Sign-changing solutions of singular Dirichlet boundary value problems *

Irena Rachůnková and Svatoslav Staněk

Department of Mathematical Analysis, Faculty of Science,
Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: rachunko@risc.upol.cz stanek@risc.upol.cz

Abstract: The singular Dirichlet problem $(r(x)x')' = \mu q(t)f(t,x)$, $x(0) = x(T) = 0$ is considered. Here f is singular at the point $x = 0$ of the phase variable x. Effective conditions for the existence of a solution in the class $C^1([0,T])$ to the above problem which changes its sign exactly ones in $(0,T)$ are presented. Existence proofs are based on "gluing" of positive and negative parts of solutions and on smoothing them.

Keywords: Singular Dirichlet problem, sign-changing solution, existence.
Mathematics Subject Classification: 34B16, 34B18.

1 Introduction, lemmas

Let T be a positive number. We will consider the singular Dirichlet boundary value problem

$$(r(x(t))x'(t))' = \mu q(t)f(t,x(t)), \tag{1.1}$$

$$x(0) = x(T) = 0,\ \max\{x(t) : 0 \le t \le T\} \cdot \min\{x(t) : 0 \le t \le T\} < 0, \tag{1.2}$$

where μ is a positive parameter and f is singular at the point $x = 0$ of the phase variable x in the following sense

$$\lim_{x\to 0^-} f(t,x) = -\infty,\ \lim_{x\to 0^+} f(t,x) = \infty \quad \text{for } t \in [0,T]. \tag{1.3}$$

We say that a function $x \in C^1([0,T])$ is a *solution of problem* (1.1), (1.2) if x has precisely one zero t_0 in $(0,T)$, $r(x)x' \in C^1((0,T) \setminus \{t_0\})$, x fulfils (1.2) and there exists $\mu_0 > 0$ such that (1.1) is satisfied for $\mu = \mu_0$ and $t \in (0,T) \setminus \{t_0\}$.

In this paper, we are interested in finding effective conditions imposed on the functions r, q and f for the existence of solutions to problem (1.1), (1.2).

*Supported by grant no. 201/01/1451 of the Grant Agency of Czech Republic and by the Czech–Slovak project No. 25(022)/2000

Any such solution goes through the singularity of f. As far as we know, this case has not been solved yet. Up till now, only positive (negative) solutions on $(0,T)$ of the Dirichlet problem with the singularity at the point $x=0$ of the phase variable x in nonlinearities of considered second-order differential equations have been studied (see, e.g., [1]–[9], [11]–[17] and references therein). Solutions were considered either in the class $C^0([0,T]) \cap C^2((0,T))$ ([1]–[3], [8], [9], [13], [14]) or $C^1([0,T]) \cap C^2((0,T))$ ([4], [9], [11]–[14], [17]) or $C^0([0,T]) \cap AC^1_{loc}((0,T))$ ([5]–[7], [15], [16]). Here $AC^1_{loc}((0,T))$ denotes the set of functions having absolutely continuous first derivatives on each $[a,b] \subset (0,T)$. The nonlinearities of equations are usually nonpositive ([1]–[5], [8], [9], [11]–[15], [17]) but in ([3], [6], [7], [16]) this assumption is overcome.

According to our above definition, solutions of problem (1.1), (1.2) belong to the class $C^1([0,T])$. The character of smoothness of solutions is very important for the consideration of their existence. We note that if we study solutions of our problem only in the class of functions having continuous first derivatives on $[0,T]$ except of zeros of solutions in $(0,T)$, we can get the existence result as well as the exact multiplicity result easier, see [10].

Since our solutions have to go through the singularity of f and they have to be smooth there, we will develop a new approach to prove their existence. This approach is based on "gluing" of positive and negative parts of solutions and on smoothing them.

Throughout the paper we assume that

(H1) $r \in C^0(\mathbb{R})$, $r(x) \geq r_0 > 0$ for $x \in \mathbb{R}$,

(H2) $q \in C^0((0,T))$, $q(t) < 0$ for $t \in (0,T)$ and

$$Q = \sup\{|q(t)| : t \in [0,T]\} < \infty,$$

(H3) $f \in C^0([0,T] \times D)$, where $D = (-\infty,0) \cup (0,\infty)$, $f(t,\cdot)$ is nonincreasing on D for $t \in [0,T]$ and

$$0 < k(t) \leq f(t,x)\operatorname{sign} x \leq g(x) \quad \text{on } [0,T] \times D,$$

where $k \in C^0([0,T])$, $g \in C^0(D)$,

$$\int^0 g(x)\,dx < \infty, \quad \int_0 g(x)\,dx < \infty.$$

Remark 1.1. From our next considerations it follows that the assumptions imposed on k in (H3) can be weakened and formulated locally. It is sufficient to assume that for any $M > 0$ there is a function $k_M \in C^0([0,T])$ such that

$$0 < k_M(t) \leq f(t,x)\operatorname{sign} x \quad \text{on } [0,T] \times [-M,0) \cup (0,M].$$

Then all results which are proved here are valid.

Let as note that since $f(t,\cdot)$ is nonincreasing on D for $t \in [0,T]$, we can assume without loss of generality that g is nondecreasing on $(-\infty,0)$, nonincreasing on $(0,\infty)$ and

$$\lim_{|x|\to\infty} g(x) \in (0,\infty). \tag{1.4}$$

Suppose that $A \in (0,\infty)$, $B \in (-\infty,0)$ and a, $b \in [0,T]$, $a < b$. We will need the following auxiliary boundary conditions

$$x(a) = x(b) = 0,\ x(t) > 0 \text{ for } t \in (a,b), \tag{1.5}$$

$$x(a) = x(b) = 0,\ x(t) < 0 \text{ for } t \in (a,b), \tag{1.6}$$

$$x(a) = x(b) = 0,\ x(t) > 0 \text{ for } t \in (a,b),\ \max\{x(t) : a \le t \le b\} \le A, \tag{1.7}$$

$$x(a) = x(b) = 0,\ x(t) < 0 \text{ for } t \in (a,b),\ \min\{x(t) : a \le t \le b\} \ge B. \tag{1.8}$$

Let $j \in \{5,6,7,8\}$ and μ be a positive fixed number. We say that $x \in C^1([a,b])$ is a *solution of problem* (1.1), (1.j), if x satisfies (1.j), $r(x)x' \in C^1((a,b))$ and (1.1) with this fixed μ is fulfilled for $t \in (a,b)$.

Let the functions $r^* : \mathbb{R} \to [r_0,\infty)$, $f^* : [0,T] \times D \to \mathbb{R}$ and $g^* : D \to \mathbb{R}$ be defined by

$$r^*(x) = r(-x), \quad f^*(t,x) = -f(t,-x), \quad g^*(x) = g(-x).$$

Then (H1) and (H2) imply that $r^* \in C^0(\mathbb{R})$, $f^* \in C^0([0,T] \times D)$, $f^*(t,\cdot)$ is nonincreasing on D for $t \in [0,T]$, $g^* \in C^0(D)$ and

$$0 < k(t) \le f^*(t,x)\,\mathrm{sign}\,x \le g^*(x) \quad \text{on } [0,T] \times D,$$

$$\int^0 g^*(x)\,dx < \infty, \quad \int_0 g^*(x)\,dx < \infty.$$

Consider the differential equation

$$(r^*(x(t))x'(t))' = \mu q(t) f^*(t,x(t)), \tag{1.9}$$

where μ is a positive parameter. Let $A \in (0,\infty)$ and $B = -A$ in (1.8). We can check that a function u is a solution of problem (1.1), (1.5) and (1.1), (1.7), if and only if the function $u^* = -u$ on $[a,b]$ is a solution of problem (1.9), (1.6) and (1.9), (1.8), respectively.

