Resonance and Multiplicity in Periodic Boundary Value Problems with Singularity

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Summary. The paper deals with the boundary value problem

$$u'' + k\,u = g(u) + e(t), \quad u(0) = u(2\pi),\ u'(0) = u'(2\pi),$$

where $k \in \mathbb{R}$, $g : (0,\infty) \mapsto \mathbb{R}$ is continuous, $e \in \mathbb{L}[0,2\pi]$ and $\lim\limits_{x\to 0+}\int_x^1 g(s)\,\mathrm{d}s = \infty$. In particular, the existence and multiplicity results are obtained using the method of lower and upper functions which are constructed as solutions of related auxiliary linear problems.

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1 . Introduction

In this paper we consider the periodic boundary value problems of the form

$$u'' + k\,u = g(u) + e(t),\ u(0) = u(2\pi),\ u'(0) = u'(2\pi), \tag{1.1}$$

where

$$g \in \mathbb{C}(0,\infty),\ \ e \in \mathbb{L}[0,2\pi],\ \ k \in \mathbb{R} \tag{1.2}$$

and g has the strong singularity at 0, i.e.

$$\lim_{x\to 0+}\int_x^1 g(s)\,\mathrm{d}s = \infty. \tag{1.3}$$

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This problem was studied by many authors, starting from Lazer and Solimini [6], where (1.1) with $k = 0$ and g positive was considered. Later, this work has been generalized or extended e.g. by del Pino, Manásevich and Montero [1], Fonda [2], Fonda, Manásevich and Zanolin[3], Mawhin [7], Ge and Mawhin [4], Omari and Ye [9], Rachůnková [10], Rachůnková and Tvrdý [12], Rachůnková, Tvrdý and Vrkoč [14], Yan and Zhang [16], Zhang [17] and others. Particularly, the problems having nonlinearities with the asymptotic behaviour at $+\infty$ which corresponds to $k > 0$ and g bounded below in (1.1) were solved by some of the above authors. For example, the papers [2], [9], [10],[12], [14] and [17] dealt with the problems characterized by a positive k which was less than the first positive Dirichlet eigenvalue μ_1 of $x'' + \mu\, x = 0$, while the cases corresponding to k lying between two neighbour higher eigenvalues were investigated in [1] or [16]. The existence results in the resonance case $k = \mu_1$ were reached in [14]. In [2] or [3] we can also find multiplicity results for subharmonics. Theorems about more solutions of (1.1) provided $k = 0$ are proved in [10].

Here we bring new results about the existence of one or two positive solutions of (1.1) for both nonresonance and resonance values of k. Moreover, our Theorem 4.4 generalizes for $n = 3$ Theorem 3.5 in [10] (even in the case $k = 0$), because the condition (3.4) in Theorem 4.4 is weaker than the corresponding condition (3.16) in [10, Theorem 3.5] which requires (for $i = 1$) $g(x) + \overline{e} > 0$ on $[a_1, b_1] \subset (0, \infty)$, where $b_1 - a_1 = \frac{\pi}{3}\, \|e - \overline{e}\|_1$.

Let J be a (possibly unbounded) subinterval of $\mathbb{R}$. We say that $f : [0, 2\pi] \times J \mapsto \mathbb{R}$ fulfils the Carathéodory conditions on $[0, 2\pi] \times \mathbb{R}$, if f has the following properties: (i) for each $x \in J$ the function $f(., x)$ is measurable on $[0, 2\pi]$; (ii) for almost every $t \in [0, 2\pi]$ the function $f(t, .)$ is continuous on J; (iii) for each compact set $\mathrm{K} \subset J$ the function $m_{\mathrm{K}}(t) = \sup_{x \in \mathrm{K}} |f(t, x)|$ is Lebesgue integrable on $[0, 2\pi]$. The set of functions satisfying the Carathéodory conditions on $[0, 2\pi] \times J$ is denoted by $\mathrm{Car}([0, 2\pi] \times J)$.

For a given subinterval J of $\mathbb{R}$ (possibly unbounded) $\mathbb{C}(J)$ denotes the set of functions continuous on J, $\mathbb{L}[0, 2\pi]$ stands for the set of functions (Lebesgue) integrable on $[0, 2\pi]$, $\mathbb{L}_2[0, 2\pi]$ is the set of functions square integrable on $[0, 2\pi]$, $\mathbb{L}_\infty[0, 2\pi]$ is the set of functions essentially bounded on $[0, 2\pi]$, $\mathbb{AC}[0, 2\pi]$ denotes the set of functions absolutely continuous on $[0, 2\pi]$, $\mathbb{AC}^1[0, 2\pi]$ is the set of functions $u \in \mathbb{AC}[0, 2\pi]$ with first derivative absolutely continuous on $[0, 2\pi]$ and $\mathbb{BV}[0, 2\pi]$ denotes the set of functions of bounded variation on $[0, 2\pi]$. For $x \in \mathbb{L}_\infty[0, 2\pi]$, $y \in \mathbb{L}[0, 2\pi]$ and $z \in \mathbb{L}_2[0, 2\pi]$, we denote $\|x\|_\infty = \sup\,\mathrm{ess}_{t \in [0, 2\pi]} |x(t)|$,

$$\overline{y} = \frac{1}{2\pi} \int_0^{2\pi} y(s)\, \mathrm{d}s, \quad \|y\|_1 = \int_0^{2\pi} |y(t)|\, \mathrm{d}t \quad \text{and} \quad \|z\|_2 = \Big(\int_0^{2\pi} z^2(t)\, \mathrm{d}t \Big)^{\frac{1}{2}}.$$

Furthermore, $\mathbb{C}^1[0, 2\pi]$ is the space of functions from $\mathbb{C}[0, 2\pi]$ having a continuous first derivative on $[0, 2\pi]$ equipped with the norm $x \in \mathbb{C}^1[0, 2\pi] \mapsto \|x\|_\infty + \|x'\|_\infty$.

If $x \in \mathbb{BV}[0,2\pi]$, $s \in (0,2\pi]$ and $t \in [0,2\pi)$, then the symbols $x(s-)$, $x(t+)$ and $\Delta^+ x(t)$ are respectively defined by

$$x(s-) = \lim_{\tau \to s-} x(\tau), \quad x(t+) = \lim_{\tau \to t+} x(\tau) \text{ and } \Delta^+ x(t) = x(t+) - x(t).$$

Furthermore, x^{ac} and x^{sing} stand for the absolutely continuous part of x and the singular part of x, respectively. We suppose $x^{\text{sing}}(0) = 0$. For $x \in \mathbb{L}[0,2\pi]$, the symbols x^+ and x^- denote its nonnegative and nonpositive parts.

Besides (1.1) we will also consider more general problem

$$x'' = f(t,x), \; x(0) = x(2\pi), \; x'(0) = x'(2\pi), \tag{1.4}$$

where $f \in \text{Car}([0,2\pi] \times J)$ and $J \subset \mathbb{R}$.

1.1. Definition. By a *solution of* (1.4) we understand a function $x\colon [0,2\pi] \mapsto \mathbb{R}$ such that $x' \in \mathbb{AC}[0,2\pi]$, $x(0) = x(2\pi)$, $x'(0) = x'(2\pi)$ and $x(t) \in J$ and $x''(t) = f(t,x(t))$ hold for a.e. $t \in [0,2\pi]$.

1.2. Definition. The functions $(\sigma,\rho) \in \mathbb{AC}[0,2\pi] \times \mathbb{BV}[0,2\pi]$ are *lower functions* of (1.4) if $\sigma(t) \in J$ for a.e. $t \in [0,2\pi]$, the singular part ρ^{sing} of ρ is nondecreasing on $[0,2\pi]$, $\sigma'(t) = \rho(t)$ and $\rho'(t) \geq f(t,\sigma(t))$ for a.e. $t \in [0,2\pi]$, $\sigma(0) = \sigma(2\pi)$ and $\rho(0+) \geq \rho(2\pi-)$.

Similarly, the functions $(\sigma,\rho) \in \mathbb{AC}[0,2\pi] \times \mathbb{BV}[0,2\pi]$ are *upper functions of* (1.4) if $\sigma(t) \in J$ for a.e. $t \in [0,2\pi]$, ρ^{sing} is nonincreasing on $[0,2\pi]$, $\sigma'(t) = \rho(t)$ and $\rho'(t) \leq f(t,\sigma(t))$ for a.e. $t \in [0,2\pi]$, $\sigma(0) = \sigma(2\pi)$ and $\rho(0+) \leq \rho(2\pi-)$.

1.3. Remark. If $J = \mathbb{R}$, then Definitions 1.1 and 1.2 reduce to those given for a regular case in [11].

If (1.2) is true and $J = (0,\infty)$, then each solution and each upper (lower) functions must be positive a.e. on $[0,2\pi]$.

Our proofs will be based on the following theorem which is contained in [11, Theorems 4.1, 4.2 and 4.3] and which concerns the nonsingular case with $J = \mathbb{R}$.

