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ON THE EXISTENCE OF MORE POSITIVE SOLUTIONS OF PERIODIC BVPs WITH SINGULARITY

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Abstract We consider periodic boundary value problems for ordinary second order differential equations of the form $u'' = f(t, u, u')$, where f satisfies the (local) Carathéodory conditions and can have a singularity in the second variable. Writing our problem in an operator form we seek for proper sets which the topological degree of the corresponding operator can be computed on. These sets are not convex, in general. Using the degree theory we get at least one fixed point of the operator at each such set which leads to the existence and localization of more solutions of the related periodic boundary value problem. Our results are based on the generalized lower and upper functions method from [15].

KEY WORDS: second order nonlinear ordinary differential equation, periodic solution, topological degree, lower and upper functions, strong repulsive singularity, multiplicity.

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1. Introduction

We will study the existence and localization of more solutions to the problem

$$u'' = f(t, u, u'), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \tag{1.1}$$

First, supposing that f satisfies the Carathéodory conditions on $[0, 2\pi] \times \mathbf{R}^2$ we modify theorems of [15] concerning a connection between the existence of lower and upper functions of (1.1) and properties of the topological degree of the operator corresponding to (1.1) on sets which are defined by means of these lower and upper functions. (Theorems 1.3, 1.4.)

Using these results we find two disjoint sets (one convex and the second non-convex) and prove that each of them contains at least one solution to (1.1). (Theorems 2.1, 2.2.) This leads to the existence and localization of three solutions to (1.1). (Theorem 2.3.) Finally we show an application to periodic boundary value problems with singularities and for any $n \in \mathbf{N}$, $n \geq 3$, we present conditions which guarantee $n-1$ different positive solutions. (Theorems 3.3, 3.5.)

We say that $f : [0, 2\pi] \times \mathbf{R}^2 \to \mathbf{R}$ fulfils the Carathéodory conditions on $[0, 2\pi] \times \mathbf{R}^2$, if f has the following properties: (i) for each $x, y \in \mathbf{R}$ the function $f(\cdot, x, y)$ is measurable on $[0, 2\pi]$; (ii) for almost every $t \in [0, 2\pi]$ the function $f(t, \cdot, \cdot)$ is continuous on $\mathbf{R}^2$; (iii) for each compact set $\mathrm{K} \subset \mathbf{R}^2$ the function $m_K(t) = \sup_{(x,y)\in \mathrm{K}} |f(t, x, y)|$ is Lebesgue integrable on $[0, 2\pi]$. For the set of functions satisfying the Carathéodory conditions on $[0, 2\pi] \times \mathbf{R}^2$ we write $\mathrm{Car}([0, 2\pi] \times \mathbf{R}^2)$.

We will work with the Banach spaces $\mathbf{C}[0, 2\pi]$ (the space of functions x continuous on $[0, 2\pi]$ with the norm $\|x\|_{\mathbf{C}} = \max_{t\in[0,2\pi]} |x(t)|$), $\mathbf{C}^1[0, 2\pi]$ (the space of functions x having continuous first derivatives on $[0, 2\pi]$ with the norm $\|x\|_{\mathbf{C}^1} = \max_{t\in[0,2\pi]} \{|x(t)| + |x'(t)|\}$), $\mathbf{L}[0, 2\pi]$ (the space of functions y Lebesgue integrable on $[0, 2\pi]$ with the norm $\|y\|_1 = \int_0^{2\pi} |y(t)| \mathrm{d}t$) and $\mathbf{L}_2[0, 2\pi]$ (the space of functions y square Lebesgue integrable on $[0, 2\pi]$ with the norm $\|y\|_2 = \left(\int_0^{2\pi} y^2(t) \mathrm{d}t\right)^{\frac{1}{2}}$).

$\mathbf{AC}[0, 2\pi]$ ($\mathbf{AC}^1[0, 2\pi]$) denotes the set of functions absolutely continuous on $[0, 2\pi]$ (having absolutely continuous first derivatives on $[0, 2\pi]$) and $\mathbf{BV}[0, 2\pi]$ is the set of functions of bounded variation on $[0, 2\pi]$. If $x \in \mathbf{BV}[0, 2\pi]$, $s \in (0, 2\pi]$ and $t \in [0, 2\pi)$, then

The following definition is taken from [15].

Definition 1.1. Functions $(\sigma_1, \rho_1) \in \mathbf{AC}[0, 2\pi] \times \mathbf{BV}[0, 2\pi]$ are said to be *lower functions* of (1.1), if ρ_1^{sing} is nondecreasing on $[0, 2\pi]$,

$$\sigma_1'(t) = \rho_1(t), \quad \rho_1'(t) \ge f(t, \sigma_1(t), \rho_1(t)) \quad \text{for a.e.} \quad t \in [0, 2\pi],$$
$$\sigma_1(0) = \sigma_1(2\pi), \quad \rho_1(0+) \ge \rho_1(2\pi-).$$

Similarly, functions $(\sigma_2, \rho_2) \in \mathbf{AC}[0, 2\pi] \times \mathbf{BV}[0, 2\pi]$ are said to be *upper functions* of (1.1), if ρ_2^{sing} is nonincreasing on $[0, 2\pi]$,

$$\sigma_2'(t) = \rho_2(t), \quad \rho_2'(t) \le f(t, \sigma_2(t), \rho_2(t)) \quad \text{for a.e.} \quad t \in [0, 2\pi],$$
$$\sigma_2(0) = \sigma_2(2\pi), \quad \rho_2(0+) \le \rho_2(2\pi-).$$

