Singular Nonlinear Problem for Ordinary Differential Equation of the Second Order

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Abstract
The paper deals with the singular nonlinear problem
\[ u''(t) + f(t, u(t), u'(t)) = 0, \]
\[ u(0) = 0, \quad u'(T) = \psi(u(T)), \]
where \( f \in \text{Car}((0, T) \times D), D = (0, \infty) \times \mathbb{R}. \) We prove the existence of a positive solution on \((0, T]\) to this problem under the assumption that the function \( f(t, x, y) \) is nonnegative and can have time singularities at \( t = 0, t = T \) and space singularity at \( x = 0. \) The proof is based on the Schauder fixed point theorem and on the method of a priori estimates.

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1 Introduction
We will study a singular boundary value problem with nonlinear boundary conditions
\[ u''(t) + f(t, u(t), u'(t)) = 0 \quad \text{for a. e. } t \in [0, T], \]
\[ u(0) = 0, \quad u'(T) = \psi(u(T)), \]
where \( f \) satisfies the Carathéodory conditions on \((0, T) \times D, [0, T] \subset \mathbb{R}, D = (0, \infty) \times \mathbb{R}. \) The function \( f(t, x, y) \) is allowed to have time singularities at \( t = 0, t = T \) and space singularity at \( x = 0, \) the function \( \psi \) is continuous on \([0, \infty). \)

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For a given interval \([a, b] \subset \mathbb{R}\) assume that \(L^1[a, b]\) denotes the set of all measurable functions defined \(a.\ e.\) on \([a, b]\) which are Lebesgue integrable on \([a, b]\), equipped with the norm
\[
\|u\|_1 = \int_a^b |u(t)|\,dt \quad \text{for each } u \in L^1[a, b];
\]
\(C^0[a, b]\) (or \(C^1[a, b]\)) denotes the set of all functions which are continuous (or have continuous first derivatives) on \([a, b]\), with the norm \(\|u\|_\infty = \max\{|u(t)| : t \in [a, b]\}\) (or \(\|u\|_{C^1[a, b]} = \|u\|_\infty + \|u'|\|_\infty\)); \(AC^1[a, b]\) denotes the set of all functions which have absolutely continuous first derivatives on \([a, b]\). We say that \(f : [a, b] \times D \to \mathbb{R}, D \subset \mathbb{R}^2\) satisfies the Carathéodory conditions on \([a, b] \times D\) if \(f\) has the following properties: (i) for each \((x, y) \in D\) the function \(f(\cdot, x, y)\) is measurable on \([a, b]\); (ii) for almost each \(t \in [a, b]\) the function \(f(t, \cdot, \cdot)\) is continuous on \(D\); (iii) for each compact set \(K \subset D\) there exists a function \(m_K(t) \in L^1[a, b]\) such that \(|f(t, x, y)| \leq m_K(t)\) for \(a.\ e.\ t \in [a, b]\) and all \((x, y) \in K\). For the set of functions satisfying the Carathéodory conditions on \([a, b] \times D\) we write \(\text{Car}([a, b] \times D)\). By \(f \in \text{Car}((0, T) \times D)\) we mean that \(f \in \text{Car}([a, b] \times D)\) for each \([a, b] \subset (0, T)\) and \(f \notin \text{Car}([0, T] \times D)\).

Singular problems have been studied by many authors (see [1] – [6] and references therein). For instance a similar problem is considered in [3], where the non–linear singular problem with mixed boundary conditions
\[
u'' + f(t, u, u') = 0, \quad u'(0) = 0, \quad u(T) = 0
\]
is investigated.

**Definition 1** Let \(f \in \text{Car}((0, T) \times D),\) where \(D = (0, \infty) \times \mathbb{R}\). We say that \(f\) has a time singularity at \(t = 0\) and/or at \(t = T\). If there exists \((x_1, y_1)\) \(\in D\) and/or \((x_2, y_2)\) \(\in D\) such that
\[
\int_0^\epsilon |f(t, x_1, y_1)|\,dt = \infty \quad \text{and/or} \quad \int_{T-\epsilon}^T |f(t, x_2, y_2)|\,dt = \infty
\]
for each sufficiently small \(\epsilon > 0\). The point \(t = 0\) and/or \(t = T\) will be called a singular point of \(f\).