In what follows we will use this fact in formulations of "dual" results which will be signed by * and which will not be proved.

In our considerations we will use the function $H : [0,\infty) \to [0,\infty)$ defined by

$$H(u) = \int_0^u r(s)\,ds, \tag{1.10}$$

where r is the function from (H1). Of course, H is continuous increasing function. The inverse functions to H is denoted by $H^{-1} : [0,\infty) \to [0,\infty)$.

Our further studies are based on some results proved in [12] which are slightly modified here.

Lemma 1.2. *For each $\mu > 0$ and a, $b \in [0,T]$, $a < b$, there exists a solution u of problem* (1.1), (1.5) *such that* $\max\{u(t) : a \le t \le b\} \le A$, *where $A > 0$ is an arbitrary number satisfying the inequality*

$$\mu \le \frac{2\Big(\displaystyle\int_0^A r(s)\,ds\Big)^2}{(b-a)^2 Q \displaystyle\int_0^A r(s)g(s)\,ds}. \tag{1.11}$$

Moreover,

$$u(t) \ge \begin{cases} H^{-1}\Big(\frac{2\mu K(t-a)}{b-a}\Big) & \textit{for } t \in [a, \frac{a+b}{2}] \\ H^{-1}\Big(\frac{2\mu K(b-t)}{b-a}\Big) & \textit{for } t \in (\frac{a+b}{2}, b], \end{cases} \tag{1.12}$$

where $K = \min\Big\{ \displaystyle\int_a^{\frac{a+b}{2}} (s-a)q(s)k(s)\,ds,\ \int_{\frac{a+b}{2}}^b (b-s)q(s)k(s)\,ds \Big\}$.

Lemma 1.3. *Let $\mu > 0$ and a, $b \in [0,T]$, $a < b$, $A > 0$ be such that* (1.11) *is true. Suppose that $u \in C^0([a,b])$ satisfies* (1.7), $r(u)u' \in C^1((a,b))$ *and* (1.1) *is fulfilled for $t \in (a,b)$. Then*

$$|r(u(t))u'(t)| \le \sqrt{2\mu Q \int_0^A r(s)g(s)\,ds} \quad \textit{for } t \in (a,b) \tag{1.13}$$

and for each ξ, $t \in [a,b]$ the inequality

$$\Big|\int_\xi^t q(s)f(s,u(s))\,ds\Big| \le 2\sqrt{\frac{2Q}{\mu}\int_0^A r(s)g(s)\,ds} \tag{1.14}$$

is valid.

The paper is organized as follows. Sec. 2 - 4 contain the existence and uniqueness results to problems (1.1), (1.j), $j \in \{5,6,7,8\}$, and results concerning analytic properties of some auxiliary functions. The main results about the existence of solutions to problem (1.1), (1.2) are given in Sec. 5.

2 Uniqueness and monotonicity

In order to construct sign-changing solutions to problem (1.1), (1.2) we first need existence and uniqueness for solutions to auxiliary problems (1.1), (1.j),

$j \in \{5,6,7,8\}$. These results are presented in Theorems 2.1, 2.2*, 2.5 and 2.6*. Further needful results about a dependence of such solutions on values of the parameter μ are contained in Lemmas 2.3 and 2.4. Finally, a dependence of the solutions on the length of intervals $[a,b] \subset [0,T]$ is described in Lemma 2.7.

Theorem 2.1. *Let a, $b \in [0,T]$, $a < b$. Then for each $\mu > 0$ problem* (1.1), (1.5) *has a unique solution. Suppose moreover that $A > 0$ and put*

$$m_+(a,b;A) = \frac{2\Big(\int_0^A r(s)\,ds\Big)^2}{(b-a)^2 Q \int_0^A r(s)g(s)\,ds}. \tag{2.1}$$

Then for each $\mu \in (0, m_+(a,b;A)]$ problem (1.1), (1.7) *has a unique solution.*
Proof. Lemma 1.2 implies the existence both for problem (1.1), (1.5) if $\mu > 0$ and for problem (1.1), (1.7) provided $\mu \in (0, m_+(a,b;A)]$. To prove the uniqueness, we assume that for some fixed $\mu > 0$ there exist two different solutions u_1, u_2 of (1.1), (1.5) or of (1.1), (1.7). Then there exist $a \le \alpha < \beta \le b$ such that $u_1(\alpha) \ge u_2(\alpha)$, $u_2'(\alpha) < u_1'(\alpha)$, $u_1(\beta) = u_2(\beta)$, $u_2(t) < u_1(t)$ for $t \in (\alpha,\beta)$. Then according to (H3) we have $f(t,u_2(t)) \ge f(t,u_1(t))$ for $t \in (\alpha,\beta)$. Let us put

$$p(t) = \int_{u_1(t)}^{u_2(t)} r(s)\,ds \quad \text{for } t \in [\alpha,\beta]. \tag{2.2}$$

Then

$$p(\alpha) \le p(\beta) = 0. \tag{2.3}$$

Further, for $t \in (\alpha,\beta)$, we get $p''(t) \le 0$ and $p'(\alpha) < 0$. Therefore

$$p'(t) < 0 \quad \text{for } t \in (\alpha,\beta], \tag{2.4}$$

which contradicts (2.3). □

Theorem 2.2.* *Let a, $b \in [0,T]$, $a < b$. Then for each $\mu > 0$ problem* (1.1), (1.6) *has a unique solution. Suppose moreover that $B < 0$ and put*

$$m_-(a,b;B) = \frac{2\Big(\int_B^0 r(s)\,ds\Big)^2}{(b-a)^2 Q \int_B^0 r(s)g(s)\,ds}. \tag{2.5}$$

Then for each $\mu \in (0, m_-(a,b;B)]$ problem (1.1), (1.8) *has a unique solution.*

Lemma 2.3. *Let $0 < \mu_1 < \mu_2$, a, $b \in [0,T]$, $a < b$, and let u_i be a (unique) solution of problem* (1.1), (1.5) *with $\mu = \mu_i$, $i = 1, 2$. Then*

$$u_1(t) < u_2(t) \quad \text{for } t \in (a,b). \tag{2.6}$$

Proof. First, let us prove the inequality

$$u_1(t) \le u_2(t) \quad \text{for } t \in [a,b]. \tag{2.7}$$

We can follow the proof of Theorem 2.1. Let us set $p(t) = \int_{u_1(t)}^{u_2(t)} r(s)\,ds$ for $t \in [a,b]$. Suppose $u_1(\alpha) = u_2(\alpha),\ u_1(\beta) = u_2(\beta),\ u_2(t) < u_1(t)$ for $t \in (\alpha,\beta)$ with some $a \le \alpha < \beta \le b$. Then the function p fulfils

$$p(\alpha) = p(\beta) = 0. \tag{2.8}$$

Further, for $t \in (\alpha,\beta)$, we get $p''(t) = q(t)(\mu_2 f(t,u_2(t)) - \mu_1 f(t,u_1(t))) < 0$ and $p'(\alpha) \le 0$. Thus (2.4) holds, which contradicts (2.8). So, we have proved (2.7) which means that

$$p(t) \ge 0 \quad \text{for } t \in [a,b]. \tag{2.9}$$

Let us suppose that there exists $\xi \in (a,b)$ such that $u_1(\xi) = u_2(\xi)$. Then, by (2.7), $u_1'(\xi) = u_2'(\xi)$. Therefore $p(\xi) = 0,\ p'(\xi) = 0,\ p''(\xi) < 0$. This implies that there exists $\varepsilon > 0$ such that $p'(t) < 0$ for $t \in (\xi, \xi+\varepsilon]$, contrary to (2.9). □

