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Boundary value problems with nonlinear conditions

IRENA RACHŮNKOVÁ

Abstract. We study the nonlinear second order differential equation \( x'' = f(t, x, x') \) with a Carathéodory nonlinearity \( f \) and nonlinear boundary conditions \( g_1(x(a), x'(a)) = 0, \ g_2(x(b), x'(b)) = 0, \ [a, b] \subset \mathbb{R} \). Using the topological degree method we find conditions for the existence of solutions to the above problem in terms of upper and lower solutions.

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(Dedicated to the memory of Svatopluk Fučík)

1 Introduction

In the paper we consider the problem

\[
 x'' = f(t, x, x'),
\]

\[
g_1(x(a), x'(a)) = 0, \quad g_2(x(b), x'(b)) = 0,
\]

where \( J = [a, b] \subset \mathbb{R}, \ f : J \times \mathbb{R}^2 \to \mathbb{R} \) satisfies the Carathéodory conditions, \( g_i : \mathbb{R}^2 \to \mathbb{R}, \ i = 1, 2, \) are continuous (generally nonlinear) functions. By a solution of problem (1.1), (1.2) we mean a function \( u \in AC^1(J) \) (having an absolutely continuous first derivative on \( J \)) and satisfying conditions (1.2) and equation (1.1) for a.e. \( t \in J \).

We find conditions for the existence of solutions of (1.1), (1.2). Such questions were studied for example in [1], [2], [3], [6]. But in [2] the appropriate linear part of (1.2) was required and in [3] the monotonicity of \( g_1, g_2 \) was supposed. Our approach is close to [1], where problem (1.1), (1.2) is studied for a continuous right hand side \( f \) satisfying the Bernstein-Nagumo conditions and \( g_1, g_2 \) monotonomous in the second variable.

Here, neither of monocity of \( g_1, g_2 \), nor growth conditions for \( f \) are required. Our results generalize the earlier ones of [6] and they are obtained by means of topological degree method together with upper and lower solutions method.

Our proofs are based on the following theorem:

Continuation Theorem. [1, p. 40]. Let \( X, Y \) be Banach spaces, \( L : \text{dom} \ L \subset X \to Y \) a Fredholm map of index 0 and \( \Omega \subset X \) an open bounded set. Let \( N : X \to \mathbb{R} \) be
Y be L-compact on $\Omega$, $Q : X \to Y$ a continuous projector with $\text{Ker} \, Q = \text{Im} \, L$ and $J : \text{Im} \, Q \to \text{Ker} \, L$ an isomorphism. Suppose

1. for each $\lambda \in (0, 1)$ every solution $x$ of $Lx = \lambda Nx$ is such that $x \notin \partial \Omega$;
2. $QN x \neq 0$ for each $x \in \text{Ker} \, L \cap \partial \Omega$ and
3. the Brouwer degree $d[N_0, \Omega \cap \text{Ker} \, L, 0] \neq 0$, where $N_0 = JQN : \text{Ker} \, L \to \text{Ker} \, L$. Then the equation $Lx = Nx$ has at least one solution in $\text{dom} \, L \cap \Omega$.

Let us remind that functions $\alpha, \beta \in AC^1(J)$ are called lower and upper solutions for (1.1), (1.2), respectively, if they fulfill

$$g_1(\alpha(t), \alpha'(t)) \geq 0, \quad g_1(\beta(t), \beta'(t)) \leq 0,$$

$$g_2(\alpha(t), \alpha'(t)) \leq 0, \quad g_2(\beta(t), \beta'(t)) \geq 0.$$ 

\section{The existence results for bounded nonlinearity}

First we will prove the existence of solutions to (1.1), (1.2) provided $f$ is bounded by an Lebesgue integrable function $\varphi$. We cannot apply the Continuation Theorem onto (1.1), (1.2) directly, but we have to construct a sequence of auxiliary problems

$$x'' = \lambda f_n(t, x, x'), \quad \lambda \in [0, 1],$$

$$g_{1n}(x(a), x'(a)) = 0, \quad g_{2n}(x(b), x'(b)) = 0, \quad (2n, \lambda)$$

where $n \in \mathbb{N}$, $r \in (0, \infty)$,

$$f_n = \begin{cases} f(t, r, 0) & \text{for } x \geq r + 1/n, \\ f(t, r, y) + [f(t, r, 0) - f(t, r, y)]n(x - r) & \text{for } r \leq x < r + 1/n, \\ f(t, x, y) & \text{for } -r \leq x \leq r, \\ f(t, -r, y) - [f(t, -r, 0) - f(t, -r, y)]n(x + r) & \text{for } -r - 1/n < x < -r, \\ f(t, -r, 0) & \text{for } x \leq -r - 1/n. \end{cases}$$

