beta(t,y)=\begin{cases} \beta_2(t) & \text{for}\quad y>\beta_2(t)\\ y & \text{for}\quad \beta_1(t)\le y\le\beta_2(t)\\ \beta_1(t) & \text{for}\quad y<\beta_1(t)\end{cases},$$

$$f^*(t,x,y) = f(t,\rho(x),\beta(t,y)),$$

and consider the equation (11). We can see that f^* satisfies (2) with $M > \max\{|f(t,x,y)| : (t,x,y) \in J \times [s_2,s_1] \times [\beta_1(t),\beta_2(t)]\}$ and σ_1,σ_2 are strict lower and upper solutions for the problem (11),(k), k$\in$ $\{3,\ 4,\ 5\}$. Let us define the set

$$\begin{aligned} \Omega &= \{x \in C^1(J) : \|x\|_{\max} < A,\ \|x'\|_{\max} < B, \\ \exists t_x &\in J : \sigma_2(t_x) < x(t_x) < \sigma_1(t_x)\} \end{aligned}$$

with $B = 2(b-a)M + \|\gamma\|_{\max},\ A = \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max} + (b-a)B$ for k=3, 4 and $B = 2(b-a)M + \|\gamma\|_{\max} + \|\sigma_2'\|_{\max},\ A = \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max} + (b-a)B$ for k=5.

Further we define the operator N^* like in the proof of Theorem 2.1 and using Theorem 1.2 we get

$$d_L(L+N^*,\Omega) = -1.$$

We can follow the proof of Theorem 2.1 and using (17) - (20), we prove for any solution u of (11),(k), k$\in$ $\{3,\ 4,\ 5\}$

$$u \in \Omega \Longrightarrow \beta_1(t) < u'(t) < \beta_2(t),\ \text{for all } t \in J.$$

Integrating the last inequality we get

$$s_2 < u(t) < s_1,\ \text{for all } t \in J,$$

i.e. $u \in \Omega_5$. By the excision property of the degree we get

$$d_L(L+N^*,\Omega_5) = -1$$

and since $N = N^*$ on Ω_5, Theorem 2.3 is proved.

Corollary 2.4 *Suppose that there exist real numbers* $r_1 > r_2$, $c_1 < 0 < c_2$, *such that (13) and(14) are satisfied for all* $(t,x) \in J \times [r_2 + c_1(b-a), r_1 + c_2(b-a)]$. *If k=5, we suppose that (15), (16) are satisfied for* $x \in [r_2 + c_1(b-a), r_1 + c_2(b-a)]$.
Then

$$d_L(L+N,\Omega_6) = -1,$$

where

$$\begin{aligned} \Omega_6 = \{x \in C^1(J) : r_2 + c_1(b-a) < x(t) < r_1 + c_2(b-a), \\ c_1 < x'(t) < c_2, \forall\, t \in J \\ \exists t_x \in J : r_2 < x(t_x) < r_1\}. \end{aligned}$$

Example. Suppose $f_1, f_2, f_3 \in C(J), k, m \in \mathbf{N}$. The function

$$f(t,x,y) = f_1(t)x^{2k+1} + f_2(t)y^{2m+1} + f_3(t)$$

satisfies the conditions of Corollary 2.2, if $f_1, f_2 > 0$ on J, and it satisfies the conditions of Corollary 2.4, if $f_1 < 0, f_2 > 0$ on J and either $m > k$ or $m = k$, $f_2(t) > \|f_1\|_{\max} (b-a)^{2k+1}$ for all $t \in J$.

3 One-sided growth conditions

Other type of conditions which can be used instead of (2) in Theorem 1.1 and Theorem 1.2 are one-sided growth conditions which were used by Kiguradze [7] in some existence theorems.

