Singular Nonlinear Problem for Ordinary Differential Equation of the Second Order*

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Abstract

The paper deals with the singular nonlinear problem

$$u''(t) + f(t, u(t), u'(t)) = 0,$$

$$u(0) = 0, \quad u'(T) = \psi(u(T)),$$

where $f \in Car((0,T) \times D)$, $D = (0, \infty) \times \mathbb{R}$. We prove the existence of a positive solution on $(0,T]$ to this problem under the assumption that the function $f(t,x,y)$ is nonnegative and can have time singularities at $t = 0$, $t = T$ and space singularity at $x = 0$. The proof is based on the Schauder fixed point theorem and on the method of a priori estimates.

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1 Introduction

We will study a singular boundary value problem with nonlinear boundary conditions

$$u''(t) + f(t, u(t), u'(t)) = 0 \quad \text{for a. e. } t \in [0,T], \tag{1}$$

$$u(0) = 0, \quad u'(T) = \psi(u(T)), \tag{2}$$

where f satisfies the Carathéodory conditions on $(0,T) \times D$, $[0,T] \subset \mathbb{R}$, $D = (0,\infty) \times \mathbb{R}$. The function $f(t,x,y)$ is allowed to have time singularities at $t = 0$, $t = T$ and space singularity at $x = 0$, the function ψ is continuous on $[0, \infty)$.

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For a given interval $[a,b] \subset \mathbb{R}$ assume that $L^1[a,b]$ denotes the set of all measurable functions defined a. e. on $[a,b]$ which are Lebesgue integrable on $[a,b]$, equipped with the norm

$$\|u\|_1 = \int_a^b |u(t)|\,\mathrm{d}t \quad \text{for each } u \in L^1[a,b];$$

$C^0[a,b]$ (or $C^1[a,b]$) denotes the set of all functions which are continuous (or have continuous first derivatives) on $[a,b]$, with the norm $\|u\|_\infty = \max\{|u(t)|$ $t \in [a,b]\}$ (or $\|u\|_{C^1[a,b]} = \|u\|_\infty + \|u'\|_\infty$); $AC^1[a,b]$ denotes the set of all functions which have absolutely continuous first derivatives on $[a,b]$. We say that $f : [a,b] \times D \to \mathbb{R}$, $D \subset \mathbb{R}^2$ satisfies the *Carathéodory conditions* on $[a,b] \times D$ if f has the following properties: (i) for each $(x,y) \in D$ the function $f(\cdot,x,y)$ is measurable on $[a,b]$; (ii) for almost each $t \in [a,b]$ the function $f(t,\cdot,\cdot)$ is continuous on D; (iii) for each compact set $K \subset D$ there exists a function $m_K(t) \in L^1[a,b]$ such that $|f(t,x,y)| \le m_K(t)$ for a. e. $t \in [a,b]$ and all $(x,y) \in K$. For the set of functions satisfying the *Carathéodory conditions* on $[a,b] \times D$ we write $Car([a,b] \times D)$. By $f \in Car((0,T) \times D)$ we mean that $f \in Car([a,b] \times D)$ for each $[a,b] \subset (0,T)$ and $f \notin Car([0,T] \times D)$.

Singular problems have been studied by many authors (see [1] – [6] and references therein). For instance a similar problem is considered in [3], where the right–hand side function is continuous and it is allowed to change its sign. Moreover, the singularity of f is possible in space variable x. In this work, we consider the function f, which is non–negative and can have both time and space singularities. Here, we found effective necessary conditions for solvability of the problem (1), (2). The arguments are based on the ideas of the paper [5], where the non–linear singular problem with mixed boundary conditions

$$u'' + f(t,u,u') = 0, \qquad u'(0) = 0, \quad u(T) = 0$$

is investigated.