Lemma 2.4. *Let* $a,\ b \in [0,T],\ a < b,\ \{\mu_n\} \subset (0,\infty),\ \lim_{n\to\infty} \mu_n = \mu_0 > 0$. *Let* u_n *be a solution of problem* (1.1), (1.5) (*or* (1.1), (1.6)) *with* $\mu = \mu_n$, $n \in \mathbb{N} \cup \{0\}$. *Then*

$$\lim_{n\to\infty} u_n(t) = u_0(t) \quad \textit{uniformly on } [a,b]. \tag{2.10}$$

Proof. It is sufficient to consider solutions of (1.1), (1.5) and to assume that $\{\mu_n\}$ is strictly monotonous, for example decreasing. Then, by Lemma 2.3,

$$u_0(t) < u_{n+1}(t) < u_n(t) \quad \text{for } t \in (a,b),\ n \in \mathbb{N}. \tag{2.11}$$

Further, by Lemma 1.3,

$$|r(u_n(t))u_n'(t)| \le \sqrt{2\mu_1 Q \int_0^{\|u_1\|} r(s)g(s)\,ds} \quad \text{for } t \in [a,b],\ n \in \mathbb{N}$$

and thus

$$|u_n'(t)| \le \frac{1}{r_0}\sqrt{2\mu_1 Q \int_0^{\|u_1\|} r(s)g(s)\,ds} \quad \text{for } t \in [a,b],\ n \in \mathbb{N},$$

where $\|\cdot\|$ stands for the sup-norm in $C^0([a,b])$. Therefore $\{u_n'(t)\}$ is uniformly bounded on $[a,b]$ which together with the monotonicity of $\{u_n(t)\}$ gives

$$\lim_{n\to\infty} u_n(t) = w(t)\ (\ge u_0(t)) \quad \text{uniformly on } [a,b]. \tag{2.12}$$

We are going to prove that $u_0(t) = w(t)$ for $t \in [a, b]$. From (H3), (2.11) and (2.12) it follows that $f(t, u_{n+1}(t)) \ge f(t, u_n(t))$ for $t \in (a, b)$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} f(t, u_n(t)) = f(t, w(t))$ for $t \in (a, b)$. Since $q(t)f(t, u_n(t))$ is integrable on $[a, b]$ for each $n \in \mathbb{N}$ (see (1.14) with $u = u_n$), the Levi theorem yields

$$\lim_{n\to\infty} \int_a^t q(s)f(s, u_n(s))\, ds = \int_a^t q(s)f(s, w(s))\, ds \quad \text{for } t \in [a, b]. \tag{2.13}$$

Now, integrating the equalities

$$r(u_n(t))u_n'(t) = r(0)u_n'(a) + \mu_n \int_a^t q(s)f(s, u_n(s))\, ds, \quad t \in [a, b],\ n \in \mathbb{N},$$

from a to $t \in (a, b]$, we get

$$\begin{aligned} \int_0^{u_n(t)} r(s)\, ds &= r(0)u_n'(a)(t-a) \\ &+\mu_n \int_a^t \int_a^s q(\tau)f(\tau, u_n(\tau))\, d\tau\, ds \quad \text{for } t \in [a, b],\ n \in \mathbb{N}. \end{aligned} \tag{2.14}$$

Since $\{u_n'(a)\}$ is bounded, we can assume without loss of generality that $\lim_{n\to\infty} u_n'(a) = C \in \mathbb{R}$. By the limit process for $n \to \infty$ in (2.14), we get according to (2.13)

$$\int_0^{w(t)} r(s)\, ds = Cr(0)(t-a) + \mu_0 \int_a^t \int_a^s q(\tau)f(\tau, w(\tau))\, d\tau\, ds \quad \text{for } t \in [a, b]$$

and thus (for $t \in [a, b]$)

$$w(t) = H^{-1}\Big(Cr(0)(t-a) + \mu_0 \int_a^t \int_a^s q(\tau)f(\tau, w(\tau))\, d\tau\, ds\Big) \tag{2.15}$$

where H is defined by (1.10) and H^{-1} is its inverse. Since

$$(H^{-1}(u))' = \frac{1}{H'(H^{-1}(u))} = \frac{1}{r(H^{-1}(u))} \quad \text{for } u \in [0, \infty),$$

(2.15) implies that $w \in C^1([a, b])$ and

$$r(w(t))w'(t) = Cr(0) + \mu_0 \int_a^t q(s)f(s, w(s))\, ds \quad \text{for } t \in [a, b],$$

which gives $r(w)w' \in C^1((a, b))$. Moreover w fulfils (1.1) with $\mu = \mu_0$ for $t \in (a, b)$. Finally, by (2.12), $w(a) = w(b) = 0$ and $w(t) > 0$ for $t \in (a, b)$. Using Theorem 2.1 on uniqueness, we obtain $w(t) = u_0(t)$ for $t \in [a, b]$.

The case that $\{\mu_n\}$ is increasing is considered similarly and so we omit it.

To prove the assertion for solutions of (1.1), (1.6) we use the dual consideration. □

Theorem 2.5. *For each $A > 0$ and $a,\ b \in [0, T]$, $a < b$, there exists just one value $\mu_0 \in [m_+(a, b; A), \infty)$ of the parameter μ, where $m_+(a, b; A)$ is given by* (2.1)*, such that problem* (1.1), (1.5) *with $\mu = \mu_0$ has a solution u satisfying*

$$\max\{u(t) : a \le t \le b\} = A.$$

This solution is unique (for $\mu = \mu_0$).

Proof. Theorem 2.1 implies that for each $\mu > 0$ there exists just one solution $u(t, \mu)$ of (1.1), (1.5). Let us put $p(\mu) = \max\{u(t, \mu) : a \le t \le b\}$ and suppose that

$$p(\mu) < A \quad \text{for } \mu > 0. \tag{2.16}$$

By Lemma 1.2, $u(\cdot, \mu)$ satisfies (1.12) for $\mu > 0$. Thus, putting $t = \frac{a+b}{2}$ in (1.12), we get $p(\mu) \ge H^{-1}(K\mu)$ for $\mu > 0$. Since $\lim_{\mu\to\infty} H^{-1}(K\mu) = \infty$, we obtain a contradiction to (2.16). Therefore there is a $\mu_1 > 0$ such that $p(\mu_1) > A$. On the other hand if $\mu \in (0, m_+(a, b; A)]$ then, due to Theorem 2.1, $p(\mu) \le A$. Lemmas 2.3 and 2.4 imply that p is increasing and continuous on $(0, \infty)$, and so there is a unique $\mu_0 \in [m_+(a, b; a), \mu_1)$ such that $A = p(\mu_0) = \max\{u(t, \mu_0) : a \le t \le b\}$. □

Theorem 2.6.* *For each $B < 0$ and $a,\ b \in [0, T]$, $a < b$, there exists just one value $\mu^* \in [m_-(a, b; B), \infty)$ of the parameter μ with $m_-(a, b; B)$ from* (2.5) *such that problem* (1.1), (1.6) *with $\mu = \mu^*$ has a solution u^* satisfying*

$$\min\{u^*(t) : a \le t \le b\} = B.$$

This solution is unique (for $\mu = \mu^*$).