1. The one-sided Bernstein-Nagumo condition:

$$\exists \omega \in C(\mathbf{R}_+),\ \omega \text{ positive},\ \int_0^\infty \frac{ds}{\omega(s)} = \infty \text{ and}$$

$$f(t,x,y) \leq \omega(|y|) \cdot (1 + |y|) \tag{21}$$
$$\forall (t,x) \in J \times [\sigma_1(t), \sigma_2(t)] \times \mathbf{R}.$$

2. The one-sided linear growth condition:

$$\exists a_1, a_2 \in (0,\infty), \rho \in C(J \times \mathbf{R}),\ \text{non-negative and non-decreasing in the second argument such that}$$

$$f(t,x,y) \leq a_1|x| + a_2|y| + \rho(t, |x| + |y|) \tag{22}$$
$$\forall (t,x,y) \in J \times \mathbf{R}^2,$$

where

$$a_1(b-a)^2 + a_2(b-a) < 1 \tag{23}$$

and

$$\lim_{z\to\infty} \frac{1}{z} \int_a^b \rho(t,z)dt = 0.$$

Note. Let us remember that if f satisfies (22) it satisfies (21) as well.

First, we will prove lemmas on a priori estimates for solutions of the problems (1),(k), k$\in \{3,4,5\}$.

Lemma 3.1 *Suppose*

$$\sigma_1(t) < \sigma_2(t) \quad \textit{for all } t \in J.$$

Let (21) be satisfied. If k=5, suppose moreover

$$\lim_{y\to\infty} g_1(r_2, y) > 0, \ \lim_{y\to-\infty} g_2(r_2, y) < 0, \tag{24}$$

$r_1 = min\{\sigma_1(t) : t \in J\}$, $r_2 = max\{\sigma_2(t) : t \in J\}$.
Then there exists $\mu^* \in (0, \infty)$ *such that for any solution* u *of the problem (1),(k), the implication*

$$\sigma_1(t) < u(t) < \sigma_2(t) \ \textit{on } J \Longrightarrow \|u'\|_{\max} < \mu^* \tag{25}$$

is valid.

Proof. Let u be a solution of (1),(k) and let

$$\sigma_1(t) < u(t) < \sigma_2(t) \text{ for all } t \in J. \tag{26}$$

Let us put $r = \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max}$, $\mu = \max\{|u'(t)| : t \in J\}$. The condition (21) implies that

$$u'' \le \omega(|u'(t)|)(1 + |u'(t)|), \forall t \in J. \tag{27}$$

1. Let k=3. Then we can find $t_0 \in (a, b)$ such that $u'(t_0) = 0$. From (21) it follows that there exist $\mu_1, \mu^* \in (1, \infty)$, $\mu_1 < \mu^*$, such that

$$\int_1^{\mu_1} \frac{ds}{\omega(s)} = K > 2r, \int_1^{\mu^*} \frac{ds}{\omega(s)} > K + 2r. \tag{28}$$

(a) Suppose that there exists $t_1 \in (t_0, b]$ such that

$$\max\{u'(t) : t \in [t_0, b]\} = u'(t_1) = c_1 > 1.$$

Then we can find $\alpha_1 \in (t_0, t_1)$ such that

$$u'(\alpha_1) = 1 \text{ and } u'(t) > 1 \ \forall t \in (\alpha_1, t_1].$$

Integrating (27) from α_1 to t_1 we get

$$\int_{\alpha_1}^{t_1} \frac{u''(t)dt}{\omega(u'(t))} \le 2\int_{\alpha_1}^{t_1} u'(t)dt,$$

thus

$$\int_1^{c_1} \frac{ds}{\omega(s)} \leq 2r, \text{ which gives } c_1 < \mu_1.$$

Therefore $u'(t) < \mu_1$ for all $t \in [t_0, b]$, $u'(a) < \mu_1$. Further suppose that there exists $t_2 \in [a, t_0)$ such that

$$\max\{u'(t) : t \in [a, t_0]\} = u'(t_2) = c_2 > \mu_1.$$

Then we can find $\alpha_2 \in [a, t_2)$ such that

$$u'(\alpha_2) = \mu_1 \text{ and } u'(t) > \mu_1 \text{ for all } t \in (\alpha_2, t_2].$$

