Singular discrete and continuous mixed boundary value problems

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Abstract. For each $n \in \mathbb{N}$, $n \geq 2$ we prove the existence of a positive solution of the singular discrete problem

$$\frac{1}{h^2}\Delta^2 u_{k-1} + f(t_k, u_k) = 0, \quad k = 1, \ldots, n-1,$$

$$\Delta u_0 = 0, \quad u_n = 0,$$

where $T \in (0, \infty)$, $h = \frac{T}{n}$, $t_k = hk$, $f\colon [0,T] \times (0,\infty)$ is continuous and has a singularity at $x = 0$. We prove that for $n \to \infty$ the sequence of solutions of the above discrete problems converges to a solution y of the corresponding continuous boundary value problem

$$y''(t) + f(t, y(t)) = 0,$$
$$y'(0) = 0, \quad y(T) = 0.$$

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1 Introduction

Let $T \in (0, \infty)$, $n \in \mathbb{N}$, $n \geq 2$ and $h = \frac{T}{n}$. We investigate the singular discrete mixed boundary value problem

$$\frac{1}{h^2}\Delta^2 u_{k-1} + f(t_k, u_k) = 0, \quad k = 1, \ldots, n-1, \tag{1.1}$$

$$\Delta u_0 = 0, \quad u_n = 0, \tag{1.2}$$

where $f\colon [0,T] \times (0,\infty) \to \mathbb{R}$ is continuous and $f(t,x)$ has a singularity at $x = 0$, i.e. we assume

$$f \in C([0,T] \times (0,\infty)), \quad \limsup_{x \to 0+} |f(t,x)| = \infty \quad \text{for each } t \in (0,T). \tag{1.3}$$

Here

$$t_k = hk, \quad \Delta u_{k-1} = u_k - u_{k-1} \quad \text{for } k = 0, \dots, n. \tag{1.4}$$

Definition 1.1 A vector $(u_0, \dots, u_n) \in \mathbb{R}^{n+1}$ satisfying equation (1.1) and the mixed boundary conditions (1.2) is called *a solution* of problem (1.1), (1.2). If $u_k > 0$ for $k = 0, \dots, n-1$, the solution is called *positive*.

The continuous version of problem (1.1), (1.2) has the form

$$y''(t) + f(t, y(t)) = 0, \tag{1.5}$$

$$y'(0) = 0, \quad y(T) = 0. \tag{1.6}$$

Definition 1.2 A function $y \in C[0,T] \cap C^2[0,T)$ satisfying equation (1.5) for $t \in [0,T)$ and fulfilling the mixed boundary conditions (1.6) is called *a solution* of problem (1.5), (1.6). If $y(t) > 0$ for $t \in [0,T)$, the solution is called *positive*.

Discrete boundary value problems arise in the study of solid state physics, chemical reaction, population dynamics and in many other areas, see [1], [13], [32]. Besides, they are also natural consequences of the discretization of differential boundary value problems. Solvability of discrete second order boundary value problems is investigated in the monographs [1], [4], [5], [19] and in many papers, e.g. [3], [6], [8], [9], [10], [14], [16], [20], [21], [22]. The lower and upper functions method for regular discrete problems is used in [7], [11], [12], [17], [25], [30], [31]. In this paper we extend this method for singular discrete problem (1.1), (1.2). It is of interest to note that singular problems for differential equations have been intensively studied in literature. For the second order singular differential equations we can refer to the monographs [18], [23], [26]. However there are only few results for its discrete analogue, see [2], [5], [24].

Here we provide conditons which imply that for each $n \in \mathbb{N}$, $n \geq 2$ the singular discrete problem (1.1), (1.2) has a positive solution $(u_0, \dots, u_n)$. Then we construct an approximate function $S^{[n]} \in C[0,T]$ satisfying

$$S^{[n]}(t_k) = u_k, \quad k = 0, \dots, n.$$

Having the sequence $\{S^{[n]}\}$ we prove that there is a subsequence $\{S^{[m]}\}$ locally uniformly converging on $[0,T)$ for $m \to \infty$ to a positive solution y of the singular continuous problem (1.5), (1.6). Similar results about discrete approximation of regular problems can be found in [15], [27], [28], [29].

2 Lower and upper functions method

It is known that if problem (1.1), (1.2) is regular, i.e. f is continuous on $[0,T]\times\mathbb{R}$ and if there exist well ordered lower and upper functions of problem (1.1), (1.2), then this problem has a solution lying between these functions. Here we extend this result for the singular problem (1.1), (1.2).