$$g_{in}(x, y) = \begin{cases} g_i(r, 0) & \text{for } x \geq r + 1/n, \\ g_i(r, y) + [g_i(r, 0) - g_i(r, y)]n(x - r) & \text{for } r \leq x < r + 1/n, \\ g_i(x, y) & \text{for } -r \leq x \leq r, \\ g_i(-r, y) - [g_i(-r, 0) - g_i(-r, y)]n(x + r) & \text{for } -r - 1/n < x < -r, \\ g_i(-r, 0) & \text{for } x \leq -r - 1/n. \end{cases}$$

$i = 1, 2.$
If we put for fixed \( n \)

\[
X = C^1([a, b]), \quad Y = L(a, b) \times \mathbb{R}^2, \quad \text{dom} \, L = AC^1([a, b]) \subset X,
\]

\[
L : \text{dom} \, L \to Y, \quad x \to (x'', 0, 0),
\]

\[
N : X \to Y, \quad x \to (f_n(\cdot, x(\cdot), x'(\cdot)),
\]

\[
g_{1n}(x(a), x'(a)), \quad g_{2n}(x(b), x'(b)),
\]

then problem \((2n, \lambda)\) can be written in the form

\[
Lx = \lambda Nx
\]

Using the Continuation Theorem for problems \((2n, \lambda)\), for any \( n \in \mathbb{N} \), we get the existence of solutions \( u_n \) of \((2n, 1)\), and by the Arzelà-Ascoli Theorem we prove the existence of limit \( u \) of the appropriate subsequence of \((u_n)_{n=1}^{\infty} \). One can see that \( u \) is a solution of \((1.1), (1.2)\). These considerations lead to the following theorem.

**Theorem 2.1.** Let \( r \in (0, \infty) \) and \( \varphi \in L(J) \) be such that for a.e. \( t \in J \) and each \( x \in [-r, r] \)

\[
g_1(-r, 0)g_1(r, 0) < 0, \quad (2.1)
\]

\[
g_2(-r, 0)g_2(r, 0) < 0, \quad (2.2)
\]

\[
f(t, -r, 0) < 0, \quad f(t, r, 0) > 0 \quad (2.3)
\]

\[
|f(t, x, y)| \leq \varphi(t) \text{ for each } y \in \mathbb{R}. \quad (2.4)
\]

Then problem \((1.1), (1.2)\) has a solution \( u \) with

\[
-r \leq u(t) \leq r \text{ for each } t \in J. \quad (2.5)
\]

For detailed proof of Theorem 2.1 see [6].

Now, using lower and upper solutions method, we can get results more general than those in Theorem 2.1.

**Theorem 2.2.** Let \( \alpha, \beta \) be lower and upper solutions of \((1.1), (1.2)\) with \( \alpha(t) \leq \beta(t) \) for each \( t \in J \) and \( \alpha'', \beta'' \in L_\infty(J) \). Further let \( \varphi \in L(J) \) be such that for a.e. \( t \in J \) and each \( x \in [\alpha(t), \beta(t)] \), \( y \in \mathbb{R} \) the condition \((2.4)\) is satisfied.

Then problem \((1.1), (1.2)\) has a solution \( u \) with

\[
\alpha(t) \leq u(t) \leq \beta(t) \text{ for each } t \in J. \quad (2.6)
\]

**Proof:** Let us fix \( n \in \mathbb{N} \) and put

\[
w(\beta) = [f(t, \beta(t), \beta'(t)) - f(t, \beta(t), y)]n(x - \beta(t)),
\]

\[
w(\alpha) = [f(t, \alpha(t), \alpha'(t)) - f(t, \alpha(t), y)]n(x - \alpha(t)),
\]
The functions \( f_n, g_{1n}, g_{2n} \) satisfy the conditions of Theorem 2.1 for \( r \geq 1 + \max\{|\alpha(t)|, |\beta(t)| : t \in J\} + \text{esssup}\{|\alpha''(t)|, |\beta''(t)| : t \in J\} \). Thus there exists solution \( u_n \) of problem

\[
\begin{align*}
x'' &= f_n(t, x, x'), \\
g_{1n}(x(a), x'(a)) &= 0, \\
g_{2n}(x(b), x'(b)) &= 0
\end{align*}
\]

satisfying (2.5).