Definition 1 Let $f \in Car((0,T) \times D)$, where $D = (0,\infty) \times \mathbb{R}$. We say that f has a time singularity at $t = 0$ and/or at $t = T$, if there exists $(x_1,y_1) \in D$ and/or $(x_2,y_2) \in D$ such that

$$\int_0^\epsilon |f(t,x_1,y_1)|\,\mathrm{d}t = \infty \quad \text{and/or} \quad \int_{T-\epsilon}^T |f(t,x_2,y_2)|\,\mathrm{d}t = \infty$$

for each sufficiently small $\epsilon > 0$. The point $t = 0$ and/or $t = T$ will be called a singular point of f.
We say that f has a space singularity at $x = 0$ if

$$\limsup_{x \to 0+} |f(t,x,y)| = \infty \quad \text{for a. e. } t \in [0,T] \text{ and for some } y \in \mathbb{R}.$$

Here, we will treat with following definition of the solution of the problem (1), (2).

Definition 2 By a solution of the problem (1), (2) we understand a function $u \in AC^1[0,T]$ satisfying the differential equation (1) and the boundary conditions (2).

2 Regular problem, lower and upper function

In order to prove the main result we need the existence theorem for regular boundary value problems. Let us consider a problem

$$u'' + h(t, u, u') = 0, \qquad g_1(u(0), u'(0)) = 0, \quad g_2(u(T), u'(T)) = 0, \qquad (3)$$

where $h \in Car([0,T] \times \mathbb{R}^2)$, g_1, $g_2 : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

Definition 3 A function $u \in AC^1[0,T]$ which satisfies the differential equation in (3) a. e. in $[0,T]$ and fulfils the boundary conditions in (3) is called a solution of the problem (3).

In the existence theorem the concept of upper and lower function will be needed.

Definition 4 A function $\sigma \in AC^1[0,T]$ is called a lower function of the problem (3) if

$$\sigma''(t) + h(t, \sigma(t), \sigma'(t)) \geq 0 \qquad \text{for a. e. } t \in [0,T]$$

and

$$g_1(\sigma(0), \sigma'(0)) \geq 0, \qquad g_2(\sigma(T), \sigma'(T)) \geq 0.$$

If these inequalities are reversed, the function σ is called an upper function of the problem (3).

For σ_1, $\sigma_2 \in AC^1[0,T]$ such that $\sigma_1 \leq \sigma_2$ on $[0,T]$ we can define a function $\gamma : [0,T] \times \mathbb{R} \to \mathbb{R}$ by

$$\gamma(t, x) = \max\{\sigma_1(t), \min\{x, \sigma_2(t)\}\} \qquad \text{for each } t \in [0,T], \ x \in \mathbb{R}. \qquad (4)$$

Now, we introduce the following result [7, Lemma 2]. It is fundamental in the proof of Lemma 6.

Lemma 5 *For $u \in C^1[0,T]$ the two following properties hold:*

a) $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t, u(t))$ *exists for a. e.* $t \in [0,T]$.

b) If $u_m \in C^1[0,T]$ *and* $u_m \to u$ *in* $C^1[0,T]$*, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t, u_m(t)) \to \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t, u(t)) \quad \textit{for a. e. } t \in [0,T].$$

Lemma 6 *Let $h \in Car([0,T] \times \mathbb{R}^2)$, g_1, $g_2 : \mathbb{R}^2 \to \mathbb{R}$ be continuous functions and σ_1, σ_2 be lower and upper function of the problem (3), respectively, such that*

$$\sigma_1(t) \leq \sigma_2(t) \qquad \textit{for each } t \in [0,T].$$

Further, assume that there exists $\varphi \in L^1[0,T]$ such that

$$|h(t, x, y)| \leq \varphi(t)$$

for a. e. $t \in [0,T]$, each $x \in [\sigma_1(t), \sigma_2(t)]$ and each $y \in \mathbb{R}$, g_1 is nondecreasing in the second variable and g_2 is nonincreasing in the second variable. Then there exists a solution u of the problem (3) such that