1.4. Theorem. *Let* (σ_1,ρ_1) *and* (σ_2,ρ_2) *be respectively lower and upper functions of the problem* (1.4), *where* $J = \mathbb{R}$. *Furthermore, assume that there is* $m \in \mathbb{L}[0,2\pi]$ *such that* $f(t,x) \geq m(t)$ *for a.e.* $t \in [0,2\pi]$ *and all* $x \in \mathbb{R}$ (*or* $f(t,x) \leq m(t)$ *for a.e.* $t \in [0,2\pi]$ *and all* $x \in \mathbb{R}$). *Then* (1.4) *has a solution* x *such that* $\|x'\|_\infty \leq \|m\|_1$. *Moreover, if*

$$\sigma_1(t) \leq \sigma_2(t) \;\; \text{for all} \;\; t \in [0,2\pi], \tag{1.5}$$

then $\sigma_1(t) \leq x(t) \leq \sigma_2(t)$ *is true for all* $t \in [0,2\pi]$ *and if* (1.5) *does not hold, then there is* $t_x \in [0,2\pi]$ *such that* $\sigma_2(t_x) \leq x(t_x) \leq \sigma_1(t_x)$.

In [12] and [13] we have presented conditions ensuring the existence and localization of lower and upper functions of (1.4). As an immediate consequence of these results we get propositions for the following special case of the problem (1.4):

$$x'' = h(x) + e(t),\ x(0) = x(2\pi),\ x'(0) = x'(2\pi), \tag{1.6}$$

where

$$h \in \mathbb{C}(J), \quad e \in \mathbb{L}[0, 2\pi] \quad \text{and} \quad J \text{ is an open subinterval in } \mathbb{R}. \tag{1.7}$$

1.5. Proposition. *Suppose that* (1.7) *holds. Further, let $A \in J$ and let the inequality*

$$h(x) + \overline{e} \le 0 \ \textit{ for } x \in [A, B] \quad (\, h(x) + \overline{e} \ge 0 \ \textit{ for } x \in [A, B]\,) \tag{1.8}$$

be fulfilled, where $B \in J$ and

$$B - A \ge \frac{\pi}{3}\, \|e - \overline{e}\|_1. \tag{1.9}$$

Then there exist lower (*upper*) *functions* (σ, ρ) *of* (1.6) *and*

$$\sigma(t) \in [A, B] \ \textit{ for all } t \in [0, 2\pi]. \tag{1.10}$$

Proof. If $J = \mathbb{R}$, then the proof is given in [12, Propositions 2.4 and 2.5]. In the case that $J \ne \mathbb{R}$, we put

$$f(t, x) = e(t) + \begin{cases} h(A) & \text{if } x < A, \\ h(x) & \text{if } x \in [A, B], \\ h(B) & \text{if } x > B. \end{cases} \tag{1.11}$$

Since $f \in \mathrm{Car}([0, 2\pi] \times \mathbb{R})$, we get by [12, Propositions 2.4 and 2.5] the existence of lower (upper) functions (σ, ρ) of (1.4) fulfilling (1.10). The conditions (1.10) and (1.11) guarantee that (σ, ρ) are lower (upper) functions of (1.6), as well. □

1.6. Proposition. *Suppose that* (1.7) *holds. Further, let $A \in J$, $k \ne n^2$ for all $n \in \mathbb{N}$ and let the inequality*

$$h(x) + k\,x + \overline{e} \le k\,\frac{A + B}{2} \ \textit{ for } x \in [A, B] \tag{1.12}$$

$$(h(x) + k\,x + \overline{e} \ge k\,\frac{A + B}{2} \ \textit{ for } x \in [A, B]),$$

be fulfilled, where $B \in J$ and

$$B - A = 2\,\Phi(k)\, \|e - \overline{e}\|_1 \tag{1.13}$$

and

$$
(1.14) \qquad \Phi(k) = \begin{cases} \min\{\dfrac{\pi}{6}, \dfrac{\coth(\sqrt{|k|}\,\pi)}{4\sqrt{|k|}}\} & \text{if } k < 0, \\ \min\{\dfrac{\pi}{6\,(1-k)}, \dfrac{1}{4k}\sin(\sqrt{k}\pi)\} & \text{if } k \in (0, \frac{1}{4}], \\ \min\{\dfrac{\pi}{6\,(1-k)}, \dfrac{1}{2\sqrt{k}\sin(\sqrt{k}\pi)}\} & \text{if } k \in (\frac{1}{4}, 1), \\ \dfrac{1}{2\sqrt{k}|\sin(\sqrt{k}\pi)|} & \text{if } k > 1, k \neq n^2, n \in \mathbb{N}. \end{cases}
$$

Then there exist lower (upper) functions (σ, ρ) of (1.6) satisfying (1.10).

Proof. If $J = \mathbb{R}$, then the proof follows from [13, Theorems 3.1-3.4], where we put $a = k\,\dfrac{A+B}{2}$. In the case $J \neq \mathbb{R}$, we can use the same arguments as in the proof of Proposition 1.5. □

The next lemma will be helpful in what follows.

1.7. Lemma. *Let (1.7) be true and let x be an arbitrary solution of (1.6). Further, let $t_1 \in [0, 2\pi]$ be such that $x(t_1) = \max_{t\in[0,2\pi]} x(t)$. Then the inequality*

$$
\int_{x(t_0)}^{A} h(s)\,\mathrm{d}s \le \|e\|_1 \|x'\|_\infty + \int_A^{x(t_1)} |h(s)|\,\mathrm{d}s
$$

holds for all $t_0 \in [0, 2\pi]$ and all $A \in J$ such that $x(t_1) \ge A$.

Proof. In virtue of the periodicity of x, we have $x'(t_1) = 0$. Consequently, multiplying the equality $x''(t) = h(x(t)) + e(t)$ a.e. on $[0, 2\pi]$ by $x'(t)$ and integrating from t_0 to t_1, we get

$$
0 \ge -\frac{(x'(t_0))^2}{2} = \int_{t_0}^{t_1} x''(t)\,x'(t)\,\mathrm{d}t = \int_{x(t_0)}^{x(t_1)} h(s)\,\mathrm{d}s + \int_{t_0}^{t_1} e(t)\,x'(t)\mathrm{d}t.
$$

Hence,

$$
\begin{aligned}
\int_{x(t_0)}^{A} h(s)\,\mathrm{d}s &= \int_{x(t_0)}^{x(t_1)} h(s)\,\mathrm{d}s - \int_A^{x(t_1)} h(s)\,\mathrm{d}s \\
&\le -\int_{t_0}^{t_1} e(t)\,x'(t)\,\mathrm{d}t - \int_A^{x(t_1)} h(s)\,\mathrm{d}s \le \|e\|_1\,\|x'\|_\infty + \int_A^{x(t_1)} |h(s)|\,\mathrm{d}s.
\end{aligned}
$$

□

The proof of the next lemma is an easy modification of that of [11, Lemma 1.1].

1.8. Lemma. *Suppose that (1.7) is true. Furthermore, let $I \subset J$ and let $h_* \in \mathbb{R}$ be such that $h(x) \ge h_*$ for all $x \in I$. Then*

$$
\|x'\|_\infty \le \|e\|_1 + 2\pi\,|h_*|
$$

holds for each solution x of (1.6) with the property $x(t) \in I$ for all $t \in [0, 2\pi]$. □

2 . Existence theorem

The main result of this section is Theorem 2.1 which gives an existence principle for the problem (1.1) in terms of lower and upper functions. More effective results can be obtained if we replace the assumption of the existence of lower and upper functions with the proper conditions from Propositions 1.5 and 1.6.

2.1. Theorem. *Suppose that* (1.2), (1.3),

$$\liminf_{x\to\infty} \frac{g(x)}{x} > k - \frac{1}{4} \tag{2.1}$$

and

$$\liminf_{x\to 0+} g(x) > -\infty \tag{2.2}$$

are satisfied. Further, let there are lower functions (σ_1, ρ_1) *and upper functions* (σ_2, ρ_2) *of* (1.1) *such that*

$$\sigma_2(t) > 0 \quad on \ [0, 2\pi]. \tag{2.3}$$

Then the problem (1.1) *has a positive solution.*

Before proving Theorem 2.1 we will prove several auxiliary assertions. In particular, Lemmas 2.2 and 2.4 give a priori estimates for solutions of (1.6). The proof of Theorem 2.1 follows from Proposition 2.9.