Integrating (27) from α_2 to t_2 we get

$$\int_1^{c_2} \frac{ds}{\omega(s)} \leq K + 2r, \text{ which gives } c_2 < \mu^*.$$

Thus we have proven

$$u'(t) < \mu^* \text{ for all } t \in J. \tag{29}$$

(b) Now, we will estimate u' from below. Suppose that there exists $t_3 \in [a, t_0)$ such that

$$\min\{u'(t) : t \in [a, t_0]\} = u'(t_3) = -c_3 < -1.$$

If we put $v'(t) = -u'(t)$, we can prove like in (a) that $c_3 < \mu_1$, i.e. $u'(t) > -\mu_1$ on $[a, t_0]$, $u'(b) > -\mu_1$. Supposing that

$$\min\{u'(t) : t \in [t_0, b]\} = u'(t_4) = -c_4 < -\mu_1,$$

we can also use for $v' = -u'$ the same argument like in (a) and get $c_4 < \mu^*$, i.e.

$$-\mu^* < u'(t) \text{ for all } t \in J. \tag{30}$$

2. Let k=4. From (21) it follows that there exists $\mu^* \in (1, \infty)$ such that

$$\int_1^{\mu^*} \frac{ds}{\omega(s)} > 2r. \tag{31}$$

Now, we can use the same considerations like for the periodic problem (1),(3), but instead of proving the estimate on $[t_0, b]$ and then on $[a, t_0]$, in the step (a), and on $[a, t_0]$ and then on $[t_0, b]$, in the step (b), we can put $t_0 = a$, in

the step (a), and get (29) and then put $t_0 = b$, in the step (b), and get (30).
3. Let k=5. Then (24) guarantees the existence of $\gamma \in (1, \infty)$ such that for any solution x of (1),(5) satisfying (26)

$$x'(a) < \gamma,\ x'(b) > -\gamma. \tag{32}$$

Otherwise, we could find a sequence of solutions $\{x_n\}_1^\infty$ of (1),(5) satisfying (26) with $x_n'(a) \to \infty$ or $x_n'(b) \to -\infty$ for $n \to \infty$. So, there exists $n_0 \in \mathbf{N}$ such that $g_1(x_{n_0}(a), x_{n_0}'(a)) > 0$ or $g_2(x_{n_0}(b), x_{n_0}'(b)) < 0$, a contradiction. Further, from (21) it follows that there exists $\mu^* \in (\gamma, \infty)$ such that

$$\int_1^{\mu^*} \frac{ds}{\omega(s)} > 2r + \int_1^{\gamma} \frac{ds}{\omega(s)}. \tag{33}$$

Then we can argue like for k=4 and using (33) we get (29), in the step (a), and (30), in the step (b).

Lemma 3.2 *Let $r_1, r_2 \in \mathbf{R}, r_1 < r_2$ and let (22) be satisfied. If k=5, suppose moreover*

$$\lim_{y\to\infty} g_1(x,y) > 0,\ \lim_{y\to-\infty} g_2(x,y) < 0, \tag{34}$$

uniformly for $x \in \mathbf{R}_+$.
Then there exists $\nu^* \in (0, \infty)$ such that for any solution u of the problem (1),(k), the implication

$$\exists t_u \in J : r_1 < u(t_u) < r_2 \Longrightarrow \|u'\|_{\max} < \nu^* \tag{35}$$

is valid.

Proof . Let x be a solution of (1),(k), k$\in\{3,4,5\}$ and let there exist a $t_x \in J$ such that $r_1 < u(t_x) < r_2$. Let us put $r = |r_1| + |r_2|$, $\mu =$max$\{|x'(t)| : t \in J\}$. The condition (22) implies that

$$x''(t) \le a_1|x(t)| + a_2|x'(t)| + \rho(t, |x| + |x'|)\ \ \forall t \in J. \tag{36}$$

1. Let k=3. We have $|x(t)| \le \mu(b-a) + r$ for all $t \in J$ and we can find $t_0 \in (a, b)$ such that $x'(t_0) = 0$.

