# Localization of Nonsmooth Lower and Upper Functions for Periodic Boundary Value Problems

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**Summary.** In this paper we present conditions ensuring the existence and localization of lower and upper functions of the periodic boundary value problem $u'' + k\,u = f(t,u), u(0) = u(2\,\pi),\ u'(0) = u'(2\pi),\ k \in \mathbb{R},\ k \neq 0$. These functions are constructed as solutions of some related generalized linear problems and they can be nonsmooth, in general.

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# 1 . Introduction

Theorems about the existence of solutions of boundary value problems for ordinary differential equations often suppose the existence of lower and upper functions to the studied problem. For such theorems concerning periodic boundary value problems we can refer e.g. to the papers [1], [2], [3], [5], [6], [8], [9], [10], [11], [12], [13], [14] and our forthcoming paper [15]. We can decide whether the problem has constant lower and upper functions (see e.g. [2], [5]) and to find them if they exist. In general, however, it is easy neither to find nonconstant lower and upper functions nor to prove their existence which can make difficult the application of such theorems. One possibility how to get nonconstant and possibly nonsmooth lower and upper functions to the periodic boundary value problem

$$u'' + k\,u = f(t,u), \quad u(0) = u(2\pi),\ u'(0) = u'(2\pi), \quad k \in \mathbb{R},\ k \neq 0 \tag{1.1}$$

is shown here. (The case $k = 0$ is solved in [13] and [14].) We make use of fairly general definitions of these notions introduced in [12] and we construct them as solutions of generalized periodic boundary value problems for linear differential equations in sections 2 and 3. (Essentially they are solutions of linear generalized differential

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equations in the sense of J. Kurzweil, cf. e.g. [4], [17], [18] and [19].) Our next paper [15] will show new effective existence criteria for the problem (1.1). The proofs of them are based on the theorems about the existence and localization of lower and upper functions from this paper.

We say that $f : [0, 2\pi] \times \mathbb{R} \mapsto \mathbb{R}$ fulfils the Carathéodory conditions on $[0, 2\pi] \times \mathbb{R}$, if $f$ has the following properties: (i) for each $x \in \mathbb{R}$ the function $f(., x)$ is measurable on $[0, 2\pi]$; (ii) for almost every $t \in [0, 2\pi]$ the function $f(t, .)$ is continuous on $\mathbb{R}$; (iii) for each compact set $\mathrm{K} \subset \mathbb{R}$ the function $m_{\mathrm{K}}(t) = \sup_{x \in \mathrm{K}} |f(t, x)|$ is Lebesgue integrable on $[0, 2\pi]$.

The set of functions satisfying the Carathéodory conditions on $[0, 2\pi] \times \mathbb{R}$ is denoted by $\mathrm{Car}([0, 2\pi] \times \mathbb{R})$.

Throughout the paper we assume $f \in \mathrm{Car}([0, 2\pi] \times \mathbb{R})$.

For a given subinterval $J$ of $\mathbb{R}$ (possibly unbounded) $\mathbb{C}(J)$ denotes the set of functions continuous on $J$, $\mathbb{L}[0, 2\pi]$ stands for the set of functions Lebesgue integrable on $[0, 2\pi]$, $\mathbb{L}_2[0, 2\pi]$ is the set of functions square Lebesgue integrable on $[0, 2\pi]$, $\mathbb{L}_\infty[0, 2\pi]$ is the set of functions essentially bounded on $[0, 2\pi]$, $\mathbb{AC}[0, 2\pi]$ denotes the set of functions absolutely continuous on $[0, 2\pi]$ and $\mathbb{BV}[0, 2\pi]$ is the set of functions of bounded variation on $[0, 2\pi]$. For $x \in \mathbb{L}[0, 2\pi]$, $y \in \mathbb{L}_2[0, 2\pi]$ and $z \in \mathbb{L}_\infty[0, 2\pi]$, we denote

$$\overline{x} = \frac{1}{2\pi} \int_0^{2\pi} x(s)\,\mathrm{d}s, \quad \|x\|_1 = \int_0^{2\pi} |x(t)|\,\mathrm{d}t$$

$$\|y\|_2 = \Big( \int_0^{2\pi} y^2(t)\,\mathrm{d}t \Big)^{\frac{1}{2}} \quad \text{and} \quad \|z\|_\infty = \sup_{t \in [0, 2\pi]} \operatorname{ess}\ |z(t)|.$$

If $x \in \mathbb{BV}[0, 2\pi]$, $s \in (0, 2\pi]$ and $t \in [0, 2\pi)$, then the symbols $x(s-)$, $x(t+)$ and $\Delta^+ x(t)$ are respectively defined by

$$x(s-) = \lim_{\tau \to s-} x(\tau), \quad x(t+) = \lim_{\tau \to t+} x(\tau) \text{ and } \Delta^+ x(t) = x(t+) - x(t)$$

and $x^{\mathrm{ac}}$ and $x^{\mathrm{sing}}$ stand for the absolutely continuous part of $x$ and the singular part of $x$, respectively. We suppose $x^{\mathrm{sing}}(0) = 0$.

$\mathbb{L}^n[0, 2\pi]$ and $\mathbb{L}^{n \times n}[0, 2\pi]$ are respectively the sets of column $n$-vector valued and of $n \times n$-matrix valued functions with elements from $\mathbb{L}[0, 2\pi]$; $\mathbb{AC}^n[0, 2\pi]$ and $\mathbb{AC}^{n \times n}[0, 2\pi]$ are respectively the sets of $n$-vector valued and of $n \times n$-matrix valued functions whose elements are absolutely continuous on $[0, 2\pi]$ and $\mathbb{BV}^n[0, 2\pi]$ is the set of $n$-vector valued functions whose elements have a bounded variation on $[0, 2\pi]$.