2.2. Lemma. *Let* $g \in \mathbb{C}(0,\infty)$, $k \in \mathbb{R}$ *and suppose that* (1.3) *is true. Further, let* $E, K \in [0,\infty)$, $A \in (0,\infty)$ *and* $R \in [A,\infty)$. *Then there is* $\varepsilon^* > 0$ *such that* $\min_{t\in[0,2\pi]} x(t) > \varepsilon^*$ *holds for each* $h \in \mathbb{C}(\mathbb{R})$ *satisfying*

$$h(x) = g(x) - k\,x \quad on \ [\varepsilon^*, R], \tag{2.4}$$

for each $e \in \mathbb{L}[0,2\pi]$ *with* $\|e\|_1 \le E$ *and for each solution* x *of* (1.6) *fulfilling*

$$\|x'\|_\infty \le K \quad and \quad \max_{t\in[0,2\pi]} x(t) \in [A, R]. \tag{2.5}$$

Proof. Put

$$K^* = E\,K + \int_A^R |g(s) - k\,s|\,\mathrm{d}s.$$

In view of (1.3), there is $\varepsilon^* > 0$ such that

$$\int_{\varepsilon^*}^A (g(s) - k\,s)\,\mathrm{d}s > K^*. \tag{2.6}$$

Let $h \in \mathbb{C}(\mathbb{R})$ fulfil (2.4), let $e \in \mathbb{L}[0,2\pi]$ be such that $\|e\|_1 \le E$ and let x be a solution of (1.6) verifying (2.5). Suppose that $\min_{t\in[0,2\pi]} x(t) \le \varepsilon^*$ and denote by t_0 and t_1 the points in $[0,2\pi]$ such that $t_0 < t_1$, $x(t_0) = \varepsilon^*$, $x(t) \ge \varepsilon^*$ on $[t_0,t_1]$ and $x(t_1) = \max_{t\in[0,2\pi]} x(t)$. With respect to (2.5) we have $x(t_1) \in [A,R]$. Thus, taking into account (2.4) and (2.6) and using Lemma 1.7, we get

$$K^* < \int_{\varepsilon^*}^{A} (g(s) - k\,s)\,\mathrm{d}s = \int_{x(t_0)}^{A} h(s)\,\mathrm{d}s$$
$$\le E\,K + \int_A^R |h(s)|\,\mathrm{d}s = E\,K + \int_A^R |g(s) - k\,s|\,\mathrm{d}s = K^*,$$

a contradiction. □

In particular, we have:

2.3. Corollary. *Suppose* (1.2) *and* (1.3). *Then each solution of* (1.1) *is positive on* $[0,2\pi]$.

2.4. Lemma. *Let* $E,\ C \in [0,\infty)$, $\eta \in (0,\frac{1}{4})$ *and* $(0,\infty) \subset J \subset \mathbb{R}$. *Then for any* $B \in (0,\infty)$ *there is* $R \in (B,\infty)$ *such that the estimate*

$$\max_{t\in[0,2\pi]} x(t) \le R \tag{2.7}$$

is valid for each $h \in \mathbb{C}(J)$ *satisfying*

$$h(x)\,x + (\frac{1}{4} - \eta)\,x^2 \ge -C\,|x| \quad \textit{for all } x \in J,$$

for each $e \in \mathbb{L}[0,2\pi]$ *with* $\|e\|_1 \le E$ *and each solution* x *of* (1.6) *fulfilling*

$$x(t) \in J \ \ \textit{for all } t \in [0,2\pi] \quad \textit{and} \quad \min_{t\in[0,2\pi]} x(t) \le B. \tag{2.8}$$

Proof. Suppose that such R does not exist. Then for any $\ell \in \mathbb{N}$ we can find $h_\ell \in \mathbb{C}(J)$, $e_\ell \in \mathbb{L}[0,2\pi]$ and a solution x_ℓ of

$$x'' = h_\ell(x) + e_\ell(t),\ x(0) = x(2\pi),\ x'(0) = x'(2\pi)$$

such that $\|e\|_1 \le E$,

$$h_\ell(x)\,x + (\frac{1}{4} - \eta)\,x^2 \ge -C\,|x| \quad \text{for all } x \in J \tag{2.9}$$

and

$$\min_{t\in[0,2\pi]} x_\ell(t) \le B \quad \text{and} \quad \max_{t\in[0,2\pi]} x_\ell(t) > \ell. \tag{2.10}$$

For $\ell \in \mathbb{N}$, denote by t_ℓ the point in $[0, 2\pi]$ for which $x_\ell(t_\ell) = B$. Furthermore, let us extend x_ℓ and e_ℓ to functions 2π-periodic on $\mathbb{R}$. We have

$$x_\ell''(t) = h_\ell(x_\ell(t)) + e_\ell(t) \quad \text{for a.e. } t \in \mathbb{R}.$$

Multiplying this equality by $x_\ell(t)$, integrating from t_ℓ to $t_\ell + 2\pi$ and making use of (2.9), we obtain

$$\begin{aligned}\|x_\ell'\|_2^2 &= -\int_{t_\ell}^{t_\ell+2\pi} h_\ell(x_\ell(t))\, x_\ell(t)\,\mathrm{d}t - \int_{t_\ell}^{t_\ell+2\pi} e_\ell(t)\, x_\ell(t)\,\mathrm{d}t \\ &\le (\frac{1}{4} - \eta)\, \|x_\ell\|_2^2 + C\, \|x_\ell\|_1 + E\, \|x_\ell\|_\infty.\end{aligned}$$

On the other hand, in virtue of (2.10) we have

$$\|x_\ell\|_\infty \le |x_\ell(t_\ell)| + \int_{t_\ell}^{t_\ell+2\pi} |x_\ell'(t)|\,\mathrm{d}t \le B + \sqrt{2\pi}\, \|x_\ell'\|_2. \tag{2.11}$$

Thus,

$$\big(\|x_\ell'\|_2 - E\sqrt{\frac{\pi}{2}}\big)^2 \le (\frac{1}{4} - \eta)\, \|x_\ell\|_2^2 + \sqrt{2\pi}\, C\, \|x_\ell\|_2 + E\, B + \frac{\pi}{2}\, E^2. \tag{2.12}$$

Inserting $x_\ell(t) = v_\ell(t) + B$ on $\mathbb{R}$ into (2.12), we obtain

$$\frac{(\|v_\ell'\|_2 - c)^2}{\|v_\ell\|_2^2} \le \frac{1}{4} - \eta + \frac{a}{\|v_\ell\|_2} + \frac{b}{\|v_\ell\|_2^2}, \tag{2.13}$$

where $a, b, c \in \mathbb{R}$ do not depend on ℓ. Now, (2.10), (2.11) and (2.12) yield

$$\lim_{\ell\to\infty} \|v_\ell'\|_2 = \infty \quad \text{and} \quad \lim_{\ell\to\infty} \|v_\ell\|_2 = \infty. \tag{2.14}$$

Since $v_\ell(t_\ell) = v_\ell(t_\ell + 2\pi) = 0$, by Scheeffer's inequality [15, p. 207] (see also [8, II.2]) we have $\|v_\ell\|_2^2 \le 4\, \|v_\ell'\|_2^2$ and

$$\frac{(\|v_\ell'\|_2 - c)^2}{\|v_\ell\|_2^2} \ge \frac{(\|v_\ell'\|_2 - c)^2}{4\|v_\ell'\|_2^2}.$$

Finally, by virtue of (2.13) and (2.14), we have

$$\frac{1}{4} = \lim_{\ell\to\infty} \frac{(\|v_\ell'\|_2 - c)^2}{4\|v_\ell'\|_2^2} \le \lim_{\ell\to\infty} \Big(\frac{1}{4} - \eta + \frac{a}{\|v_\ell\|_2} + \frac{b}{\|v_\ell\|_2^2}\Big) = \frac{1}{4} - \eta,$$

a contradiction. □

Let $g \in \mathbb{C}(0,\infty)$ fulfil (1.3), (2.1) and (2.2). Denote

$$g_0(x) = g(x) - k\,x \quad \text{for } x \in (0,\infty). \tag{2.15}$$

Then we have $g_0 \in \mathbb{C}(0,\infty)$,

$$\lim_{x\to 0+} \int_x^1 g_0(s)\,\mathrm{d}s = \infty \tag{2.16}$$

and

$$\liminf_{x\to\infty} \frac{g_0(x)}{x} > -\frac{1}{4}. \tag{2.17}$$

Furthermore, (2.2) and (2.16) imply

$$\inf_{x\in(0,R]} g_0(x) \in \mathbb{R} \quad \text{for each} \quad R \in (0,\infty) \tag{2.18}$$

and

$$\limsup_{x\to 0+} g_0(x) = \infty. \tag{2.19}$$

Moreover, in view of (2.17) and (2.18), there exist $\eta \in (0,\frac{1}{4})$ and $C \in [0,\infty)$ such that

$$g_0(x) + (\frac{1}{4} - \eta)\,x \ge -C. \quad \text{for all } x \in (0,\infty) \tag{2.20}$$

2.5. Lemma. *Assume* (1.2), (1.3) *and* (2.2) *and let* (σ,ρ) *be lower functions of* (1.1). *Then* $\min_{t\in[0,2\pi]} \sigma(t) > 0$.

Proof. In view of (2.15), (2.16) and (2.18), there are $\delta \in (0,\infty)$ and $M \in [0,\infty)$ such that

$$\lim_{x\to 0+} \int_x^{\delta'} g_0(s)\,\mathrm{d}s = \infty \quad \text{for all } \delta' \in (0,\delta) \tag{2.21}$$

and

$$g_0(x) \ge -M \quad \text{for all } x \in (0,\delta). \tag{2.22}$$

In view of Definition 1.2 we have $\sigma(t) > 0$ a.e. on $[0,2\pi]$. Thus, for any $\varepsilon > 0$, there is $t_0 \in (0,\varepsilon]$ such that $\sigma(t_0) > 0$. Choose an arbitrary $t_0 \in (0,2\pi)$ with this property and put $t^* = \sup\{t \in [t_0,2\pi] : \sigma(s) > 0 \text{ on } [t_0,t]\}$. Notice that $\sigma(t^*) = 0$ holds whenever $t^* < 2\pi$.

