Existence and uniqueness of damped solutions of singular IVPs with ϕ-Laplacian

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Abstract. We study analytical properties of a singular nonlinear ordinary differential equation with a ϕ-Laplacian. In particular we investigate solutions of the initial value problem

$$(p(t)\phi(u'(t)))' + p(t)f(\phi(u(t))) = 0,\ t > 0,\ u(0) = u_0 \in [L_0, L],\ u'(0) = 0$$

on the half-line $[0, \infty)$. Here, f is a continuous function with three zeros $\phi(L_0) < 0 < \phi(L)$, function p is positive on $(0, \infty)$ and the problem is singular in the sense that $p(0) = 0$ and $1/p(t)$ may not be integrable on $[0, 1]$. The main goal of the paper is to prove the existence of damped solutions defined as solutions u satisfying $\sup\{u(t), t \in [0, \infty)\} < L$. Moreover, we study the uniqueness of damped solutions. Since the standard approach based on the Lipschitz property is not applicable here in general, the problem is more challenging. We also discuss the uniqueness of other types of solutions.

1 Introduction

We study the equation

$$(p(t)\phi(u'(t)))' + p(t)f(\phi(u(t))) = 0,\ t > 0 \qquad (1.1) \quad \texttt{\{ppphi\}}$$

with the initial conditions

$$u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L], \qquad (1.2) \quad \texttt{\{ic\}}$$

where

$$\phi \in C^1(\mathbb{R}), \quad \phi'(x) > 0 \text{ for } x \in (\mathbb{R} \setminus \{0\}), \qquad (1.3) \quad \texttt{\{phi\_1\}}$$

$$\phi(\mathbb{R}) = \mathbb{R}, \quad \phi(0) = 0, \qquad (1.4) \quad \texttt{\{phi\_2\}}$$

$$L_0 < 0 < L, \quad f(\phi(L_0)) = f(0) = f(\phi(L)) = 0, \qquad (1.5) \quad \texttt{\{f\_1\}}$$

$$f \in C[\phi(L_0), \phi(L)], \quad xf(x) > 0 \text{ for } x \in ((\phi(L_0), \phi(L)) \setminus \{0\}), \qquad (1.6) \quad \texttt{\{f\_2\}}$$

$$p \in C[0, \infty) \cap C^1(0, \infty), \quad p'(t) > 0 \text{ for } t \in (0, \infty), \quad p(0) = 0. \qquad (1.7) \quad \texttt{\{p\_1\}}$$

A model example of (1.1), (1.2) is a problem with the α-Laplacian described below.

Example 1.1. Consider $\phi(x) = |x|^\alpha \operatorname{sgn} x$, $x \in \mathbb{R}$, where $\alpha \geq 1$. Then $\phi'(x) = \alpha|x|^{\alpha-1}$ and conditions (1.3) and (1.4) are fulfilled. If we take $p(t) = t^\beta$, $t \in [0, \infty)$, where $\beta > 0$, then p fulfils (1.7). As an example of f satisfying conditions (1.5) and (1.6) we can take $f(x) = x\,(x - \phi(L_0))\,(\phi(L) - x)$, $\quad x \in \mathbb{R}$. {ex1}

A special case of equation (1.1), which has the form

$$\left(t^{n-1}u'(t)\right)' + t^{n-1}f(u(t)) = 0,$$

arises in many areas. For example in the study of phase transition of Van der Waals fluids [9], in population genetics, where it serves as a model for the spatial distribution of the genetic composition of a population [8], in the homogeneous nucleation theory [1], in the relativistic cosmology for description of particles which can be treated as domains in the universe [13], or in the nonlinear field theory, in particular, when describing bubbles generated by scalar fields of the Higgs type in the Minkowski spaces [7]. The equation

$$\left(p(t)u'(t)\right)' + q(t)f(u(t)) = 0,$$

without ϕ-Laplacian, was investigated for $p \equiv q$ in [14]–[19] and for $p \not\equiv q$ in [5], [6], [20] and [21]. Other problems close to (1.1), (1.2) can be found in [2]–[4], [10]–[12].

Definition 1.2. A function $u \in C^1[0,\infty)$ with $\phi(u') \in C^1(0,\infty]$ which satisfies equation (1.1) for every $t \in [0,\infty)$ is called a *solution* of equation (1.1). If moreover u satisfies the initial conditions (1.2), then u is called a *solution* of problem (1.1), (1.2).

Definition 1.3. Consider a solution u of problem (1.1), (1.2) with $u_0 \in (L_0, L)$ and denote {def2}

$$u_{sup} = \sup\{u(t) \colon t \in [0,\infty)\}.$$

If $u_{sup} < L$, then u is called a *damped solution* of problem (1.1), (1.2).
If $u_{sup} = L$, then u is called a *homoclinic solution* of problem (1.1), (1.2).
The homoclinic solution is called a *regular homoclinic solution*, if $u(t) < L$ for $t \in [0,\infty)$ and a *singular homoclinic solution*, if there exists $t_0 > 0$ such that $u(t_0) = L$.
If $u_{sup} > L$, then u is called an *escape solution* of problem (1.1), (1.2).

