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Boundary value problems with nonlinear conditions

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Abstract. We study the nonlinear second order differential equation $x'' = f(t, x, x')$ with a Carathédory nonlinenarity f and nonlinear boundary conditions $g_1(x(a), x'(a)) = 0$, $g_2(x(b), x'(b)) = 0$, $[a, b] \subset \mathbb{R}$. Using the topological degree method we find conditions for the existence of solutions to the above problem in terms of upper and lower solutions.

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(Dedicated to the memory of Svatopluk Fučík)

1 Introduction

In the paper we consider the problem

$$x'' = f(t, x, x'), \tag{1.1}$$

$$g_1(x(a), x'(a)) = 0, \quad g_2(x(b), x'(b)) = 0, \tag{1.2}$$

where $J = [a, b] \subset \mathbb{R}$, $f : J \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Carathéodory conditions, $g_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$, are continuous (generally nonlinear) functions. By a solution of problem (1.1), (1.2) we mean a function $u \in AC^1(J)$ (having an absolutely continuous first derivative on J) and satisfying conditions (1.2) and equation (1.1) for a. e. $t \in J$.

We find conditions for the existence of solutions of (1.1), (1.2). Such questions were studied for example in [1], [2], [3], [6]. But in [2] the appropriate linear part of (1.2) was required and in [3] the monotonocity of g_1, g_2 was supposed. Our approach is close to [1], where problem (1.1), (1.2) is studied for a continuous right hand side f satisfying the Bernstein-Nagumo conditions and g_1, g_2 monotonous in the second variable.

Here, neither of monocity of g_1, g_2, nor growth conditions for f are required. Our results generalize the earlier ones of [6] and they are obtained by means of topological degree method together with upper and lower solutions method.

Our proofs are based on the following theorem:

Continuation Theorem. *[1, p. 40]. Let X, Y be Banach spaces, L : $\operatorname{dom} L \subset X \to Y$ a Fredholm map of index 0 and $\Omega \subset X$ an open bounded set. Let $N : X \to$*

Y be L-compact on $\overline{\Omega}$, $Q : X \to Y$ a continuous projector with $\operatorname{Ker} Q = \operatorname{Im} L$ *and* $J : \operatorname{Im} Q \to \operatorname{Ker} L$ *an isomorphism. Suppose*

1. *for each $\lambda \in (0, 1)$ every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*
2. *$QNx \neq 0$ for each $x \in \operatorname{Ker} L \cap \partial\Omega$ and*
3. *the Brouwer degree $d[N_0, \Omega \cap \operatorname{Ker} L, 0] \neq 0$, where $N_0 = JQN : \operatorname{Ker} L \to \operatorname{Ker} L$. Then the equation $Lx = Nx$ has at least one solution in* $\operatorname{dom} L \cap \overline{\Omega}$.

Let us remind that functions $\alpha, \beta \in AC^1(J)$ are called lower and upper solutions for (1.1), (1.2), respectively, if they fulfill $\alpha''(t) \geq f(t, \alpha(t), \alpha'(t))$, $\beta''(t) \leq f(t, \beta(t), \beta'(t))$ for a. e. $t \in J$ and

$$g_1(\alpha(a), \alpha'(a)) \geq 0, \qquad g_1(\beta(a), \beta'(a)) \leq 0,$$
$$g_2(\alpha(b), \alpha'(b)) \leq 0, \qquad g_2(\beta(b), \beta'(b)) \geq 0.$$