Assume $\sigma(t^*) = 0$. Then there is $t' \in (t_0,t^*)$ such that

$$\sigma(t) \in [0,\delta) \quad \text{for all } t \in [t',t^*]. \tag{2.23}$$

As $\rho \in \mathbb{BV}[0,2\pi]$, it is $r = \|\rho\|_\infty + 1 < \infty$ and, with respect to Definition 1.2, we get that

$$\rho'(t)\,(\rho(t)-r) \le g_0(\sigma(t))\,(\rho(t)-r) + e(t)\,(\rho(t)-r)$$

is satisfied for a.e. $t \in [0,2\pi]$. Let $t_n \in (t',t^*)$ be an increasing sequence such that $\lim_{n\to\infty} t_n = t^*$. Then

$$\lim_{n\to\infty} \sigma(t_n) = \sigma(t^*) = 0 \tag{2.24}$$

and

$$\begin{aligned}
&\int_{t'}^{t_n} \rho'(t)\,(\rho(t)-r)\,\mathrm{d}t \\
&\quad\le \int_{t'}^{t_n} g_0(\sigma(t))\,(\rho(t)-r)\,\mathrm{d}t + \int_{t'}^{t_n} e(t)\,(\rho(t)-r)\,\mathrm{d}t \\
&\quad= -\int_{\sigma(t_n)}^{\sigma(t')} g_0(s)\,\mathrm{d}s - r\int_{t'}^{t_n} g_0(\sigma(t))\,\mathrm{d}t + \int_{t'}^{t_n} e(t)\,(\rho(t)-r)\,\mathrm{d}t.
\end{aligned}$$

In virtue of (2.22) and (2.23), for any $n \in \mathbb{N}$ we have

$$-\int_{t'}^{t_n} g_0(\sigma(t))\,\mathrm{d}t \le 2\pi\,M \quad\text{and}\quad \Big|\int_{t'}^{t_n} e(t)\,(\rho(t)-r)\,\mathrm{d}t\Big| \le 2\,r\,\|e\|_1.$$

Moreover, by (2.21) and (2.24)

$$\lim_{n\to\infty}\int_{\sigma(t_n)}^{\sigma(t')} g_0(s)\,\mathrm{d}s = \infty.$$

This implies that

$$\lim_{n\to\infty}\int_{t'}^{t_n} \rho'(t)\,(\rho(t)-r)\,\mathrm{d}t = -\infty.$$

On the other hand,

$$\Big|\int_{t'}^{t_n} \rho'(t)\,(\rho(t)-r)\,\mathrm{d}t\Big| \le 2\,r\int_0^{2\pi} |\rho'(t)|\,\mathrm{d}t \le 2\,r\,\|\rho\|_{\mathbb{BV}} < \infty,$$

a contradiction. Thus, $t^* = 2\pi$ and $\sigma(t^*) = \sigma(2\pi) > 0$. In particular, we have shown that $\sigma(t)$ is positive on any interval $(\varepsilon, 2\pi]$, $\varepsilon > 0$, and, as with respect to the periodicity condition we also have $\sigma(0) = \sigma(2\pi) > 0$, this completes the proof. □

2.6. Remark. If g satisfies the assumptions of Lemma 2.5 and, in addition $g(x) \geq k$ on $(0,\infty)$, $e(t) = (k-1)\sin t$ on $[0,2\pi]$, $\sigma(t) = 1+\sin t$ and $\rho(t) = \cos t$ on $[0,2\pi]$, then (σ,ρ) are upper functions of (1.1) and $\min_{t\in[0,2\pi]} \sigma(t) = 0$. This shows that for upper functions the analogue of Lemma 2.5 does not hold.

2.7. Definition. Let $g \in \mathbb{C}(0,\infty)$ and $k \in \mathbb{R}$. Then, for given E, $A \in (0,\infty)$ and $B \in [A,\infty)$, we denote by $\mathcal{E}(E,A,B)$ the set of functions $e \in \mathbb{L}[0,2\pi]$ with $\|e\|_1 \leq E$ and such that (1.1) has lower functions (σ_1,ρ_1) and upper functions (σ_2,ρ_2) fulfilling

$$A \leq \sigma_i(t) \leq B \ \text{ on } \ [0,2\pi] \ \text{ for } \ i = 1,2. \tag{2.25}$$

2.8. Remark. Let $g \in \mathbb{C}(0,\infty)$ satisfy (1.3), (2.1) and (2.2) and let g_0 be given by (2.15). Then g_0 fulfils (2.16)-(2.19) and we can choose a sequence $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0,1)$ with the properties

$$\begin{cases} \varepsilon_{n+1} < \varepsilon_n \quad \text{and} \quad g_0(\varepsilon_n) > 0 \ \text{ for all } \ n \in \mathbb{N}, \\ \lim\limits_{n\to\infty} \varepsilon_n = 0 \quad \text{and} \quad \lim\limits_{n\to\infty} g_0(\varepsilon_n) = \infty \end{cases} \tag{2.26}$$

and define functions $g_{n,m} \in \mathbb{C}(\mathbb{R})$, $n,m \in \mathbb{N}$, in such a way that the relations

$$g_{n,m}(x) = g_0(x) \ \text{ for } \ x \in [\varepsilon_n, m], \tag{2.27}$$

$$g_{n,m}(x)\,x + (\frac{1}{4} - \eta)\,x^2 \geq -C\,|x| \ \text{ for all } \ x \in \mathbb{R} \tag{2.28}$$

and

$$g_* := \inf_{\substack{x\in\mathbb{R}\\ n,m\in\mathbb{N}}} g_{n,m}(x) \in \mathbb{R} \tag{2.29}$$

are valid for all $n,m \in \mathbb{N}$. Indeed, for given $n,m \in \mathbb{N}$, we can put e.g.

$$g_{n,m}(x) = \begin{cases} 0 & \text{if } \ x \leq 0, \\ g_0(\varepsilon_n)\,\dfrac{x}{\varepsilon_n} & \text{if } \ x \in [0,\varepsilon_n], \\ g_0(x) & \text{if } \ x \in [\varepsilon_n, m], \\ g_0(m) & \text{if } \ x \geq m. \end{cases}$$

In what follows, having a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ which satisfies (2.27) and (2.28) and functions $g_{n,m}$, $n,m \in \mathbb{N}$, satisfying (2.27)-(2.29), we will often work with the auxiliary regular boundary value problems

$$x'' = g_{n,m}(x) + e(t), \ x(0) = x(2\pi), \ x'(0) = x'(2\pi). \tag{2.30}$$

2.9. Proposition. *Suppose that* $g \in \mathbb{C}(0,\infty)$, $k \in \mathbb{R}$, E, $A \in (0,\infty)$, $B \in [A,\infty)$ *and let* (1.3), (2.1), (2.2) *be satisfied. Then there are* R, $\varepsilon^* \in (0,\infty)$, *and* $K \in [0,\infty)$ *such that for each* $e \in \mathcal{E}(E,A,B)$ *the problem* (1.1) *has a solution* u *satisfying*

$$\varepsilon^* \leq u(t) \leq R \ \text{ on } \ [0,2\pi] \quad \text{and} \quad \|u'\|_\infty \leq K. \tag{2.31}$$

Moreover, if (σ_1,ρ_1) *and* (σ_2,ρ_2) *are respectively lower and upper functions of* (1.1) *with the property* (2.25), *then*

$$A \le \min\{\sigma_1(t_u),\sigma_2(t_u)\} \le u(t_u) \le \max\{\sigma_1(t_u),\sigma_2(t_u)\} \le B \quad \text{for some } \ t_u \in [0,2\pi]. \tag{2.32}$$

Proof. Let ε_n and $g_{n,m}$, $n,m \in \mathbb{N}$, be chosen in such a way that (2.26)-(2.29) are true and

$$\varepsilon_n < A \ \text{ for all } \ n \in \mathbb{N}. \tag{2.33}$$

By Lemma 2.4, there is $R \in \mathbb{N} \cap (B,\infty)$ such that the estimate (2.7) is valid for all $n,m \in \mathbb{N}$, for each $e \in \mathbb{L}[0,2\pi]$ with $\|e\|_1 \le E$ and each solution x of (2.30) fulfilling (2.8) with $J=\mathbb{R}$. For $n \in \mathbb{N}$, consider the problems

$$x'' = g_{n,R}(x) + e(t),\ x(0)=x(2\pi),\ x'(0)=x'(2\pi). \tag{2.34}$$

In view of (2.29), we have $g_{n,R}(x) \ge g_* \in \mathbb{R}$ for all $x \in \mathbb{R}$, i.e.

$$g_{n,R}(x) + e(t) \ge g_* + e(t) \ \text{ for a.e. } \ t \in [0,2\pi] \ \text{ and all } \ x \in \mathbb{R} \quad \text{and} \quad n \in \mathbb{N}.$$

Let $e \in \mathcal{E}(E,A,B)$ and let (σ_1,ρ_1) and (σ_2,ρ_2) be respectively lower and upper functions of (1.1) fulfilling (2.25). In view of (2.33), (σ_1,ρ_1) and (σ_2,ρ_2) are lower and upper functions of (2.34) for all $n \in \mathbb{N}$, respectively. Thus, by Theorem 1.4 and Lemma 1.8, for any $n \in \mathbb{N}$, the problem (2.34) has a solution x_n such that $\|x_n'\|_\infty \le E + 2\pi\,|g_*| = K$ and

$$A \le \min\{\sigma_1(t_n),\sigma_2(t_n)\} \le x_n(t_n) \le \max\{\sigma_1(t_n),\sigma_2(t_n)\} \le B$$

for some $t_n \in [0,2\pi]$. In particular, we have

$$\max_{t\in[0,2\pi]} x_n(t) \in [A,R] \ \text{ for all } \ n \in \mathbb{N}, n \ge n_1.$$

Now, let $\varepsilon^* > 0$ correspond to E, K, A and R by Lemma 2.2 and let $n_0 \in \mathbb{N}$ be such that $\varepsilon_{n_0} < \varepsilon^*$. Then, since $g_{n_0,R}(s) = g_0(s)$ on $[\varepsilon_{n_0}, R]$, Lemma 2.2 yields

$$\min_{t\in[0,2\pi]} x_{n_0}(t) > \varepsilon^* > \varepsilon_{n_0}.$$

Therefore, $u = x_{n_0}$ is a solution to (1.1) with the properties (2.31) and (2.32). ☐

Proof of Theorem 2.1. In virtue of (2.3) and of Lemma 2.5 we can find $A \in (0,\infty)$ and $B \in [A,\infty)$ such that $e \in \mathcal{E}(\|e\|_1, A, B)$ and the assertion of Theorem 2.1 follows from Proposition 2.9. ☐

The next result shows that the assumption of the existence of upper functions for (1.1) in Theorem 2.1 can be replaced with (2.35).