Let us choose an arbitrary number $\mu \in (-\infty, 0)$ and define operators

$$L_\mu : \operatorname{dom} L_\mu \to \mathbf{L}[0, 2\pi], \qquad x \mapsto x'' + \mu x,$$
$$N_\mu : \mathbf{C}^1[0, 2\pi] \to \mathbf{L}[0, 2\pi], \qquad x \mapsto f(\cdot, x(\cdot), x'(\cdot)) + \mu\, x,$$

where

$$\operatorname{dom} L_\mu = \{x \in \mathbf{AC}^1[0, 2\pi] : x(0) = x(2\pi),\, x'(0) = x'(2\pi)\}.$$

The linear bounded operator L_μ has its bounded inverse $L_\mu^{-1} : \mathbf{L}[0, 2\pi] \to \operatorname{dom} L_\mu$ and if we denote $L_\mu^+ = iL_\mu^{-1}$ where $i : \mathbf{AC}^1[0, 2\pi] \to \mathbf{C}^1[0, 2\pi]$ is the embedding, the operator $L_\mu^+ N_\mu$ is compact and the problem (1.1) is equivalent to an operator equation

$$(I - L_\mu^+ N_\mu)x = 0. \tag{1.5}$$

The degree theory implies that if for some open bounded set $\Omega \subset \mathbf{C}^1[0, 2\pi]$ the relation

$$\deg(I - L_\mu^+ N_\mu, \Omega) \ne 0$$

is true, then the operator $L_\mu^+ N_\mu$ has a fixed point in Ω which means that the problem (1.1) has a solution in Ω. Such set Ω can be found by means of strict lower and upper functions of (1.1) which are defined in the following way.

Definition 1.2. Lower functions (σ_1, ρ_1) of (1.1) are called *strict* if σ_1 does not satisfy the equation in (1.1) a.e. on $[0, 2\pi]$ and if there is $\varepsilon > 0$ such that

$$\rho_1'(t) \ge f(t, x, y) \quad \text{for a.e. } t \in [0, 2\pi] \text{ and all } (x, y) \in [\sigma_1(t), \sigma_1(t)+\varepsilon] \times [\rho_1(t)-\varepsilon, \rho_1(t)+\varepsilon].$$

Similarly, upper functions (σ_2, ρ_2) of (1.1) are called *strict* if σ_2 does not satisfy the equation in (1.1) a.e. on $[0, 2\pi]$ and if there is $\varepsilon > 0$ such that

$$\rho_2'(t) \le f(t, x, y) \quad \text{for a.e. } t \in [0, 2\pi] \text{ and all } (x, y) \in [\sigma_2(t)-\varepsilon, \sigma_2(t)] \times [\rho_2(t)-\varepsilon, \rho_2(t)+\varepsilon].$$

Theorem 1.3. *Let* (σ_1, ρ_1) *and* (σ_2, ρ_2) *be strict lower and upper functions of* (1.1) *with*

$$\sigma_1(t) < \sigma_2(t) \quad \text{on } [0, 2\pi], \tag{1.6}$$

and let there exist $m \in \mathbf{L}[0, 2\pi]$ *such that*

$$f(t, x, y) > m(t) \quad (\text{or } \ \mathrm{f(t,x,y) < m(t)}), \tag{1.7}$$

for a.e. $t \in [0,2\pi]$ *and for all* $(x,y) \in [\sigma_1(t),\sigma_2(t)] \times [-M_2,M_2]$ *with* M_2 *by* (1.2). *Further, let*

$$\Omega_1 = \{x \in \mathbf{C}^1[0,2\pi] : \sigma_1(t) < x(t) < \sigma_2(t) \quad \textit{on } [0,2\pi],\ \|x'\|_{\mathbf{C}} < \|m\|_1\}$$

and let $L_\mu^+ N_\mu$ *be the operator from* (1.5) *with* $\mu < 0$. *Then*

$$\deg(I - L_\mu^+ N_\mu, \Omega_1) = 1.$$

Proof. We can argue as in the proof of Theorem 2.4 [15] with the following differences. Here, we have $w(y) = y$, so the consideration in [15] concerning w and $\widetilde{w}$ can be omitted and we can suppose that f is bounded either above or below - see (1.7). Moreover, it suffices to assume that (1.7) is fulfilled for $y \in [-M_2,M_2]$ instead of $y \in \mathbf{R}$ which can be seen in the mentioned proof if we put $M_2 = c$. Finally, the results and proofs in [15] remain valid if we work with an arbitrary negative μ instead of -1. Then, this μ has to appear in auxiliary equations used in [15], for example we take the equation $u'' + \mu u = f(t,u,u') + \mu u$ instead of $u'' - u = f(t,u,u') - u$. □

Theorem 1.4. *Let* (σ_1,ρ_1) *and* (σ_2,ρ_2) *be strict lower and upper functions of* (1.1) *with*

$$\sigma_2(t) < \sigma_1(t) \quad \textit{on } [0,2\pi] \tag{1.8}$$

and let there exist $m \in \mathbf{L}[0,2\pi]$ *such that* (1.7) *is true for a.e.* $t \in [0,2\pi]$ *and for all* $(x,y) \in [A_2,B_1] \times [-M_2,M_2]$, *where* M_2 *is given by* (1.2) *and* A_2, B_1 *by* (1.4). *Further, let*

$$\begin{aligned}\Omega_2 \;=\; & \{x \in \mathbf{C}^1[0,2\pi] : A_2 < x(t) < B_1 \textit{ on } [0,2\pi],\ \|x'\|_{\mathbf{C}} < \|m\|_1,\\ & \sigma_2(t_x) < x(t_x) < \sigma_1(t_x) \textit{ for some } t_x \in [0,2\pi]\}\end{aligned}$$

and let $L_\mu^+ N_\mu$ *be the operator from* (1.5) *with* $\mu < 0$. *Then*

$$\deg(I - L_\mu^+ N_\mu, \Omega_2) = -1.$$

Proof. Let us put $\widetilde{f}(t,x,y) = f(t,\alpha(x),\beta_2(y))$, where

$$\alpha(x) = \begin{cases} B_1 & \text{if } x > B_1, \\ x & \text{if } A_2 \le x \le B_1, \\ A_2 & \text{if } x < A_2 \end{cases}$$

and β_2 is given by (1.3). Then we can argue as in the proof of Theorem 2.5 [15] working with $\widetilde{f}$ instead of f. □

If we use Theorems 1.3, 1.4 and similar arguments as in [15] we get the following existence results.