We say that \(f\) has a space singularity at \(x = 0\) if
\[
\lim_{x \to 0^+} |f(t, x, y)| = \infty \quad \text{for } a.\ e.\ t \in [0, T] \text{ and for some } y \in \mathbb{R}.
\]

Here, we will treat with following definition of the solution of the problem (1), (2).

**Definition 2** By a solution of the problem (1), (2) we understand a function \(u \in AC^1[0, T]\) satisfying the differential equation (1) and the boundary conditions (2).
2 Regular problem, lower and upper function

In order to prove the main result we need the existence theorem for regular boundary value problems. Let us consider a problem

\[ u'' + h(t, u, u') = 0, \quad g_1(u(0), u'(0)) = 0, \quad g_2(u(T), u'(T)) = 0, \]

where \( h \in \text{Car}([0, T] \times \mathbb{R}^2), g_1, g_2 : \mathbb{R}^2 \to \mathbb{R} \) are continuous functions.

**Definition 3** A function \( u \in AC^1[0, T] \) which satisfies the differential equation in (3) a.e. in \([0, T]\) and fulfils the boundary conditions in (3) is called a solution of the problem (3).

In the existence theorem the concept of upper and lower function will be needed.

**Definition 4** A function \( \sigma \in AC^1[0, T] \) is called a lower function of the problem (3) if

\[ \sigma''(t) + h(t, \sigma(t), \sigma'(t)) \geq 0 \quad \text{for a.e. } t \in [0, T] \]

\[ g_1(\sigma(0), \sigma'(0)) \geq 0, \quad g_2(\sigma(T), \sigma'(T)) \geq 0. \]

If these inequalities are reversed, the function \( \sigma \) is called an upper function of the problem (3).

For \( \sigma_1, \sigma_2 \in AC^1[0, T] \) such that \( \sigma_1 \leq \sigma_2 \) on \([0, T]\) we can define a function \( \gamma : [0, T] \times \mathbb{R} \to \mathbb{R} \) by

\[ \gamma(t, x) = \max\{\sigma_1(t), \min\{x, \sigma_2(t)\}\} \quad \text{for each } t \in [0, T], \quad x \in \mathbb{R}. \]  

(4)

Now, we introduce the following result [7, Lemma 2]. It is fundamental in the proof of Lemma 6.

**Lemma 5** For \( u \in C^1[0, T] \) the two following properties hold:

a) \( \frac{d}{dt} \gamma(t, u(t)) \) exists for a.e. \( t \in [0, T] \).

b) If \( u_m \in C^1[0, T] \) and \( u_m \to u \) in \( C^1[0, T] \), then

\[ \frac{d}{dt} \gamma(t, u_m(t)) \to \frac{d}{dt} \gamma(t, u(t)) \quad \text{for a.e. } t \in [0, T]. \]

**Lemma 6** Let \( h \in \text{Car}([0, T] \times \mathbb{R}^2), g_1, g_2 : \mathbb{R}^2 \to \mathbb{R} \) be continuous functions and \( \sigma_1, \sigma_2 \) be lower and upper function of the problem (3), respectively, such that

\[ \sigma_1(t) \leq \sigma_2(t) \quad \text{for each } t \in [0, T]. \]

Further, assume that there exists \( \varphi \in L^1[0, T] \) such that

\[ |h(t, x, y)| \leq \varphi(t) \]

for a.e. \( t \in [0, T] \), each \( x \in [\sigma_1(t), \sigma_2(t)] \), and each \( y \in \mathbb{R} \), \( g_1 \) is nondecreasing in the second variable and \( g_2 \) is nonincreasing in the second variable. Then there exists a solution \( u \) of the problem (3) such that

\[ \sigma_1 \leq u \leq \sigma_2 \quad \text{on } [0, T]. \]  

(5)
Proof. Let us define functionals $A, B : C^1[0, T] \to \mathbb{R}$ by

$$A(u) = \gamma(0, u(0) + g_1(u(0), u'(0))),$$
$$B(u) = \gamma(T, u(T) + g_2(u(T), u'(T)))$$

for each $u \in C^1[0, T]$. Lemma 5 allows us to define for each $u \in C^1[0, T]$ a function $\tilde{h}_u : [0, T] \to \mathbb{R}$ such that

$$\tilde{h}_u(t) = h(t, \gamma(t, u(t)), \frac{d}{dt}(t, u(t))) \quad \text{for a. e. } t \in [0, T].$$