$$\sigma_1 \leq u \leq \sigma_2 \qquad \textit{on } [0,T]. \qquad (5)$$

Proof. Let us define functionals A, $B : C^1[0,T] \to \mathbb{R}$ by

$$A(u) = \gamma(0, u(0) + g_1(u(0), u'(0))),$$

$$B(u) = \gamma(T, u(T) + g_2(u(T), u'(T)))$$

for each $u \in C^1[0,T]$. Lemma 5 allows us to define for each $u \in C^1[0,T]$ a function $\tilde{h}_u : [0,T] \to \mathbb{R}$ such that

$$\tilde{h}_u(t) = h(t, \gamma(t, u(t)), \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t, u(t))) \quad \text{for a. e. } t \in [0,T].$$

Obviously, there exists $\bar{h} \in L^1[0,T]$ such that

$$|\tilde{h}_u(t)| \le \bar{h}(t) \qquad \text{for a. e. } t \in [0,T] \text{ and each } u \in C^1[0,T].$$

Consider an auxiliary problem

$$\left\{ \begin{array}{rcll} u''(t) & = & -\tilde{h}_u(t) & \text{a. e. } t \in [0,T], \\ u(0) & = & A(u), & \\ u(T) & = & B(u). & \end{array} \right. \tag{6}$$

Let us define a mapping $F : C^1[0,T] \to C^1[0,T]$ by

$$(Fu)(t) = -\int_0^T G(t,s)\tilde{h}_u(s)\,\mathrm{d}s + \frac{T-t}{T}A(u) + \frac{t}{T}B(u)$$

for each $u \in C^1[0,T]$ and $t \in [0,T]$, where

$$G(t,s) = \left\{ \begin{array}{ll} \frac{t(s-T)}{T} & \text{for} \quad 0 \le t \le s \le T, \\ \frac{s(t-T)}{T} & \text{for} \quad 0 \le s < t \le T. \end{array} \right.$$

We can check that each fixed point of the operator F is a solution of the problem (6). Using the Schauder Fixed Point Theorem we will prove that there exists a fixed point u of the operator F satisfying the inequalities (5) and such that u is a solution of the problem (3).
It is easy to see that

$$\|Fu\|_\infty \le T\|\bar{h}\|_1 + 2(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty)$$

and

$$\|(Fu)'\|_\infty \le \|\bar{h}\|_1 + \frac{2}{T}(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty),$$

i. e. that there exist $K > 0$ and $\Omega = \{u \in C^1[0,T] : \|u\|_{C^1[0,T]} \le K\}$, such that $F(\Omega) \subset \Omega$. It suffices to prove that the set

$$F' = \{(Fu)' : u \in \Omega\}$$

is relatively compact in $C^0[0,T]$. Obviously, for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $u \in \Omega$ and s_1, $s_2 \in [0,T]$, $|s_2 - s_1| < \delta$, relations

$$|(Fu)'(s_2) - (Fu)'(s_1)| = |\int_{s_1}^{s_2} \tilde{h}_u(s)\,\mathrm{d}s| \le |\int_{s_1}^{s_2} \bar{h}(s)\,\mathrm{d}s| < \epsilon$$

are valid. Now, applying Arzelà–Ascoli Theorem we get that $F(\Omega)$ is relatively compact in $C^1[0,T]$. Thus, there exists a fixed point u of the operator F and $u \in AC^1[0,T]$. We will prove that relations (5) are satisfied. From boundary conditions in (6) it follows that

$$\sigma_1(0) \le u(0) \le \sigma_2(0) \quad \text{and} \quad \sigma_1(T) \le u(T) \le \sigma_2(T).$$

Assume that there exists $\tau \in (0,T)$ such that $u(\tau) < \sigma_1(\tau)$. Then there exist $\xi \in (0,T)$ and $\delta > 0$ such that

$$(u-\sigma_1)(\xi) = \min_{t\in[0,T]} (u-\sigma_1)(t) < 0$$

and

$$0 > (u-\sigma_1)(t) > (u-\sigma_1)(\xi) \qquad \text{for each } t \in (\xi, \xi+\delta). \tag{7}$$