Theorem 1.5. *The assumptions of Theorem* 1.3 *with possibly nonstrict lower and upper functions and nonstrict inequalities in* (1.6) *and* (1.7) *imply the existence of a solution of* (1.1) *in the set* $\mathrm{cl}(\Omega_1)$. □

Theorem 1.6. *The assumptions of Theorem* 1.4 *with possibly nonstrict lower and upper functions and nonstrict inequalities in* (1.8) *and* (1.7) *imply the existence of a solution of* (1.1) *in the set* $\mathrm{cl}(\Omega_2)$. □

2. Multiplicity results

Here, we will prove theorems about the existence and localization of two and three solutions to (1.1). Suppose that $\eta \in [0,1]$, $n \in \mathbf{N}$, $f \in \mathrm{Car}([0,2\pi]\times\mathbf{R}^2)$, $(\sigma_i,\rho_i) \in \mathbf{AC}([0,2\pi]) \times \mathbf{BV}([0,2\pi])$, $i=1,\dots,n$, and define functions

$$\omega_i(t,\eta) = \sup\{|f(t,\sigma_i(t),\rho_i(t)) - f(t,\sigma_i(t),z)| : |\rho_i(t)-z| \le \eta\},\ i=1,\dots,n. \tag{2.1}$$

We can see that $\omega_i \in \mathrm{Car}([0,2\pi]\times[0,1])$, $i=1,\dots,n$, are non-negative, non-decreasing in the second variable and $\omega_i(t,0)=0$ a.e. on $[0,2\pi]$, $i=1,\dots,n$.

Theorem 2.1. *Let $\varepsilon \in (0,\infty)$ and let (σ_1,ρ_1), (σ_3,ρ_3) be lower functions of* (1.1), *(σ_2,ρ_2), $(\sigma_2+\varepsilon,\rho_2)$ be upper functions of* (1.1), *and*

$$\sigma_1(t) \le \sigma_2(t) < \sigma_2(t)+\varepsilon \le \sigma_3(t) \quad \text{on } [0,2\pi]. \tag{2.2}$$

Suppose that there exists $m \in \mathbf{L}([0,2\pi])$ such that

$$f(t,x,y) \ge m(t) \quad (\text{or} \ \ \le) \tag{2.3}$$

for a.e. $t \in [0,2\pi]$, all $(x,y) \in [\sigma_1(t),B_3]\times[-M_3,M_3]$, where M_3, B_3 are given by (1.2), (1.4). *Further, let Ω_1 be the set from Theorem* 1.3 *and*

$$\begin{aligned}\Omega_3 \;=\; &\{x \in \mathbf{C}^1[0,2\pi] : \sigma_1(t) < x(t) < B_3 \quad \text{on } [0,2\pi],\ \|x'\|_{\mathbf{C}} < \|m\|_1,\\ &\sigma_2(t_x)+\varepsilon < x(t_x) < \sigma_3(t_x) \text{ for some } t_x \in [0,2\pi]\}.\end{aligned}$$

Then (1.1) *has at least two different solutions u, v such that $u \in \mathrm{cl}(\Omega_1)$ and $v \in \mathrm{cl}(\Omega_3)$.*

Proof. Let us put

$$g(t,x,y) = \begin{cases} f(t,\sigma_1(t),\beta_3(y)) + x - \sigma_1(t) - \omega_1(t,\frac{\sigma_1 - x}{\sigma_1 - x + 1}) & \text{if } x < \sigma_1,\\ f(t,x,\beta_3(y)) & \text{if } \sigma_1 \le x \le B_3,\\ f(t,B_3,\beta_3(y)) & \text{if } x > B_3,\end{cases}$$

where β_3 and ω_1 are defined in (1.3) and (2.1), respectively. By Theorem 1.5, the problem

$$u'' = g(t,u,u'),\ u(0)=u(2\pi),\ u'(0)=u'(2\pi) \tag{2.4}$$

has a solution $u \in \mathrm{cl}(\Omega_1)$. Moreover, we can apply Theorem 1.6 on the problem (2.4) with the lower functions (σ_3,ρ_3) and the upper functions $(\sigma_2+\varepsilon,\rho_2)$ and get a solution v satisfying

$$\begin{cases} \varepsilon + A_2 \le v(t) \le B_3 \quad \text{on } [0,2\pi], \quad \|v'\|_{\mathbf{C}} \le \|m\|_1,\\ \sigma_2(t_v)+\varepsilon \le v(t_v) \le \sigma_3(t_v) \quad \text{for some } t_v \in [0,2\pi].\end{cases} \tag{2.5}$$

Therefore, if we prove $\sigma_1(t) \le v(t)$ on $[0,2\pi]$, we get $v \in \mathrm{cl}(\Omega_3)$. Suppose on the contrary that

$$\max_{t\in[0,2\pi]} \sigma_1(t) - v(t) = \sigma_1(t_0) - v(t_0) > 0. \tag{2.6}$$