For a subset $M$ of $\mathbb{R}$, $\chi_M$ denotes the characteristic function of $M$. For $x \in \mathbb{L}[0, 2\pi]$ and a.e. $t \in [0, 2\pi]$, we write

$$x^+(t) = \max\{x(t), 0\} \quad \text{and} \quad x^-(t) = \max\{-x(t), 0\}.$$

By a *solution of* (1.1) we understand a function $u : [0, 2\pi] \mapsto \mathbb{R}$ such that $u' \in \mathbb{AC}[0, 2\pi]$, $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and

$$u''(t) + k\, u(t) = f(t, u(t)) \quad \text{for a.e.} \quad t \in [0, 2\pi].$$

**1.1. Definition.** Functions $(\sigma_1, \rho_1) \in \mathbb{AC}[0, 2\pi] \times \mathbb{BV}[0, 2\pi]$ are said to be *lower functions of the problem* (1.1), if the singular part $\rho_1^{\text{sing}}$ of $\rho_1$ is nondecreasing on $[0, 2\pi]$,

$$\sigma_1'(t) = \rho_1(t), \quad \rho_1'(t) + k\, \sigma_1(t) \geq f(t, \sigma_1(t)) \quad \text{for a.e.} \quad t \in [0, 2\pi]$$

and

$$\sigma_1(0) = \sigma_1(2\pi), \quad \rho_1(0+) \geq \rho_1(2\pi-). \tag{1.2}$$

Similarly, functions $(\sigma_2, \rho_2) \in \mathbb{AC}[0, 2\pi] \times \mathbb{BV}[0, 2\pi]$ are said to be *upper functions of the problem* (1.1), if the singular part $\rho_2^{\text{sing}}$ of $\rho_2$ is nonincreasing on $[0, 2\pi]$,

$$\sigma_2'(t) = \rho_2(t), \quad \rho_2'(t) = k\, \sigma_2(t) \leq f(t, \sigma_2(t)) \quad \text{for a.e.} \quad t \in [0, 2\pi]$$

and

$$\sigma_2(0) = \sigma_2(2\pi), \quad \rho_2(0+) \leq \rho_2(2\pi-). \tag{1.3}$$

# 2 . Periodic solutions of certain generalized linear differential problems

We want to show that if for a.e. $t \in [0, 2\pi]$ and all $x \in I_t$, where $I_t$ is a subinterval of $\mathbb{R}$, the function $f$ fulfils a condition of the form

$$f(t, x) \leq \beta(t) \tag{2.1}$$

or

$$f(t, x) \geq \beta(t), \tag{2.2}$$

with $\beta \in \mathbb{L}[0, 2\pi]$, then it is possible to construct lower or upper functions for the problem (1.1), respectively.

It is known that if $k \neq n^2$ for all $n \in \mathbb{N} \cup \{0\}$, then the problem

$$\sigma' = \rho, \quad \rho' + k\, \sigma = \beta(t) \quad \text{a.e. on} \quad [0, 2\pi], \tag{2.3}$$

$$\sigma(0) = \sigma(2\pi), \quad \rho(0) = \rho(2\pi) \tag{2.4}$$

possesses a unique solution $(\sigma,\rho) \in \mathbb{AC}[0,2\pi] \times \mathbb{AC}[0,2\pi]$ for any $\beta \in \mathbb{L}[0,2\pi]$. Consequently, if we have in addition

$$\sigma(t) \in I_t \quad \text{for all } t \in [0,2\pi], \tag{2.5}$$

then the functions $(\sigma,\rho)$ are lower or upper functions of (1.1) (according to whether (2.1) or (2.2) is satisfied). In general the relation (2.5) need not be true, of course. However, if we admit a more general notion of a solution to the linear problem (2.3), (2.4) and if the intervals $I_t$ of validity of (2.1) or (2.2) are large enough, we can always use the problem (2.3), (2.4) for a construction of lower or upper functions for (1.1).

To show this, let us consider a linear differential system on $[0,2\pi]$

$$\xi' = P(t)\,\xi + q(t), \tag{2.6}$$

where $P \in \mathbb{L}^{n\times n}[0,2\pi]$ and $q \in \mathbb{L}^n[0,2\pi]$. By a solution of (2.6) on $[0,2\pi]$ we mean a function $\xi \in \mathbb{AC}^n[0,2\pi]$ satisfying (2.6) a.e. on $[0,2\pi]$. The corresponding normalized fundamental matrix solution of the system

$$\xi' = P(t)\,\xi \tag{2.7}$$

is denoted by $X$, i.e. $X \in \mathbb{AC}^{n\times n}[0,2\pi]$ and

$$X(t) = \mathrm{I} + \int_0^t P(s)\,X(s)\,\mathrm{d}s \quad \text{on } [0,2\pi],$$

where I stands, as usual, for the identity $n \times n$-matrix. Its inverse matrix $X^{-1}(t)$ is defined for any $t \in [0,2\pi]$, $X^{-1} \in \mathbb{AC}^{n\times n}[0,2\pi]$ and if

$$\det\left(X^{-1}(2\pi) - \mathrm{I}\right) \neq 0 \tag{2.8}$$

holds, then for any $q \in \mathbb{L}^n[0,2\pi]$ there is a unique solution $\xi \in \mathbb{AC}^n[0,2\pi]$ of (2.6) on $[0,2\pi]$ such that

$$\xi(0) = \xi(2\pi). \tag{2.9}$$

This solution can be written in the form

$$\xi(t) = \int_0^{2\pi} G(t,s)\,q(s)\,\mathrm{d}s \quad \text{on } [0,2\pi],$$

where

$$G(t,s) = X(t) \left\{ \begin{array}{ll} \left(X^{-1}(2\pi) - \mathrm{I}\right)^{-1} & \text{for } t \le s, \\ \mathrm{I} + \left(X^{-1}(2\pi) - \mathrm{I}\right)^{-1} & \text{for } s < t \end{array} \right\} X^{-1}(s) \tag{2.10}$$

is the Green function of the problem (2.7), (2.9).