Remark 1.4. Equation (1.1) has the constant solutions $u(t) \equiv L$, $u(t) \equiv 0$ and $u(t) \equiv L_0$. {kons}

Our goal in this paper is to prove new existence and uniqueness results for equation (1.1) with ϕ-Laplacian. The presence of ϕ-Laplacian in equation (1.1) brings difficulties in the study of the uniqueness. For example if $\phi(x) = |x|^\alpha \operatorname{sgn} x$ and $\alpha > 1$, then ϕ fulfils the Lipschitz condition on $\mathbb{R}$. Since $\phi^{-1} = |x|^{\frac{1}{\alpha}} \operatorname{sgn} x$ and $\left(\phi^{-1}\right)'(x) = \frac{1}{\alpha}|x|^{\frac{1}{\alpha}-1}$, we get $\lim_{x\to 0}\left(\phi^{-1}\right)'(x) = \infty$ and the function ϕ^{-1} does not fulfil the Lipschitz condition in the neighbourhood of 0. Since both ϕ and ϕ^{-1} must be present in the operator form of problem (1.1), (1.2), (compare with (4.2)), we cannot use the standard approach with some Lipschitz constant to prove the uniqueness near 0. Therefore we develop a different approach near 0 and show conditions which guarantee the uniqueness of damped and regular homoclinic solutions of problem (1.1), (1.2) (Theorem 5.5) and the uniqueness of escape solutions (Theorem 6.5) of the auxiliary problem (2.1), (1.2) introduced in Section 2.

We also present conditions sufficient for the existence of solutions of problem (1.1), (1.2). The existence of damped solutions of problem (1.1), (1.2) is proved here (Theorem 5.2). The complicated questions about the existence of escape and homoclinic solutions and about nonexistence of singular homoclinic solutions remain open and they will be studied in our next paper.

2 Properties of solutions of auxiliary equation (2.1)

{sec2}

In this section we introduce an auxiliary equation with a bounded nonlinearity and we describe properties of its solutions. By means of these results we proceed to a priori estimates of solutions, existence and continuous dependence of solutions on initial values in next sections. The auxiliary equation has the form

$$(p(t)\phi(u'(t)))' + p(t)\tilde{f}(\phi(u(t))) = 0, \tag{2.1}$$ {vppphi}

where

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in [\phi(L_0), \phi(L)], \\ 0 & \text{for } x < \phi(L_0), \quad x > \phi(L). \end{cases} \tag{2.2}$$ {vf}

Properties of solutions of (2.1) are derived in the next lemmas.

Lemma 2.1. *Let (1.3)–(1.7) hold and let u be a solution of equation (2.1).* {lemma1}

a) *Assume that there exists $a \geq 0$ such that $u(a) \in (0, L)$ and $u'(a) = 0$. Then $u'(t) < 0$ for $t \in (a, \theta]$, where θ is the first zero of u on (a,∞). If such θ does not exist, then $u'(t) < 0$ for $t \in (a,\infty)$.*

b) *Assume that there exists $b \geq 0$ such that $u(b) \in (L_0, 0)$ and $u'(b) = 0$. Then $u'(t) > 0$ for $t \in (b, \theta]$, where θ is the first zero of u on (b,∞). If such θ does not exist, then $u'(t) > 0$ for $t \in (b,\infty)$.*

Proof.

a) Let $a \geq 0$ be such that $u(a) \in (0, L)$ and $u'(a) = 0$. First, we assume that there exists $\theta > a$ satisfying $u(t) > 0$ on (a, θ) and $u(\theta) = 0$. Assume that there exists $\tau \in (a, \theta)$ such that $u'(\tau) \geq 0$, $u(t) \in [u(a), L)$ for $t \in (a, \tau]$. Integrate (2.1) from a to τ and obtain

$$p(\tau)\phi(u'(\tau)) = -\int_a^\tau p(s)\tilde{f}(\phi(u(s)))\,\mathrm{d}s < 0.$$

Hence, by (1.3) and (1.7), $u'(\tau) < 0$, a contradiction. Therefore $u' < 0$ on (a, θ). Furthermore, integrating (2.1) over (a, θ), we get

$$p(\theta)\phi(u'(\theta)) = -\int_a^\theta p(s)\tilde{f}(\phi(u(s)))\,\mathrm{d}s < 0.$$

Thus, by (1.3) and (1.7), $u'(\theta) < 0$ and we have $u' < 0$ on $(a, \theta]$. If u is positive on $[a, \infty)$, we obtain as before $u' < 0$ on (a, ∞).