Obviously, there exists $\tilde{h} \in L^1[0, T]$ such that

$$|\tilde{h}_u(t)| \leq \tilde{h}(t) \quad \text{for a. e. } t \in [0, T] \text{ and each } u \in C^1[0, T].$$

Consider an auxiliary problem

$$\begin{cases}
  u''(t) = -\tilde{h}_u(t) \quad \text{a. e. } t \in [0, T],
  
  u(0) = A(u),
  
  u(T) = B(u).
\end{cases} \tag{6}$$

Let us define a mapping $F : C^1[0, T] \to C^1[0, T]$ by

$$(Fu)(t) = -\int_0^T G(t, s)\tilde{h}_u(s) \, ds + \frac{T-t}{T} A(u) + \frac{t}{T} B(u)$$

for each $u \in C^1[0, T]$ and $t \in [0, T]$, where

$$G(t, s) = \begin{cases}
  \frac{t(s-T)}{T} & \text{for } 0 \leq t \leq s \leq T, \\
  \frac{s(t-T)}{T} & \text{for } 0 \leq s < t \leq T.
\end{cases}$$

We can check that each fixed point of the operator $F$ is a solution of the problem (6). Using the Schauder Fixed Point Theorem we will prove that there exists a fixed point $u$ of the operator $F$ satisfying the inequalities (5) and such that $u$ is a solution of the problem (3).

It is easy to see that

$$\|Fu\|_\infty \leq T\|\tilde{h}\|_1 + 2(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty)$$

and

$$\|(Fu)'\|_\infty \leq \|\tilde{h}\|_1 + \frac{2}{T}(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty),$$

i. e. that there exist $K > 0$ and $\Omega = \{u \in C^1[0, T] : \|u\|_{C^1[0, T]} \leq K\}$, such that $F(\Omega) \subset \Omega$. It suffices to prove that the set

$$F' = \{(Fu)' : u \in \Omega\}$$

is relatively compact in $C^0[0, T]$. Obviously, for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $u \in \Omega$ and $s_1, s_2 \in [0, T], |s_2 - s_1| < \delta$, relations

$$|(Fu)'(s_2) - (Fu)'(s_1)| = \left| \int_{s_1}^{s_2} \tilde{h}_u(s) \, ds \right| \leq \left| \int_{s_1}^{s_2} \tilde{h}(s) \, ds \right| < \epsilon$$
are valid. Now, applying Arzelà–Ascoli Theorem we get that $F(\Omega)$ is relatively compact in $C^1[0,T]$. Thus, there exists a fixed point $u$ of the operator $F$ and $u \in AC^1[0,T]$. We will prove that relations (5) are satisfied. From boundary conditions in (6) it follows that 

$$\sigma_1(0) \leq u(0) \leq \sigma_2(0) \quad \text{and} \quad \sigma_1(T) \leq u(T) \leq \sigma_2(T).$$

Assume that there exists $\tau \in (0,T)$ such that $u(\tau) < \sigma_1(\tau)$. Then there exist $\xi \in (0,T)$ and $\delta > 0$ such that 

$$(u - \sigma_1)(\xi) = \min_{t \in [0,T]} (u - \sigma_1)(t) < 0$$

and 

$$0 > (u - \sigma_1)(t) > (u - \sigma_1)(\xi) \quad \text{for each } t \in (\xi, \xi + \delta).$$

(7)

Obviously, $(u - \sigma_1)'(\xi) = 0$ and $u(t) < \sigma_1(t)$ for each $t \in (\xi, \xi + \delta)$. According to the definition of $\tilde{h}_u$, we have 

$$(u - \sigma_1)'(t) \leq \int_{\xi}^{t} \left[ -\tilde{h}_u(s) + h(s, \sigma_1(s), \sigma_1'(s)) \right] ds = 0,$$

for each $t \in (\xi, \xi + \delta)$, which contradicts (7). Similarly, we can prove that $u \leq \sigma_2$ on $[0,T]$. From (5) it follows that $u$ satisfies the differential equation in (3). It suffices to prove that $u$ satisfies boundary conditions in (3), i.e. according to (5) and definition of $\gamma$, to prove inequalities 

$$\sigma_1(0) \leq u(0) + g_1(u(0), u'(0)) \leq \sigma_2(0)$$

(8)

and 

$$\sigma_1(T) \leq u(T) + g_2(u(T), u'(T)) \leq \sigma_2(T).$$

Let the first inequality in (8) be not satisfied. Then according to (5) we have 

$$u(0) = \sigma_1(0), \quad 0 > g_1(\sigma_1(0), u'(0)) \quad \text{and} \quad u'(0) \geq \sigma_1'(0).$$

Using the monotonicity of $g_1$ we have $0 > g_1(\sigma_1(0), \sigma_1'(0))$, which contradicts the definition of a lower function. The remaining inequalities can be proven in a similar way. \qed

3 Main result

Now, we are ready to prove the existence theorem for singular problem (1), (2).