**2.1. Definition.** Let $\tau \in [0, 2\pi)$ and $d \in \mathbb{R}^n$ be given. By a solution of the problem (2.6), (2.9),

$$\Delta^+\xi(\tau) = d \tag{2.11}$$

we mean a function $\xi \in \mathbb{BV}^n[0, 2\,\pi]$ such that the relations (2.9) and

$$\xi'(t) = P(t)\,\xi(t) + q(t) \quad \text{a.e. on } [0, 2\,\pi] \tag{2.12}$$

are satisfied and $\xi - d\,\chi_{(\tau,2\pi]} \in \mathbb{AC}^n[0, 2\,\pi]$.

**2.2. Proposition.** *Assume* (2.8). *Then for any* $\tau \in [0, 2\pi)$, *any* $d \in \mathbb{R}^n$ *and any* $q \in \mathbb{L}^n[0, 2\,\pi]$, *the problem* (2.6), (2.9), (2.11) *possesses a unique solution* $\xi$ *and this solution is given by*

$$\xi(t) = G(t, \tau)\,d + \int_0^{2\pi} G(t, s)\,q(s)\,\mathrm{d}s \quad \textit{on } [0, 2\,\pi], \tag{2.13}$$

*where* $G$ *is defined by* (2.10).

*Proof.* For any $c \in \mathbb{R}^n$, the functions

$$x(t) = X(t)\,c + X(t)\int_0^t X^{-1}(s)\,q(s)\,\mathrm{d}s, \quad t \in [0, 2\,\pi],$$

and

$$y(t) = X(t)X^{-1}(2\pi)\,c - X(t)\int_t^{2\pi} X^{-1}(s)\,q(s)\,\mathrm{d}s \quad t \in [0, 2\,\pi],$$

are the unique solutions of (2.6) on $[0, 2\,\pi]$, such that $x(0) = c$ and $y(2\pi) = c$, respectively. Define $\xi(t) = x(t)$ for $0 \le t \le \tau$ and $\xi(t) = y(t)$ for $\tau < t \le 2\pi$. Then $\xi \in \mathbb{BV}^n[0, 2\,\pi]$ fulfils (2.9), (2.12) and

$$\Delta^+\xi(\tau) = X(\tau)\,[X^{-1}(2\pi) - \mathrm{I}]\,c - X(\tau)\int_0^{2\pi} X^{-1}(s)\,q(s)\,\mathrm{d}s.$$

Consequently, if we put

$$c = M^{-1}\Big(X^{-1}(\tau)\,d + \int_0^{2\pi} X^{-1}(s)\,q(s)\,\mathrm{d}s\Big),$$

where $M = X^{-1}(2\pi) - \mathrm{I}$, then $\xi$ verifies (2.11). Moreover, $\xi(t) - d\,\chi_{(\tau,2\pi]}(t) = x(t)$ holds on $[0, 2\,\pi]$ and hence $\xi - d\,\chi_{(\tau,2\pi]} \in \mathbb{AC}^n[0, 2\,\pi]$. Finally, using the relation $X^{-1}(2\pi)\,M^{-1} = \mathrm{I} + M^{-1}$, we get

$$\xi(t) = X(t)\Big(M^{-1}\big[X^{-1}(\tau)\,d + \int_0^{2\pi} X^{-1}(s)\,q(s)\,\mathrm{d}s\big] + \int_0^t X^{-1}(s)\,q(s)\,\mathrm{d}s\Big)$$

for $0 \le t \le \tau$ and

$$\xi(t) = X(t)\left([\mathrm{I} + M^{-1}]\left[X^{-1}(\tau)\,d + \int_0^{2\pi} X^{-1}(s)\,q(s)\,\mathrm{d}s\right] - \int_t^{2\pi} X^{-1}(s)\,q(s)\,\mathrm{d}s\right)$$

for $\tau < t \le 2\pi$, wherefrom the representation (2.13) of $\xi$ follows. □

**2.3. Remark.** Clearly, for any solution $\xi$ of (2.6), (2.9), (2.11) we have $\xi^{\mathrm{ac}} = \xi - d\,\chi_{(\tau,2\pi]}$, $\xi^{\mathrm{sing}} = d\,\chi_{(\tau,2\pi]}$ and $\xi$ is left-continuous on $(0, 2\pi]$.

**2.4. Remark.** The problem (2.6), (2.9), (2.11) can be rewritten as the integral equation

$$\xi(t) = \xi(0) + \int_0^t P(s)\,\xi(s)\,\mathrm{d}s + h(t) - h(0), \quad t \in [0, 2\,\pi],$$

where

$$h(t) = d\,\chi_{(\tau,2\pi]}(t) + \int_0^t q(s)\,\mathrm{d}s \ \text{ on } [0, 2\,\pi].$$

This equation is a very special case of generalized differential equations introduced by J. Kurzweil in [4].

Now, we will apply Proposition 2.2 on the problem (2.3), (2.4) generalized in the sense of Definition 2.1. In the case $k = -\alpha^2$ we get the following result:

**2.5. Corollary.** *Let $k = -\alpha^2$, $\alpha \in (0, \infty)$, $\tau \in [0, 2\pi)$, $\delta \in \mathbb{R}$ and $\beta \in \mathbb{L}[0, 2\,\pi]$. Then the problem*

$$\begin{aligned} &\sigma' = \rho, && \rho' + k\,\sigma = \beta(t), \\ &\sigma(0) = \sigma(2\pi), && \rho(0) = \rho(2\pi), \qquad \Delta^+\sigma(\tau) = 0, \quad \Delta^+\rho(\tau) = \delta \end{aligned} \tag{2.14}$$

*possesses a unique solution $(\sigma, \rho)$. Moreover, $\sigma \in \mathbb{AC}[0, 2\,\pi]$, $\rho^{sing} = \delta\,\chi_{(\tau,2\pi]}$ and*

$$\sigma(t) = g(|t - \tau|)\,\delta + \int_0^{2\pi} g(|t - s|)\,\beta(s)\,\mathrm{d}s, \quad \textit{on } [0, 2\,\pi], \tag{2.15}$$

*where*

$$g(x) = -\frac{\cosh(\alpha\,(\pi - x))}{2\,\alpha\,\sinh(\alpha\pi)} \quad \textit{for } x \in [0, 2\,\pi]. \tag{2.16}$$

*Proof.* The fundamental matrix solution $X$ of the corresponding homogeneous system $\sigma' = \rho$, $\rho' = \alpha^2\,\sigma$, is given by

$$X(t) = \begin{pmatrix} \cosh(\alpha\,t) & \dfrac{\sinh(\alpha\,t)}{\alpha} \\ \alpha\,\sinh(\alpha t) & \cosh(\alpha\,t) \end{pmatrix} \quad \text{on } [0, 2\,\pi]$$

and $\det\left(X^{-1}(2\pi) - \mathrm{I}\right) = -4\,\sinh(\alpha\pi) \neq 0$.