2.10. Corollary. *Suppose that* (1.2), (1.3), (2.1), (2.2) *and*

$$\inf_{t\in[0,2\pi]} \operatorname{ess}\, e(t) > -\infty \tag{2.35}$$

are satisfied. Further, let (σ_1, ρ_1) *be lower functions of* (1.1). *Then the problem* (1.1) *has a positive solution* u *such that*

$$u(t_u) \le \sigma_1(t_u) \quad \textit{for some} \quad t_u \in [0, 2\pi]. \tag{2.36}$$

Proof. By (1.3) we have $\limsup_{x\to 0+} g(x) = \infty$ and, in virtue of (2.35), there is $\varepsilon^* \in (0,1)$ such that

$$\varepsilon^* < \min_{t\in[0,2\pi]} \sigma_1(t) \quad \text{and} \quad g(\varepsilon^*) - k\,\varepsilon^* + e(t) \ge g(\varepsilon^*) - k\,\varepsilon^* + \inf_{t\in[0,2\pi]} \operatorname{ess}\, e(t) \ge 0.$$

Therefore the functions $(\sigma_2(t), \rho_2(t)) = (\varepsilon^*, 0)$ on $[0, 2\pi]$ are upper functions of (1.1) and Theorem 2.1 yields the existence of a positive solution u which fulfils (2.36). □

2.11. Remark. Assume (1.2), while (1.3) need not be satisfied, and let u be a solution of (1.1). Further, suppose that g has a singularity at 0, which means that g is unbounded at 0, i.e.

$$\limsup_{x\to 0+} g(x) = \infty. \tag{2.37}$$

Definition 1.1 requires that any solution u of (1.1) is positive a.e. on $[0, 2\pi]$. In particular, u can touch the singularity point $x = 0$. However it can vanish only on the set of the zero measure. Corollary 2.3 says that this is impossible provided the singularity is strong, i.e. if (1.3) is satisfied. Therefore, the problem (1.1) can possess nonnegative solutions with at least one zero only if $\lim_{x\to 0+} \int_x^1 g(s)\,\mathrm{d}s \in \mathbb{R}$. If this together with (2.37) hold, the singularity $x = 0$ of g is called *weak*.

3 . Existence criteria

The main result of this section is Theorem 3.3 which gives more effective existence criterion without the a priori assumption of the existence of lower and upper functions. For its proof the following lemmas will be helpful.

3.1. Lemma. *Suppose* (1.7). *Furthermore, let* $(d, A_0] \subset J$ *and*

$$h(x) - k\,x + \overline{e} > 0 \quad \textit{for all} \quad x \in (d, A_0]. \tag{3.1}$$

Then $\max_{t\in[0,2\pi]} x(t) > A_0$ *holds for each solution* x *of* (1.6) *such that* $x(t) \in J$ *for all* $t \in [0, 2\pi]$ *and* $\min_{t\in[0,2\pi]} x(t) > d$.

Proof. Let x be a solution of (1.6) and let $\min\limits_{t\in[0,2\pi]} x(t) > d$ and $x(t) \in J$ for all $t \in [0,2\pi]$. Integrating the equality

$$x''(t) = h(x(t)) - k\,x(t) + e(t) \quad \text{a.e. on } [0,2\pi]$$

over $[0,2\pi]$ and taking into account the periodicity of x, we get

$$\int_0^{2\pi} (h(x(t) - k\,x(t))\,\mathrm{d}t + 2\pi\,\overline{e} = 0. \tag{3.2}$$

On the other hand, if it were $x(t) \le A_0$ for all $t \in [0,2\pi]$, then using (3.1) we would have

$$\int_0^{2\pi} (h(x(t)) - k\,x(t))\,\mathrm{d}t + 2\pi\,\overline{e} > 0,$$

a contradiction to (3.2). □

3.2. Lemma. *Let $A \in (0,\infty)$, $B \in [A,\infty)$, $E \in [0,\infty)$, $k \in \mathbb{R}$ and let $g \in \mathbb{C}(0,\infty)$ fulfil* (1.3), (2.1) *and* (2.2). *Then there are R, $\varepsilon^* \in (0,\infty)$ and $K \in [0,\infty)$ such that for each $e \in \mathbb{L}[0,2\pi]$ with $\|e\|_1 \le E$ and each solution u of* (1.1) *satisfying*

$$u(t_u) \in [A,B] \quad \textit{for some} \;\; t_u \in [0,2\pi], \tag{3.3}$$

the estimates (2.31) *are true.*

Proof. Let g_0 be given by (2.15). Then $g_0 \in \mathbb{C}(0,\infty)$ fulfils (2.20) with some $\eta \in (0,\frac{1}{4})$ and $C \in [0,\infty)$. By Lemma 2.4, for each $e \in \mathbb{L}[0,2\pi]$ with $\|e\|_1 \le E$ and each solution u of (1.1) fulfilling (3.3) the estimate $u(t) \le R$ on $[0,2\pi]$ is true. Furthermore, as in view of (2.18) $g_* := \inf_{x\in(0,R]} g_0(x) \in \mathbb{R}$, according to Corollary 2.3 and Lemma 1.8 we have $u(t) > 0$ on $[0,2\pi]$ and $\|u'\|_\infty \le E + 2\pi\,|g_*|$ for all such solutions. Thus, if we put $K = E + 2\pi\,|g_*|$, then we can complete the proof by means of Lemma 2.2. □

3.3. Theorem. *Suppose that* (1.2), (1.3), (2.1) *and*

$$\liminf_{x\to 0+} g(x) > -\overline{e} \tag{3.4}$$

are satisfied. Further, let $A \in (0,\infty)$ be such that

$$g(x) - k\,x \le -\overline{e} \;\; \textit{for all} \;\; x \in [A,B], \tag{3.5}$$

where B fulfils (1.9). *Then the problem* (1.1) *has a positive solution u such that*

$$u(t_u) \le B \quad \textit{for some} \;\; t_u \in [0,2\pi]. \tag{3.6}$$

Proof. i) First, assume that (3.5) is satisfied with the strict inequality, i.e.

$$g(x) - k\,x + \overline{e} < 0 \quad \text{for all} \quad x \in [A, B]. \tag{3.7}$$

For $n \in \mathbb{N}$ define

$$e_n(t) = \max\{e(t), -n\} \quad \text{a.e. on} \quad [0, 2\pi] \tag{3.8}$$

and consider the problems

$$x'' + k\,x = g(x) + e_n(t),\; x(0) = x(2\pi),\; x'(0) = x'(2\pi). \tag{3.9}$$

There is $E \in (0, \infty)$ such that $\|e_n\|_1 \le E$ for all $n \in \mathbb{N}$. Furthermore,

$$\inf_{t\in[0,2\pi]} \text{ess}\; e_n(t) \ge -n \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to\infty} \overline{e_n} = \overline{e}. \tag{3.10}$$

In virtue of (3.4), we can choose $A_0 \in (0, \frac{A}{2})$ in such a way that $g(x) - k\,x + \overline{e_n} \ge g(x) - k\,x + \overline{e} > 0$ holds for all $x \in (0, A_0]$ and all $n \in \mathbb{N}$. By Lemma 3.1 this implies

$$\max_{t\in[0,2\pi]} x(t) > A_0 \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and all positive solutions} \quad x \text{ of } (3.9). \tag{3.11}$$

Let $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0, A_0)$ be an arbitrary decreasing sequence which tends to 0 as $n \to \infty$ and satisfies the relation $g(\varepsilon_n) - k\,\varepsilon_n > n$ for any $n \in \mathbb{N}$. In particular, with respect to (3.10), the functions $(\sigma_{2,n}(t), \rho_{2,n}(t)) = (\varepsilon_n, 0)$ are upper functions for (3.9). Furthermore, in view of (3.7), (3.8) and (1.9) we can find $n_0 \in \mathbb{N}$ and $\nu \in (0, \frac{A}{2})$ such that $g(x) - k\,x + \overline{e_n} \le 0$ and $\|e_n - \overline{e_n}\|_1 \le \frac{3}{\pi}(B - A + \nu)$ hold for all $x \in [A - \nu, B]$ and all $n \ge n_0$. By Proposition 1.5, this means that, for each $n \ge n_0$, the problem (3.9) has lower functions $(\sigma_{1,n}, \rho_{1,n})$ such that $\sigma_{1,n}(t) \in [A - \nu, B]$ for all $t \in [0, 2\pi]$.