2 The existence results for bounded nonlinearity

First we will prove the existence of solutions to (1.1), (1.2) provided f is bounded by an Lebesgue integrable function φ. We cannot apply the Continuation Theorem onto (1.1), (1.2) directly, but we have to construct a sequence of auxiliary problems

$$\begin{gathered} x'' = \lambda f_n(t, x, x'), \quad \lambda \in [0, 1], \\ g_{1n}(x(a), x'(a)) = 0, \quad g_{2n}(x(b), x'(b)) = 0, \end{gathered} \tag{2n.λ}$$

where $n \in \mathbb{N}$, $r \in (0, \infty)$,

$$f_n = \begin{cases} f(t, r, 0) & \text{for } x \geq r + 1/n \\ f(t, r, y) + [f(t, r, 0) - f(t, r, y)]n(x - r) & \text{for } r < x < r + 1/n \\ f(t, x, y) & \text{for } -r \leq x \leq r \\ f(t, -r, y) - [f(t, -r, 0) - f(t, -r, y)]n(x + r) & \\ \quad \text{for } -r - 1/n < x < -r & \\ f(t, -r, 0) & \text{for } x \leq -r - 1/n, \end{cases}$$

$$g_{in}(x, y) = \begin{cases} g_i(r, 0) & \text{for } x \geq r + 1/n \\ g_i(r, y) + [g_i(r, 0) - g_i(r, y)]n(x - r) & \text{for } r < x < r + 1/n \\ g_i(x, y) & \text{for } -r \leq x \leq r \\ g_i(-r, y) - [g_i(-r, 0) - g_i(-r, y)]n(x + r) & \\ \quad \text{for } -r - 1/n < x < -r & \\ g_i(-r, 0) & \text{for } x \leq -r - 1/n, \end{cases}$$

$i = 1, 2$.

If we put for fixed n

$$X = C^1([a, b]), \quad Y = L(a, b) \times \mathbb{R}^2, \quad \operatorname{dom} L = AC^1([a, b]) \subset X,$$
$$L : \operatorname{dom} L \to Y, \quad x \to (x'', 0, 0),$$
$$N : X \to Y, \quad x \to (f_n(\cdot, x(\cdot), x'(\cdot)),$$
$$g_{1n}(x(a), x'(a)), \quad g_{2n}(x(b), x'(b))),$$

then problem $(2n.\lambda)$ can be written in the form

$$Lx = \lambda Nx$$

Using the Continuation Theorem for problems $(2n.\lambda)$, for any $n \in \mathbb{N}$, we get the existence of solutions u_n of $(2n.1)$, and by the Arzelà-Ascoli Theorem we prove the existence of limit u of the appropriate subsequence of $(u_n)_1^\infty$. One can see that u is a solution of (1.1), (1.2). These considerations lead to the following theorem.

Theorem 2.1. *Let $r \in (0, \infty)$ and $\varphi \in L(J)$ be such that for a. e. $t \in J$ and each $x \in [-r, r]$*

$$g_1(-r, 0)g_1(r, 0) < 0, \tag{2.1}$$

$$g_2(-r, 0)g_2(r, 0) < 0, \tag{2.2}$$

$$f(t, -r, 0) < 0, \quad f(t, r, 0) > 0 \tag{2.3}$$

$$|f(t, x, y)| \le \varphi(t) \text{ for each } y \in \mathbb{R}. \tag{2.4}$$

Then problem (1.1), (1.2) has a solution u with

$$-r \le u(t) \le r \text{ for each } t \in J. \tag{2.5}$$

For detailed proof of Theorem 2.1 see [6].

Now, using lower and upper solutions method, we can get results more general than those in Theorem 2.1.

Theorem 2.2. *Let α, β be lower and upper solutions of (1.1), (1.2) with $\alpha(t) \le \beta(t)$ for each $t \in J$ and α'', $\beta'' \in L_\infty(J)$. Further let $\varphi \in L(J)$ be such that for a. e. $t \in J$ and each $x \in [\alpha(t), \beta(t)]$), $y \in \mathbb{R}$ the condition (2.4) is satisfied.*