(a) Integrating (36) from t_0 to $t \in (t_0, b]$, we get

$$x'(t) \le A(\mu, t_0, b)\ \forall t \in [t_0, b] \text{ and } x'(a) \le A(\mu, t_0, b), \tag{37}$$

where

$$A(\mu, t_0, b) = a_1[\mu(b-a)+r](b-t_0) + a_2\mu(b-t_0) + \int_{t_0}^{b} \rho(s, \mu(b-a+1)+r)ds.$$

Integrating (36) from a to $t \in (a, t_0]$ and using (37), we get

$$x'(t) \leq A(\mu, t_0, b) + A(\mu, a, t_0) = A(\mu, a, b)\, \forall t \in [a, t_0].$$

Thus

$$x'(t) \leq A(\mu, a, b) \text{ for all } t \in J. \tag{38}$$

(b) Now, let us integrate (36) from $t \in [a, t_0)$ to t_0 :

$$-x'(t) \leq A(\mu, a, t_0)\, \forall t \in [a, t_0] \text{ and } -x'(b) \leq A(\mu, a.t_0). \tag{39}$$

Finally, using (39) and integrating (36) from $t \in [t_0, b)$ to b, we have

$$-x'(t) \leq A(\mu, a, t_0) + A(\mu, t_0, b) = A(\mu, a, b)\, \forall t \in \{t_0, b].$$

Therefore

$$-x'(t) \leq A(\mu, a, b) \text{ for all } t \in J. \tag{40}$$

(38) and (40) give

$$\mu \leq A(\mu, a, b). \tag{41}$$

2. Let k=4. Then we can put $t_0 = a$, in the step (a), and $t_0 = b$, in the step (b), and get (41).
3. Let k=5. We can show like in the proof of Lemma 3.1, part 3, that (32) is valid for any solution x of (1),(5). For proving the estimation of the first derivatives of the solutions we can argue like for k=4 and get

$$\mu \leq \gamma + A(\mu, a, b). \tag{42}$$

Let us show that there exists $\nu^* \in (0, \infty)$ such that

$$\max\{|x'(t)| : t \in J\} = \mu < \nu^*$$

for any solution x of the problem (1),(k), k∈{3,4,5}. Suppose that such constant ν^* does not exist. Then we can find a sequence of solutions $\{x_n\}_1^\infty$ of the problem (1),(k) and the associated sequence of $\{\mu_n\}_1^\infty$ such that

$\lim_{n\to\infty}\mu_n = \infty$. If k$\in$\{3,4\}, we get, according to (41), $\mu_n \le A(\mu_n,a,b)$, i.e.

$$\begin{aligned} 1 \ \le\ & \frac{1}{\mu_n}A(\mu_n,a,b) = a_1(b-a)^2 + a_2(b-a) + \\ & \frac{1}{\mu_n}a_1 r(b-a) + \frac{1}{\mu_n}\int_a^b \rho(s,\mu_n(1+b-a)+r)ds \end{aligned}$$

and if k=5, we get from (42)

$$1 \le \frac{1}{\mu_n}\gamma + \frac{1}{\mu_n}A(\mu_n,a,b).$$

Provided $\mu_n \to \infty$ we get for k=\{3,4,5\}

$$1 \le a_1(b-a)^2 + a_2(b-a),$$

which contradicts (23). So, the implication (35) is valid and Lemma 3.2 is proved.