To summarize, we have $e_n \in \mathcal{E}(E, \varepsilon_n, B)$ for all $n \ge n_0$. Thus, due to (3.11) Proposition 2.9 implies that for each $n \ge n_0$ the problem (3.9) has a positive solution x_n such that

$$x_n(t_n) \in [A_0, B] \quad \text{for some} \quad t_n \in [0, 2\pi]. \tag{3.12}$$

Hence, we can use Lemma 3.2 with $A = A_0$ to get that there are $R, \varepsilon^* \in (0, \infty)$ and $K \in [0, \infty)$ such that the relations $\varepsilon^* \le x_n(t) \le R$ on $[0, 2\pi]$ and $\|x_n'\|_\infty \le K$ hold for each $n \ge n_0$. This means that the set $\{g(x_n(t)) - k\,x_n(t) \colon t \in [0, 2\pi],\, n \ge n_0\}$ is bounded and it follows that the sequence $\{x_n\}_{n=n_0}^{\infty}$ is equi-bounded and equi-continuous in $\mathbb{C}^1[0, 2\pi]$ and so, by the Arzelà - Ascoli Theorem, we can assume without loss of generality that $\{x_n\}_{n=n_0}^{\infty}$ converges in $\mathbb{C}^1[0, 2\pi]$ to some function $u \in \mathbb{C}^1[0, 2\pi]$. Consequently, for each $t \in [0, 2\pi]$ we have

$$\lim_{n\to\infty} \left[x_n'(t) - x_n'(0) + k \int_0^t x_n(s)\,\mathrm{d}s\right] = \lim_{n\to\infty} \int_0^t \left(g(x_n(s)) + e_n(s)\right) \mathrm{d}s,$$

wherefrom, using the Lebesgue Dominated Convergence Theorem we get

$$u'(t) - u'(0) + k \int_0^t u(s)\,\mathrm{d}s = \int_0^t (g(u(s)) + e(s))\,\mathrm{d}s,$$

which means that $u \in \mathbb{AC}^1[0,2\pi]$ and u is a solution to (1.1). Moreover, with respect to (3.12) we have (3.6).

ii) It remains to get rid of the assumption (3.7). For $n \in \mathbb{N}$, let us define

$$p_n(t) = e(t) - \frac{1}{n} \quad \text{a.e. on } [0,2\pi].$$

For each $n \in \mathbb{N}$ and $x \in [A,B]$, we have $g(x) - k\,x + \overline{p_n} < 0$. Further, in view of (3.4) there are $n_0 \in \mathbb{N}$ and $A_1 \in (0, \frac{A}{2})$ such that

$$g(x) - k\,x + \overline{p_n} > 0 \quad \text{for all } x \in (0, A_1) \quad \text{and} \quad n \geq n_0.$$

Thus, by the first part of this proof, we get a sequence $\{x_n\}_{n=n_0}^{\infty}$ of solutions of the problems

$$x'' + k\,x = g(x) + p_n(t),\ x(0) = x(2\pi),\ x'(0) = x'(2\pi). \tag{3.13}$$

Since $\|p_n\|_1 \leq E + 1$ for $n \geq n_0$, we get that the solutions x_n of (3.13) satisfy $\varepsilon^* \leq x_n(t) \leq R$ on $[0,2\pi]$ and $\|x_n'\|_\infty \leq K$, where the constants R, K and ε^* are now determined for A_1 A, B and $E+1$ instead of A_0, A, B, E. Therefore, we can use the limiting process as in the first part of this proof and get the desired solution $u(t) = \lim_{n\to\infty} x_n(t)$ to (1.1) with the property (3.6). □

The proof of the following theorem can be done as the previous one with the only difference that instead of Proposition 1.5 we will use Proposition 1.6.

3.4. Theorem. *Suppose that* $k \neq n^2$ *for all* $n \in \mathbb{N}$ *and replace condition* (3.5) *in Theorem* 3.3 *with*

$$g(x) + \overline{e} \leq k\,\frac{A+B}{2} \quad \textit{for all } x \in [A,B], \tag{3.14}$$

where $A > 0$ *and* B *fulfil* (1.13) *and* (1.14). *Then the problem* (1.1) *has a positive solution.* □

4 . Multiplicity results

In this section we present sufficient conditions for the existence of at least two positive solutions of (1.1). First, we will give some necessary auxiliary assertions.

4.1. Lemma. *Assume* (1.2) *and let* $A \in (0,\infty)$ *and* $B \in (A,\infty)$ *be such that* (3.7) *and* (1.9) *are satisfied. Then there are* $\gamma : [0,2\pi]\times[0,2\pi] \mapsto \mathbb{R}$ *continuous on* $[0,2\pi]\times[0,2\pi]$ *and* $\sigma_1 \in \mathbb{AC}^1[0,2\pi]$ *such that* (σ_1,σ_1') *are lower functions of* (1.1),

$$\sigma_1(t) = A + \frac{\pi}{6}\,\|e-\overline{e}\|_1 + \int_0^{2\pi} \gamma(t,s)\,(e(s)-\overline{e})\,\mathrm{d}s \quad on \ [0,2\pi] \tag{4.1}$$

and $\sigma_1(t) \in [A,B]$ *for all* $t \in [0,2\pi]$.

Furthermore, there is $\nu_0 > 0$ *such that for each* $\nu \in [-\nu_0,\nu_0]$ *the functions* $(\sigma_1+\nu,\sigma_1')$ *are lower functions of* (1.1).

Proof. Due to (3.7), we can find $\nu_0 > 0$ in such a way that

$$g(x) - k\,x + \overline{e} \le 0 \quad \text{for} \ x \in [A-\nu_0, B+\nu_0]. \tag{4.2}$$

Let $\gamma_0(t,s)$ be the Green function of the problem $x''=0$, $x(0)=x(2\pi)=0$ and let

$$\sigma_0(t) = \int_0^{2\pi} \gamma(t,s)\,(e(t)-\overline{e})\,\mathrm{d}s, \quad \text{for} \ t \in [0,2\pi],$$

where

$$\gamma(t,s) = \gamma_0(t,s) - \frac{1}{2\pi}\int_0^{2\pi} \gamma_0(\tau,s)\,\mathrm{d}\tau \quad \text{for} \ t,s \in [0,2\pi]. \tag{4.3}$$

It is easy to verify that $\sigma_0 \in \mathbb{AC}^1[0,2\pi]$,

$$\sigma_0''(t) = e(t) - \overline{e} \quad \text{a.e. on} \ [0,2\pi], \quad \sigma_0(0)=\sigma_0(2\pi), \quad \sigma_0'(0)=\sigma_0'(2\pi). \tag{4.4}$$

Moreover, $\overline{\sigma_0} = 0$ and therefore by the proof of [12, Proposition2.4] we have

$$\|\sigma_0\|_\infty \le \frac{\pi}{6}\,\|e-\overline{e}\|_1. \tag{4.5}$$

In particular, we have

$$\sigma_1(t) = \sigma_0(t) + A + \frac{\pi}{6}\,\|e-\overline{e}\|_1 \quad \text{on} \ [0,2\pi]. \tag{4.6}$$

Now, choose an arbitrary $\nu \in [-\nu_0,\nu_0]$ and put

$$\sigma(t) = \sigma_1(t) + \nu \quad \text{for} \ t \in [0,2\pi]. \tag{4.7}$$

Obviously, σ and $\sigma_1 \in \mathbb{AC}^1[0,2\pi]$ fulfil (4.4) in place of σ_0. Furthermore, (4.5)-(4.7) imply that

$$\sigma_1(t) \in [A,B] \quad \text{and} \quad \sigma(t) \in [A-\nu_0, B+\nu_0] \quad \text{for all} \ t \in [0,2\pi].$$

Finally, in view of (4.2) we have $k\,\sigma(t) - \overline{e} \ge g(\sigma(t))$ on $[0,2\pi]$ and, consequently,

$$\sigma''(t) + k\,\sigma(t) = e(t) - \overline{e} + k\,\sigma(t) \ge e(t) + g(\sigma(t)) \quad \text{for a.e.} \ t \in [0,2\pi],$$

i.e. $(\sigma_1+\nu,\sigma_1')$ are lower functions of (1.1) for each $\nu \in [-\nu_0,\nu_0]$. □

4.2. Proposition. *Suppose* (1.2) *and let* (σ_1, ρ_1) *and* (σ_2, ρ_2) *be respectively lower and upper functions of* (1.1) *such that* $\sigma_1(t) \le \sigma_2(t)$ *on* $[0, 2\pi]$. *Then there is a solution* u *of* (1.1) *such that*

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \quad on \ [0, 2\pi]. \tag{4.8}$$

Proof. Choose arbitrarily $f \in \mathrm{Car}([0, 2\pi] \times \mathbb{R})$ in such a way that $f(t, x) = g(x) - k\,x + e(t)$ for a.e. $t \in [0, 2\pi]$ and all $x \in [\sigma_1(t), \sigma_2(t)]$. Then, Theorem 1.4 ensures the existence of a solution u of (1.4) with $J = \mathbb{R}$ satisfying the estimates (4.8), which means that u is a solution to (1.1), as well. □

4.3. Theorem. *Suppose that* (1.2), (1.3), (2.2) *and* (2.35) *hold and let* $A \in (0, \infty)$ *and* $B \in (A, \infty)$ *be such that* (3.7) *and* (1.9) *are true. Further, assume that there are upper functions* (σ_2, ρ_2) *of* (1.1) *such that* $\sigma_2(t) \ge B$ *for all* $t \in [0, 2\pi]$. *Then the problem* (1.1) *has at least two positive solutions.*

Proof. First, notice that by Lemma 4.1 and Proposition 4.2 the problem (1.1) has lower functions (σ_1, σ_1') and a solution u for which (4.8) is true. Moreover, we have $\sigma_1(t) \in [A, B]$ on $[0, 2\pi]$ and there is $\nu \in (0, A)$ such that $(\sigma_1 - \nu, \sigma_1')$ are also lower functions of (1.1).