Obviously, $(u-\sigma_1)'(\xi) = 0$ and $u(t) < \sigma_1(t)$ for each $t \in (\xi, \xi+\delta)$. According to the definition of $\tilde{h}_u$, we have

$$(u-\sigma_1)'(t) \le \int_\xi^t [-\tilde{h}_u(s) + h(s, \sigma_1(s), \sigma_1'(s))]\, \mathrm{d}s = 0,$$

for each $t \in (\xi, \xi+\delta)$, which contradicts (7). Similarly, we can prove that $u \le \sigma_2$ on $[0,T]$. From (5) it follows that u satisfies the differential equation in (3). It suffices to prove that u satisfies boundary conditions in (3), i. e. according to (5) and definition of γ, to prove inequalities

$$\sigma_1(0) \le u(0) + g_1(u(0), u'(0)) \le \sigma_2(0) \tag{8}$$

and

$$\sigma_1(T) \le u(T) + g_2(u(T), u'(T)) \le \sigma_2(T).$$

Let the first inequality in (8) be not satisfied. Then according to (5) we have

$$u(0) = \sigma_1(0), \quad 0 > g_1(\sigma_1(0), u'(0)) \quad \text{and} \quad u'(0) \ge \sigma_1'(0).$$

Using the monotonicity of g_1 we have $0 > g_1(\sigma_1(0), \sigma_1'(0))$, which contradicts the definition of a lower function. The remaining inequalities can be proven in a similar way. □

3 Main result

Now, we are ready to prove the existence theorem for singular problem (1), (2).

Theorem 7 *Assume that $f \in Car((0,T)\times D)$, where $T > 0$, $D = (0,\infty)\times\mathbb{R}$, with possible time singularities at $t = 0$ and/or $t = T$ and a space singularity at $x = 0$. Further assume that there exist $\epsilon \in (0,1)$, $\nu \in (0,T)$, $c \in (\nu,\infty)$ and $\epsilon_0 \in (0,\infty)$ such that*

$$f(t, ct, c) = 0 \qquad \text{for a. e. } t \in [0,T], \tag{9}$$

$$0 \leq f(t,x,y) \qquad \textit{for a. e. } t \in [0,T], \ \textit{each } x \in (0,ct], \ y \in [\min_{t\in[0,cT]} \psi(t), c], \tag{10}$$

$$\epsilon \leq f(t,x,y) \qquad \textit{for a. e. } t \in [T-\nu,T], \ \textit{each } x \in (0,ct], \ y \in (-\epsilon_0,\nu], \tag{11}$$

$$0 = \psi(0), \quad \psi(cT) \leq c \tag{12}$$

hold. Then there exists a solution u *of the problem* (1), (2) *such that*

$$0 < u(t) \leq ct \tag{13}$$

for each $t \in (0,T]$.

Proof. Step 1. Let $k \in \mathbb{N}$, $k \geq 3/T$. We define

$$\alpha_k(t,x) = \begin{cases} c/k & \text{for} \quad x < c/k, \\ x & \text{for} \quad c/k \leq x \leq ct, \\ ct & \text{for} \quad x > ct, \end{cases}$$

for each $t \in [1/k, T-1/k]$, $x \in \mathbb{R}$,

$$\beta(y) = \begin{cases} \min_{t\in[0,cT]} \psi(t) & \text{for} \quad y < \min_{t\in[0,cT]} \psi(t), \\ y & \text{for} \quad \min_{t\in[0,cT]} \psi(t) \leq y \leq c, \\ c & \text{for} \quad y > c, \end{cases}$$

and

$$\gamma(y) = \begin{cases} \epsilon & \text{for} \quad y < \nu, \\ \epsilon\frac{c-y}{c-\nu} & \text{for} \quad \nu \leq y \leq c, \\ 0 & \text{for} \quad y > c, \end{cases}$$