Lemma 2.7. *Let $\mu > 0$, $a = 0$, $b_1,\ b_2 \in (0, T]$, $b_1 < b_2$, and let u_i be a (unique) solution of problem* (1.1), (1.5) *with $b = b_i$, $i = 1, 2$. Then*

$$u_1(t) \le u_2(t) \quad \textit{for } t \in [0, b_1]. \tag{2.17}$$

Proof. Since $u_1(0) = u_2(0) = 0$ and $u_1(b_1) = 0 < u_2(b_1)$, there is a $b_0 \in [0, b_1)$ such that $u_1(t) < u_2(t)$ for $t \in (b_0, b_1]$ and $u_1(b_0) = u_2(b_0)$. If $b_0 = 0$, then (2.17) is true. Let us suppose that $b_0 > 0$ and that there exist $0 \le \alpha < \beta \le b_0$ such that (2.3) is true with p given by (2.2). Then we can argue as in the proof of Theorem 2.1 and get a contradiction, which completes this proof. □

3 Continuous dependence of parameter values on endpoints of solutions domains

Let $A > 0$. Then, by Theorem 2.5, for each $c \in (0, T]$ there exists just one value of the parameter μ, which will be denoted by $\mu(c)$, such that the problem

$$\begin{gathered} (r(x(t))x'(t))' = \mu(c)q(t)f(t, x(t)),\ t \in (0, c) \\ x(0) = x(c) = 0,\ x(t) > 0 \text{ on } (0, c),\ \max\{x(t) : 0 \le t \le c\} = A \end{gathered} \tag{3.1}$$

has a (unique) solution which we will denote by u_c. In such a way we get the function

$$\mu : (0, T] \to (0, \infty). \tag{3.2}$$

This section is devoted to the study of analytic properties of μ. This function will play an important role in the next consideration of a behaviour of derivatives of solutions of (3.1) at endpoints. Some properties of μ are presented in the following proposition.

Proposition 3.1. *The function $\mu(c)$ is continuous and nonincreasing on $(0, T]$.*

Proof. First, let us prove that $\mu(c)$ in nonincreasing on $(0, T]$. Let $0 < c_1 < c_2 \le T$ and let y be a solution of equation (1.1) for $\mu = \mu(c_1)$ satisfying the conditions $y(0) = y(c_2) = 0$, $y(t) > 0$ for $t \in (0, c_2)$. Then, by Lemma 2.7, $u_{c_1}(t) \le y(t)$ for $t \in [0, c_1]$ and thus $\max\{y(t) : 0 \le t \le c_2\} \ge A$. Using Lemma 2.3 and Theorem 2.5 we deduce that $\mu(c_2) \le \mu(c_1)$. Now, we will prove that $\mu(c)$ is continuous on $(0, T]$. Suppose that μ is discontinuous from the left at some $c_0 \in (0, T]$, i.e. that

$$\mu_0 = \lim_{c \to c_0 -} \mu(c) \ne \mu(c_0). \tag{3.3}$$

Since $\mu(c)$ is nonincreasing, we have $\mu_0 > \mu(c_0) > 0$. Let $\{c_n\} \subset (0, c_0)$ be an increasing sequence and $\lim_{n\to\infty} c_n = c_0$. Consider the corresponding sequence $\{u_{c_n}\}$ of solutions of problems (3.1) for $c = c_n, n \in \mathbb{N}$. According to (3.1) we have

$$0 \le u_{c_n}(t) \le A \quad \text{for } t \in [0, c_n], n \in \mathbb{N}. \tag{3.4}$$

Further, by (H1) and Lemma 1.3, we get $r_0|u'_{c_n}(t)| \le |r(u_{c_n}(t))u'_{c_n}(t)| \le L_1$, where $L_1 = \sqrt{2\mu(c_1)Q\int_0^A g(s)r(s)ds}$, therefore

$$|u'_{c_n}(t)| \le \frac{L_1}{r_0} \quad \text{for } t \in [0, c_n], n \in \mathbb{N}. \tag{3.5}$$

Using the Arzelà-Ascoli theorem we can suppose without loss of generality that $\{u_{c_n}(t)\}$ uniformly converges on each interval $[0, c_0 - \varepsilon] \subset [0, c_0)$, where $\varepsilon \in (0, c_0)$. Thus

$$\lim_{n\to\infty} u_{c_n}(t) = u(t) \quad \text{locally uniformly on } [0, c_0). \tag{3.6}$$

Then $u \in C^0([0, c_0))$ and $u(0) = 0$. Let us denote for $n \in \mathbb{N} \cup \{0\}$

$$K_n = \min\left\{\int_0^{\frac{c_n}{2}} sq(s)k(s)ds, \int_{\frac{c_n}{2}}^{c_n} (c_n - s)q(s)k(s)ds\right\}. \tag{3.7}$$

Then, by Lemma 1.2,

$$u_{c_n}(t) \geq \begin{cases} H^{-1}\left(\frac{2\mu(c_n)K_n t}{c_n}\right) & \text{for } t \in [0, \frac{c_n}{2}] \\ H^{-1}\left(\frac{2\mu(c_n)K_n(c_n - t)}{c_n}\right) & \text{for } t \in (\frac{c_n}{2}, c_n], \end{cases}$$

and thus, according to (3.3) and (3.6),

$$u(t) \geq \begin{cases} H^{-1}\left(\frac{2\mu_0 K_0 t}{c_0}\right) & \text{for } t \in [0, \frac{c_0}{2}] \\ H^{-1}\left(\frac{2\mu_0 K_0(c_0 - t)}{c_0}\right) & \text{for } t \in (\frac{c_0}{2}, c_0), \end{cases}$$

which means that $u(t) > 0$ for $t \in (0, c_0)$. Moreover $f(t, u_{c_n}(t)) > 0$ for $t \in (0, c_n)$, $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} f(t, u_{c_n}(t)) = f(t, u(t)) \quad \text{for } t \in (0, c_0). \tag{3.8}$$

According to Lemma 1.3 we get

$$\left|\int_0^t q(s)f(s, u_{c_n}(s)ds\right| \leq 2\sqrt{\frac{2Q}{\mu(c_n)}\int_0^A g(s)r(s)ds} \quad \text{for } t \in [0, c_n],\ n \in \mathbb{N},$$

which implies, by means of the Fatou theorem, that $q(t)f(t, u(t))$ is integrable on each compact interval which is contained in $[0, c_0)$. Let us choose $\varepsilon > 0$ such that $I_\varepsilon = [\varepsilon, c_0 - \varepsilon] \subset (0, c_0)$. Let $\xi \in (\varepsilon, c_0 - \varepsilon)$. Then we have for sufficiently large $n \in \mathbb{N}$

$$\begin{aligned} \int_{u_{c_n}(\xi)}^{u_{c_n}(t)} r(s)ds &= r(u_{c_n}(\xi))u'_{c_n}(\xi)(t - \xi) \\ &\quad + \mu(c_n)\int_\xi^t \int_\xi^s q(\tau)f(\tau, u_{c_n}(\tau)d\tau ds \quad \text{for } t \in I_\varepsilon, \end{aligned} \tag{3.9}$$

and

$$0 < f(t, u_{c_n}(t)) \leq f(t, C_\varepsilon) \quad \text{for } t \in I_\varepsilon, \tag{3.10}$$

where $C_\varepsilon = \frac{1}{2}\min\{u(t) : t \in I_\varepsilon\} > 0$. In view of (3.4) and (3.5) the sequence $\{r(u_{c_n}(\xi))u'_{c_n}(\xi)\}$ is bounded and thus we can suppose that it is convergent, i.e. $\lim_{n\to\infty} r(u_{c_n}(\xi))u'_{c_n}(\xi) = V$. Now, having in mind (3.8) and (3.10), we can use the Lebesque dominated converegence theorem. Letting $n \to \infty$ in (3.9) we get

$$\int_{u(\xi)}^{u(t)} r(s)ds = V(t-\xi) + \mu_0 \int_\xi^t \int_\xi^s q(\tau)f(\tau, u(\tau))d\tau ds \quad \text{for } t \in I_\varepsilon. \quad (3.11)$$