Thus, we can apply Proposition 2.2 to the problem (2.6), (2.9), (2.11) with

$$\xi(t) = \begin{pmatrix} \sigma \\ \rho \end{pmatrix}, \quad q(t) = \begin{pmatrix} 0 \\ \beta \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} 0 \\ \delta \end{pmatrix},$$

to obtain that the problem (2.14) possesses a unique solution $(\sigma, \rho)$. Since, in particular, $\Delta^+\sigma(\tau) = 0$ and $\Delta^+\rho(\tau) = \delta$, it follows from Definition 2.1 that $\sigma \in \mathbb{AC}[0, 2\,\pi]$ and $\rho - \delta\,\chi_{(\tau,2\pi]} \in \mathbb{AC}[0, 2\,\pi]$ (i.e. $\rho^{\mathrm{sing}} = \delta\,\chi_{(\tau,2\pi]}$). Furthermore, substituting for $X$ into (2.10), we get

$$G(t,s) = \begin{cases} \begin{pmatrix} -\dfrac{\sinh(\alpha(\pi+t-s))}{2\,\sinh(\alpha\pi)} & -\dfrac{\cosh(\alpha(\pi+t-s))}{2\,\alpha\,\sinh(\alpha\pi)} \\ -\dfrac{\alpha\,\cosh(\alpha(\pi+t-s))}{2\,\sinh(\alpha\pi)} & -\dfrac{\sinh(\alpha(\pi+t-s))}{2\,\sinh(\alpha\pi)} \end{pmatrix} \\ \qquad\qquad \text{if} \quad 0 \le t \le s \le 2\pi, \\ \begin{pmatrix} \dfrac{\sinh(\alpha(\pi+s-t))}{2\,\sinh(\alpha\pi)} & -\dfrac{\cosh(\alpha(\pi+s-t))}{2\,\alpha\,\sinh(\alpha\pi)} \\ -\dfrac{\alpha\,\cosh(\alpha(\pi+s-t))}{2\,\sinh(\alpha\pi)} & \dfrac{\sinh(\alpha(\pi+s-t))}{2\,\sinh(\alpha\pi)} \end{pmatrix} \\ \qquad\qquad \text{if} \quad 0 \le s < t \le 2\pi, \end{cases}$$

which implies that $\sigma$ has the form (2.15), where $g$ is defined in (2.16). □

**2.6. Remark.** We can verify that for any $\alpha \in (0, \infty)$, the Green function $g$ from (2.16) satisfies the relations

$$-\frac{\coth(\alpha\pi)}{2\,\alpha} \le g(x) \le -\frac{1}{2\,\alpha\,\sinh(\alpha\pi)} < 0 \qquad \text{for } x \in [0, 2\,\pi] \tag{2.17}$$

and

$$\int_0^{2\pi} g(|t-s|)\,\mathrm{d}s = -\frac{1}{\alpha^2} \qquad \text{for } t \in [0, 2\,\pi]. \tag{2.18}$$

The next result concerns the case $k = \alpha^2$.

**2.7. Corollary.** *Let $k = \alpha^2$, $\alpha \in (0, \infty) \setminus \mathbb{N}$, $\tau \in [0, 2\pi)$, $\delta \in \mathbb{R}$ and $\beta \in \mathbb{L}[0, 2\,\pi]$. Then the problem* (2.14) *possesses a unique solution $(\sigma, \rho)$. Moreover, $\sigma \in \mathbb{AC}[0, 2\,\pi]$, $\rho^{sing} = \delta\,\chi_{(\tau,2\pi]}$ and $\sigma$ has the form* (2.15)*, where*

$$g(x) = \frac{\cos(\alpha\,(\pi - x))}{2\,\alpha\,\sin(\alpha\pi)} \quad \textit{for } x \in [0, 2\,\pi]. \tag{2.19}$$

*Proof.* Substituting $X$ in (2.10) by

$$X(t) = \begin{pmatrix} \cos(\alpha\, t) & \dfrac{\sin(\alpha\, t)}{\alpha} \\ -\alpha\, \sin(\alpha\, t) & \cos(\alpha\, t) \end{pmatrix}, \quad t \in [0, 2\,\pi],$$

we get

$$G(t,s) = \begin{cases} \begin{pmatrix} -\dfrac{\sin(\alpha(\pi+t-s))}{2\,\sin(\alpha\pi)} & \dfrac{\cos(\alpha(\pi+t-s))}{2\,\alpha\,\sin(\alpha\pi)} \\ -\alpha\,\dfrac{\cos(\alpha(\pi+t-s))}{2\,\sin(\alpha\pi)} & -\dfrac{\sin(\alpha(\pi+t-s))}{2\,\sin(\alpha\pi)} \end{pmatrix} \\ \qquad\qquad \text{if } \ 0 \le t \le s \le 2\pi, \\ \begin{pmatrix} \dfrac{\sin(\alpha(\pi+s-t))}{2\,\sin(\alpha\pi)} & \dfrac{\cos(\alpha(\pi+s-t))}{2\,\alpha\,\sin(\alpha\pi)} \\ -\dfrac{\alpha\,\cos(\alpha(\pi+s-t))}{2\,\sin(\alpha\pi)} & \dfrac{\sin(\alpha(\pi+s-t))}{2\,\sin(\alpha\pi)} \end{pmatrix} \\ \qquad\qquad \text{if } \ 0 \le s < t \le 2\pi \end{cases}$$

and since under our assumptions we have $\det\left(X^{-1}(2\pi) - \mathrm{I}\right) = 4\,\sin^2(\alpha\pi) \neq 0$, the proof follows from Proposition 2.2 similarly as the proof of Corollary 2.5. □