**Theorem 7** Assume that $f \in Car((0,T) \times D)$, where $T > 0$, $D = (0,\infty) \times \mathbb{R}$, with possible time singularities at $t = 0$ and/or $t = T$ and a space singularity at $x = 0$. Further assume that there exist $\epsilon \in (0,1)$, $\nu \in (0,T)$, $c \in (\nu,\infty)$ and $\epsilon_0 \in (0,\infty)$ such that 

$$f(t,ct,c) = 0 \quad \text{for a.e. } t \in [0,T],$$

(9)
\[ 0 \leq f(t, x, y) \quad \text{for a.e. } t \in [0, T], \text{ each } x \in (0, ct], \text{ } y \in \left[ \min_{t \in [0, cT]} \psi(t), c \right], \quad (10) \]
\[ \epsilon \leq f(t, x, y) \quad \text{for a.e. } t \in [T - \nu, T], \text{ each } x \in (0, ct], \text{ } y \in (-\epsilon_0, \nu], \quad (11) \]
\[ 0 = \psi(0), \quad \psi(cT) \leq c \quad (12) \]

hold. Then there exists a solution \( u \) of the problem (1), (2) such that
\[ 0 < u(t) \leq ct \quad (13) \]

for each \( t \in (0, T] \).

Proof. Step 1. Let \( k \in \mathbb{N}, k \geq 3/T \). We define
\[ \alpha_k(t, x) = \begin{cases} 
  c/k & \text{for } \ x < c/k, \\
  x & \text{for } \ c/k \leq x \leq ct, \\
  ct & \text{for } \ x > ct,
\end{cases} \]
for each \( t \in [1/k, T - 1/k], x \in \mathbb{R} \),
\[ \beta(y) = \begin{cases} 
  \min_{t \in [0, cT]} \psi(t) & \text{for } \ y < \min_{t \in [0, cT]} \psi(t), \\
  y & \text{for } \ \min_{t \in [0, cT]} \psi(t) \leq y \leq c, \\
  c & \text{for } \ y > c,
\end{cases} \]
and
\[ \gamma(y) = \begin{cases} 
  \epsilon & \text{for } \ y < \nu, \\
  \frac{\epsilon - c}{\epsilon - \nu} & \text{for } \ \nu \leq y \leq c, \\
  0 & \text{for } \ y > c,
\end{cases} \]
for each \( y \in \mathbb{R} \) and
\[ f_k(t, x, y) = \begin{cases} 
  0 & \text{for } \ t \in [0, 1/k), \\
  f(t, \alpha_k(t, x), \beta(y)) & \text{for } \ t \in [1/k, T - 1/k], \\
  \gamma(y) & \text{for } \ t \in (T - 1/k, T],
\end{cases} \]
for each \( t \in \mathbb{R} \). Obviously, \( f_k \in \text{Car}([0, T] \times \mathbb{R}^2) \) and
\[ f_k(t, x, y) \geq 0 \quad \text{for a.e. } t \in [0, T] \text{ and each } x, y \in \mathbb{R}. \quad (14) \]

Let us define regular problem
\[ u'' + f_k(t, u, u') = 0, \quad u(0) = 0, \quad u'(T) = \psi(u(T)). \quad (15) \]

From relations (9), (12) and (14) it follows that \( \sigma_1(t) = 0 \) and \( \sigma_2(t) = ct \) for \( t \in [0, T] \) are lower and upper functions of problems (15), respectively. From Lemma 6 we get a solution \( u_k \) of the problem (15) (where we put \( h = f_k, \quad g_1(x, y) = -x, \quad g_2(x, y) = \psi(x) - y \) such that
\[ 0 \leq u_k(t) \leq ct \quad t \in [0, T]. \quad (16) \]

Obviously, it is valid
\[ u_k'(0) \geq 0 \quad \text{and} \quad u_k'(0) = \lim_{t \to 0^+} \frac{u_k(t)}{t} \leq c. \quad (17) \]
From (14) it follows that $u'_k$ is nonincreasing on $[0, T]$. These facts, (15) and (16) imply

$$\min_{s \in [0, T]} \psi(s) \leq \psi(u_k(T)) = u'_k(T) \leq u'_k(t) \leq u'_k(0) \leq c$$

for every $t \in [0, T]$.