Since ε is an arbitrary small positive number and the function $q(t)f(t,u(t))$ is integrable on $[0, c_0 - \varepsilon) \subset [0, c_0)$, we can deduce that (3.11) is valid for $t \in [0, c_0)$. Then

$$u'(t) = \frac{1}{r(u(t))}\left(V + \mu_0 \int_\xi^t q(s)f(s,u(s))ds\right) \quad \text{for } t \in [0, c_0), \quad (3.12)$$

i.e. $u \in C^1([0, c_0))$, and further

$$(r(u(t))u'(t))' = \mu_0 q(t)f(t,u(t)) \quad \text{for } t \in (0, c_0), \quad (3.13)$$

and so $r(u)u' \in C^1((0, c_0))$. By (3.5), $u_{c_n}(t) \le \frac{L_1}{r_0}(c_n - t) < \frac{L_1}{r_0}(c_0 - t)$ for $t \in [0, c_n]$, $n \in \mathbb{N}$, which yields $0 < u(t) = \lim_{n\to\infty} u_{c_n}(t) \le \frac{L_1}{r_0}(c_0 - t)$ for $t \in (0, c_0)$. Set $u(c_0) = \lim_{c \to c_0-} u(c)$. Then $u(c_0) = 0$ and also $\max\{u(t) : 0 \le t \le c_0\} = A$. Further, by Lemma 1.3 (see (1.14)), the function $q(t)f(t,u(t))$ is integrable on $[0, c_0]$, which means that (3.11) and (3.12) are valid on $[0, c_0]$ and $u \in C^1([0, c_0])$. We have proved that u is a solution of problem (3.1) with $c = c_0$ and $\mu(c) = \mu_0$. But with respect to the definition of the function μ we get $\mu_0 = \mu(c_0)$, which contradicts (3.3). Therefore μ is continuous from the left on $(0, T]$.

We can argue similarly to prove that μ is continuous from the right on $(0, T)$. □

Let $B < 0$. Then, by Theorem 2.6*, for each $c \in [0, T)$ there exists just one value of the parameter μ, which will be denoted by $\mu^*(c)$, such that the problem

$$\begin{gathered}(r(x(t))x'(t))' = \mu^*(c)q(t)f(t,x(t)),\ t \in (c, T) \\ x(c) = x(T) = 0, \quad x(t) < 0 \ \text{ on } (c, T), \ \ \min\{x(t) : c \le t \le T\} = B\end{gathered} \quad (3.14)$$

has a (unique) solution which we will denote by u_c^*. This defines the function

$$\mu^* : [0, T) \to (0, \infty), \quad (3.15)$$

whose properties are "dual" to those of μ.

Proposition 3.2.* *The function $\mu^*(c)$ is continuous and nondecreasing on $[0, T)$.*

4 Continuous dependence of derivatives of solutions on endpoints of their domains

Let $A > 0, c \in (0, T]$ and let $\mu(c)$ and $u_c(t)$ be the corresponding (uniquely determined) parameter and solution of problem (3.1), respectively. Let us define the function $\Lambda_A : (0, T] \to (-\infty, 0)$ by the formula

$$\Lambda_A(c) = u_c'(c). \tag{4.1}$$

Here $u_c'(c)$ means the left derivative of u_c at c. The gluing and smoothing process described in next Sec. 5 is based on the analytic properties of the function Λ_A and functions Φ_A, Λ_B^*, Φ_B^* defined by formulas (4.13), (4.17), (4.20). These properties are presented in propositions of this section.

Proposition 4.1. *The function Λ_A defined by* (4.1) *is continuous on* $(0, T]$ *and satisfies the inequality*

$$\Lambda_A(c) < -\frac{Ar_0}{Tr(0)} \quad \text{for } c \in \left[\frac{T}{2}, T\right]. \tag{4.2}$$

Proof. Let us suppose that Λ_A is discontinuous from the left at some $c_0 \in (0, T]$. Then there exists an increasing sequence $\{c_n\} \subset (0, c_0), \lim_{n\to\infty} c_n = c_0$, such that $\lim_{n\to\infty} \Lambda_A(c_n) \neq \Lambda_A(c_0)$, i.e.

$$\lim_{n\to\infty} u_{c_n}'(c_n) \neq u_{c_0}'(c_0), \tag{4.3}$$

where $\{u_{c_n}\}$ is the corresponding sequence of solutions of problems (3.1) for $c = c_n, n \in \mathbb{N} \cup \{0\}$. From Proposition 3.1 and its proof we get that

$$\lim_{n\to\infty} \mu(c_n) = \mu(c_0) \tag{4.4}$$

and

$$\lim_{n\to\infty} u_{c_n}(t) = u_{c_0}(t) \quad \text{locally uniformly on } [0, c_0). \tag{4.5}$$

Let $\max\{u_{c_n}(t) : 0 \le t \le c_n\} = u_{c_n}(\xi_n) = A$ for $n \in \mathbb{N}$. Then $\xi_n \in (0, c_n)$ and $u_{c_n}'(\xi_n) = 0$ for $n \in \mathbb{N}$. The sequence $\{\xi_n\}$ is bounded and thus we can write without loss of generality

$$\lim_{n\to\infty} \xi_n = \xi_0 \in (0, c_0),\ u_{c_0}'(\xi_0) = 0, \tag{4.6}$$

because $\lim_{n\to\infty} u_{c_n}(\xi_n) = u_{c_0}(\xi_0) = A$. Using (4.5), (4.6) and the fact that $u_{c_0}(c_0) = 0$, we can find $n_0 \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that for $n \in \mathbb{N}, n \ge n_0$,

$$c_n > c_0 - \varepsilon_0, \quad \xi_0 < c_0 - \varepsilon_0, \quad \xi_n < c_0 - \varepsilon_0,$$

and consequently for $j \in \mathbb{N} \cup \{0\}$

$$u_{c_j}'(t) < 0 \quad \text{for } t \in [c_0 - \varepsilon_0, c_j]. \tag{4.7}$$

Now, choose $\varepsilon \in (0, \varepsilon_0]$ and integrate the equalities (for $j \in \mathbb{N} \cup \{0\}$)

$$(r(u_{c_j}(t))u'_{c_j}(t))' = \mu(c_j)q(t)f(t, u_{c_j}(t))$$

from ξ_j to $c_0 - \varepsilon$. We get

$$r(u_{c_j}(c_0 - \varepsilon))u'_{c_j}(c_0 - \varepsilon) = \mu(c_j) \int_{\xi_j}^{c_0 - \varepsilon} q(t)f(t, u_{c_j}(t))dt.$$

According to (3.8) and (3.10) with $u = u_{c_0}$, we obtain

$$\lim_{n\to\infty} \int_{\xi_n}^{\xi_0} q(t)f(t, u_{c_n}(t))dt = 0$$

and

$$\lim_{n\to\infty} \int_{\xi_0}^{c_0-\varepsilon} q(t)f(t, u_{c_n}(t))dt = \int_{\xi_0}^{c_0-\varepsilon} q(t)f(t, u_{c_0}(t))dt,$$

which imply that for each $\varepsilon \in (0, \varepsilon_0]$ we have $\lim_{n\to\infty} u'_{c_n}(c_0 - \varepsilon)) = u'_{c_0}(c_0 - \varepsilon))$. In view of (4.3) we can assume without loss of generality that there is a $\delta > 0$ such that either

$$r(0)u'_{c_n}(c_n) \geq r(0)u'_{c_0}(c_0) + \rho \quad \text{for } n \in \mathbb{N}, n \geq n_0, \tag{4.8}$$

or

$$r(0)u'_{c_n}(c_n) \leq r(0)u'_{c_0}(c_0) - \rho \quad \text{for } n \in \mathbb{N}, n \geq n_0, \tag{4.9}$$

is true. First, suppose that (4.8) occurs. Then, since $r(u_{c_n})u'_{c_n}$ is decreasing on $[0, c_n]$, we get $r(0)u'_{c_0}(c_0) + \rho \leq r(u_{c_0}(c_0 - \varepsilon))u'_{c_0}(c_0 - \varepsilon)$ for $\varepsilon \in (0, \varepsilon_0]$, which contradicts the continuity of the function $r(u_{c_0})u'_{c_0}$ on $[0, c_0]$. Now, let (4.9) be valid. Then we can find $\varepsilon_1 \in (0, \varepsilon_0]$ such that

$$\int_0^{u_{c_0}(c_0-\varepsilon_1)} g(s)r(s)ds < \frac{\rho^2}{2\mu(c_0)Q}. \tag{4.10}$$