**2.8. Remark.** For the function $g$ from (2.19) and any $\alpha \in (0,\infty) \setminus \mathbb{N}$, we have

$$\|g^-\|_\infty \le \|g^+\|_\infty = \|g\|_\infty = \frac{1}{2\,\alpha\,|\sin(\alpha\pi)|} \tag{2.20}$$

and

$$\int_0^{2\pi} g(|t-s|)\,\mathrm{d}s = \frac{1}{\alpha^2} \qquad \text{for } t \in [0, 2\,\pi]. \tag{2.21}$$

Furthermore, if $\alpha \in (0, \frac{1}{2}]$, then

$$0 \le \frac{\operatorname{cotan}(\alpha\pi)}{2\,\alpha} \le g(x) \le \frac{1}{2\,\alpha\,\sin(\alpha\pi)} \quad \text{for } x \in [0, 2\,\pi]. \tag{2.22}$$

In the rest of this section we will derive some estimates for solutions of the problem

$$\sigma'' + k\,\sigma = b(t), \quad \sigma(0) = \sigma(2\pi),\ \sigma'(0) = \sigma'(2\pi) \tag{2.23}$$

which will be useful for the proofs of Section 3.

**2.9. Lemma.** *Let $p \in \mathbb{L}_\infty[0, 2\pi]$, $b \in \mathbb{L}[0, 2\pi]$ and $\overline{b} = 0$. Then*

$$\Big| \int_0^{2\pi} p(s)\, b(s)\, \mathrm{d}s \Big| \le p^* \, \|b\|_1, \quad \textit{where} \quad p^* = \frac{\|p^+\|_\infty + \|p^-\|_\infty}{2}.$$

*Proof.* First, notice that $\overline{b^+} = \overline{b^-}$ and $\|b\|_1 = 4\pi\overline{b^+} = 4\,\pi\overline{b^-}$ holds whenever $b \in \mathbb{L}[0, 2\pi]$ and $\overline{b} = 0$. Thus, if $p(s) \ge 0$ a.e. on $[0, 2\pi]$, then

$$\Big| \int_0^{2\pi} p(s)\, b(s)\, \mathrm{d}s \Big| \le \|p\|_\infty \, 2\pi\, \overline{b^+} = \|p^+\|_\infty \frac{\|b\|_1}{2}.$$

In a general case, we have

$$\begin{aligned} \Big| \int_0^{2\pi} p(s)\, b(s)\, \mathrm{d}s \Big| &= \Big| \int_0^{2\pi} p^+(s)\, b(s)\, \mathrm{d}s + \int_0^{2\pi} p^-(s)\, (-b(s))\, \mathrm{d}s \Big| \\ &\le (\|p^+\|_\infty + \|p^-\|_\infty)\, \frac{\|b\|_1}{2} = p^* \, \|b\|_1. \end{aligned}$$

□

**2.10. Lemma.** *Let $k \ne 0$, $b \in \mathbb{L}[0, 2\pi]$ and $\overline{b} = 0$. Then $\overline{\sigma} = 0$ holds for any solution $\sigma$ of* (2.23).

*Proof.* In virtue of the periodicity conditions we have $\overline{\sigma''} = 0$. Thus, integrating the differential equation from (2.23) over $[0, 2\pi]$, we get $0 = \overline{\sigma''} = -k\,\overline{\sigma} + \overline{b}$, i.e. $\overline{\sigma} = 0$. □

**2.11. Lemma.** *Let $k = -\alpha^2$, $\alpha \in (0, \infty)$, $b \in \mathbb{L}[0, 2\pi]$, $\overline{b} = 0$ and let $\sigma$ be a solution of* (2.23). *Then*

$$\|\sigma\|_\infty \le \Phi(\alpha)\, \|b\|_1, \tag{2.24}$$

*where*

$$\Phi(\alpha) = \min\{\frac{\pi}{6}, \frac{\coth(\alpha\pi)}{4\,\alpha}\}. \tag{2.25}$$

*If $b \in \mathbb{L}_2[0, 2\pi]$, then $\sigma$ fulfils moreover*

$$\|\sigma\|_\infty \le \sqrt{\frac{\pi}{6}}\, \|b\|_2. \tag{2.26}$$

*Proof.* Multiplying the relation

$$-\sigma''(t) = -\alpha^2\, \sigma(t) - b(t) \ \text{ for a.e. } t \in [0, 2\pi] \tag{2.27}$$

by $\sigma(t)$, integrating it over $[0, 2\,\pi]$ and using the Hölder inequality, we get

$$\|\sigma'\|_2^2 = -\alpha^2\,\|\sigma\|_2^2 + \int_0^{2\pi} b(t)\,\sigma(t)\,\mathrm{d}t \le \|b\|_1\,\|\sigma\|_\infty.$$

Furthermore, by Lemma 2.10 we have $\overline{\sigma} = 0$, and thus $\sigma$ satisfies the Sobolev inequality (see [7, Proposition 1.3])

$$\|\sigma\|_\infty \le \sqrt{\frac{\pi}{6}}\,\|\sigma'\|_2. \tag{2.28}$$

Hence

$$\|\sigma'\|_2 \le \sqrt{\frac{\pi}{6}}\,\|b\|_1$$

and, using the Sobolev inequality (2.28) once more, we obtain

$$\|\sigma\|_\infty \le \frac{\pi}{6}\,\|b\|_1. \tag{2.29}$$

On the other hand, by Corollary 2.5,

$$\sigma(t) = \int_0^{2\pi} g(|t-s|)\,b(s)\,\mathrm{d}s \quad \text{on } [0, 2\,\pi] \tag{2.30}$$

with $g$ defined by (2.16). Hence, according to Lemma 2.9 and (2.17) we also have

$$\|\sigma\|_\infty \le \frac{\coth(\alpha\pi)}{4\,\alpha}\,\|b\|_1.$$

This together with (2.29) completes the proof of the first assertion of the lemma.