for each $y \in \mathbb{R}$ and

$$f_k(t,x,y) = \begin{cases} 0 & \text{for} \quad t \in [0,1/k), \\ f(t,\alpha_k(t,x),\beta(y)) & \text{for} \quad t \in [1/k, T-1/k], \\ \gamma(y) & \text{for} \quad t \in (T-1/k, T], \end{cases}$$

for each x, $y \in \mathbb{R}$. Obviously, $f_k \in Car([0,T] \times \mathbb{R}^2)$ and

$$f_k(t,x,y) \geq 0 \qquad \text{for a. e. } t \in [0,T] \text{ and each } x, y \in \mathbb{R}. \tag{14}$$

Let us define regular problem

$$u'' + f_k(t,u,u') = 0, \qquad u(0) = 0, \quad u'(T) = \psi(u(T)). \tag{15}$$

From relations (9), (12) and (14) it follows that $\sigma_1(t) = 0$ and $\sigma_2(t) = ct$ for $t \in [0,T]$ are lower and upper functions of problems (15), respectively. From Lemma 6 we get a solution u_k of the problem (15) (where we put $h = f_k$, $g_1(x,y) = -x$, $g_2(x,y) = \psi(x) - y$) such that

$$0 \leq u_k(t) \leq ct \qquad t \in [0,T]. \tag{16}$$

Obviously, it is valid

$$u_k'(0) \geq 0 \qquad \text{and} \qquad u_k'(0) = \lim_{t\to 0+} \frac{u_k(t)}{t} \leq c.$$

From (14) it follows that u_k' is nonincreasing on $[0,T]$. These facts, (15) and (16) imply

$$\min_{s\in[0,cT]} \psi(s) \le \psi(u_k(T)) = u_k'(T) \le u_k'(t) \le u_k'(0) \le c$$

for every $t \in [0,T]$.

STEP 2. (A priori estimates) Consider a sequence $\{u_k\}$ from STEP 1. We will prove the relation

$$\liminf_{k\to\infty} u_k(T) > 0. \tag{17}$$

Let (17) be not valid, i. e. $\liminf_{k\to\infty} u_k(T) = 0$. From the continuity of ψ and (12) it follows that for each arbitrarily small $\epsilon_1 > 0$ ($\epsilon_1 \le \epsilon_0$ and $\epsilon_1 \le \nu$) there exists $\delta > 0$ (we can choose it such that $\delta \le \epsilon_1$) such that for every $x \in \mathbb{R}$ the implication

$$0 \le x \le \delta \qquad \Longrightarrow \qquad |\psi(x)| < \epsilon_1$$

holds. Then there exists $l \in \mathbb{N}$ such that

$$0 \le u_l(T) < \delta \le \epsilon_1 \qquad \text{and} \qquad |u_l'(T)| = |\psi(u_l(T))| < \epsilon_1. \tag{18}$$

particularly, $-\epsilon_0 \le -\epsilon_1 < u_l'(T) \le u_l'(t)$ for each $t \in [0,T]$ and $u_l'(T) < \epsilon_1 < \nu$. Then there exists $t_l \in (0,T)$ such that $-\epsilon_0 \le u_l'(t) \le \nu$ for every $t \in (t_l,T]$. There are two possibilities. If $t_l \le T-\nu$, then integrating the differential equation from (15) we get

$$\begin{aligned} u_l'(T) - u_l'(t) &= \int_t^T u_l''(s)\,\mathrm{d}s \\ &= -\int_t^T f_l(s,u_l(s),u_l'(s))\,\mathrm{d}s \le -\int_t^T \epsilon\,\mathrm{d}s = -\epsilon(T-t) \end{aligned} \tag{19}$$