**Step 2.** (A priori estimates) Consider a sequence $\{u_k\}$ from Step 1. We will prove the relation

$$\liminf_{k \to \infty} u_k(T) > 0. \quad (17)$$

Let (17) be not valid, i.e. $\liminf_{k \to \infty} u_k(T) = 0$. From the continuity of $\psi$ and (12) it follows that for each arbitrarily small $\epsilon_1 > 0$ ($\epsilon_1 \leq \epsilon_0$ and $\epsilon_1 < \nu$) there exists $\delta > 0$ (we can choose it such that $\delta \leq \epsilon_1$) such that for every $x \in \mathbb{R}$ the implication

$$0 \leq x \leq \delta \implies |\psi(x)| < \epsilon_1$$

holds. Then there exists $l \in \mathbb{N}$ such that

$$0 \leq u_l(T) < \delta \leq \epsilon_1 \quad \text{and} \quad |u'_l(T)| = |\psi(u_l(T))| < \epsilon_1. \quad (18)$$

particularly, $-\epsilon_0 \leq -\epsilon_1 < u'_l(T) \leq u'_l(t)$ for each $t \in [0, T]$ and $u'_l(T) < \epsilon_1 < \nu$. Then there exists $t_l \in (0, T)$ such that $-\epsilon_0 \leq u'_l(t) \leq \nu$ for every $t \in (t_l, T]$. There are two possibilities. If $t_l \leq T - \nu$, then integrating the differential equation from (15) we get

$$u'_l(T) - u'_l(t) = \int_t^T u''_l(s) \, ds$$

$$= -\int_t^T f_i(s, u_l(s), u'_l(s)) \, ds \leq -\int_t^T \epsilon \, ds = -\epsilon(T - t) \quad (19)$$

for every $t \in [T - \nu, T]$. If $t_l > T - \nu$ and $u'_l(t) > \nu$ for every $t \in [T - \nu, t_l)$, then (19) is valid for each $t \in [t_l, T]$. Since $\nu \geq \epsilon(T - t)$ for $t \in [T - \nu, t_l)$, it follows that $u'_l(t) \geq \epsilon(T - t)$ for each $t \in [T - \nu, t_l)$. In both cases we have the inequality

$$u'_l(t) \geq -\epsilon_1 + \epsilon(T - t)$$

for $t \in [T - \nu, T]$. Integrating this relation over the interval $[T - \nu, T]$ we get

$$u_l(T) - u_l(T - \nu) \geq -\epsilon_1 \nu + \frac{\epsilon \nu^2}{2}$$

and according to (16) and (18) (and since $u_l(T - \nu) \geq 0$) we have

$$\frac{\epsilon \nu^2}{2} < \epsilon_1 (\nu + 1).$$

Taking $\epsilon_1$ sufficiently small we get a contradiction. Hence (17) is valid. According to the concavity of $u_k$ and (17), there exists $\omega > 0$ such that

$$u_k(t) \geq \omega t \quad \text{for every} \ t \in [0, T], \ \text{a. e.} \ k \in \mathbb{N}. \quad (20)$$
STEP 3. (Convergence of the sequence \{u_k\}) Let \(u_k\) be a solution of the problem (15) for each \(k \in \mathbb{N}, k \geq 3/T\) and \([a, b] \subset (0, T)\) be a compact interval. Then (20) implies that there exists \(k_0 \in \mathbb{N}\) such that for every \(t \in [a, b]\) and \(k \geq k_0\)

\[
\frac{c}{k_0} \leq u_k(t) \leq ct.
\]

There exists \(\varphi \in L^1[a, b]\) such that

\[
|f_k(t, u_k(t), u_k'(t))| \leq \varphi(t) \quad \text{for a. e. } t \in [a, b]
\]

From Arzelà–Ascoli Theorem and diagonalization principle it follows that there exists \(u \in C^0[0, T]\) such that \(u'\) is continuous on \((0, T)\) and a subsequence \(\{u_{n_k}\}\) such that

\[
\begin{align*}
&u_{n_k} \to u \quad \text{uniformly on } [0, T], \\
&u_{n_k}' \to u' \quad \text{locally uniformly on } (0, T), \\
&u_{n_k}'(T) \to \psi(u(T))
\end{align*}
\]

and \(u(0) = 0\). Without any loss of generality we assume that \(\{n_k\} = \{k\}\).