Then problem (1.1), (1.2) has a solution u with

$$\alpha(t) \le u(t) \le \beta(t) \text{ for each } t \in J. \tag{2.6}$$

Proof: Let us fix $n \in \mathbb{N}$ and put

$$w(\beta) = [f(t, \beta(t), \beta'(t)) - f(t, \beta(t), y)]n(x - \beta(t)),$$
$$w(\alpha) = [f(t, \alpha(t), \alpha'(t)) - f(t, \alpha(t), y)]n(x - \alpha(t)),$$

$$f_n(t,\,x,\,y)=\begin{cases} f(t,\,\beta(t),\,\beta'(t))+x-\beta(t)-1/n \\ \qquad \text{for } x\ge\beta(t)+1/n \\ f(t,\,\beta(t),\,y)+w(\beta) & \text{for } \beta(t)<x<\beta(t)+1/n \\ f(t,\,x,\,y) & \text{for } \alpha(t)\le x\le\beta(t) \\ f(t,\,\alpha(t),\,y)-w(\alpha) & \text{for } \alpha(t)-1/n<x<\alpha(t) \\ f(t,\alpha(t),\alpha'(t))+x-\alpha(t)+1/n \\ \qquad \text{for } x\le\alpha(t)-1/n \end{cases}$$

$$g_{1n}(x,\,y)=\begin{cases} g_1(\beta(a),\,\beta'(a))+\beta(a)+1/n-x \\ \qquad \text{for } x\ge\beta(a)+1/n \\ g_1(\beta(a),y)+[g_1(\beta(a),\beta'(a))-g_1(\beta(a),y)]n(x-\beta(a)) \\ \qquad \text{for } \beta(a)<x<\beta(a)+1/n \\ g_1(x,\,y) \\ \qquad \text{for } \alpha(a)\le x\le\beta(a) \\ g_1(\alpha(a),y)-[g_1(\alpha(a),\alpha'(a))-g_1(\alpha(a),y)]n(x-\alpha(a)) \\ \qquad \text{for } \alpha(a)-1/n<x<\alpha(a) \\ g_1(\alpha(a),\,\alpha'(a))+\alpha(a)-1/n-x \\ \qquad \text{for } x\le\alpha(a)-1/n \end{cases}$$

$$g_{2n}(x,\,y)=\begin{cases} g_2(\beta(b),\,\beta'(b))+x-\beta(b)-1/n \\ \qquad \text{for } x\ge\beta(b)+1/n \\ g_2(\beta(b),y)+[g_2(\beta(b),\beta'(b))-g_2(\beta(b),y)]n(x-\beta(b)) \\ \qquad \text{for } \beta(b)<x<\beta(b)+1/n \\ g_2(x,\,y) \\ \qquad \text{for } \alpha(b)\le x\le\beta(b) \\ g_2(\alpha(b),y)-[g_2(\alpha(b),\alpha'(b))-g_2(\alpha(b),y)]n(x-\alpha(b)) \\ \qquad \text{for } \alpha(b)-1/n<x<\alpha(b) \\ g_2(\alpha(b),\,\alpha'(b))+x-\alpha(b)+1/n \\ \qquad \text{for } x\le\alpha(b)-1/n \end{cases}$$

The functions f_n, g_{1n}, g_{2n} satisfy the conditions of Theorem 2.1 for $r\ge 1+\max\{|\alpha(t)|,\,|\beta(t)|\,:\,t\in J\}+\operatorname{esssup}\{|\alpha''(t)|,\,|\beta''(t)|\,:\,t\in J\}$. Thus there exists solution u_n of problem

$$x''=f_n(t,\,x,\,x'),$$
$$g_{1n}(x(a),\,x'(a))=0,\quad g_{2n}(x(b),\,x'(b))=0$$

satisfying (2.5).

Let us show that u_n fulfills also the inequalities

$$-1/n+\alpha(t)\le u_n(t)\le\beta(t)+1/n \text{ for each } t\in J. \tag{2.7}$$

Put $v(t)=u_n(t)-\alpha(t)+1/n$ and suppose that $\min\{v(t)\,:\,t\in J\}=v(\bar t)<0$. Let $\bar t\in(a,\,b)$. Then we can find $\delta>0$ and $t_0\ge\bar t$ such that $v'(t_0)=0$, $v'(t)\ge 0$ and $v(t)<0$ for each $t\in(t_0,\,t_0+\delta]\subset J$. Thus $\int_{t_0}^{t_0+\delta}v''(\tau)\mathrm{d}\tau=v'(t_0+\delta)-v'(t_0)\ge 0$, and on the other hand $\int_{t_0}^{t_0+\delta}v''(\tau)\mathrm{d}\tau\le\int_{t_0}^{t_0+\delta}v(\tau)\mathrm{d}\tau<0$, a contradiction. Let