Since $\sigma_1(0)-v(0) = \sigma_1(2\pi)-v(2\pi)$, we can restrict ourselves on the case $t_0 \in [0,2\pi)$. Let $t_0 \in (0,2\pi)$. Then, according to (2.6),

$$\rho_1(t_0+) - v'(t_0) \le 0 \le \rho_1(t_0-) - v'(t_0).$$

On the other hand since ρ_1^{sing} is nondecreasing, we get $\rho_1(t_0-) \le \rho_1(t_0+)$. Therefore $\rho_1(t_0+) - v'(t_0) = 0$. If $t_0 = 0$, then $\sigma_1 - v$ has the maximum at 2π as well and

$$\rho_1(0+) - v'(0) \le 0 \le \rho_1(2\pi-) - v'(2\pi).$$

On the other hand, by Definition 1.1, $\rho_1(0+) \ge \rho_1(2\pi-)$. Thus $\rho_1(0+) - v'(0) = 0$. Hence, we have proved

$$\rho_1(t_0+) - v'(t_0) = 0. \tag{2.7}$$

In view of (2.6) and (2.7) we can find $\varepsilon_1 \in [0,1]$ and $\delta > 0$ such that for all $t \in [t_0, t_0+\delta] \subset [0, 2\pi]$

$$|\rho_1(t) - v'(t)| \le \varepsilon_1 < \frac{\sigma_1(t) - v(t)}{\sigma_1(t) - v(t) + 1}.$$

Therefore, by Definition 1.1, (2.1) and (2.4), we have for a.e. $t \in [t_0, t_0+\delta]$

$$\begin{aligned}\rho_1'(t) - v''(t) &\ge f(t, \sigma_1(t), \rho_1(t)) - g(t, v, v')\\ &= f(t, \sigma_1(t), \rho_1(t)) - f(t, \sigma_1(t), \beta_3(v')) - v(t) + \sigma_1(t) + \omega_1(t, \frac{\sigma_1 - v}{\sigma_1 - v + 1})\\ &\ge -\omega_1(t, |\rho_1 - v'|) + \omega_1(t, \frac{\sigma_1 - v}{\sigma_1 - v + 1}) + \sigma_1(t) - v(t) \ge \sigma_1(t) - v(t) > 0.\end{aligned}$$

Hence, for all $t \in [t_0, t_0+\delta]$ we get

$$\begin{aligned}0 &< \int_{t_0}^{t} (\rho_1'(s) - v''(s))\mathrm{d}s \le \rho_1(t) - v'(t) - (\rho_1(t_0+) - v'(t_0))\\ &= \rho_1(t) - v'(t)\end{aligned}$$

and

$$0 < \int_{t_0}^{t} (\rho_1(s) - v'(s))\mathrm{d}s = \sigma_1(t) - v(t) - (\sigma_1(t_0) - v(t_0),$$

which contradicts (2.6). So, we have proved that the problem (2.4) has a solution $u \in \mathrm{cl}(\Omega_1)$ and a solution $v \in \mathrm{cl}(\Omega_3)$. Since $g = f$ on $\mathrm{cl}(\Omega_1) \cup \mathrm{cl}(\Omega_3)$, u, v are solutions of (1.1) and the relation $\mathrm{cl}(\Omega_1) \cap \mathrm{cl}(\Omega_3) = \emptyset$ quarantees that u and v are different. □

Theorem 2.2. *Let $\varepsilon \in (0, \infty)$ and let (σ_1, ρ_1), (σ_3, ρ_3) be upper functions of* (1.1), *(σ_2, ρ_2), $(\sigma_2 + \varepsilon, \rho_2)$ be lower functions of* (1.1), *and let* (2.2) *be valid. Suppose that there exists $m \in \mathbf{L}([0, 2\pi])$ such that* (2.3) *is satisfied for a.e. $t \in [0, 2\pi]$, all $(x, y) \in [A_1, \sigma_3(t)] \times [-M_3, M_3]$, where M_3, A_1 are given by* (1.2), (1.4). *Further, let*

$$\Omega_4 = \{x \in \mathbf{C}^1[0, 2\pi] : \sigma_2(t) + \varepsilon < x(t) < \sigma_3(t) \quad \text{on } [0, 2\pi],\ \|x'\|_{\mathbf{C}} < \|m\|_1\}$$

and

$$\begin{aligned}\Omega_5 \quad = \quad &\{x \in \mathbf{C}^1[0, 2\pi] : A_1 < x(t) < \sigma_3(t) \quad \text{on } [0, 2\pi], \quad \|x'\|_{\mathbf{C}} < \|m\|_1,\\ &\sigma_1(t_x) < x(t_x) < \sigma_2(t_x) \text{ for some } \ t_x \in [0, 2\pi]\}.\end{aligned}$$

Then (1.1) *has at least two different solutions u, v such that $u \in \mathrm{cl}(\Omega_4)$ and $v \in \mathrm{cl}(\Omega_5)$.*

Proof. Since Theorem 2.2 is dual to Theorem 2.1, we can put

$$g(t, x, y) = \begin{cases} f(t, A_1, \beta_3(y)) & \text{if } x < A_1,\\ f(t, x, \beta_3(y)) & \text{if } A_1 \le x \le \sigma_3(t)\\ f(t, \sigma_3(t), \beta_3(y)) + x - \sigma_3(t) + \omega_3(t, \frac{x - \sigma_3}{x - \sigma_3 + 1}) & \text{if } x > \sigma_3(t)\end{cases}$$

and use similar arguments as in the proof of Theorem 2.1. □

Theorem 2.3. *Let* $\varepsilon \in (0,\infty)$ *and let* (σ_1,ρ_1), (σ_3,ρ_3), $(\sigma_3+\varepsilon,\rho_3)$ *be lower functions of* (1.1), (σ_2,ρ_2), $(\sigma_2+\varepsilon,\rho_2)$, (σ_4,ρ_4) *be upper functions of* (1.1) *and*

$$\sigma_1(t) \le \sigma_2(t) < \sigma_2(t)+\varepsilon \le \sigma_3(t) < \sigma_3(t)+\varepsilon \le \sigma_4(t) \quad on\ [0,2\pi].$$