for every $t \in [T-\nu,T]$. If $t_l > T-\nu$ and $u_l'(t) > \nu$ for every $t \in [T-\nu,t_l)$, then (19) is valid for each $t \in [t_l,T]$. Since $\nu \ge \epsilon(T-t)$ for $t \in [T-\nu,t_l)$, it follows that $u_l'(t) \ge \epsilon(T-t)$ for each $t \in [T-\nu,t_l)$. In both cases we have the inequality

$$u_l'(t) \ge -\epsilon_1 + \epsilon(T-t)$$

for $t \in [T-\nu,T]$. Integrating this relation over the interval $[T-\nu,T]$ we get

$$u_l(T) - u_l(T-\nu) \ge -\epsilon_1\nu + \frac{\epsilon\nu^2}{2}$$

and according to (16) and (18) (and since $u_l(T-\nu) \ge 0$) we have

$$\frac{\epsilon\nu^2}{2} < \epsilon_1(\nu+1).$$

Taking ϵ_1 sufficiently small we get a contradiction. Hence (17) is valid. According to the concavity of u_k and (17), there exists $\omega > 0$ such that

$$u_k(t) \ge \omega t \qquad \text{for every } t \in [0,T], \text{ a. e. } k \in \mathbb{N}. \tag{20}$$

STEP 3. (Convergence of the sequence $\{u_k\}$) Let u_k be a solution of the problem (15) for each $k \in \mathbb{N}$, $k \geq 3/T$ and $[a,b] \subset (0,T)$ be a compact interval. Then (20) implies that there exists $k_0 \in \mathbb{N}$ such that for every $t \in [a,b]$ and $k \geq k_0$

$$\frac{c}{k_0} \leq u_k(t) \leq ct.$$

There exists $\varphi \in L^1[a,b]$ such that

$$|f_k(t,u_k(t),u_k'(t))| \leq \varphi(t) \qquad \text{for a. e. } t \in [a,b]$$

From Arzelà–Ascoli Theorem and diagonalization principle it follows that there exists $u \in C^0[0,T]$ such that u' is continuous on $(0,T)$ and a subsequence $\{u_{n_k}\}$ such that

$$\left.\begin{array}{ll} u_{n_k} \to u & \text{uniformly on } [0,T], \\ u_{n_k}' \to u' & \text{locally uniformly on } (0,T), \quad u_{n_k}'(T) \to \psi(u(T)) \end{array}\right\} \tag{21}$$

and $u(0)=0$. Without any loss of generality we assume that $\{n_k\} = \{k\}$.

STEP 4. (Convergence of the approximate problems) Let us take $\xi \in (0,T)$ such that $f(\xi,\cdot,\cdot)$ is continuous on $(0,\infty) \times \mathbb{R}$. Then there exists a compact interval $J^\star \subset (0,T)$ and $k^\star \in \mathbb{N}$ such that $\xi \in J^\star$ and for each $k \geq k_0$

$$u_k(\xi) > \frac{c}{k^\star}, \qquad J^\star \subset \left[\frac{1}{k}, T - \frac{1}{k}\right].$$

Then $f_k(\xi,u_k(\xi),u_k'(\xi)) = f(\xi,u_k(\xi),u_k'(\xi))$. We get assertion

$$\lim_{k\to\infty} f_k(t,u_k(t),u_k'(t)) = f(t,u(t),u'(t)) \qquad \text{for a. e. } t \in (0,T). \tag{22}$$

Let $t \in (0,T)$. Then there exists a compact interval $[a,b] \subset (0,T)$ and $\varphi \in L^1[a,b]$ such that $t \in [a,b]$, $T/2 \in [a,b]$ and

$$|f_k(s,u_k(s),u_k'(s))| \leq \varphi(s) \qquad \text{for a. e. } s \in [a,b]. \tag{23}$$

Obviously,

$$u_k'\left(\frac{T}{2}\right) - u_k'(t) = \int_{\frac{T}{2}}^t f_k(s,u_k(s),u_k'(s))\,\mathrm{d}s.$$