Consider the function g_0 from (2.15). By (2.19) and (2.35) there is $A_0 \in (0, A - \nu)$ such that

$$g_0(A_0) + e(t) \ge g_0(A_0) + \inf_{t \in [0, 2\pi]} \mathrm{ess}\; e(t) \ge 0 \quad \text{for a.e. } t \in [0, 2\pi]. \tag{4.9}$$

This means that $(A_0, 0)$ are upper functions of (1.1). Furthermore, for a.e. $t \in [0, 2\pi]$, put

$$m(t) = e(t) + \min\{0, \inf_{s \in (0, \sigma_2(t)]} g_0(s)\}. \tag{4.10}$$

Then, in view of (2.18), $m \in \mathbb{L}[0, 2\pi]$. Denote

$$R = \|\sigma_2\|_\infty, \quad K = \|m\|_1 \quad \text{and} \quad K^* = K\,\|e\|_1 + \int_{A_0}^{R} |g_0(s)|\, \mathrm{d}s.$$

Due to (2.16), we can choose $\varepsilon^* \in (0, A_0)$ in such a way that $g_0(\varepsilon^*) > 0$ and

$$\int_{\varepsilon^*}^{A_0} g_0(s)\, \mathrm{d}s > K^*. \tag{4.11}$$

Now, for a.e. $t \in [0, 2\pi]$, define

$$f(t, x) = e(t) + \begin{cases} \widetilde{g}_0(x) & \text{if } x \le \sigma_2(t), \\[2mm] \widetilde{g}_0(\sigma_2(t)) + \dfrac{x - \sigma_2(t)}{x - \sigma_2(t) + 1} & \text{if } x > \sigma_2(t), \end{cases} \tag{4.12}$$

where

$$\widetilde{g}_0(x) = \begin{cases} 0 & \text{if } x < 0, \\ g_0(\varepsilon^*)\, \dfrac{x}{\varepsilon^*} & \text{if } x \in [0, \varepsilon^*), \\ g_0(x) & \text{if } x \geq \varepsilon^*. \end{cases} \tag{4.13}$$

Consider the problem (1.4) with $J = \mathbb{R}$. We have $f \in \mathrm{Car}([0, 2\pi] \times \mathbb{R})$. Further, since $\varepsilon^* < A_0$ and (4.9) are valid, the couple $(A_0, 0)$ defines upper functions of (1.4). Similarly, since $\sigma_1(t) - \nu < \sigma_2(t)$ on $[0, 2\pi]$, the functions $(\sigma_1 - \nu, \sigma_1')$ are lower functions for (1.4). Finally, (4.10) and (4.12) imply

$$f(t, x) \geq m(t) \quad \text{for all } \ x \in \mathbb{R} \quad \text{and a.e. } \ t \in [0, 2\pi].$$

Therefore we can use Theorem 1.4 and obtain a solution v of (1.4) such that

$$A_0 \leq v(t_v) \leq \sigma_1(t_v) - \nu \quad \text{for some } \ t_v \in [0, 2\pi]. \tag{4.14}$$

Relations (4.8) and (4.14) ensure that u and v are different. It remains to prove that v is a solution to (1.1). To this aim we need to show that the inequalities

$$\varepsilon^* \leq v(t) \leq \sigma_2(t) \ \text{ on } \ [0, 2\pi] \tag{4.15}$$

are valid. First, let us put $z(t) = v(t) - \sigma_2(t)$ for $t \in [0, 2\pi]$ and suppose that $\max_{t \in [0,2\pi]} z(t) = z(\tau_1) > 0$. Due to the periodic conditions we can assume that $\tau_1 \in [0, 2\pi)$ and $z'(\tau_1) = 0$. Moreover, there is $\tau_2 \in (\tau_1, 2\pi]$ such that $z(t) > 0$ on $[\tau_1, \tau_2)$ and $z'(\tau_2) \leq 0$. Then, since (σ_2, ρ_2) are lower functions of (1.4), taking into account (4.12) we get

$$z''(t) \geq \frac{z(t)}{z(t)+1} > 0 \quad \text{a.e. on } [\tau_1, \tau_2] \quad \text{and so } \ 0 < \int_{\tau_1}^{\tau_2} z''(t)\, \mathrm{d}t = z'(\tau_2) \leq 0,$$

a contradiction. This proves that

$$v(t) \leq \sigma_2(t) \ \text{ on } \ [0, 2\pi] \tag{4.16}$$

and, in particular, we have

$$\max_{t \in [0,2\pi]} v(t) = v(t_1) \in [A_0, R]. \tag{4.17}$$

Now, assume that $\min_{t \in [0,2\pi]} v(t) = v(t_0) < \varepsilon^*$. With respect to (4.12), (4.13) and (4.16), we have

$$v''(t) = \widetilde{g}_0(v(t)) + e(t) \quad \text{a.e. on } \ [0, 2\pi].$$

Therefore, using (4.11), (4.13), (4.17) and Lemma 1.7 we obtain

$$K^* < \int_{\varepsilon^*}^{A_0} g_0(s)\, \mathrm{d}s \leq \int_{v(t_0)}^{A_0} \widetilde{g}_0(s)\, \mathrm{d}s \leq \|e\|_1 K + \int_{A_0}^{R} |g_0(s)|\, \mathrm{d}s = K^*,$$

a contradiction. This proves that $v(t_0) \geq \varepsilon^*$, wherefrom, with respect to (4.16), the relations (4.15) follow. □

4.4. Theorem. *Theorem* 4.3 *remains valid if* (3.4) *is assumed instead of* (2.2) *and* (2.35).

Proof. Due to Lemma 4.1, the functions (σ_1, σ_1') with σ_1 given by (4.1) are lower functions of (1.1). Therefore, by Proposition 4.2, the problem (1.1) has a solution u satisfying (4.8).

Let g_0 and e_n, $n \in \mathbb{N}$, be given by (2.15) and (3.8), respectively. Recall that the sequence $\{e_n\}_{n=1}^{\infty}$ is nonincreasing for a.e. $t \in [0, 2\pi]$, the relations (3.10) are true and

$$\lim_{n\to\infty} \|e_n - e\|_1 = 0. \tag{4.18}$$

Consequently, there is $E \in [0, \infty)$ such that $\|e_n\|_1 \le E$ for all $n \in \mathbb{N}$. Due to (3.4) and (3.10) there is $A_0 \in (0, \frac{A}{2})$ such that

$$g_0(x) + \overline{e_n} \ge g_0(x) + \overline{e} > 0 \quad \text{for all } x \in (0, A_0] \quad \text{and} \quad n \in \mathbb{N}. \tag{4.19}$$

Choose a sequence $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0, A_0)$ in such a way that (2.26) and

$$g_0(\varepsilon_n) \ge n \quad \text{for all } n \in \mathbb{N} \tag{4.20}$$

are satisfied. Now, for $n \in \mathbb{N}$ and a.e. $t \in [0, 2\pi]$, define

$$f_n(t,x) = e_n(t) + \begin{cases} \widetilde{g}_n(x) & \text{if } x < \sigma_2(t), \\ g_0(\sigma_2(t)) + \dfrac{x - \sigma_2(t)}{x - \sigma_2(t) + 1} & \text{if } x \ge \sigma_2(t) \end{cases} \tag{4.21}$$

and

$$\widetilde{g}_n(x) = \begin{cases} |\overline{e}| + 1 & \text{if } x < 0, \\ g_0(\varepsilon_n)\,\frac{x}{\varepsilon_n} + (|\overline{e}| + 1)\,\dfrac{\varepsilon_n - x}{\varepsilon_n} & \text{if } x \in [0, \varepsilon_n), \\ g_0(x) & \text{if } x \ge \varepsilon_n \end{cases} \tag{4.22}$$

and consider the problems

$$x'' = f_n(t,x), \quad x(0) = x(2\pi),\ x'(0) = x'(2\pi). \tag{4.23}$$

We will show that for all n sufficiently large the problem (4.23) verifies the assumptions of Theorem 1.4. Indeed, put $R = \|\sigma_2\|_\infty$. Then

$$g_* = \min\{|\overline{e}| + 1, \inf_{x \in (0,R]} g_0(x)\} \in \mathbb{R}$$

and $f_n(t,x) \ge e_n(t) + g_*$ for all $n \in \mathbb{N}$, $x \in \mathbb{R}$ and a.e. $t \in [0, 2\pi]$. Furthermore, from (3.10) and (4.20) we get that $f_n(t, \varepsilon_n) \ge g_0(\varepsilon_n) - n \ge 0$ holds for a.e. $t \in [0, 2\pi]$ and each $n \in \mathbb{N}$. This implies that $(\varepsilon_n, 0)$ are upper functions of (4.23). Finally, in view of (3.7), (3.10) and (4.18) there are $n_0 \in \mathbb{N}$ and $\nu \in (0, \frac{A}{2})$ such that the relations

$$g_0(x) + \overline{e_n} < 0 \text{ for all } x \in [A - \nu, B] \quad \text{and} \quad \frac{\pi}{3} \|e_n - \overline{e_n}\|_1 < \frac{\pi}{3} \|e - \overline{e}\|_1 + \frac{\nu}{2}$$

hold for all $n \geq n_0$. Moreover, by (1.9) we have

$$(B - \frac{\nu}{2}) - (A - \nu) = B - A + \frac{\nu}{2} > \frac{\pi}{3} \|e_n - \overline{e_n}\|_1 \text{ for } n \geq n_0.$$