Suppose that there exists $m \in \mathbf{L}([0,2\pi])$ *such that (2.3) is fulfilled for a.e.* $t \in [0,2\pi]$, *all* $(x,y) \in [\sigma_1(t),\sigma_4(t)] \times [-M_4,M_4]$, *where* M_4 *is given by* (1.2). *Further, let* Ω_1 *be the set from Theorem* 1.3 *and*

$$\begin{aligned}\Omega_6 &= \{x \in \mathbf{C}^1[0,2\pi] :\ \sigma_1(t) < x(t) < \sigma_4(t)\ on\ [0,2\pi],\ \|x'\|_{\mathbf{C}} < \|m\|_1,\\ &\qquad \sigma_2(t_x)+\varepsilon < x(t_x) < \sigma_3(t_x)\ for\ some\ \ t_x \in [0,2\pi]\},\\ \Omega_7 &= \{x \in \mathbf{C}^1[0,2\pi] : \sigma_3(t)+\varepsilon < x(t) < \sigma_4(t)\ on\ [0,2\pi],\ \|x'\|_{\mathbf{C}} < \|m\|_1|\}.\end{aligned}$$

Then (1.1) *has at least three different solutions* u, v *and* w *such that* $u \in \mathrm{cl}(\Omega_1)$, $v \in \mathrm{cl}(\Omega_6)$ *and* $w \in \mathrm{cl}(\Omega_7)$.

Proof. Let us put

$$g(t,x,y) = \begin{cases} f(t,\sigma_1(t),\beta_4(y)) + x - \sigma_1(t) - \omega_1(t,\frac{\sigma_1-x}{\sigma_1-x+1}) & \text{if } x < \sigma_1,\\ f(t,x,\beta_4(y)) & \text{if } \sigma_1 \le x \le \sigma_4,\\ f(t,\sigma_4(t),\beta_4(y)) + x - \sigma_4(t) + \omega_4(t,\frac{x-\sigma_4}{x-\sigma_4+1}) & \text{if } x > \sigma_4,\end{cases}$$

where β_4 and ω_1, ω_4 are defined in (1.3) and (2.1), respectively. By Theorem 1.5, the problem (2.4) has a solution $u \in \mathrm{cl}(\Omega_1)$ and a solution $w \in \mathrm{cl}(\Omega_7)$. Further, as in the proof of Theorem 2.1, we can apply Theorem 1.6 on the problem (2.4) and get a solution v satisfying (2.5). Finally, arguing similarly as in the proof of Theorem 2.1 we get $\sigma_1(t) \le v(t) \le \sigma_4(t)$ on $[0,2\pi]$, which together with (2.5) imply that $v \in \mathrm{cl}(\Omega_6)$. Since $g = f$ on $\mathrm{cl}(\Omega_1) \cup \mathrm{cl}(\Omega_6) \cup \mathrm{cl}(\Omega_7)$ and these three sets are disjoint, we get three different solutions of (1.1). □

3. Periodic problem with a singularity

Here, we suppose that the function f in (1.1) has the form

$$f(t,x,y) = g(x) + e(t),$$

and consider the problem

$$u'' = g(u) + e(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \tag{3.1}$$

where

$$g \in \mathbf{C}(0,\infty) \quad \text{and} \quad e \in \mathbf{L}[0,2\pi]. \tag{3.2}$$

We denote by $\overline{e}$ the mean value of a function e, i.e. $\overline{e} = \frac{1}{2\pi}\int_0^{2\pi} e(t)\mathrm{d}t$. The function g can have a singularity at 0, i.e. g need not be bounded at 0.

Under the assumption

$$\lim_{x\to 0+} \int_x^1 g(\xi)\mathrm{d}\xi = \infty, \tag{3.3}$$

the existence of positive solutions to (3.1) has been studied by many authors starting from the paper [8] by Lazer and Solimini. Their results have been extended for example by [2], [3], [5], [10], [13], [16], [17], [18] and [23]. Here we bring conditions which guarantee multiplicity results for (3.1) and generalize some of the existence results mentioned above.

First, we present two lemmas which are taken from [17] and which will be useful in what follows.

Lemma 3.1. *Let* $g \in \mathbf{C}(0,\infty)$ *satisfy* (3.3). *Then there exists a sequence* $\{\varepsilon_m\}_{m=1}^{\infty} \subset (0,1)$ *such that*

$$g(\varepsilon_m) > 0 \text{ for all } m \in \mathbf{N}, \quad \lim_m \varepsilon_m = 0, \quad \lim_m g(\varepsilon_m) = \infty.$$

□

Lemma 3.2. *Let us suppose that* g *and* ε_m, $m \in \mathbf{N}$, *are from Lemma* 3.1 *and let* g *fulfil*

$$\liminf_{x\to\infty} \frac{g(x)}{x} > -\frac{1}{4}. \tag{3.4}$$

Let us put

$$g_m(x) = \begin{cases} 0 & \text{if } x < 0, \\ g(\varepsilon_m)\frac{x}{\varepsilon_m} & \text{if } x \in [0,\varepsilon_m], \\ g(x) & \text{if } x > \varepsilon_m. \end{cases}$$

Then for any $r > 0$ *and any* $e \in \mathbf{L}[0,2\pi]$ *there exists* $R > 0$ *such that*

$$u(t) \le R \quad \text{on } [0,2\pi]$$

holds for all $m \in \mathbf{N}$ *and all solutions* u *of*

$$u'' = g_m(u) + e(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \quad \min_{t\in[0,2\pi]} u(t) \le r.$$

□

Theorem 3.3. *Suppose that* (3.2) *is fulfilled. Further, let there exist* $n \in \mathbf{N}$, $n \ge 3$, *and* $a_1,\dots,a_n \in (0,\infty)$ *such that*

$$(g(x) + \overline{e})\,(-1)^i > 0 \quad \text{for all } \; x \in [a_i,b_i], \; i = 1,\dots,n, \tag{3.5}$$

where

$$b_i = a_i + \frac{\pi}{3}\,\|e-\overline{e}\|_1, \; i = 1,\dots,n, \quad \text{and} \quad a_i > b_{i-1}, \, i = 2,\dots,n. \tag{3.6}$$

If n *is odd, suppose moreover that either*

$$\limsup_{x\to\infty} g(x) < \infty, \tag{3.7}$$

or (3.4) *is satisfied. Then the problem* (3.1) *has at least* $n-1$ *different positive solutions.*

Let us note that we need not the assumption (3.3) in Theorem 3.3. In fact, the behaviour of g can be arbitrary in a right neighbourhood of 0. Therefore we use the following little modification of Lemma 3.2.