Now, suppose $b \in \mathbb{L}_2[0, 2\,\pi]$. Using the Schwarz inequality, we deduce from (2.27) that

$$\|\sigma'\|_2^2 \le \|b\|_2\,\|\sigma\|_2.$$

Since $\sigma$ satisfies the Wirtinger inequality (see [7, Proposition 1.3])

$$\|\sigma\|_2 \le \|\sigma'\|_2, \tag{2.31}$$

we get $\|\sigma'\|_2 \le \|b\|_2$. This together with the Sobolev inequality (2.28) implies the estimate (2.26). □

**2.12. Remark.** As the function $\varphi(\alpha) = \dfrac{\coth(\alpha\pi)}{4\,\alpha}$ is decreasing on $(0,\infty)$, $\varphi(0+) = \infty$ and $\lim_{x\to\infty}\varphi(x) = 0$, there is exactly one $\alpha^* \in (0,\infty)$ ($\alpha^* \approx 0.51624$) such that $\varphi(\alpha^*) = \frac{\pi}{6}$, while $\varphi(\alpha) > \frac{\pi}{6}$ for $\alpha \in (0,\alpha^*)$ and $\varphi(\alpha) < \frac{\pi}{6}$ for $\alpha \in (\alpha^*,\infty)$. This means that the function $\Phi$ from (2.25) can be described by

$$\Phi(\alpha) = \begin{cases} \dfrac{\pi}{6} & \text{if } \ \alpha \in (0,\alpha^*], \\ \dfrac{\coth(\alpha\pi)}{4\,\alpha} & \text{if } \ \alpha \in (\alpha^*,\infty). \end{cases}$$

**2.13. Lemma.** *Let $k = \alpha^2$, $\alpha \in (0,\infty)\setminus\mathbb{N}$, $b \in \mathbb{L}[0,2\,\pi]$, $\overline{b} = 0$ and let $\sigma$ be a solution of* (2.23). *Then*

$$\|\sigma\|_\infty \le \Psi(\alpha)\,\|b\|_1, \tag{2.32}$$

*where*

$$\Psi(\alpha) = \begin{cases} \min\{\dfrac{\pi}{6(1-\alpha^2)}, \dfrac{1}{4\,\alpha\,\sin(\alpha\pi)}\} & \text{if } \ \alpha \in (0,\frac{1}{2}], \\ \min\{\dfrac{\pi}{6(1-\alpha^2)}, \dfrac{1}{2\,\alpha\,\sin(\alpha\pi)}\} & \text{if } \ \alpha \in (\frac{1}{2},1), \\ \dfrac{1}{2\,\alpha\,|\sin(\alpha\pi)|} & \text{if } \ \alpha \in (1,\infty)\setminus\mathbb{N}. \end{cases} \tag{2.33}$$

*If $b \in \mathbb{L}_2[0,2\,\pi]$ and $\alpha \in (0,1)$, then $\sigma$ fulfils moreover*

$$\|\sigma\|_\infty \le \sqrt{\frac{\pi}{6}}\,\frac{\|b\|_2}{1-\alpha^2}. \tag{2.34}$$

*Proof.* We have

$$-\sigma''(t) = \alpha^2\,\sigma(t) - b(t) \quad \text{a.e. on } \ [0,2\,\pi]. \tag{2.35}$$

According to Lemma 2.10, $\sigma$ satisfies both the Sobolev inequality (2.28) and the Wirtinger inequality (2.31). Thus, proceeding similarly as in the proof of Lemma 2.11, we get

$$\|\sigma'\|_2^2 \le \alpha^2\,\|\sigma'\|_2^2 + \sqrt{\frac{\pi}{6}}\,\|b\|_1\,\|\sigma'\|_2 \quad \text{i.e.} \quad \|\sigma'\|_2\,(1-\alpha^2) \le \sqrt{\frac{\pi}{6}}\,\|b\|_1.$$

If $\alpha \in (0,1)$, then using (2.28) once more, the relation

$$\|\sigma\|_\infty \le \frac{\pi}{6\,(1-\alpha^2)}\,\|b\|_1$$

follows.

Further, $\sigma$ has the form (2.30), where $g$ is given by (2.19) and satisfies (2.22) if $\alpha \in (0, \frac{1}{2}]$ and (2.20) for $\alpha \in (\frac{1}{2}, \infty) \setminus \mathbb{N}$. Therefore Lemma 2.9 implies that

$$\|\sigma\|_\infty \le \frac{\|b\|_1}{4\,\alpha\,\sin(\alpha\pi)} \quad \text{if } \alpha \in (0, \frac{1}{2}]$$

and

$$\|\sigma\|_\infty \le \frac{\|b\|_1}{2\,\alpha\,|\sin(\alpha\pi)|} \quad \text{if } \alpha \in (\frac{1}{2}, \infty) \setminus \mathbb{N}.$$

Finally, let $b \in \mathbb{L}_2[0, 2\,\pi]$ and $\alpha \in (0, 1)$. Then we can argue as in the proof of Lemma 2.11 and derive from (2.35) the inequalities

$$\|\sigma'\|_2^2 \le \alpha^2\,\|\sigma'\|_2^2 + \|b\|_2\|\sigma\|_2$$

and

$$\|\sigma'\|_2\,(1-\alpha^2) \le \|b\|_2,$$

wherefrom (2.34) follows. □

**2.14. Remark.** For $\alpha \in (0, 1)$, denote

$$\psi_1(\alpha) = \frac{\pi}{6(1-\alpha^2)} \quad \text{and} \quad \psi_2(\alpha) = \frac{1}{4\,\alpha\,\sin(\alpha\pi)}.$$