Definition 2.1 The vector $(\alpha_0,\ldots,\alpha_n)\in\mathbb{R}^{n+1}$ is called a *lower function* of problem (1.1), (1.2) if

$$\frac{1}{h^2}\Delta^2\alpha_{k-1}+f(t_k,\alpha_k)\geq 0,\quad k=1,\ldots,n-1, \tag{2.1}$$

$$\Delta\alpha_0\geq 0,\quad \alpha_n\leq 0. \tag{2.2}$$

Definition 2.2 The vector $(\beta_0,\ldots,\beta_n)\in\mathbb{R}^{n+1}$ is called an *upper function* of problem (1.1), (1.2) if

$$\frac{1}{h^2}\Delta^2\beta_{k-1}+f(t_k,\beta_k)\leq 0,\quad k=1,\ldots,n-1, \tag{2.3}$$

$$\Delta\beta_0\leq 0,\quad \beta_n\geq 0. \tag{2.4}$$

Theorem 2.3 (Lower and upper functions method). *Assume that conditon* (1.3) *holds. Let* $(\alpha_0,\ldots,\alpha_n)$ *and* $(\beta_0,\ldots,\beta_n)$ *be a lower function and an upper function of problem* (1.1), (1.2) *such that* $0<\alpha_k\leq\beta_k$, $k=1,\ldots,n-1$. *Then problem* (1.1), (1.2) *has a solution* $(u_0,\ldots,u_n)$ *satisfying*

$$\alpha_k\leq u_k\leq\beta_k,\quad k=0,\ldots,n. \tag{2.5}$$

Proof. For $k\in\{1,\ldots,n-1\}$, $x\in\mathbb{R}$ define a function

$$\tilde f(t_k,x)=\begin{cases} f(t_k,\beta_k)-\frac{x-\beta_k}{x-\beta_k+1} & \text{if } x>\beta_k,\\ f(t_k,x) & \text{if } \alpha_k\leq x\leq\beta_k,\\ f(t_k,\alpha_k)+\frac{\alpha_k-x}{\alpha_k-x+1} & \text{if } x<\alpha_k.\end{cases} \tag{2.6}$$

We see that $\tilde f(t_k,\cdot)\colon\mathbb{R}\to\mathbb{R}$ is continuous for $k=1,\ldots,n-1$ and there exists $M>0$ such that

$$|\tilde f(t_k,x)|\leq M\quad \text{for } k=1,\ldots,n-1,\ x\in\mathbb{R}.$$

Consider the auxiliary difference equation

$$\frac{1}{h^2}\Delta^2u_{k-1}+\tilde f(t_k,u_k)=0,\quad k=1,\ldots,n-1. \tag{2.7}$$

Denote

$$E=\{\mathbf{v}=(v_0,\ldots,v_n)\in\mathbb{R}^{n+1}\colon \Delta v_0=0,\ v_n=0\}$$

and define $\|\mathbf{v}\| = \max\{|v_k|: k = 1, \ldots, n-1\}$. Then E is a Banach space with $\dim E = n - 1$. Define an operator $\mathcal{F}: E \to E$ by

$$(\mathcal{F}\mathbf{v})_k = -\sum_{i=1}^{n-1} G(t_k, s_i)\tilde{f}(s_i, v_i), \quad k = 0, \ldots, n, \tag{2.8}$$

where G is the Green function of the homogeneous linear problem

$$\frac{1}{h^2}\Delta^2 u_{k-1} = 0, \quad \Delta u_0 = 0, \quad u_n = 0. \tag{2.9}$$

Then

$$G(t_k, s_i) = h \begin{cases} t_k - T & \text{for } 0 < s_i \le t_k \le T, \\ s_i - T & \text{for } 0 \le t_k < s_i \le T, \end{cases}$$

where $s_i = hi$, $t_k = hk$, $k = 0, \ldots, n$, $i = 1, \ldots, n$. Since

$$-hT < G(t_k, s_i) < 0 \quad \text{for } i = 1, \ldots, n-1, \; k = 0, \ldots, n-1,$$

we get

$$\|\mathcal{F}\mathbf{v}\| < nhTM = T^2M.$$

Therefore if we denote $r^* = T^2M$ and consider the closed ball $\overline{\mathcal{K}(r^*)} = \{\mathbf{v} \in E: \|\mathbf{v}\| \le r^*\}$, we see that $\mathcal{F}$ maps $\overline{\mathcal{K}(r^*)}$ into itself. Since $\mathcal{F}$ is continuous, the Brouwer fixed point theorem yields a fixed point $\mathbf{u} \in \overline{\mathcal{K}(r^*)}$ of the operator $\mathcal{F}$. So, we have $\mathbf{u} = \mathcal{F}\mathbf{u}$ and consequently, by (2.8),

$$u_k = -\sum_{i=1}^{n-1} G(t_k, s_i)\tilde{f}(s_i, u_i), \quad k = 0, \ldots, n.$$

Since G is the Green function of problem (2.9), the vector $\mathbf{u} = (u_0, \ldots, u_n)$ is a solution of problem (2.7), (1.2).