From (3.1), (4.7) and (H3) we derive the inequalities

$$\begin{aligned} &2(r(u_{c_n}(t))u'_{c_n}(t))'r(u_{c_n}(t))u'_{c_n}(t) \\ &\qquad \leq -2\mu(c_n)Qg(u_{c_n}(t))r(u_{c_n}(t))u'_{c_n}(t) \end{aligned} \tag{4.11}$$

for $t \in [c_0 - \varepsilon_1, c_n]$ and for a sufficiently large $n \in \mathbb{N}$. Choose $\varepsilon \in (0, \varepsilon_1]$ and integrate (4.11) from $c_0 - \varepsilon$ to c_n. Letting $n \to \infty$, we obtain

$$\begin{aligned} &\Big(r(u_{c_0}(c_0 - \varepsilon))u'_{c_0}(c_0 - \varepsilon)\Big)^2 \geq \Big(r(0)u'_{c_0}(c_0) - \rho\Big)^2 \\ &\quad -2\mu(c_0)Q \int_0^{u_{c_0}(c_0-\varepsilon_1)} g(s)r(s)ds > \Big(r(0)u'_{c_0}(c_0)\Big)^2 - 2\rho r(0)u'_{c_0}(c_0) \end{aligned}$$

and thus for each $\varepsilon \in (0, \varepsilon_1]$ the inequality

$$\Big(r(u_{c_0}(c_0-\varepsilon))u'_{c_0}(c_0-\varepsilon)\Big)^2 > \Big(r(0)u'_{c_0}(c_0)\Big)^2 - 2\rho r(0)u'_{c_0}(c_0)$$

is fulfilled. But this is impossible because $2\rho r(0)u'_{c_0}(c_0) < 0$ and the function $r(u_{c_0})u'_{c_0}$ is continuous on $[0, c_0]$. This completes the proof of the continuity of Λ_A from the left on $(0, T]$.

To prove that Λ_A is continuous from the right on $(0, T)$ we can argue in the same way as before with the only difference that $\{c_n\} \subset (c_0, T)$ is decreasing and the convergence in (4.5) is uniform on $[0, c_0]$, now.

It remains to prove estimate (4.2). Let $c \in [\frac{T}{2}, T]$. Then there exists $\xi \in (0, c)$ such that $\max\{u_c(t) : 0 \le t \le c\} = u_c(\xi) = A$, $u'_c(\xi) = 0$. Since $A = u_c(\xi) - u_c(c) = u'_c(\nu)(\xi - c)$, where $\nu \in (\xi, c)$, we get $u'_c(\nu) = A/(\xi - c) < -A/T$. Having in mind that $r(u_c)u'_c$ is decreasing on $[0, c]$, we obtain $-Ar_0/T > r_0 u'_c(\nu) \ge r(u_c(\nu))u'_c(\nu) > r(0)u'_c(c)$, which gives (4.2). □

Now, consider $A > 0, c \in (0, T)$ and the corresponding parameter $\mu(c)$. By Theorem 2.2*, for $\mu = \mu(c)$, $a = c$, $b = T$, there exists exactly one solution of the problem

$$\begin{gathered} (r(x(t))x'(t))' = \mu(c)q(t)f(t, x(t)),\ t \in (c, T) \\ x(c) = x(T) = 0,\ x(t) < 0 \text{ on } (c, T), \end{gathered} \tag{4.12}$$

which we denote by v_c. Let us define the function $\Phi_A : (0, T) \to (-\infty, 0)$ by the formula

$$\Phi_A(c) = v'_c(c), \tag{4.13}$$

where $v'_c(c)$ means the right derivative of v_c at c.

Proposition 4.2. *The function Φ_A defined by* (4.13) *is continuous on* $(0, T)$ *and*

$$\lim_{c \to T-} \Phi_A(c) = 0. \tag{4.14}$$

Proof. To prove the continuity we can follow the proof of Proposition 4.1 doing only small modifications.

It remains to prove (4.14). Suppose on the contrary that (4.14) falls. Then there exists an increasing sequence $\{c_n\} \subset (\frac{T}{2}, T)$, $\lim_{n\to\infty} c_n = T$, such that

$$\lim_{n\to\infty} v'_{c_n}(c_n) = V < 0, \tag{4.15}$$

where $\{v_{c_n}\}$ is the corresponding sequence of solutions of problems (4.12) for $c = c_n, n \in \mathbb{N}$. By Proposition 3.1, the sequence $\{\mu(c_n)\}$ is nonincreasing and $0 < \mu(c_n) \le \mu(c_1)$ for $n \in \mathbb{N}$. Further, there exist $\xi_n \in (c_n, T)$ such that

$$\min\{v_{c_n}(t) : c_n \le t \le T\} = v_{c_n}(\xi_n) = B_n < 0 \quad \text{for } n \in \mathbb{N}.$$

Then $v'_{c_n}(\xi_n) = 0$ for $n \in \mathbb{N}$ and the sequence $\{B_n\} \subset [B_1, 0)$ is nondecreasing. Therefore $\lim_{n\to\infty} B_n = \beta \leq 0$. Fix $n \in \mathbb{N}$. Then, by Theorem 2.6*, there exists just one μ_n^* such that the problem (4.12) with $c = c_n$ has a (unique) solution u_n^* satisfying $\min\{u_n^*(t) : c_n \leq t \leq T\} = B_n$. This implies that $\mu_n^* = \mu(c_n)$ and $u_n^* = v_{c_n}$ for $n \in \mathbb{N}$ and, by Theorems 2.2* and 2.6*, the relation

$$0 < m_-(c_n, T; B_n) \leq \mu(c_1) \tag{4.16}$$

is true. Let $\beta < 0$. Then, by (2.5) and (4.16), we get

$$\frac{2\left(\int_{B_n}^0 r(s)ds\right)^2}{\int_{B_n}^0 g(s)r(s)ds} \leq \mu(c_1)(T - c_n)^2 Q \quad \text{for } n \in \mathbb{N}.$$

By the limitting process for $n \to \infty$ we obtain a contradiction. So, we have proved $\lim_{n\to\infty} B_n = \lim_{n\to\infty} v_n(\xi_n) = 0$. Similarly as in the proof of Proposition 4.1, we compute that

$$\lim_{n\to\infty} \left(v'_{c_n}(c_n)\right)^2 \leq \frac{2\mu(c_1)Q}{(r(0))^2} \lim_{n\to\infty} \int_{B_n}^0 g(s)r(s)ds = 0,$$

which contradicts (4.15). □

Now, let us consider the "dual" situation. Let $B < 0, c \in [0, T)$ and let $\mu^*(c)$ and $u_c^*(t)$ be the corresponding (uniquely determined) parameter and solution of problem (3.14), respectively. Let us define the function $\Lambda_B^* : [0, T) \to (-\infty, 0)$ by the formula

$$\Lambda_B^*(c) = (u_c^*)'(c). \tag{4.17}$$

Here $(u_c^*)'(c)$ means the right derivative of u_c^* at c.