STEP 4. (Convergence of the approximate problems) Let us take \(\xi \in (0, T)\) such that \(f(\xi, \cdot, \cdot)\) is continuous on \((0, \infty) \times \mathbb{R}\). Then there exists a compact interval \(J^* \subset (0, T)\) and \(k^* \in \mathbb{N}\) such that \(\xi \in J^*\) and for each \(k \geq k_0\)

\[
u_k(\xi) > \frac{c}{k^*}, \quad J^* \subset \left[\frac{1}{k^*}T - \frac{1}{k^*}\right].
\]

Then \(f_k(\xi, u_k(\xi), u_k'(\xi)) = f(\xi, u_k(\xi), u_k'(\xi))\). We get assertion

\[
\lim_{k \to \infty} f_k(t, u_k(t), u_k'(t)) = f(t, u(t), u'(t)) \quad \text{for a. e. } t \in (0, T). \quad (22)
\]

Let \(t \in (0, T)\). Then there exists a compact interval \([a, b] \subset (0, T)\) and \(\varphi \in L^1[a, b]\) such that \(t \in [a, b], T/2 \in [a, b]\) and

\[
|f_k(s, u_k(s), u_k'(s))| \leq \varphi(s) \quad \text{for a. e. } s \in [a, b]. \quad (23)
\]

Obviously,

\[
u_k\left(\frac{T}{2}\right) - u_k(t) = \int_{T/2}^t f_k(s, u_k(s), u_k'(s)) \, ds.
\]

In view of this fact, (21), (22), (23) and Lebesgue Dominated Convergence Theorem we have

\[
u'(\frac{T}{2}) - u'(t) = \int_{T/2}^t f(s, u(s), u'(s)) \, ds.
\]

Obviously, this inequality is valid for every \(t \in (0, T)\). It means that \(u'\) is continuous on each compact subinterval of the interval \((0, T)\) and

\[
u''(t) + f(t, u(t), u'(t)) = 0 \quad \text{for a. e. } t \in (0, T).
\]

For \(k \geq 3/T\) we have

\[
\int_0^T f_k(s, u_k(s), u_k'(s)) \, ds = u_k'(0) - u_k'(T) = u_k'(0) - \psi(u_k(T)) \leq c - \min_{s \in [0, cT]} \psi(s)
\]
From this fact, (14) and Fatou Lemma it follows that \( f(\cdot, u(\cdot), u'(\cdot)) \in L^1[0, T] \) and obviously \( u \in AC^1[0, T] \). It remains to prove the last boundary condition in (2). For \( k \geq 3/T \) and \( t \in (0, T) \) we have

\[
|u'_k(t) - u'_k(T)| \leq \int_t^T |f(s, u(s), u'(s))| \, ds \\
+ \int_t^T |f_k(s, u_k(s), u'_k(s)) - f(s, u(s), u'(s))| \, ds.
\]

This inequality and (21) imply that for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for every \( t \in (T - \delta, T) \) there exists \( k_0 = k_0(\epsilon, t) \in \mathbb{N} \) such that

\[
|u'(t) - \psi(u(T))| \leq |u'(t) - u'_{k_0}(t)| + |u'_{k_0}(t) - u'_{k_0}(T)| + |u'_{k_0}(T) - \psi(u(T))| < \epsilon.
\]

Thus, \( u'(T) = \lim_{t \to T^-} u'(t) = \psi(u(T)) \). This completes the proof. \( \square \)

**Example 8** Let \( \alpha, \beta \in (0, \infty) \). Then, by Theorem 7 the problem

\[
u'' + (u^{-\alpha} + u^\beta + t^2 + 1)(1 - (u')^3) = 0, \quad u(0) = 0, \quad u'(1) = -(u(1))^2
\]

has a solution \( u \in AC^1[0, 1] \) such that

\[
0 < u(t) \leq t \quad \text{for each } t \in (0, 1].
\]

**References**