Now we will prove estimate (2.5). Let us put $z_k = \alpha_k - u_k$ and assume

$$\max\{z_k: k = 0, \ldots, n\} = z_\ell > 0. \tag{2.10}$$

Then $\ell \in \{1, \ldots, n-1\}$ because $z_n = \alpha_n - u_n \le 0$ and $\Delta z_0 = z_1 - z_0 = \alpha_1 - \alpha_0 - (u_1 - u_0) \ge 0$. Consequently $z_1 \ge z_0$. Since z_ℓ is maximal, we have

$$z_{\ell-1} \le z_\ell, \quad z_\ell \ge z_{\ell+1}.$$

Therefore

$$\Delta u_{\ell-1} \le \Delta\alpha_{\ell-1}, \quad \Delta u_\ell \ge \Delta\alpha_\ell.$$

This leads to

$$\Delta^2 u_{\ell-1} \ge \Delta^2\alpha_{\ell-1}. \tag{2.11}$$

On the other hand, by Definition 2.1 and formula (2.6), we obtain

$$\frac{1}{h^2}(\Delta^2\alpha_{\ell-1} - \Delta^2 u_{\ell-1}) = \frac{1}{h^2}\Delta^2\alpha_{\ell-1} + \tilde{f}(t_\ell, u_\ell)$$

$$= \frac{1}{h^2}\Delta^2\alpha_{\ell-1} + f(t_\ell, u_\ell) + \frac{\alpha_\ell - u_\ell}{\alpha_\ell - u_\ell + 1} \geq \frac{z_\ell}{z_\ell + 1} > 0,$$

contrary to (2.11). So, we have proved $\alpha_k \leq u_k$, $k = 0, \dots, n$. The estimate $u_k \leq \beta_k$, $k = 0, \dots, n$ can be proved similarly. Therefore $(u_0, \dots, u_n)$ satisfies (2.5) and hence $(u_0, \dots, u_n)$ is also a solution of problem (1.1), (1.2). $\square$

3 Approximate functions

Assume that conditions (1.3) and

$$\alpha, \beta \in C[0,T], \quad 0 < \alpha(t) \leq \beta(t) \quad \text{for } t \in (0,T) \tag{3.1}$$

hold and denote

$$\alpha_k = \alpha(t_k), \quad \beta_k = \beta(t_k), \quad k = 0, \dots, n. \tag{3.2}$$

Let us choose $n \in \mathbb{N}$, $n \geq 2$ and suppose that $(\alpha_0, \dots, \alpha_n)$ and $(\beta_0, \dots, \beta_n)$ are a lower function and an upper function of problem (1.1), (1.2). By Theorem 2.3 there exists a solution $(u_0, \dots, u_n)$ of problem (1.1), (1.2) fulfilling estimate (2.5). By means of the substitution

$$\frac{\Delta u_k}{h} = v_k, \quad k = 0, \dots, n-1 \tag{3.3}$$

in equation (1.1) we get

$$\frac{\Delta v_{k-1}}{h} = -f(t_k, u_k), \quad k = 1, \dots, n-1. \tag{3.4}$$

Since $\Delta u_0 = v_0 = 0$, equations (3.3) and (3.4) can be written in the form

$$u_{k+1} = u_0 + h\sum_{i=1}^{k} v_i, \quad k = 1, \dots, n-1 \tag{3.5}$$

and

$$v_k = -h\sum_{i=1}^{k} f(t_i, u_i), \quad k = 1, \dots, n-1. \tag{3.6}$$

Let us put

$$S^{[n]}(t) = u_k + v_k(t - t_k), \quad t \in [t_k, t_{k+1}], \quad k = 0, \dots, n-1. \tag{3.7}$$

Then

$$S^{[n]} \in C[0,T], \quad S^{[n]}(t_k) = u_k, \quad k = 0, \dots, n.$$

Let us put

$$\begin{cases} P^{[n]}(t) = v_k + \dfrac{\Delta v_k}{h}(t - t_k), \quad t \in [t_k, t_{k+1}], \quad k = 0, \dots, n-2, \\ P^{[n]}(t) = v_{n-1}, \quad t \in [t_{n-1}, t_n]. \end{cases} \tag{3.8}$$

Then

$$P^{[n]} \in C[0,T], \quad P^{[n]}(t_k) = v_k, \quad k = 0, \dots, n-1.$$

By (3.5) and (3.6) we get

$$S^{[n]}(t_{k+1}) = S^{[n]}(0) + h \sum_{i=1}^{k} P^{[n]}(t_i), \quad k = 1, \dots, n-1 \tag{3.9}$$

and

$$P^{[n]}(t_k) = -h \sum_{i=1}^{k} f(t_i, S^{[n]}(t_i)), \quad k = 1, \dots, n-1. \tag{3.10}$$

The main result of the paper is contained in the following theorem.