Proposition 4.3.* *The function* Λ_B^* *defined by* (4.17) *is continuous on* $[0, T)$ *and satisfies the inequality*

$$\Lambda_B^*(c) < \frac{Br_0}{Tr(0)} \quad for\ c \in \left[0, \frac{T}{2}\right]. \tag{4.18}$$

Now, let $c \in (0, T)$. By Theorem 2.1 for $\mu = \mu^*(c), a = 0, b = c$, there exists exactly one solution of the problem

$$\begin{gathered}(r(x(t))x'(t))' = \mu^*(c)q(t)f(t, x(t)),\ t \in (0, c)\\ x(0) = x(c) = 0,\ x(t) > 0 \text{ on } (0, c),\end{gathered} \tag{4.19}$$

which is denoted by v_c^*. Let us define the function $\Phi_B^* : (0, T) \to (-\infty, 0)$ by

$$\Phi_B^*(c) = (v_c^*)'(c), \tag{4.20}$$

where $(v_c^*)'(c)$ means the left derivative of v_c^* at c.

Proposition 4.4.[*] *Φ_B^* defined by* (4.20) *is continuous on* $(0,T)$ *and*

$$\lim_{c\to 0+} \Phi_B^*(c) = 0. \tag{4.21}$$

5 Main results

Theorem 5.1. *For each $A \in (0,\infty)$ there exists a solution x of problem* (1.1), (1.2) *with the unique zero $t_0 \in (0,T)$ such that*

$$\max\{x(t) : 0 \le t \le T\} = \max\{x(t) : 0 \le t \le t_0\} = A \quad if \quad t_0 \in \Big[\frac{T}{2}, T\Big)$$

and

$$\max\{x(t) : 0 \le t \le T\} = \max\{x(t) : 0 \le t \le t_0\} \le A \quad if \quad t_0 \in \Big(0, \frac{T}{2}\Big).$$

Proof. Fix $A \in (0,\infty)$. For $c \in (0,T)$ suppose that $u_c(t)$ and $v_c(t)$ are the corresponding (uniquely determined) solutions of (3.1) and (4.12), respectively, with the corresponding parameter $\mu(c)$. Let Λ_A and Φ_A be the functions defined by (4.1) and (4.13), respectively. Then three cases can occur.

1) Let

$$\Lambda_A\left(\frac{T}{2}\right) = \Phi_A\left(\frac{T}{2}\right). \tag{5.1}$$

Then the function

$$x(t) = \begin{cases} u_{\frac{T}{2}}(t) & \text{for } t \in [0, \frac{T}{2}] \\ v_{\frac{T}{2}}(t) & \text{for } t \in (\frac{T}{2}, T] \end{cases}$$

is a solutions of problem (1.1), (1.2) with $\mu = \mu(\frac{T}{2})$. Moreover, $t_0 = \frac{T}{2}$ and $\max\{x(t) : 0 \le t \le T\} = \max\{u_{\frac{T}{2}}(t) : 0 \le t \le \frac{T}{2}\} = A$.

2) Let

$$\Lambda_A\left(\frac{T}{2}\right) > \Phi_A\left(\frac{T}{2}\right). \tag{5.2}$$

By Proposition 4.1, the function $\Lambda_A(c)$ is continuous on $(0,T]$ and $\Lambda_A(c) < -\frac{Ar_0}{Tr(0)}$ for $c \in [\frac{T}{2}, T]$. According to Proposition 4.2, the function $\Phi_A(c)$ is continuous on $(0,T)$ and $\lim_{c\to T-} \Phi_A(c) = 0$. Therefore there exists at least one $c_0 \in (\frac{T}{2}, T)$ such that $\Lambda_A(c_0) = \Phi_A(c_0)$. Then the function

$$x(t) = \begin{cases} u_{c_0}(t) & \text{for } t \in [0, c_0] \\ v_{c_0}(t) & \text{for } t \in (c_0, T] \end{cases}$$

is a solution of problem (1.1), (1.2) with $\mu = \mu(c_0)$. Thus $t_0 = c_0$ and $\max\{x(t) : 0 \le t \le T\} = \max\{u_{c_0}(t) : 0 \le t \le c_0\} = A$.

3) Let

$$\Lambda_A\left(\frac{T}{2}\right) < \Phi_A\left(\frac{T}{2}\right). \tag{5.3}$$

Let us put $B = \min\{v_{\frac{T}{2}}(t) : \frac{T}{2} \le t \le T\} < 0$ and consider the "dual" functions $u_c^*(t)$ and $v_c^*(t)$ which are (uniquely determined) solutions of (3.14) and (4.19), respectively, with the corresponding parameter $\mu^*(c)$. Then

$$u^*_{\frac{T}{2}}(t) = v_{\frac{T}{2}} \quad \text{for } t \in \left[\frac{T}{2}, T\right], \quad v^*_{\frac{T}{2}}(t) = u_{\frac{T}{2}} \quad \text{for } t \in \left(0, \frac{T}{2}\right]. \tag{5.4}$$

Let Λ_B^* and Φ_B^* be the functions defined by (4.17) and (4.20), respectively. (5.4) implies that $\Lambda_B^*(\frac{T}{2}) = \Phi_A(\frac{T}{2})$ and $\Phi_B^*(\frac{T}{2}) = \Lambda_A(\frac{T}{2})$ which, by (5.3) gives

$$\Lambda_B^*\left(\frac{T}{2}\right) > \Phi_B^*\left(\frac{T}{2}\right). \tag{5.5}$$

Since $\mu^*(\frac{T}{2}) = \mu(\frac{T}{2})$, we get by Propositions 3.1 and 3.2*

$$\mu^*(c) \le \mu(c) \quad \text{for } c \in \left(0, \frac{T}{2}\right]. \tag{5.6}$$

According to Proposition 4.3*, the function $\Lambda_B^*(c)$ is continuous on $[0, T)$ and $\Lambda_B^*(c) < \frac{Br_0}{Tr(0)} < 0$ for $c \in \left[0, \frac{T}{2}\right]$. Using Proposition 4.4*, we have that the function $\Phi_B^*(c)$ is continuous on $(0, T)$ and $\lim_{c\to 0+} \Phi_B^*(c) = 0$. This together with (5.5) gurantee the existence of at least one $c_1 \in (0, \frac{T}{2})$ such that $\Lambda_B^*(c_1) = \Phi_B^*(c_1)$. Then the function

$$x(t) = \begin{cases} v_{c_1}^*(t) & \text{for } t \in [0, c_1] \\ u_{c_1}^*(t) & \text{for } t \in (c_1, T] \end{cases}$$

is a solution of problem (1.1), (1.2) with $\mu = \mu^*(c_1)$ and $t_0 = c_1$. Let us apply Lemma 2.3 for $a = 0, b = c_1, \mu_1 = \mu^*(c_1), \mu_2 = \mu(c_1), u_1(t) = v_{c_1}^*(t), u_2(t) = u_{c_1}(t)$ for $t \in [0, c_1]$. Then we get by (5.6) that $v_{c_1}^*(t) \le u_{c_1}(t)$ for $t \in [0, c_1]$, and so

$$\begin{aligned} \max\{x(t) : 0 \le t \le T\} &= \max\{v_{c_1}^*(t) : 0 \le t \le c_1\} \\ &\le \max\{u_{c_1}(t) : 0 \le t \le c_1\} = A. \end{aligned}$$

□

Theorem 5.2.* *For each $B \in (-\infty, 0)$ there exists a solution x of problem (1.1), (1.2) with the unique zero $t_0 \in (0, T)$ such that*

$$\min\{x(t) : 0 \le t \le T\} = \min\{x(t) : t_0 \le t \le T\} = B \quad \textit{if} \quad t_0 \in \left(0, \frac{T}{2}\right]$$

and

$$\min\{x(t) : 0 \le t \le T\} = \min\{x(t) : t_0 \le t \le T\} \ge B \quad if \quad t_0 \in \Big(\frac{T}{2}, T\Big).$$