$a = t_0$. Then $v(a) < 0$, i.e. $u_n(a) < \alpha(a) - 1/n$ and $g_{1n}(u_n(a), u_n'(a)) = g_1(\alpha(a), \alpha'(a)) - v(a) > 0$, a contradiction. Similary for $b = t_0$.

The second inequality in (2.7) we prove analogously putting $v(t) = \beta(t) - u_n(t) + 1/n$.

For each $n \in \mathbb{N}$ we get a solution u_n by this way and so we have a sequence $(u_n)_1^\infty$ of equibounded and equicontinuous functions together with their derivatives. Applying Arzelà-Ascoli Theorem we get a subsequence uniformly converging to u. We can see that u satisfies (2.6) and (1.1), (1.2). □

3 The existence results for unbounded nonlinearity

Theorem 3.1. *Let α, β be lower and upper solutions of (1.1), (1.2) with $\alpha(t) \le \beta(t)$ for each $t \in J$ and α'', $\beta'' \in L_\infty(J)$. Further let μ, $\nu \in AC(J)$ be such that $\mu(t) \le \alpha'(t) \le \nu(t)$, $\mu(t) \le \beta'(t) \le \nu(t)$ for each $t \in J$ and for a. e. $t \in J$ and each $x \in [\alpha(t), \beta(t)]$ the conditions*

$$f(t, x, \nu(t)) \ge \nu'(t), \quad f(t, x, \mu(t)) \le \mu'(t), \tag{3.1}$$

$$g_2(x, \nu(t)) \ge 0, \quad g_2(x, \mu(t)) \le 0 \tag{3.2}$$

are fulfilled.

Then problem (1.1), (1.2) has at least one solution u satisfying (2.6) and

$$\mu(t) \le \mu'(t) \le \nu(t) \text{ for each } t \in J. \tag{3.3}$$

PROOF: Let us put

$$\tilde{f}(t, x, y) = \begin{cases} f(t, x, \nu(t)) + (y - \nu(t))/(y - \nu(t) + 1) & \text{for } y > \nu(t) \\ f(t, x, y) & \text{for } \mu(t) \le y \le \nu(t) \\ f(t, x, \mu(t)) + (y - \mu(t))/(|y - \mu(t)| + 1) & \text{for } y < \mu(t) \end{cases}$$

$$\tilde{g}_2(x, y) = \begin{cases} g_2(x, \nu(t)) + y - \nu(t)) & \text{for } y > \nu(t) \\ g_2(x, y) & \text{for } \mu(t) \le y \le \nu(t) \\ g_2(x, \mu(t)) + y - \mu(t) & \text{for } y < \mu(t) \end{cases}$$

and consider the problem

$$x'' = \tilde{f}(t, x, x') \tag{3.4}$$

$$g_1(x(a), x'(a)) = 0, \quad \tilde{g}_2(x(b), x'(b)) = 0. \tag{3.5}$$

The functions $\tilde{f}$, g_1, $\tilde{g}_2$ fulfill the conditions of Theorem 2.2 with $\varphi(t) = \sup\{|f(t, x, y)| : x \in [\alpha(t), \beta(t)], y \in [\mu(t), \nu(t)]\} + 1$. So, problem (3.4), (3.5) has a solution u with $\alpha(t) \le u(t) \le \beta(t)$ on J.