Theorem 3.1 *Assume that conditions* (1.3), (3.1) *and* (3.2) *hold. Let for each* $n \geq 2$ *the vectors* $(\alpha_0, \dots, \alpha_n)$ *and* $(\beta_0, \dots, \beta_n)$ *be a lower function and an upper function of problem* (1.1), (1.2) *and let* $\alpha_0 > 0$, $\beta_n = 0$. *Then for each* $n \geq 2$ *problem* (1.1), (1.2) *has a solution* $(u_0, \dots, u_n)$, *a sequence* $\{S^{[n]}\}$ *can be given by* (3.7) *and there exists a subsequence* $\{S^{[m]}\} \subset \{S^{[n]}\}$ *which converges locally uniformly on* $[0,T)$ *to a solution* $y \in C[0,T] \cap C^2[0,T)$ *of problem* (1.5), (1.6).

If, in addition,

$$|f(t,x)| \leq g_0(t,x) + g_1(t,x) \quad \text{for } t \in [0,T], \ x \in (0,\infty), \tag{3.11}$$

where $g_0 \in C([0,T] \times (0,\infty))$ *is nonincreasing in its second variable with*

$$\int_0^T g_0(t, \alpha(t)) \mathrm{d}t < \infty \tag{3.12}$$

and $g_1 \in C([0,T] \times [0,\infty))$, *then moreover* $y \in C^1[0,T]$.

To prove Theorem 3.1 we use the next two lemmas.

Lemma 3.2 *Let the assumptions of Theorem* 3.1 *hold. Assume that* $S^{[n]}$ *and* $P^{[n]}$ *are given by* (3.7) *and* (3.8) *and choose an arbitrary interval* $[0,b] \subset [0,T)$. *Then the sequences* $\{S^{[n]}\}$ *and* $\{P^{[n]}\}$ *are bounded and equicontinuous on* $[0,b]$.

Proof. Let us choose $c \in (b,T)$ and denote

$$A = \min\{\alpha(t) \colon t \in [0,c]\},$$

$$B = \max\{\beta(t)\colon t \in [0,c]\},$$
$$M = \max\{|f(t,x)|\colon t \in [0,c],\ x \in [A,B]\}.$$

There exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ we can choose $b_n \in \{1,\dots,n\}$ such that

$$t_{b_n-1} \leq b < t_{b_n} < t_{b_n+1} < c. \tag{3.13}$$

Clearly

$$\lim_{n\to\infty} t_{b_n} = \lim_{n\to\infty} t_{b_n+1} = b. \tag{3.14}$$

Further we have

$$\max\{|S^{[n]}(t)|\colon t \in [0,b]\} \leq \max\{|u_k| + |\Delta u_k|\colon k = 0,\dots,b_n\}$$

and

$$\max\{|P^{[n]}(t)|\colon t \in [0,b]\} \leq \max\{|v_k| + |\Delta v_k|\colon k = 0,\dots,b_n\}.$$

By (3.1) and (2.5) we get

$$\max\{|u_k|\colon k = 0,\dots,b_n\} \leq B$$

and by (2.5) and (3.6) we obtain

$$\max\{|v_k|\colon k = 0,\dots,b_n\} \leq \frac{T}{n} b_n M < TM.$$

Condition (3.3) implies

$$\max\{|\Delta u_k|\colon k = 0,\dots,b_n\} \leq \frac{T}{n} TM < T^2 M$$

and condition (3.4) gives

$$\max\{|\Delta v_k|\colon k = 0,\dots,b_n\} \leq \frac{T}{n} M < TM.$$

Therefore we get for $n \geq n_0$

$$\max\{|S^{[n]}(t)|\colon t \in [0,b]\} \leq B + T^2 M,$$
$$\max\{|P^{[n]}(t)|\colon t \in [0,b]\} \leq 2TM.$$

We have proved that the sequences $\{S^{[n]}\}$ and $\{P^{[n]}\}$ are bounded on $[0,b]$.