Thus, with respect to Lemma 4.1, for each $n \geq n_0$ the functions $(\sigma_{1,n}, \sigma'_{1,n})$, where

$$\sigma_{1,n}(t) = A + \frac{\pi}{6} \|e_n - \overline{e_n}\|_1 + \int_0^{2\pi} \gamma(t,s)\,(e_n(s) - \overline{e_n})\,\mathrm{d}s - \nu, \quad t \in [0, 2\pi], \tag{4.24}$$

are lower functions of the problem

$$x'' = g_0(x) + e_n(t), \quad x(0) = x(2\pi), \quad x'(0) = x'(2\pi)$$

and

$$\sigma_{1,n}(t) \in [A - \nu, B - \frac{\nu}{2}] \text{ for all } t \in [0, 2\pi]. \tag{4.25}$$

Since we have

$$\varepsilon_n < \frac{A}{2} < A - \nu \leq B \leq \sigma_2(t) \text{ for all } t \in [0, 2\pi] \quad \text{and} \quad n \geq n_0,$$

it is easy to see that the functions $(\sigma_{1,n}, \sigma'_{1,n})$ are also lower functions of (4.23).

Thus, we can use Theorem 1.4 to show that for each $n \geq n_0$ the problem (4.23) has a solution x_n such that

$$\|x'_n\|_\infty \leq K = E + 2\pi\,|g_*| \tag{4.26}$$

and

$$\varepsilon_n \leq x_n(t_n) \leq \sigma_{1,n}(t_n) \quad \text{for some } t_n \in [0, 2\pi]. \tag{4.27}$$

Now, fix $n \geq n_0$ and define $z(t) = x_n(t) - \sigma_2(t)$ on $[0, 2\pi]$. Using the arguments from the proof of Theorem 4.3 (see the proof of (4.16)) we get

$$x_n(t) \leq \sigma_2(t) \text{ on } [0, 2\pi]. \tag{4.28}$$

With respect to (4.21) this means that for each $n \in \mathbb{N}$ the function x_n is a solution of

$$x'' = \widetilde{g}_n(x) + e_n(t), \quad x(0) = x(2\pi), \; x'(0) = x'(2\pi).$$

Furthermore, using (3.10), (4.19) and (4.22) we can verify that there is $n_1 \geq n_0$ such that

$$\widetilde{g}_n(x) + \overline{e_n} > 0 \quad \text{for all } x \in (-\infty, A_0] \text{ and all } n \geq n_1.$$

Therefore, by Lemma 3.1 we have

$$\max_{t\in[0,2\pi]} x_n(t) > A_0 \quad \text{for all } n \geq n_1$$

and Lemma 2.2 yields that there is $\varepsilon^* > 0$ such that

$$x_n(t) \geq \varepsilon^* \quad \text{for all } t \in [0, 2\pi] \quad \text{and} \quad n \geq n_1. \tag{4.29}$$

In view of (4.26), (4.28) and (4.29), the sequence $\{x_n\}_{n=n_1}^{\infty}$ is equibounded and equicontinuous in $\mathbb{C}^1[0, 2\pi]$ and thus we can assume without loss of generality that it converges in $\mathbb{C}^1[0, 2\pi]$ to some function v. With respect to (4.22), (4.28) and (4.29), we have

$$\varepsilon^* \leq v(t) \leq \sigma_2(t) \quad \text{on } [0, 2\pi].$$

Therefore v is a solution to (1.1).

It remains to show that v differs from u. First, notice that since $\|\gamma\|_\infty = \sup_{t,s\in[0,2\pi]} |\gamma(t,s)| < \infty$, the relations (4.1), (4.18) and (4.24) yield

$$\lim_{n\to\infty} \|\sigma_1 - \nu - \sigma_{1,n}\|_\infty \leq \lim_{n\to\infty} (\|\gamma\|_\infty + \frac{\pi}{6}) \|e - e_n\|_1 = 0. \tag{4.30}$$

Furthermore, we can choose a subsequence $\{t_{n_\ell}\}_{\ell=1}^{\infty}$ in $\{t_n\}_{n=1}^{\infty}$ in such a way that $\lim_{\ell\to\infty} t_{n_\ell} = t^* \in [0, 2\pi]$. Therefore, with respect to (4.30), (4.8) and (4.27) we have

$$v(t^*) = \lim_{\ell\to\infty} x_{n_\ell}(t_{n_\ell}) \leq \lim_{\ell\to\infty} \sigma_{1,n_\ell}(t_{n_\ell}) = \sigma_1(t^*) - \nu < \min_{t\in[0,2\pi]} u(t),$$

which completes the proof of the theorem. □

4.5. Remark. Notice that in contrast to Theorems 2.1 and 3.3, in Theorems 4.3 and 4.4 we do not need to assume (2.1).

At the close of this paper we will give two additional multiplicity results (cf. Theorem 4.7). Their proofs can be done as those of Theorems 4.3 and 4.4, only instead of Lemma 4.1 we have to use its following modification related to Proposition 1.6.

4.6. Lemma. *Assume* (1.2). *Furthermore, let* $k \neq n^2$ *for all* $n \in \mathbb{N}$ *and let* $A \in (0, \infty)$ *and* $B \in (A, \infty)$ *be such that*

$$g(x) + \overline{e} < k\,\frac{A+B}{2} \quad \textit{for } x \in [A, B], \tag{4.31}$$

(1.13) *and* (1.14) *are true. Let*

$$\sigma_1(t) = \frac{A+B}{2} + \int_0^{2\pi} \widetilde{\gamma}(t,s)\,(e(s) - \overline{e})\,\mathrm{d}s \quad \text{for } t \in [0, 2\pi],$$

where $\widetilde{\gamma}$ *is the Green function of the problem* $x'' + k\,x = 0$, $x(0) = x(2\pi)$, $x'(0) = x'(2\pi)$.

Then $\sigma_1 \in \mathbb{AC}^1[0, 2\pi]$, $\sigma_1(t) \in [A, B]$ *for all* $t \in [0, 2\pi]$ *and there is* $\nu_0 > 0$ *such that for each* $\nu \in [-\nu_0, \nu_0]$ *the functions* $(\sigma_1 + \nu, \sigma_1')$ *are lower functions of* (1.1).

Proof. Choose $\nu_0 > 0$ in such a way that

$$(4.32) \quad g(x) - k\,\nu + \overline{e} \le k\,\frac{A+B}{2} \quad \text{for } x \in [A - \nu_0, B + \nu_0] \quad \text{and} \quad \nu \in [-\nu_0, \nu_0].$$

By the proofs of [13, Theorems 3.1 and 3.2], the function

$$\sigma_0(t) = \int_0^{2\pi} \widetilde{\gamma}(t,s)\,(e(s) - \overline{e})\,\mathrm{d}s, \quad t \in [0, 2\pi],$$

possesses the following properties: $\sigma_0 \in \mathbb{AC}^1[0, 2\pi]$, $\overline{\sigma_0} = 0$,

$$\sigma_0''(t) + k\,\sigma_0(t) = e(t) - \overline{e} \quad \text{a.e. on} \quad [0, 2\pi], \quad \sigma_0(0) = \sigma_0(2\pi), \ \sigma_0'(0) = \sigma_0'(2\pi)$$

and $\|\sigma_0\|_\infty \le \Phi(k)\,\|e - \overline{e}\|_1$. In view of (1.13) we have

$$\sigma_1(t) = A + \Phi(k)\,\|e - \overline{e}\|_1 + \sigma_0(t) \quad \text{on} \quad [0, 2\pi].$$

Therefore $\sigma_1(t) \in [A, B]$ for all $t \in [0, 2\pi]$ and, with respect to (4.32), it follows that $(\sigma_1 + \nu, \sigma_1')$ are lower functions of (1.1) for each $\nu \in [-\nu_0, \nu_0]$. □

4.7. Theorem. *Suppose that* $k \ne n^2$ *for all* $n \in \mathbb{N}$ *and replace conditions* (3.7) *and* (1.9) *in Theorem* 4.3 (*Theorem* 4.4) *with* (4.31), (1.13) *and* (1.14). *Then the problem* (1.1) *has at least two positive solutions.*

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