Theorem 3.3 *Let (21) be fulfilled and let*

$$\sigma_1(t) < \sigma_2(t) \textit{ for all } t \in J.$$

If k=5, suppose moreover (24).
Then there exists $r^* \in (0,\infty)$ such that

$$d_L(L+N,\Omega_6) = 1,$$

where

$$\Omega_6 = \{x \in C^1(J) : \sigma_1(t) < x(t) < \sigma_2(t)\ \forall t \in J,\ \|x'\|_{\max} < r^*\}.$$

Proof. Let μ^* be the constant from Lemma 3.1. Put

$$r = \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max},\ r^* = r + \max\{\mu^*, \|\sigma_1'\|_{\max}, \|\sigma_2'\|_{\max}\},$$

$$\chi(s,\phi) = \begin{cases} 1 & \text{for} \quad 0 \le s \le \phi \\ 2 - s/\phi & \text{for} \quad \phi < s < 2\phi \\ 0 & \text{for} \quad s \ge 2\phi \end{cases},$$

$$f^*(t,x,y) = \chi(|x|+|y|,r^*)f(t,x,y),$$

and define for k∈{3,4}

$$N^*: C^1(J) \to C(J),\ x \longmapsto -f^*(\cdot, x(\cdot), x'(\cdot)),$$

and for k=5

$$\begin{array}{rcl} N^* & : & C^1(J) \to C(J) \times \mathbf{R}^2, \\ x & \longmapsto & (-f^*(\cdot, x(\cdot), x'(\cdot)), g_1(x(a), x'(a)), g_2(x(b), x'(b))). \end{array}$$

The differential equation

$$x'' = f^*(t, x, x') \tag{43}$$

has also σ_1, σ_2 as its strict lower and upper solutions and the function f^* satisfies (2) with $M = 1+\max\{|f^*(t,x,y)| : t \in J, |x|+|y| \le 2r^*\}$. Therefore, by Theorem 1.1,

$$d_L(L + N^*, \Omega) = 1, \tag{44}$$

where

$$\Omega = \{x \in C^1(J) : \sigma_1(t) < x(t) < \sigma_2(t), |x'(t)| < c\ \forall t \in J\}$$

with

$$c = (2M + r + 1)(b - a) + \frac{2(r+1)}{b-a} + r^*.$$

We can see that f^* fulfils (21) and thus, by Lemma 3.1, the implication (25) is valid. It means that any solution $u \in \Omega$ of the problem (43),(k) belongs to Ω_6. Thus, by the excision property of the degree

$$d_L(L + N^*, \Omega_6) = 1,$$

and since $N^* = N$ on Ω_6 Theorem 3.3 is proved.

Theorem 3.4 *Let (22) be fulfilled and let*

$$\sigma_2(t) < \sigma_1(t) \textit{ for all } t \in J.$$

If k=5, suppose moreover (34).
Then there exists $r^* \in (0, \infty)$ such that

$$d_L(L + N, \Omega_7) = -1,$$

where

$$\begin{array}{rcl} \Omega_7 & = & \{x \in C^1(J) : \|x\|_{\max} + \|x'\|_{\max} < r^*, \\ \exists t_x & \in & J : \sigma_2(t_x) < x(t_x) < \sigma_1(t_x)\}. \end{array}$$

Proof. Let ν^* be the constant from Lemma 3.2. Put $r_1 = \min\{\sigma_2(t) : t \in J\}$, $r_2 = \max\{\sigma_1(t) : t \in J\}$, $r^* = \nu^*(1+b-a) + |r_1| + |r_2| + \|\sigma_1'\|_{\max} + \|\sigma_2'\|_{\max}$. Now, we can follow the proof of Theorem 3.3, define f^* and N^* in the same way and, using Theorem 1.2, we get (44), where

$$\begin{aligned} \Omega &= \{x \in C^1(J) : \|x\|_{\max} < A, \|x'\| < B, \\ \exists t_x &\in J : \sigma_2(t_x) < x(t_x) < \sigma_1(t_x)\}, \end{aligned}$$

with $B = 2(b-a)M + r^*$, $A = \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max} + 2(b-a)^2 M$.

We can see that f^* fulfills (22) and thus, using Lemma 3.2, we can finish the proof similarly like that of Theorem 3.3.