Now, choose $\tau_1, \tau_2 \in [0,b]$, $\tau_1 < \tau_2$. By (3.13) we can find $k, \ell \in \{1,\dots,b_n\}$, $k < \ell$ such that $\tau_1 \in [t_{k-1}, t_k)$, $\tau_2 \in (t_{\ell-1}, t_\ell)$ and for each $n \geq n_0$

$$|S^{[n]}(\tau_2) - S^{[n]}(\tau_1)|$$
$$\leq \sum_{i=k+1}^{\ell-1} |S^{[n]}(t_i) - S^{[n]}(t_{i-1})| + |S^{[n]}(t_k) - S^{[n]}(\tau_1)| + |S^{[n]}(\tau_2) - S^{[n]}(t_{\ell-1})|$$

$$\le \sum_{i=k+1}^{\ell-1} |v_{i-1}|(t_i - t_{i-1}) + |v_{k-1}|(t_k - \tau_1) + |v_{\ell-1}|(\tau_2 - t_{\ell-1})$$

$$< TM(\tau_2 - \tau_1).$$

If $k+1 > \ell - 1$ we put $\sum_{i=k+1}^{\ell-1} = 0$.

Similarly, due to (3.4),

$$|P^{[n]}(\tau_2) - P^{[n]}(\tau_1)|$$

$$\le \sum_{i=k+1}^{\ell-1} \left|\frac{\Delta v_i}{h}\right| (t_i - t_{i-1}) + \left|\frac{\Delta v_{k-1}}{h}\right| (t_k - \tau_1) + \left|\frac{\Delta v_{\ell-1}}{h}\right| (\tau_2 - t_{\ell-1})$$

$$< M(\tau_2 - \tau_1).$$

We have proved that the sequences $\{S^{[n]}\}$ and $\{P^{[n]}\}$ are equicontinuous on $[0, b]$. □

Lemma 3.3 *Let the assumptions of Theorem* 3.1 *hold. Assume that* $S^{[n]}$ *and* $P^{[n]}$ *are given by* (3.7) *and* (3.8). *Then there exist subsequences* $\{S^{[m]}\} \subset \{S^{[n]}\}$ *and* $\{P^{[m]}\} \subset \{P^{[n]}\}$ *satisfying*

$$\lim_{m\to\infty} S^{[m]}(t) = S(t) \quad \textit{locally uniformly on } [0,T) \tag{3.15}$$

and

$$\lim_{m\to\infty} P^{[m]}(t) = P(t) \quad \textit{locally uniformly on } [0,T). \tag{3.16}$$

Moreover

$$0 < \alpha(t) \le S(t) \le \beta(t) \quad \textit{for } t \in [0,T). \tag{3.17}$$

Proof. Choose an interval $[0, b] \subset [0, T)$. Lemma 3.2 and the Arzelà-Ascoli theorem imply that we can choose subsequences of $\{S^{[n]}\}$ and of $\{P^{[n]}\}$ which uniformly converge on $[0, b]$. Since $[0, b]$ is an arbitrary interval in $[0, T)$, we use the diagonalization theorem (see e.g. [26]) to get that these subsequences can be chosen in such a way that they fulfil (3.15) and (3.16).

Now, choose an arbitrary $b \in [0, T]$ and assume that (3.13) holds. By (2.5) we have

$$\alpha(t_{b_n}) \le S(t_{b_n}) \le \beta(t_{b_n})$$

and letting $n \to \infty$ we get due to (3.14)

$$\alpha(b) \le S(b) \le \beta(b).$$

Since $b \in [0, T)$ is arbitrary, estimate (3.17) follows. □

Proof of Theorem 3.1. By (3.15) and (3.16) the functions S and P are continuous on $[0, T)$. Let $b \in (0, T)$, $c \in (b, T)$ and let (3.13) hold. By (3.14)–(3.16) we have

$$\lim_{m\to\infty} S^{[m]}(t_{b_m+1}) = S(b), \quad \lim_{m\to\infty} P^{[m]}(t_{b_m}) = P(b). \tag{3.18}$$