Lemma 3.4. *Suppose that the assumptions of Theorem* 3.3 *are satisfied. Then there exists* $R > b_n$ *such that*

$$u(t) \le R \quad \text{on } [0, 2\pi]$$

for any solution u *of* (3.1) *with the property*

$$a_1 \le \min_{t\in[0,2\pi]} u(t) \le b_n. \tag{3.8}$$

Proof. First, suppose that (3.7) is true. Then there is $M > 0$ such that $g(x) \le M$ for all $x \in [a_1, \infty)$. Let u be a solution of (3.1) satisfying (3.8). Then $\|u'\|_{\mathbf{C}} \le M + \|e\|_1$ and so, for all $t \in [0, 2\pi]$

$$u(t) \le b_n + 2\pi(M + \|e\|_1) = R.$$

Now, let (3.7) be false but (3.4) be true. Assume the contrary, i.e. that there is a sequence $\{u_k\}$ of solutions of (3.1) satisfying

$$a_1 \le \min_{t\in[0,2\pi]} u_k(t) \le b_n, \quad \lim_k \max_{t\in[0,2\pi]} u_k(t) = \infty.$$

In particular, for any $k \in \mathbf{N}$, there is $t_k \in [0, 2\pi]$ such that $u_k(t_k) = b_n$. Further, we can extend the functions u_k, $k \in \mathbf{N}$ and e on $\mathbf{R}$ and get that

$$u_k''(t) = g(u_k(t)) + e(t) \quad \text{for a.e. } t \in \mathbf{R}$$

is true for any $k \in \mathbf{N}$. Now, we can do the same computations as in the proof of [17, Lemma 3.4] and get a contradiction. □

Proof of Theorem 3.3. Let us put

$$g_1(x) = \begin{cases} g(R) & \text{if } x > R, \\ g(x) & \text{if } a_1 \le x \le R, \\ g(a_1) & \text{if } x < a_1, \end{cases} \tag{3.9}$$

where R is the constant from Lemma 3.4. Then $g_1 + e \in \mathrm{Car}([0, 2\pi] \times \mathbf{R})$ fulfils all conditions of Theorem 3.3 and moreover, for a.e. $t \in [0, 2\pi]$ and all $x \in \mathbf{R}$ it satisfies (2.3) with $m(t) = e(t) + \max_{x\in[a_1,R]} g(x)$. The conditions (3.5) and (3.6) imply the existence of an $\varepsilon > 0$ such that

$$(g_1(x) + \overline{e})\,(-1)^i \ge 0 \quad \text{for all } \; x \in [a_i - \varepsilon, b_i], \; i = 1, \dots, n, \tag{3.10}$$

with

$$a_i - \varepsilon > b_{i-1}, \; i = 2, \dots, n. \tag{3.11}$$

Therefore, by (3.10), we have

$$(g_1(x) + e(t))\,(-1)^i \ge (e(t) - \overline{e})\,(-1)^i \quad \text{for a.e. } t \in [0, 2\pi] \text{ and all } x \in [a_i - \varepsilon, b_i], \; i = 1, \dots, n. \tag{3.12}$$

Thus, if we put $b(t) = e(t) - \overline{e}$ and apply [16, Propositions 2.1 and 2.2] on the problem

$$u'' = g_1(u) + e(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \tag{3.13}$$

we can construct lower and upper functions (σ_i, ρ_i), $i = 1, \dots, n$, of the problem (3.13). Namely, if we put

$$\gamma(t) = -\frac{1}{2\pi} \int_0^{2\pi} \big(\int_0^{2\pi} G(t,s)\, b(s) \mathrm{d}s \big) \mathrm{d}t + \int_0^{2\pi} G(t,s)\, b(s) \mathrm{d}s + \frac{\pi}{6} \|b\|_1,$$

where G is the Green function of the problem $v'' = 0, v(0) = v(2\pi) = 0$, then the functions $(\sigma_1, \rho_1) = (\gamma(t) + a_1, \gamma'(t))$ are lower functions. It follows from (3.11) and from the fact that $\sigma_1(t) \in [a_1, b_1]$ on $[0, 2\pi]$. Similarly the functions $(\sigma_2, \rho_2) = (\gamma(t) + a_2 - \varepsilon, \gamma'(t))$ are upper functions. Moreover, having in mind that $\sigma_2(t) \in [a_2 - \varepsilon, b_2 - \varepsilon]$ on $[0, 2\pi]$ and that (3.12) is valid on $[a_2 - \varepsilon, b_2]$, we see that $(\sigma_2 + \varepsilon, \rho_2)$ are upper functions of (3.13), as well. Repeating these arguments for i odd (even), we find lower (upper) functions (σ_i, ρ_i), $(\sigma_i + \varepsilon, \rho_i)$, $i = 3, \dots, n$ to the problem (3.13). Moreover, the following ordering

$$(3.14) \quad \sigma_1 < \sigma_2 < \sigma_2 + \varepsilon < \sigma_3 < \sigma_3 + \varepsilon < \ldots < \sigma_{n-1} < \sigma_{n-1} + \varepsilon < \sigma_n \quad \text{on} \quad [0, 2\pi].$$

is valid.