4 Multiplicity results

Let us suppose that σ_1, σ_2 and σ_3 are strict lower, upper and lower solutions of (1),(k), k$\in\{3,4,5\}$. Using Theorem 2.1 and Theorem 2.3 we get the following multiplicity result:

Theorem 4.1 *Suppose that*

$$\sigma_1(t) < \sigma_2(t) < \sigma_3(t) \textit{ for all } t \in J \tag{45}$$

and that (17) and, according to k, the condition (18) or (19) or (20) are fulfilled for all $(t,x) \in J \times [\sigma_1(t), s_3]$, *where* $s_3 = max\ \{\sigma_3(t) : t \in J\} + \int_a^b \gamma(t)dt$.
Then (1),(k) has at least two different solutions u, v *satisfying*

$$\sigma_1(t) < u(t) < \sigma_2(t),\ \sigma_1(t) < v(t) \textit{ for all } t \in J,$$

$$\sigma_2(t_v) < v(t_v) < \sigma_3(t_v) \textit{ for a } t_v \in J. \tag{46}$$

Proof. Since (17)-(20) is the special case of (7)-(10), the existence of a solution u lying between σ_1 and σ_2 follows from Theorem 2.1. The existence of the second solution v satisfying (46) follows from Theorem 2.3. The inequality $\sigma_1 < v$ on J can be proven in the same way like for u in the proof of Theorem 2.1.

Similarly, by means of Theorem 3.3 and Theorem 3.4 and the fact that (22) and (34) are the special cases of (21) and (24), we get:

Theorem 4.2 *Let us suppose that (45) and (22) are fulfilled and, for k=5, suppose moreover (34). Then the assertion of Theorem 4.1 is valid.*

Now, let us consider the dual situation, where σ_3 is an upper solution of (1),(k).

Theorem 4.3 *Suppose that*

$$\sigma_3(t) < \sigma_1(t) < \sigma_2(t) \text{ for all } t \in J \tag{47}$$

and that (17) and, according to k, the condition (18) or (19) or (20) are fulfilled for all $(t,x) \in J \times [b_3, \sigma_2(t)]$, *where* $b_3 = min\ \{\sigma_3(t) : t \in J\} - \int_a^b \gamma(t)dt$.
Then (1),(k) has at least two different solutions u, v *satisfying*

$$\sigma_1(t) < u(t) < \sigma_2(t),\ v(t) < \sigma_2(t) \text{ for all } t \in J,$$

$$\sigma_3(t_v) < v(t_v) < \sigma_1(t_v) \text{ for a } t_v \in J.$$

Theorem 4.4 *Let us suppose that (47) and (22) are fulfilled and, for k=5, suppose moreover (34). Then the assertion of Theorem 4.3 is valid.*

For constant lower and upper solutions we can generalize the theorems from [12], concerning the multiplicity results of the Ambrosetti-Prodi type for the periodic problem.

Theorem 4.5 *Suppose k*$\in\{3,4\}$. *Let* $n \in \mathbf{N}, n \geq 2,\ c_1, c_2, s_1, r_1, ..., r_{n+1} \in \mathbf{R}$ *be such that*

$$r_1 < r_2 < ... < r_{n+1}, \tag{48}$$

$$c_1 < 0 < c_2,$$

$$(s_1 - f(t, r_i, 0))(-1)^i > 0 \text{ for all } t \in J,\ i \in \{1, ..., n\}, \tag{49}$$

and

$$f(t, x, c_1) < 0,\ f(t, x, c_2) > 0 \tag{50}$$

for all (t,x)$\in J \times [r_1, r^*]$, *where*

$$r^* = \begin{cases} r_{n+1} & \text{for } n \text{ odd} \\ r_{n+1} + max\{|c_1|, c_2\}(b-a) & \text{for } n \text{ even.} \end{cases} \tag{51}$$

Then there exist $s_2, s_3 \in (-\infty, s_1)$, $s_3 \leq s_2$, *such that the problem*

$$x'' + f(t,x,x') = s,\ (k) \tag{52}$$

has:
(i) at least n *different solutions* u_i, $i = 1,..,n$, *satisfying*

$$r_1 < u_i(t) < r^* \text{ for all } t \in J, i \in \{1,...,n\}; \tag{53}$$

(ii) at least $\frac{n+1}{2}$ $(\frac{n}{2})$ *solutions satisfying (53) for* $s = s_2$ *and* n *odd (even);*
(iii) provided $s_3 < s_2$ *at least one solution satysfying (53) for* $s \in [s_3, s_2)$;
(iv) no solution satifying (53) for $s < s_3$.