Due to (1.3) and (3.17) the function $f(t, S(t))$ is continuous on $[0, T)$. Let us denote

$$\varrho_m = \max\{|P^{[m]}(t) - P(t)| : t \in [0, c]\}$$

and

$$\sigma_m = \max\{|f(t, S^{[m]}(t)) - f(t, S(t))| : t \in [0, c]\}.$$

Then, by (3.15) and (3.16), we get

$$\lim_{m\to\infty} \varrho_m = 0, \quad \lim_{m\to\infty} \sigma_m = 0$$

and consequently, having $h = \frac{T}{m}$, we conclude

$$\lim_{m\to\infty} \frac{T}{m} \sum_{i=1}^{b_m} |P^{[m]}(t_i) - P(t_i)| = \lim_{m\to\infty} \frac{T}{m} b_m \varrho_m \le T \lim_{m\to\infty} \varrho_m = 0, \tag{3.19}$$

$$\begin{cases} \displaystyle\lim_{m\to\infty} \frac{T}{m} \sum_{i=1}^{b_m} |f(t_i, S^{[m]}(t_i)) - f(t_i, S(t_i))| \\ \displaystyle = \lim_{m\to\infty} \frac{T}{m} b_m \sigma_m \le T \lim_{m\to\infty} \sigma_m = 0. \end{cases} \tag{3.20}$$

Further we have

$$\lim_{m\to\infty} \frac{T}{m} \sum_{i=1}^{b_m} f(t_i, S(t_i)) = \int_0^b f(\tau, S(\tau)) \mathrm{d}\tau \tag{3.21}$$

and

$$\lim_{m\to\infty} \frac{T}{m} \sum_{i=1}^{b_m} P(t_i) = \int_0^b P(\tau) \mathrm{d}\tau. \tag{3.22}$$

Equation (3.9) yields

$$S^{[m]}(t_{b_m+1}) = S^{[m]}(0) + \frac{T}{m} \sum_{i=1}^{b_m} P^{[m]}(t_i).$$

Letting $m \to \infty$ and using (3.18), (3.19) and (3.22), we get

$$S(b) = S(0) + \int_0^b P(\tau) \mathrm{d}\tau.$$

Equation (3.10) yields

$$P^{[m]}(t_{b_m}) = -\frac{T}{m} \sum_{i=1}^{b_m} f(t_i, S^{[m]}(t_i)).$$

By (3.18), (3.20) and (3.21) we get for $m \to \infty$

$$P(b) = -\int_0^b f(\tau, S(\tau)) \mathrm{d}\tau.$$

Since $b \in (0,T)$ is arbitrary, we have

$$\begin{cases} S(t) = S(0) + \int_0^t P(\tau)\mathrm{d}\tau, \\ P(t) = -\int_0^t f(\tau, S(\tau))\mathrm{d}\tau, \quad t \in [0,T). \end{cases} \tag{3.23}$$

Let us put $y(t) = S(t)$ for $t \in [0,T)$. Then (3.23) gives

$$y'(t) = P(t) = -\int_0^t f(\tau, y(\tau))\mathrm{d}\tau, \quad t \in [0,T) \tag{3.24}$$

and hence $y \in C^2[0,T)$, $y'(0) = 0$. According to (3.17) we have

$$0 < \alpha(t) \le y(t) \le \beta(t) \quad \text{for } t \in [0,T).$$

Conditions (3.1) and (2.2) imply $\alpha_n = \alpha(T) = 0$. Hence the assumption $\beta_n = \beta(T) = 0$ leads to $\lim_{t \to T-} y(t) = 0$. So, if we put $y(T) = 0$, we get $y \in C[0,T]$ and consequently y is a solution of problem (1.5), (1.6).

Now, assume in addition that inequality (3.11) holds with g_0 and g_1 described in Theorem 3.1. Then

$$|f(t, y(t))| \le g_0(t, \alpha(t)) + g_1(t, y(t))$$

and

$$\int_0^T |f(t, y(t))|\mathrm{d}t \le \int_0^T g_0(t, \alpha(t))\mathrm{d}t + \int_0^T g_1(t, y(t))\mathrm{d}t < \infty.$$

Therefore the function $\int_0^t f(\tau, y(\tau))\mathrm{d}\tau$ is continuous for $t \in [0,T]$ and equality (3.24) yields $y \in C^1[0,T]$. $\square$

Example 1. Assume that $T \in (0,\infty)$, $n \in \mathbb{N}$, $n \ge 2$, $h = \frac{T}{n}$ and we use notation (1.4). Consider the singular difference equation

$$\frac{1}{h^2}\Delta^2 u_{k-1} + \frac{a_0}{u_k^2} - \frac{b_0}{u_k} - c_0 t_k^{2\gamma-4} = 0, \quad k = 1, \dots, n-1, \tag{3.25}$$

where $a_0, b_0, c_0 \in (0,\infty)$, $\gamma \in [2,\infty)$. Such types of equations appear in the theory of shallow membrane caps. We see that the function

$$f(t,x) = \frac{a_0}{x^2} - \frac{b_0}{x} - c_0 t^{2\gamma-4} \tag{3.26}$$

satisfies (1.3). Choose $\nu, c \in (0,\infty)$ and define

$$\alpha(t) = \nu(\nu + t)(T - t), \quad \beta(t) = c\sqrt{T^2 - t^2}, \quad t \in [0,T]. \tag{3.27}$$