Now, let us discuss the multiplicity of solutions. If $n = 3$, Theorem 2.1 gives two different solutions $u_1 \in cl(\Omega_1)$ and $u_2 \in cl(\Omega_3)$ for the problem (3.13). Provided $n = 4$, we use Theorem 2.3 and get for the problem (3.13) three different solutions $u_1 \in cl(\Omega_1)$, $u_2 \in cl(\Omega_6)$ and $u_3 \in cl(\Omega_7)$. Let $n = 5$. Having three above solutions u_1, u_2, u_3, we can use Theorem 2.1 for the string of functions

$$\sigma_3 + \varepsilon < \sigma_4 < \sigma_4 + \varepsilon < \sigma_5 \quad \text{on } [0, 2\pi]$$

and get two solutions of (3.13), the first in $\mathrm{cl}(\Omega_7)$ and the second in $cl(\Omega_8)$, where

$$\begin{aligned} \Omega_8 \;\; = \;\; & \{x \in \mathbf{C}^1[0, 2\pi] : \sigma_3(t) + \varepsilon < x(t) < B_5 \text{ on } [0, 2\pi],\ \|x'\|_{\mathbf{C}} < \|m\|_1, \\ & \sigma_4(t_x) + \varepsilon < x(t_x) < \sigma_5(t_x) \text{ for some } \ t_x \in [0, 2\pi]\}. \end{aligned}$$

(For B_5 see (1.4).) But the first solution, which is in $\mathrm{cl}(\Omega_7)$, can be the same as u_3. Therefore we can get only four different solutions $u_1 \in \mathrm{cl}(\Omega_1)$, $u_2 \in \mathrm{cl}(\Omega_6)$, $u_3 \in \mathrm{cl}(\Omega_7)$ and $u_4 \in \mathrm{cl}(\Omega_8)$ for the problem (3.13). Let $n = 6$. As before, we have four solutions u_1, u_2, u_3, u_4 and the string

$$\sigma_3 + \varepsilon < \sigma_4 < \sigma_4 + \varepsilon < \sigma_5 < \sigma_5 + \varepsilon < \sigma_6 \quad \text{on } [0, 2\pi].$$

We use Theorem 2.3 and get three solutions of (3.13), the first in $\mathrm{cl}(\Omega_7)$, the second in $\mathrm{cl}(\Omega_9)$ and the third in $\mathrm{cl}(\Omega_{10})$, where

$$\begin{aligned} \Omega_9 \;\; = \;\; & \{x \in \mathbf{C}^1[0, 2\pi] : \sigma_3(t) + \varepsilon < x(t) < \sigma_6(t) \text{ on } [0, 2\pi],\ \|x'\|_{\mathbf{C}} < \|m\|_1, \\ & \sigma_4(t_x) + \varepsilon < x(t_x) < \sigma_5(t_x) \text{ for some } \ t_x \in [0, 2\pi]\} \end{aligned}$$

and

$$\Omega_{10} = \{x \in \mathbf{C}^1[0, 2\pi] : \sigma_5(t) + \varepsilon < x(t) < \sigma_6(t) \text{ on } [0, 2\pi],\ \|x'\|_{\mathbf{C}} < \|m\|_1\}.$$

But since $u_3 \in cl(\Omega_7)$, $u_4 \in cl(\Omega_8)$ and $\mathrm{cl}(\Omega_8 \cap \mathrm{cl}(\Omega_9) \neq \emptyset$, we can guarantee only five different solutions $u_1 \in \mathrm{cl}(\Omega_1)$, $u_2 \in \mathrm{cl}(\Omega_6)$, $u_3 \in \mathrm{cl}(\Omega_7)$, $u_4 \in \mathrm{cl}(\Omega_9)$ and $u_5 \in \mathrm{cl}(\Omega_{10})$ for the problem (3.13). Since $g = g_1$ on each set $\mathrm{cl}(\Omega_i)$, $i \in \{1, \dots, 10\}$, the obtained solutions are also solutions of (3.1). For $n \geq 7$ we can use similar arguments. □

Theorem 3.5. *Suppose* (3.2), (3.3) *and*

$$(3.15) \qquad \liminf_{x \to 0+} g(x) > -\infty.$$

Further, let there exist $n \in \mathbf{N}$, $n \geq 3$, *and* $a_1, \dots, a_n \in (0, \infty)$ *such that the conditions*

$$(3.16) \qquad (g(x) + \overline{e})\,(-1)^i < 0 \quad \textit{for all} \ \ x \in [a_i, b_i],\ i = 1, \dots, n,$$

and (3.6) *are valid. If* n *is even, suppose moreover* (3.4). *Then the problem* (3.1) *has at least* $n - 1$ *different positive solutions.*

Proof. For n odd, let us put $R = b_n$. If n is even, let $R \geq b_n$ be the constant given by Lemma 3.2 for $r = b_n$. By (3.2) and (3.15) we have $g_* := \inf_{x\in(0,R]} g(x) \in \mathbf{R}$. Put $K = \|e\|_1 + |g_*|$ and

$$K^* = K\,\|e\|_1 + \int_{a_1}^{R} |g(x)|\mathrm{d}x.$$

It follows from (3.3) and Lemma 3.1 that we can choose $\varepsilon \in \{\varepsilon_m\}_{m=1}^{\infty}$ such that $\varepsilon \in (0, a_1)$ and

$$\int_{\varepsilon}^{a_1} g(\xi)\mathrm{d}\xi > K^*. \tag{3.17}$$

Define

$$g_1(x) = \begin{cases} 0 & \text{if } x < 0, \\ g(\varepsilon)\frac{x}{\varepsilon} & \text{if } x \in [0,\varepsilon), \\ g(x) & \text{if } x \in [\varepsilon, R), \\ g(R) & \text{if } x \geq R. \end{cases} \tag{3.18}$$

Then $g_1 + e \in \mathrm{Car}([0, 2\pi] \times \mathbf{R})$ fulfils (3.16) and moreover, for a. e. $t \in [0, 2\pi]$ and all $x \in \mathbf{R}$ it satisfies (2.3) with $m(t) = g_* + e(t)$. In the same way as in the proof of Theorem 3.3 we get lower and upper functions for the problem (3.13) and the ordered string (3.14). The only difference is that now (σ_1, ρ_1) are upper functions, (σ_2, ρ_2) are lower functions, and so on.