It can be verified that there is exactly one $\alpha_1^* \in (0, 1)$ ($\alpha_1^* \approx 0.412036$) such that $\psi_1(\alpha_1^*) = \psi(\alpha_1^*)$, $\psi_1(\alpha) > \psi_2(\alpha)$ for $\alpha \in (0, \alpha_1^*)$ and $\psi_1(\alpha) < \psi_2(\alpha)$ for $\alpha \in (\alpha_1^*, 1)$. Similarly, there is exactly one $\alpha_2^* \in (0, 1)$ ($\alpha_2^* \approx 0.628308$) such that $\psi_1(\alpha_2^*) = 2\,\psi(\alpha_2^*)$, $\psi_1(\alpha) < 2\,\psi_2(\alpha)$ for $\alpha \in (\frac{1}{2}, \alpha_1^*)$ and $\psi_1(\alpha) > 2\,\psi_2(\alpha)$ for $\alpha \in (\alpha_2^*, 1)$. This means that the function $\Psi$ from (2.33) can be described by

$$\Psi(\alpha) = \begin{cases} \dfrac{\pi}{6(1-\alpha^2)}, & \text{if } \alpha \in (0, \alpha_1^*], \\ \dfrac{1}{4\,\alpha\,\sin(\alpha\pi)} & \text{if } \alpha \in (\alpha_1^*, \frac{1}{2}], \\ \dfrac{\pi}{6(1-\alpha^2)}, & \text{if } \alpha \in (\frac{1}{2}, \alpha_2^*], \\ \dfrac{1}{2\,\alpha\,\sin(\alpha\pi)} & \text{if } \alpha \in (\alpha_2^*, \infty) \setminus \mathbb{N} \end{cases}$$

(cf. Fig. 1).

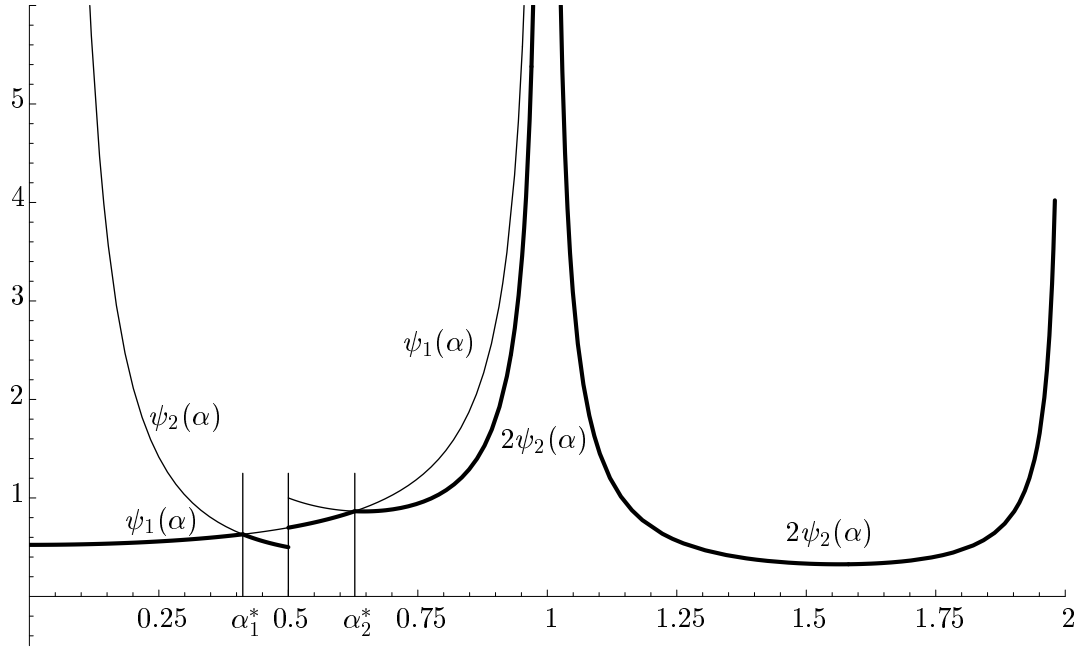


Figure 1:

# 3 . Lower and upper functions

Consider the problem (1.1) and suppose that $k > 0$ and that there exist $a \in \mathbb{R}$, $A_1 \in (0, \infty)$, $b \in \mathbb{L}_\infty[0, 2\pi]$ such that

$$f(t, x) \le a + b(t) \tag{3.1}$$

holds for a.e. $t \in [0, 2\pi]$ and all $x \in (A_1, \infty)$. Then we can find $r_1 \in (A_1, \infty)$ such that

$$k\, r_1 \ge a + \|b\|_\infty \ge f(t, r_1) \ \text{ for a.e. } t \in [0, 2\pi],$$

which means that if we put

$$\sigma_1(t) = r_1, \ \ \rho_1(t) = \sigma_1'(t) = 0 \ \text{ on } [0, 2\pi],$$

the functions $(\sigma_1, \rho_1)$ are lower functions for (1.1). Similarly, if

$$f(t, x) \ge a + b(t) \tag{3.2}$$

holds for a.e. $t \in [0, 2\pi]$ and all $x \in (-\infty, -A_2)$, we can find $r_2 \in (-\infty, -A_2)$ such that

$$k\, r_2 \le a - \|b\|_\infty \le f(t, r_2) \ \text{ for a.e. } t \in [0, 2\pi],$$

which means that the functions $(\sigma_2(t), \rho_2(t)) = (r_2, 0)$ on $[0, 2\pi]$ are upper functions of (1.1). We see that the constant function $\sigma_1$ has to be quite large and positive and the constant function $\sigma_2$ has to be sufficiently small negative. Similar observations can be done for $k < 0$.

In the case that $b$ is not essentially bounded or if we need other localization of $\sigma_1$ or $\sigma_2$, this approach falls. However we can construct and localize lower and upper functions by means of the results from Section 2.