Then for each sufficiently large c the functions α and β satisfy (3.1) and

$$\alpha(0) > 0, \quad \alpha(T) = \beta(T) = 0. \tag{3.28}$$

Now, we use notation (3.2) and get

$$\Delta\alpha_{k-1} = \alpha(t_k) - \alpha(t_{k-1}) = \nu(t_k - t_{k-1})(T - \nu - (t_k + t_{k-1}))$$

and

$$\Delta\alpha_0 = \nu t_1(T - \nu - t_1) > 0 \quad \text{for each } \nu \in \left(0, \frac{T}{2}\right).$$

Similarly

$$\Delta\beta_{k-1} = \beta(t_k) - \beta(t_{k-1}) = c\left(\sqrt{T^2 - t_k^2} - \sqrt{T^2 - t_{k-1}^2}\right)$$

and

$$\Delta\beta_0 = c\left(\sqrt{T^2 - t_1^2} - \sqrt{T^2}\right) < 0 \quad \text{for each } c > 0.$$

Further, we have

$$\frac{1}{h^2}\Delta^2\alpha_{k-1} = -2\nu, \quad k = 1, \dots, n-1$$

and

$$\begin{aligned} f(t_k, \alpha_k) &= \frac{a_0}{\alpha^2(t_k)} - \frac{b_0}{\alpha(t_k)} - c_0 t_k^{2\gamma-4} \\ &= \frac{a_0}{\nu^2(\nu + t_k)^2(T - t_k)^2} - \frac{b_0}{\nu(\nu + t_k)(T - t_k)} - c_0 t_k^{2\gamma-4} \\ &= \frac{1}{\nu^2(\nu + t_k)^2(T - t_k)^2}\varphi(t_k, \nu), \end{aligned}$$

where

$$\varphi(t_k, \nu) = a_0 - b_0\nu(\nu + t_k)(T - t_k) - c_0 t_k^{2\gamma-4}\nu^2(\nu + t_k)^2(T - t_k)^2$$

for $k = 1, \dots, n-1$. We see that

$$\lim_{\nu \to 0+} \frac{1}{\nu^2(\nu + t_k)^2(T - t_k)^2}\varphi(t_k, \nu) = \infty, \quad k = 1, \dots, n-1.$$

Hence for each sufficiently small $\nu > 0$ the vector $(\alpha_0, \dots, \alpha_n)$ fulfils (2.2) and (2.1) with f given by (3.26) and therefore this vector is a lower function of problem (3.25), (1.2).

Similarly

$$\begin{aligned} \Delta^2\beta_{k-1} &= \Delta\beta_k - \Delta\beta_{k-1} \\ &= c\left(\sqrt{T^2 - t_{k+1}^2} - \sqrt{T^2 - t_k^2}\right) - c\left(\sqrt{T^2 - t_k^2} - \sqrt{T^2 - t_{k-1}^2}\right) \\ &= -c\frac{t_{k+1}^2 - t_k^2}{\sqrt{T^2 - t_{k+1}^2} + \sqrt{T^2 - t_k^2}} + c\frac{t_k^2 - t_{k-1}^2}{\sqrt{T^2 - t_k^2} + \sqrt{T^2 - t_{k-1}^2}} \end{aligned}$$

$$= -2cTh\left(\frac{1}{\sqrt{T^2-t_{k+1}^2}+\sqrt{T^2-t_k^2}} - \frac{1}{\sqrt{T^2-t_k^2}+\sqrt{T^2-t_{k-1}^2}}\right).$$

Thus

$$\frac{1}{h^2}\Delta^2\beta_{k-1} = \frac{-2cT}{h}\left(\frac{\sqrt{T^2-t_{k-1}^2}-\sqrt{T^2-t_{k+1}^2}}{\left(\sqrt{T^2-t_{k+1}^2}+\sqrt{T^2-t_k^2}\right)\left(\sqrt{T^2-t_k^2}+\sqrt{T^2-t_{k-1}^2}\right)}\right)$$

$$\leq \frac{-2cT}{h}\frac{t_{k+1}^2-t_{k-1}^2}{\left(2\sqrt{T^2-t_{k-1}^2}\right)^3} \leq \frac{-cT^2}{(T^2-t_{k-1}^2)^{3/2}}, \quad k=1,\dots,n-1.$$