Let us discuss the multiplicity of solutions. If $n = 3$, Theorem 2.2 gives two different solutions $u_1 \in \mathrm{cl}(\Omega_4)$ and $u_2 \in \mathrm{cl}(\Omega_5)$ for the problem (3.13). Provided $n = 4$, we have solutions u_1, u_2 as before and, moreover, we can use Theorem 2.1 for the string of functions

$$\sigma_2 + \varepsilon < \sigma_3 < \sigma_3 + \varepsilon < \sigma_4 \quad \text{on } [0, 2\pi]$$

and get two solutions of (3.13), the first in $\mathrm{cl}(\Omega_4)$ and the second in $\mathrm{cl}(\Omega_{11})$, where

$$\begin{aligned}\Omega_{11} = \{& x \in \mathbf{C}^1[0, 2\pi] : \sigma_2(t) + \varepsilon < x(t) < B_4 \text{ on } [0, 2\pi],\ \|x'\|_{\mathbf{C}} < \|m\|_1, \\ & \sigma_3(t_x) + \varepsilon < x(t_x) < \sigma_4(t_x) \text{ for some } t_x \in [0, 2\pi]\}.\end{aligned}$$

(For B_4 see (1.4).) But the first solution, which is in $\mathrm{cl}(\Omega_4)$, can be the same as u_1. Therefore we can get only three different solutions $u_1 \in \mathrm{cl}(\Omega_4)$, $u_2 \in \mathrm{cl}(\Omega_5)$ and $u_3 \in \mathrm{cl}(\Omega_{11})$ for the problem (3.13). Finally, suppose that $n = 5$. (For $n \geq 6$ we can argue similarly.) As before, we have three solutions u_1, u_2, u_3 and the string

$$\sigma_2 + \varepsilon < \sigma_3 < \sigma_3 + \varepsilon < \sigma_4 < \sigma_4 + \varepsilon < \sigma_5 \quad \text{on } [0, 2\pi].$$

We use Theorem 2.3 and also get three solutions of (3.13), the first in $\mathrm{cl}(\Omega_4)$, the second in $\mathrm{cl}(\Omega_{12})$ and the third in $\mathrm{cl}(\Omega_{13})$, where

$$\begin{aligned}\Omega_{12} = \{& x \in \mathbf{C}^1[0, 2\pi] : \sigma_2(t) + \varepsilon < x(t) < \sigma_5(t) \text{ on } [0, 2\pi],\ \|x'\|_{\mathbf{C}} < \|m\|_1, \\ & \sigma_3(t_x) + \varepsilon < x(t_x) < \sigma_4(t_x) \text{ for some } t_x \in [0, 2\pi]\}\end{aligned}$$

and

$$\Omega_{13} = \{x \in \mathbf{C}^1[0, 2\pi] : \sigma_4(t) + \varepsilon < x(t) < \sigma_5(t) \text{ on } [0, 2\pi],\ \|x'\|_{\mathbf{C}} < \|m\|_1\}.$$

But since $u_1 \in \mathrm{cl}(\Omega_4)$, $u_3 \in \mathrm{cl}(\Omega_{11})$ and $\mathrm{cl}(\Omega_{11}) \cap \mathrm{cl}(\Omega_{12}) \neq \emptyset$, we can guarantee only four different solutions $u_1 \in \mathrm{cl}(\Omega_4)$, $u_2 \in \mathrm{cl}(\Omega_5)$, $u_3 \in \mathrm{cl}(\Omega_{12})$ and $u_4 \in \mathrm{cl}(\Omega_{13})$ for the problem (3.13).

From the definition of the sets Ω_j, $j \in \{4, 5, 11, 12, 13\}$ it follows that each obtained solution u_i, $i \in \{1, 2, 3, 4\}$ satisfies $\min_{t\in[0,2\pi]} u_i(t) \leq b_n$.

Now, let us suppose, that u is an arbitrary solution of (3.13) with g_1 defined by (3.18) and that $\min_{t\in[0,2\pi]} u(t) \le b_n$. We need to prove that u is a solution of the problem (3.1). Lemma 3.2 implies that $u(t) \le R$ on $[0, 2\pi]$. Let us show that $u(t) \ge \varepsilon$ holds on $[0, 2\pi]$. Let t_0 and $t_1 \in [0, 2\pi]$ be such that

$$u(t_0) = \min_{t\in[0,2\pi]} u(t) \quad \text{and} \quad u(t_1) = \max_{t\in[0,2\pi]} u(t).$$

Clearly, $a_1 \le u(t_1) \le R$. With respect to the periodic boundary conditions we have $u'(t_0) = u'(t_1) = 0$. Now, multiplying the differential relation $u''(t) = e(t) + g_1(u(t))$ by $u'(t)$ and integrating over $[t_0, t_1]$, we get

$$\int_{u(t_0)}^{u(t_1)} g_1(\xi)\mathrm{d}\xi = -\int_{t_0}^{t_1} e(t)\, u'(t)\mathrm{d}t \le K\,\|e\|_1,$$

which leads to

$$\int_{u(t_0)}^{a_1} g_1(\xi)\mathrm{d}\xi \le K\,\|e\|_1 + \int_{a_1}^{R} |g_1(\xi)|\mathrm{d}\xi = K^*.$$

On the other hand, with respect to (3.17) and (3.18),

$$\int_{\varepsilon}^{a_1} g_1(\xi)\mathrm{d}\xi > K^*.$$

Therefore $u(t_0) > \varepsilon$ and u is a solution to (3.1). □

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