Theorem 5.3. *For each $A \in (0, \infty)$ there exists a solution x of problem* (1.1), (1.2) *with the unique zero $t_0 \in (0, T)$ such that*

$$\max\{x(t) : 0 \le t \le T\} = \max\{x(t) : t_0 \le t \le T\} = A \quad if \quad t_0 \in \Big(0, \frac{T}{2}\Big]$$

and

$$\max\{x(t) : 0 \le t \le T\} = \max\{x(t) : t_0 \le t \le T\} \le A \quad if \quad t_0 \in \Big(\frac{T}{2}, T\Big).$$

Proof. To prove our theorem we replace the interval $[0, c]$ with $[c, T]$ in (3.1) and by means of the solution $y_c(t)$ of such problem we define the function

$$\Gamma_A : [0, T] \to (-\infty, 0), \quad \Gamma_A(c) = y_c'(c).$$

Then we replace the interval $[c, T]$ with $[0, c]$ in (4.12) and by means of the solution $z_c(t)$ of such problem we define the function

$$\Psi_A : (0, T) \to (-\infty, 0), \quad \Psi_A(c) = z_c'(c).$$

Analogously we introduce the "dual" functions Γ_B^* and Ψ_B^* by means of solutions of problems (3.14) and (4.19), where the intervals $[0, c]$ and $[c, T]$ are mutually replaced. Then, using similar arguments as in Sec. 4, we can prove the continuity of Γ_A, Ψ_A, Γ_A^*, Ψ_A^* and formulas $\lim_{c \to 0+} \Psi_A(c) = 0$, $\lim_{c \to T^-} \Psi_B^*(c) = 0$,

$$\Gamma_A(c) > \frac{Ar_0}{Tr(0)} \quad \text{for } c \in \Big[0, \frac{T}{2}\Big], \quad \Gamma_B^*(c) > -\frac{Br_0}{Tr(0)} \quad \text{for } c \in \Big[\frac{T}{2}, T\Big].$$

Finally, we can argue as in the proof of Theorem 5.1. □

Theorem 5.4.* *For each $B \in (-\infty, 0)$ there exists a solution x of problem* (1.1), (1.2) *with the unique zero $t_0 \in (0, T)$ such that*

$$\min\{x(t) : 0 \le t \le T\} = \min\{x(t) : 0 \le t \le t_0\} = B \quad if \quad t_0 \in \Big[\frac{T}{2}, T\Big)$$

and

$$\min\{x(t) : 0 \le t \le T\} = \min\{x(t) : 0 \le t \le t_0\} \ge B \quad if \quad t_0 \in \Big(0, \frac{T}{2}\Big).$$

Example 5.5. Let $\alpha,\ \beta \in (0,1)$, $a \in (0,\infty)$, $b \in (-\infty,0)$ and

$$p(x) = \begin{cases} \dfrac{a}{x^\alpha} & \text{for } x > 0 \\ \dfrac{b}{(-x)^\beta} & \text{for } x < 0. \end{cases}$$

Consider the differential equation

$$((1+e^{x\cos x})^\gamma x')' = \mu\Big(\sin\frac{1}{t(T-t)} - 2\Big)(h(t)\operatorname{sign} x + p(x)) \tag{5.7}$$

with $\gamma \in (0,\infty)$ and $h : [0,T] \to (0,\infty)$ continuous. The assumptions (H1)-(H3) are satisfied with $r(u) = (1+e^{u\cos u})^\gamma > 1$, $Q = 3$, $k = h$ and

$$g(x) = \max\{h(t) : 0 \le t \le T\} + \frac{\max\{a, -b\}}{\min\{|x|^\alpha,\ |x|^\beta\}}.$$

Consequently, Theorems 5.1, 5.2*, 5.3 and 5.4* can be applied to problem (5.7), (1.2). For example, by Theorem 5.1, for each $A \in (0,\infty)$ there exists a solution x of problem (5.7), (1.2). If $t_0 \in (0,T)$ denotes the unique zero of x, then $\max\{x(t) : 0 \le t \le T\} = \max\{x(t) : 0 \le t \le t_0\} = A$ provided $t_0 \in [\frac{T}{2}, T]$ and $\max\{x(t) : 0 \le t \le T\} = \max\{x(t) : 0 \le t \le t_0\} \le A$ provided $t_0 \in (0, \frac{T}{2})$.

Remark 5.6. With respect to Remark 1.1, Theorems 5.1, 5.2*, 5.3 and 5.4* can be applied to problem (5.7), (1.2), where h in (5.7) is even nonnegative on $[0,T]$. Particularly, we can consider (5.7) with $h = 0$, that is,

$$((1+e^{x\cos x})^\gamma x')' = \mu\Big(\sin\frac{1}{t(T-t)} - 2\Big)p(x).$$

In this case the functions k_M in Remark 1.1 are for example

$$k_M(t) = \min\left\{\frac{a}{M^\alpha}, \frac{|b|}{M^\beta}\right\} \qquad \text{for } t \in [0,T] \text{ and } M > 0.$$

References

[1] R. P. Agarwal, D. O'Regan, *Singular boundary value problems for superlinear second order ordinary and delay differential equations.* J. Differential Equations **130**(1996), 333-355.

[2] R. P. Agarwal, D. O'Regan, *Positive solutions to superlinear singular boundary value problems.* J. Comput. Appl. Math. **88**(1998), 129–147.

[3] R. P. Agarwal, D. O'Regan, *Some new results for singular problems with sign changing nonlinearities.* J. Comput. Appl. Math. **113**(2000), 1–15.

[4] J. A. Gatica, V. Oliker, P. Waltman, *Singular nonlinear boundary value problems for second order ordinary differential equations.* J. Differential Equations **79**(1989), 62–78.

[5] I. T. Kiguradze, B. L. Shekhter, *Singular Boundary Value Problems for Second Order Ordinary Differential Equations.* In "Current Problems in Mathematics: Newest Results", vol **30**, 105–201, Moscow, 1987 (in Russian); English translation: J. Soviet Math. **43**(1988), 2340–2417.

[6] A. Lomtatidze, *Positive solutions of boundary value problems for second order differential equations with singular points*, Differentsial'nye Uravneniya **23**(1987), 1685–1692. (in Russian)

[7] A. Lomtatidze and P. Torres, *On a two-point boundary value problem for second order singular equations*, preprint.

[8] D. O'Regan, *Theory of Singular Boundary Value problems.* World Scientific Publishing, Singapore, 1994.

[9] D. O'Regan, *Existence principles and theory for singular Dirichlet boundary value problems.* Differential Eqns. Dyn. Systems **3**(1995), 289–304.

[10] I. Rachůnková and S. Staněk, *Connection between the type of singularity and smoothness of solutions of Dirichlet BVPs*, in preparation.

[11] S. Staněk, *Positive solutions of singular positone Dirichlet boundary value problems.* Math. Computer Modelling **33**(2001), 341–351.

[12] S. Staněk, *Positive solutions of singular Dirichlet and periodic boundary value problems*, Math. Computer Modelling (accepted for publication).

[13] S. D. Taliaferro, *A nonlinear singular boundary value problem.* Nonlin. Anal. **3**(1979), 897–904.

[14] A. Tineo, *Existence theorems for a singular two-point Dirichlet problem.* Nonlin. Anal. **19**(1992), 323–333.

[15] J. Wang, *Solvability of singular nonlinear two-point boundary value problems.* Nonlin. Anal. **24**(1995), 555-561.

[16] J. Wang and W.Gao, *A note on singular nonlinear two-point boundary-value problems.* Nonlin. Anal. **39**(2000), 281–287.

[17] Z. Zengquin, *Uniqueness of positive solutions for singular nonlinear second-order boundary-value problems.* Nonlin. Anal. **23**(1994), 755-765.