Moreover

$$f(t_k,\beta_k) = \frac{a_0}{\beta_k^2} - \frac{b_0}{\beta_k} - c_0 t_k^{2\gamma-4} = \frac{a_0}{c^2(T^2-t_k^2)} - \frac{b_0}{c\sqrt{T^2-t_k^2}} - c_0 t_k^{2\gamma-4}$$

and

$$\frac{1}{h^2}\Delta^2\beta_{k-1} + f(t_k,\beta_k) \leq \frac{-cT^2}{(T^2-t_{k-1}^2)^{3/2}}\left(\frac{1}{2} - \frac{a_0(T^2-t_{k-1}^2)}{c^3T^2(T^2-t_k^2)}\sqrt{T^2-t_{k-1}^2}\right)$$

$$< \frac{-cT^2}{(T^2-t_{k-1}^2)^{3/2}}\left(\frac{1}{2} - \frac{4a_0\sqrt{T^2-t_{k-1}^2}}{c^3T^2}\right) =: \psi(c,t_{k-1}).$$

We see that

$$\lim_{c\to\infty}\psi(c,t_{k-1}) = -\infty \quad \text{for } k=1,\dots,n-1.$$

Hence for each sufficiently large c the vector $(\beta_0,\dots,\beta_n)$ fulfils (2.4) and (2.3) with f given by (3.26) and therefore it is an upper function of problem (3.25), (1.2). By Theorem 2.3, for each $n\in\mathbb{N}$, $n\geq 2$, problem (3.25), (1.2) has a solution $(u_0,\dots,u_n)$ satisfying (2.5). If we define v_k by (3.3) and $S^{[n]}$ by (3.7), Theorem 3.1 yields that there is a subsequence $\{S^{[m]}\}$ which converges locally uniformly on $[0,T)$ to a solution $y\in C[0,T]\cap C^2[0,T)$ of problem

$$y''(t) + \frac{a_0}{y^2(t)} - \frac{b_0}{y(t)} - c_0t^{2\gamma-4} = 0, \quad y'(0)=0, \quad y(T)=0.$$

Example 2. Assume that $T\in(0,\infty)$, $n\in\mathbb{N}$, $n\geq 2$, $h=\frac{T}{n}$ and we use notation (1.4) and (3.2). Consider the singular difference equation

$$\frac{1}{h^2}\Delta^2u_{k-1} + (T-t_k)^2\left(\frac{a_0}{u_k^2} - \frac{b_0}{u_k} - c_0t_k^{2\gamma-4}\right) = 0, \quad k=1,\dots,n-1, \tag{3.29}$$

where $a_0,b_0,c_0\in(0,\infty)$, $\gamma\in[2,\infty)$. Define α and β by (3.27). We can check as in Example 1 that for each sufficiently small $\nu>0$ the vector $(\alpha_0,\dots,\alpha_n)$ is a lower function of problem (3.29), (1.2) and for each sufficiently large $c > \nu$ the vector $(\beta_0, \dots, \beta_n)$ is an upper function of problem (3.29), (1.2). We see that $\alpha(0) > 0$ and $\beta(T) = 0$. Therefore, by Theorem 2.3, for each $n \in \mathbb{N}$, $n \geq 2$, problem (3.29), (1.2) has a solution $(u_0, \dots, u_n)$ satisfying (2.5). As in Example 1 we get a subsequence $\{S^{[m]}\}$ which converges locally uniformly on $[0, T)$ to a solution $y \in C[0, T] \cap C^2[0, T)$ of problem

$$y''(t) + (T-t)^2 \left(\frac{a_0}{y^2(t)} - \frac{b_0}{y(t)} - c_0 t^{2\gamma - 4} \right) = 0, \quad y'(0) = 0, \quad y(T) = 0.$$

Moreover, the function

$$f(t, x) = (T-t)^2 \left(\frac{a_0}{x^2} - \frac{b_0}{x} - c_0 t^{2\gamma - 4} \right)$$

satisfies (3.11) with

$$g_0(t, x) = (T-t)^2 \left(\frac{a_0}{x^2} + \frac{b_0}{x} \right), \quad g_1(t, x) = (T-t)^2 c_0 t^{2\gamma - 4}.$$

Since

$$\int_0^T g_0(t, \alpha(t)) \mathrm{d}t = \int_0^T \left(\frac{a_0}{\nu^2 (\nu + t)^2} + \frac{b_0 (T-t)}{\nu(\nu + t)} \right) \mathrm{d}t < \infty,$$

we get by Theorem 3.1 that moreover $y \in C^1[0, T]$.

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