If $k < 0$, we can write (1.1) in the form

$$u'' - \alpha^2 u = f(t, u), \quad u(0) = u(2\pi),\ u'(0) = u'(2\pi). \tag{3.3}$$

**3.1. Theorem.** *Let $\alpha \in (0, \infty)$ and let $a \in \mathbb{R}$, $\tau \in [0, 2\pi)$, $\delta \in [0, \infty)$, $b \in \mathbb{L}[0, 2\pi]$ be such that $\overline{b} = 0$ and (3.1) holds for a.e. $t \in [0, 2\pi]$ and all $x \in [A(t), B(t)]$, where*

$$A(t) = g(|t-\tau|)\delta - \frac{a}{\alpha^2} - \Phi(\alpha)\|b\|_1 \ \text{ and } \ B(t) = A(t) + 2\,\Phi(\alpha)\|b\|_1 \ \text{ on } [0, 2\pi] \tag{3.4}$$

*and $g$ and $\Phi$ are given by (2.16) and (2.25), respectively.*

*Then there exist lower functions $(\sigma, \rho)$ of (3.3) such that*

$$A(t) \le \sigma(t) \le B(t) \quad \text{on } [0, 2\pi]. \tag{3.5}$$

*Proof.* Let us put

$$\beta(t) = a + b(t) \quad \text{for a.e. } t \in [0, 2\pi].$$

By Corollary 2.5, there is a unique solution $(\sigma, \rho)$ of (2.14), where $k = -\alpha^2$, and, in the view of (2.18), we have

$$\sigma(t) = g(|t-\tau|)\,\delta - \frac{a}{\alpha^2} + \int_0^{2\pi} g(|t-s|)\,b(s)\,\mathrm{d}s \quad \text{on } [0, 2\pi], \tag{3.6}$$

where $g$ is given by (2.16). Denote

$$\sigma_0(t) = \int_0^{2\pi} g(|t-s|)\,b(s)\,\mathrm{d}s \quad \text{for } t \in [0, 2\pi]. \tag{3.7}$$

Then $\sigma_0$ is a solution to (2.23) and by Lemma 2.11 the estimate

$$\|\sigma_0\|_\infty \le \Phi(\alpha)\,\|b\|_1 \tag{3.8}$$

is true. Substituting (3.7) into (3.6) and using (3.4) and (3.8), we get (3.5). This together with (3.1) and (2.14) means that

$$\rho'(t) - \alpha^2\,\sigma(t) = \beta(t) = a + b(t) \ge f(t, \sigma(t))$$

is true a.e. on $[0, 2\pi]$, i.e. $(\sigma, \rho)$ are lower functions of (3.3). □

In the case $k > 0$ we will write the problem (1.1) in the form

$$u'' + \alpha^2\, u = f(t,u), \quad u(0) = u(2\pi),\ u'(0) = u'(2\pi). \tag{3.9}$$

**3.2. Theorem.** *Let* $\alpha \in (0,\infty) \setminus \mathbb{N}$ *and let* $a \in \mathbb{R}$, $\tau \in [0,2\pi)$, $\delta \in [0,\infty)$, $b \in \mathbb{L}[0,2\,\pi]$ *be such that* $\overline{b} = 0$ *and* (3.1) *holds for a.e.* $t \in [0,2\,\pi]$ *and all* $x \in [A(t),B(t)]$, *where*

$$A(t) = g(|t-\tau|)\delta + \frac{a}{\alpha^2} - \Psi(\alpha)\|b\|_1 \ \text{ and } \ B(t) = A(t) + 2\Psi(\alpha)\|b\|_1 \tag{3.10}$$

*and* $g$ *and* $\Psi$ *are given by* (2.19) *and* (2.33), *respectively.*

*Then there exist lower functions* $(\sigma,\rho)$ *of* (3.9) *fulfilling* (3.5).

*Proof.* By Corollary 2.7, the problem (2.14) with $k = \alpha^2$ and $\beta(t) = a + b(t)$ a.e. on $[0,2\,\pi]$ has a unique solution $(\sigma,\rho)$ and, with respect to (2.15), (2.21),

$$\sigma(t) = g(|t-\tau|)\,\delta + \frac{a}{\alpha^2} + \sigma_0(t) \ \text{ on } [0,2\,\pi], \tag{3.11}$$

where $\sigma_0$ and $g$ are defined by (3.7) and (2.19), respectively. Thus, $\sigma_0$ is a solution of (2.23). By Lemma 2.13 we have

$$\|\sigma_0\|_\infty \le \Psi(\alpha)\,\|b\|_1. \tag{3.12}$$

Furthermore, since (3.10)-(3.12) yield (3.5), according to (3.1) and (2.14) we have

$$\rho'(t) + \alpha^2\,\sigma(t) = a + b(t) \ge f(t,\sigma(t)) \text{ a.e. on } [0,2\,\pi],$$

i.e. $(\sigma,\rho)$ are lower functions of (3.9). □

The next two assertions are respectively dual to Theorems 3.1 and 3.2 and their proofs can be omitted.

**3.3. Theorem.** *Let* $\alpha \in (0,\infty)$ *and let* $a \in \mathbb{R}$, $\tau \in [0,2\pi)$, $\delta \in (-\infty,0]$, $b \in \mathbb{L}[0,2\,\pi]$ *be such that* $\overline{b} = 0$ *and* (3.2) *holds for a.e.* $t \in [0,2\,\pi]$ *and all* $x \in [A(t),B(t)]$, *where* $A(t)$, $B(t)$ *are defined by* (3.4), (2.16) *and* (2.25).

*Then there exist upper functions* $(\sigma,\rho)$ *of* (3.3) *fulfilling* (3.5).

**3.4. Theorem.** *Let* $\alpha \in (0,\infty) \setminus \mathbb{N}$ *and let* $a \in \mathbb{R}$, $\tau \in [0,2\pi)$, $\delta \in (-\infty,0]$, $b \in \mathbb{L}[0,2\,\pi]$ *be such that* $\overline{b} = 0$ *and* (3.2) *holds for a.e.* $t \in [0,2\,\pi]$ *and all* $x \in [A(t),B(t)]$, *where* $A(t)$, $B(t)$ *are defined by* (3.10), (2.19) *and* (2.33).

*Then there exist upper functions* $(\sigma,\rho)$ *of* (3.9) *fulfilling* (3.5).

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