Put $z(t) = u'(t) - \nu(t)$. Let $\max\{z(t) : t \in J\} = z(t_0) > 0$. First, suppose $t_0 \in [a, b)$. Then we can find $\delta > 0$ such that $0 < z(t) \leq z(t_0)$ for each $t \in (t_0, t_0 + \delta] \subset J$. On the other hand by (3.1) $\int_{t_0}^{t_0+\delta} z'(\tau)\mathrm{d}\tau = \int_{t_0}^{t_0+\delta}(\tilde{f}(\tau, u(\tau), u'(\tau)) - \nu'(\tau))\mathrm{d}\tau > 0$, a contradiction. Further, $u'(b) > \nu(b)$ implies $\tilde{g}_2(u(b), u'(b)) = g_2(u(b), \nu(b)) + u'(b) - \nu(b) > 0$. So $u'(t) \leq \nu(t)$ for each $t \in J$. The inequality $\mu(t) \leq u'(t)$ for each $t \in J$ can be proved by similar arguments. Thus (3.3) is valid and therefore u is a solution of (1.1), (1.2) as well. □

Theorem 3.2. *Let α, β be lower and upper solutions of (1.1), (1.2) with $\alpha(t) \leq \beta(t)$ for each $t \in J$ and α'', $\beta'' \in L_\infty(J)$. Further let μ, $\nu \in AC(J)$ be such that $\mu(t) \leq \alpha'(t) \leq \nu(t)$, $\mu(t) \leq \beta'(t) \leq \nu(t)$ for each $t \in J$ and for a. e. $t \in J$ and each $x \in [\alpha(t), \beta(t)]$ the conditions*

$$f(t, x, \nu(t)) \leq \nu'(t), \quad f(t, x, \mu(t)) \geq \mu'(t), \tag{3.6}$$

$$g_1(x, \nu(t)) \geq 0, \quad g_1(x, \mu(t)) \leq 0 \tag{3.7}$$

are fulfilled.

Then problem (1.1), (1.2) has at least one solution u satisfying (2.6) and (3.3).

PROOF: Theorem 3.2 can be proved similary as Theorem 3.1. □

Note. The condition (3.2) may be changed onto

$$g_2(x, \nu(t)) \leq 0, \quad g_2(x, \mu(t)) \geq 0, \tag{3.2$'$}$$

and the assertion of Theorem 3.1 keeps its validity. Namely, in this case, we can define $\tilde{g}_2$ by

$$\tilde{g}_2(x, y) = \begin{cases} g_2(x, \nu(t)) - y + \nu(t) & \text{for } y > \nu(t) \\ g_2(x, y) & \text{for } \mu(t) \leq y \leq \nu(t) \\ g_2(x, \mu(t)) - y + \mu(t) & \text{for } y < \mu(t). \end{cases}$$

Similarly the condition (3.7) can be replaced by

$$g_1(x, \nu(t)) \leq 0, \quad g_1(x, \mu(t)) \geq 0. \tag{3.7$'$}$$

As a consequence of the theorems 3.1 and 3.2 we obtain

Theorem 3.3. *[6] Let r, $R \in (0, \infty)$ be such that for a. e. $t \in J$ and each $x \in [-r, r]$ the conditions (2.1), (2.2), (2.3) and*

$$f(t, x, R) > 0, \quad f(t, x, -R) < 0,$$
$$g_2(x, R) \cdot g_2(x, -R) < 0$$

are fulfilled.

Then problem (1.1), (1.2) has at least one solution u satisfying (2.5) and

$$-R \le u'(t) \le R. \tag{3.8}$$

Theorem 3.4. *[6] Let $r,\ R \in (0,\ \infty)$ be such that for a. e. $t \in J$ and each $x \in [-r,\ r]$ the conditions (2.1), (2.2), (2.3) and*

$$f(t,\ x,\ R) < 0, \quad f(t,\ x,\ -R) > 0,$$
$$g_2(x,\ R) \cdot g_2(x,\ -R) < 0$$

are fulfilled.

Then problem (1.1), (1.2) has at least one solution u satisfying (2.5) and (3.8).

Note. At the end we would like to emphasize that the theorems 3.1 and 3.2 are an important tool for proving multiplicity results to problem (1.1), (1.2).

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