Let us show that \( u_n \) fulfills also the inequalities

\[
-1/n + \alpha(t) \leq u_n(t) \leq \beta(t) + 1/n \quad \text{for each } t \in J. \quad (2.7)
\]

Put \( v(t) = u_n(t) - \alpha(t) + 1/n \) and suppose that \( \min\{v(t) : t \in J\} = v(\bar{t}) < 0 \). Let \( \bar{t} \in (a, b) \). Then we can find \( \delta > 0 \) and \( t_0 \geq \bar{t} \) such that \( v'(t_0) = 0, v'(t) \geq 0 \) and \( v(t) < 0 \) for each \( t \in (t_0, t_0 + \delta) \subset J \). Thus \( \int_{t_0}^{t_0 + \delta} v''(\tau)\,d\tau = v'(t_0 + \delta) - v'(t_0) \geq 0 \), and on the other hand \( \int_{t_0}^{t_0 + \delta} v''(\tau)\,d\tau \leq \int_{t_0}^{t_0 + \delta} v(\tau)\,d\tau < 0 \), a contradiction. Let
a = t_0. Then \( v(a) < 0 \), i.e. \( u_n(a) < \alpha(a) - 1/n \) and \( g_1(u_n(a), u_n'(a)) = g_1(\alpha(a), \alpha'(a)) - v(a) > 0 \), a contradiction. Similarly for \( b = t_0 \).

The second inequality in (2.7) we prove analogously putting \( v(t) = \beta(t) - u_n(t) + 1/n \).

For each \( n \in \mathbb{N} \) we get a solution \( u_n \) by this way and so we have a sequence \( (u_n)_n \) of equibounded and equicontinuous functions together with their derivatives. Applying Arzelà-Ascoli Theorem we get a subsequence uniformly converging to \( u \). We can see that \( u \) satisfies (2.6) and (1.1), (1.2).

3 The existence results for unbounded nonlinearity

**Theorem 3.1.** Let \( \alpha, \beta \) be lower and upper solutions of (1.1), (1.2) with \( \alpha(t) \leq \beta(t) \) for each \( t \in J \) and \( \alpha'', \beta'' \in L_\infty(J) \). Further let \( \mu, \nu \in AC(J) \) be such that

\[
\mu(t) \leq \alpha'(t) < \nu(t), \quad \mu(t) < \beta'(t) \leq \nu(t) \quad \text{for each } t \in J \text{ and for a.e. } t \in J \text{ and each } x \in [\alpha(t), \beta(t)] \text{ the conditions}
\]

\[
f(t, x, \nu(t)) \geq \nu'(t), \quad f(t, x, \mu(t)) \leq \mu'(t),
\]

\[
g_2(x, \nu(t)) \geq 0, \quad g_2(x, \mu(t)) \leq 0
\]

are fulfilled.

Then problem (1.1), (1.2) has at least one solution \( u \) satisfying (2.6) and

\[
\mu(t) \leq \mu'(t) \leq \nu(t) \quad \text{for each } t \in J.
\]

**Proof:** Let us put

\[
\tilde{f}(t, x, y) = \begin{cases} 
  f(t, x, \nu(t)) + (y - \nu(t))/(y - \nu(t) + 1) & \text{for } y > \nu(t) \\
  f(t, x, \mu(t)) + (y - \mu(t))/(y - \mu(t) + 1) & \text{for } y < \mu(t) \\
  f(t, x, y) & \text{for } \mu(t) \leq y \leq \nu(t)
\end{cases}
\]

\[
\tilde{g}_2(x, y) = \begin{cases} 
  g_2(x, \nu(t)) + y - \nu(t) & \text{for } y > \nu(t) \\
  g_2(x, y) & \text{for } \mu(t) \leq y \leq \nu(t) \\
  g_2(x, \mu(t)) + y - \mu(t) & \text{for } y < \mu(t)
\end{cases}
\]

and consider the problem

\[
x'' = \tilde{f}(t, x, x')
\]

\[
g_1(x(a), x'(a)) = 0, \quad \tilde{g}_2(x(b), x'(b)) = 0.
\]