Proof. Let $j \in \{1,...,n+1\}$. The condition (49) implies that there exists $s_2 < s_1$ such that for j odd (even) r_j is a strict lower (upper) solution to (52) for $s \in (s_2, s_1]$. Therefore, using Theorem 4.1 we get (i). For $s = s_2$ at least one of the strict upper solutions r_j of the problem (52) becames nonstrict and so two solutions of this problem can identify. In the case where all the upper solutions became nonstrict for $s = s_2$, all neighbour pairs of solutions of (52) can be identical. Thus (ii) is true. Suppose that x is a solution of (52) satysfying (53). Put $m < \min\{f(t,x,y) : (t,x,y) \in J \times [r_1, r^*] \times [c_1, c_2]\}$. Then, integrating the equation (52) from a to b, we get $m < s$. Thus for $s \leq m$ the problem (52) has no solution satisfying (53). Suppose that for some $s^* \in (m, s_1)$ the problem (52) has a solution u^*. Then there exists a solution of (52) for all $s \in [s^*, s_1]$, because u^* is an upper solution and r_1 a lower solution of (52) for $s \in [s^*, s_1]$, and $u^*(t) > r_1$ on J. So, we can put $s_3 = \inf\{s : s < s_1,\ (52) \text{ has a solution satisfying } (53)\}$. Then $s_3 \in (m, s_2]$. If $s_3 < s_2$, we consider a sequence $\{\sigma_n\} \subset (s_3, s_2)$ converging to s_3 and the corresponding sequence of solutions $\{u_n\}$ of the problems $\{(52),\ s{=}\sigma_n\}$. This sequence is equi-bounded and equi-continuous in $C^1(J)$ and by the Arzelà-Ascoli theorem, we can choose a subsequence converging in the space $C^1(J)$ to a solution of (52) for $s = s_3$. Thus (iii) and (iv) are valid.

Similarly we can prove:

Theorem 4.6 *If we change the inequality (49) in Theorem 4.5 onto the opposite one, i.e.*

$$(s_1 - f(t, r_i, 0))(-1)^i < 0 \text{ for all } t \in J,\ i \in \{1,...,n\},$$

and suppose that (50) is fulfilled for all $(t,x) \in J \times [s^*, r^*]$, *where* r^* *is given by (51) and* $s^* = r_1 - (b-a) max\{|c_1|, c_2\}$, *then the assertions (i)-(iv) of Theorem 4.5 remain valid.*

Using one-sided growth conditions and Theorems 3.3, 3.4, 4.2 and 4.4, we get:

Theorem 4.7 *Suppose* $k \in \{3,4\}$. *Let* $n \in \mathbf{N}, n > 2$ *be odd and let* $s_1, r_1, ..., r_{n+1} \in \mathbf{R}$ *satisfy (48) and (49). Further, let (21) be fulfilled.*
Then there exists $r^* \geq r_{n+1}$ *such that (i)-(iv) of Theorem 4.5 are valid.*

Theorem 4.8 *Suppose* $k \in \{3,4\}$. *Let* $n \in \mathbf{N}, n \geq 2$ *be even and let* $s_1, r_1, ..., r_{n+1} \in \mathbf{R}$ *satisfy (48) and (49). Further let (22) be fulfilled.*
Then there exists $r^* \geq r_{n+1}$ *such that (i)-(iv) of Theorem 4.5 are valid.*

Close results concerning the existence of two or three solutions of the periodic problem can be found also in [3] and [13].

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