In view of this fact, (21), (22), (23) and Lebesgue Dominated Convergence Theorem we have

$$u'\left(\frac{T}{2}\right) - u'(t) = \int_{\frac{T}{2}}^t f(s,u(s),u'(s))\,\mathrm{d}s.$$

Obviously, this inequality is valid for every $t \in (0,T)$. It means that u' is continuous on each compact subinterval of the interval $(0,T)$ and

$$u''(t) + f(t,u(t),u'(t)) = 0 \qquad \text{for a. e. } t \in (0,T).$$

For $k \geq 3/T$ we have

$$\int_0^T f_k(s,u_k(s),u_k'(s))\,\mathrm{d}s = u_k'(0) - u_k'(T) = u_k'(0) - \psi(u_k(T)) \leq c - \min_{s\in[0,cT]} \psi(s)$$

From this fact, (14) and Fatou Lemma it follows that $f(\cdot,u(\cdot),u'(\cdot)) \in L^1[0,T]$ and obviously $u \in AC^1[0,T]$. It remains to prove the last boundary condition in (2). For $k \geq 3/T$ and $t \in (0,T)$ we have

$$|u_k'(t) - u_k'(T)| \leq \int_t^T |f(s,u(s),u'(s))| \,\mathrm{d}s + \int_t^T |f_k(s,u_k(s),u_k'(s)) - f(s,u(s),u'(s))| \,\mathrm{d}s.$$

This inequality and (21) imply that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $t \in (T-\delta,T)$ there exists $k_0 = k_0(\epsilon,t) \in \mathbb{N}$ such that

$$|u'(t) - \psi(u(T))| \leq |u'(t) - u_{k_0}'(t)| + |u_{k_0}'(t) - u_{k_0}'(T)| + |u_{k_0}'(T) - \psi(u(T))| < \epsilon.$$

Thus, $u'(T) = \lim_{t\to T-} u'(t) = \psi(u(T))$. This completes the proof. □

Example 8 Let α, $\beta \in (0,\infty)$. Then, by Theorem 7 the problem

$$u'' + (u^{-\alpha} + u^{\beta} + t^2 + 1)(1 - (u')^3) = 0, \quad u(0) = 0, \quad u'(1) = -(u(1))^2$$

has a solution $u \in AC^1[0,1]$ such that

$$0 < u(t) \leq t \qquad \text{for each } t \in (0,1].$$

References

[1] Kiguradze, I. T.: On Some Singular Boundary Value Problems for Ordinary Differential Equations. *Tbilisi Univ. Press, Tbilisi,* 1975 (in Russian).

[2] Kiguradze, I. T., Shekhter, B. L.: *Singular boundary value problems for second order ordinary differential equations.* Itogi Nauki i Tekhniki Ser. Sovrem. Probl. Mat. Nov. Dost. **30** (1987), 105–201 (in Russian), translated in J. Soviet Math. **43** (1988), 2340–2417.

[3] O'Regan, D.: *Upper and lower solutions for singular problems arising in the theory of membrane response of a spherical cap.* Nonlinear Anal. **47** (2001), 1163–1174.

[4] O'Regan, D.: Theory of Singular Boundary Value Problems. *World Scientific, Singapore*, 1994.

[5] Rachůnková, I.: *Singular mixed boundary value problem.* J. Math. Anal. Appl., **320** (2006), 611–618.

[6] Rachůnková, I.; Staněk, S.; Tvrdý, M.: Singularities and Laplacians in Boundary Value Problems for Nonlinear Ordinary Differential Equations. Handbook of Differential Equations. Ordinary Differential Equations, Ed. by A. Cañada, P. Drábek, A. Fonda, Vol. 3., pp. 607–723, Elsevier, 2006.

[7] Wang, M.; Cabada, A.; Nieto, J. J.: *Monotone method for nonlinear second order periodic boundary value problems with Carathéodory functions.* Ann. Polon. Math. **58** (1993), 3, 221–235.