The functions \( \tilde{f}, g_1, \tilde{g}_2 \) fulfill the conditions of Theorem 2.2 with \( \varphi(t) = \sup\{|f(t, x, y)| : x \in [\alpha(t), \beta(t)], y \in [\mu(t), \nu(t)]\} + 1 \). So, problem (3.4), (3.5) has a solution \( u \) with \( \alpha(t) \leq u(t) \leq \beta(t) \) on \( J \).
Put $z(t) = u'(t) - \nu(t)$. Let $\max\{z(t) : t \in J\} = z(t_0) > 0$. First, suppose $t_0 \in [a, b)$. Then we can find $\delta > 0$ such that $0 < z(t) \leq z(t_0)$ for each $t \in (t_0, t_0 + \delta] \subset J$. On the other hand by (3.1) $\int_{t_0}^{t_0 + \delta} z'(\tau) d\tau = \int_{t_0}^{t_0 + \delta} (f(\tau, u(\tau), u'(\tau)) - \nu'(\tau)) d\tau > 0$, a contradiction. Further, $u'(b) > \nu(b)$ implies $\tilde{g}_2(u(b), u'(b)) = g_2(u(b), \nu(b)) + u'(b) - \nu(b) > 0$. So $u'(t) \leq \nu(t)$ for each $t \in J$. The inequality $\mu(t) \leq u'(t)$ for each $t \in J$ can be proved by similar arguments. Thus (3.3) is valid and therefore $u$ is a solution of (1.1), (1.2) as well.

**Theorem 3.2.** Let $\alpha, \beta$ be lower and upper solutions of (1.1), (1.2) with $\alpha(t) \leq \beta(t)$ for each $t \in J$ and $\alpha^\alpha, \beta^\alpha \in L_{\infty}(J)$. Further let $\mu, \nu \in AC(J)$ be such that $\mu(t) \leq \alpha'(t) \leq \nu(t), \mu(t) \leq \beta'(t) \leq \nu(t)$ for each $t \in J$ and for a.e. $t \in J$ and each $x \in [\alpha(t), \beta(t)]$ the conditions

\[
 f(t, x, \nu(t)) \leq \nu'(t), \quad f(t, x, \mu(t)) \geq \mu'(t),
\]

\[
 g_1(x, \nu(t)) \geq 0, \quad g_1(x, \mu(t)) \leq 0
\]

are fulfilled.

Then problem (1.1), (1.2) has at least one solution $u$ satisfying (2.6) and (3.3).

**Proof:** Theorem 3.2 can be proved similarly as Theorem 3.1.

**Note.** The condition (3.2) may be changed onto

\[
 g_2(x, \nu(t)) \leq 0, \quad g_2(x, \mu(t)) \geq 0
\]

and the assertion of Theorem 3.1 keeps its validity. Namely, in this case, we can define $\tilde{g}_2$ by

\[
 \tilde{g}_2(x, y) = \begin{cases} 
 g_2(x, \nu(t)) - y + \nu(t) & \text{for } y > \nu(t) \\
 g_2(x, y) & \text{for } \mu(t) \leq y \leq \nu(t) \\
 g_2(x, \mu(t)) - y + \mu(t) & \text{for } y < \mu(t).
\end{cases}
\]

Similarly the condition (3.7) can be replaced by

\[
 g_1(x, \nu(t)) \leq 0, \quad g_1(x, \mu(t)) \geq 0
\]

As a consequence of the theorems 3.1 and 3.2 we obtain

**Theorem 3.3.** [6] Let $r, R \in (0, \infty)$ be such that for a.e. $t \in J$ and each $x \in [-r, r]$ the conditions (2.1), (2.2), (2.3) and

\[
 f(t, x, R) > 0, \quad f(t, x, -R) < 0,
\]

\[
 g_2(x, R) \cdot g_2(x, -R) < 0
\]
are fulfilled.

Then problem (1.1), (1.2) has at least one solution \( u \) satisfying (2.5) and

\[- R \leq u'(t) \leq R. \tag{3.8}\]

**Theorem 3.4.** [6] Let \( r, R \in (0, \infty) \) be such that for a.e. \( t \in J \) and each \( x \in [-r, r] \) the conditions (2.1), (2.2), (2.3) and

\[
f(t, x, R) < 0, \quad f(t, x, -R) > 0, \quad g_2(x, R) - g_2(x, -R) < 0
\]

are fulfilled.

Then problem (1.1), (1.2) has at least one solution \( u \) satisfying (2.5) and (3.8).

**Note.** At the end we would like to emphasize that the theorems 3.1 and 3.2 are an important tool for proving multiplicity results to problem (1.1), (1.2).

**References**


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