b) We argue similarly as in a). □

Lemma 2.2. *Let (1.3)–(1.7) hold and let u be a solution of equation (2.1). Assume that there exists $a \geq 0$ such that $u(a) = L$ and $u'(a) = 0$.* {lemma1a}

a) Let $\theta > a$ be the first zero of u on (a, ∞). Then there exists $a_1 \in [a, \theta)$ such that

$$u(a_1) = L, \quad u'(a_1) = 0, \quad 0 \leq u(t) < L, \; u'(t) < 0, \; t \in (a_1, \theta].$$

b) Let $u > 0$ on $[a, \infty)$ and $u \not\equiv L$ on $[a, \infty)$. Then there exists $a_1 \in [a, \infty)$ such that

$$u(a_1) = L, \quad u'(a_1) = 0, \quad 0 < u(t) < L, \; u'(t) < 0, \; t \in (a_1, \infty). \tag{2.3}$$ {lem1aa}

In the both cases $u(t) = L$ for $t \in [a, a_1]$.

Proof.

a) Assume that there exists $t^\star > a$ such that $u(t^\star) > L$. Then we can find $\tau \in [a, t^\star)$ satisfying

$$u(t) > L, \; t \in (\tau, t^\star], \qquad u(\tau) = L. \tag{2.4}$$ {lem1ab}

Hence $u'(\tau) \geq 0$. Integrating (2.1) over $[\tau, t^\star]$, we get, by (2.2),

$$p(t^\star)\phi(u'(t^\star)) = p(\tau)\phi(u'(\tau)) \geq 0,$$

which yields $u'(t^\star) \geq 0$. Therefore $u > L$ on $[t^\star, \infty)$ and u cannot have the zero θ, a contradiction. We have proved $0 < u \leq L$ on $[a, \theta)$ and deduce from (2.1) $(p(t)\phi(u'(t)))' \leq 0$ for $t \in [a, \theta]$. Consequently $u'(t) \leq 0$ and u is nonincreasing on $[a, \theta]$. Hence there exists $a_1 = [a, \theta)$ such that

$$u(a_1) = L, \quad u'(a_1) = 0, \quad 0 < u(t) < L, \; t \in (a_1, \theta).$$

Since u is monotonous on $[a, a_1]$ then $u \equiv L$ on $[a, a_1]$. Now, we can argue as in the proof of Lemma 2.1 a) with a_1 instead of a.

b) Assume as in a) that there exists $t^\star > a$ such that $u(t^\star) > L$. Then we can find $\tau \in [a, t^\star)$ satisfying (2.4). Hence $u'(\tau) \geq 0$. Integrate (2.1) over $[\tau, t]$, where $t \in (\tau, t^\star]$. We get, by (2.2),

$$p(t)\phi(u'(t)) = p(\tau)\phi(u'(\tau)), \qquad t \in (\tau, t^\star].$$

If $u'(\tau) = 0$, then $u'(t) = 0$ for $t \in (\tau, t^\star]$, which contradicts $u(\tau) = L$, $u(t^\star) > L$. Therefore $u'(\tau) > 0$. Let $\xi \in [0, \tau)$ be the minimal number fulfilling $0 < u(t) < L$, $u'(t) > 0$, $t \in (\xi, \tau)$. Since $u(\xi) < L$, $u'(\xi) \geq 0$, we obtain $\xi > a$. Integrating (2.1) over $[a, \xi]$, we derive $u'(\xi) < 0$, a contradiction. We have proved that $0 < u \leq L$ on $[a, \infty)$, and that u is nonincreasing on (a, ∞). If $u \not\equiv L$ on $[a, \infty)$, we can find $a_1 \geq a$ such that (2.3) holds using the arguments from the proof of Lemma 2.1 a). Moreover, $u \equiv L$ on $[a, a_1]$. □

In order to derive further important properties of solutions of (2.1) we need to assume

$$\exists \bar{B} \in (L_0, 0) \colon \tilde{F}\left(\bar{B}\right) = \tilde{F}(L), \quad \text{where } \tilde{F}(x) = \int_0^x \tilde{f}(\phi(s))\,\mathrm{d}s, \quad x \in \mathbb{R} \tag{2.5}$$ {f_4}

and

$$\limsup_{t \to \infty} \frac{p'(t)}{p(t)} < \infty. \tag{2.6}$$ {p_3}

Remark 2.3. According to (2.5), we have $\tilde{F} \in C^1(\mathbb{R})$, $\tilde{F}(0) = 0$, $\tilde{F}$ is positive and increasing on $[0, L]$ and positive and decreasing on $[L_0, 0]$. {pozn6}

Example 2.4. If p, ϕ and f are from Example 1.1 and in addition $L < |L_0|$, then conditions (2.5) and (2.6) are satisfied.

Remark 2.5. From (1.3) and (1.4), we get

$$x\phi(x) > 0 \quad \text{for } x \in (\mathbb{R} \setminus \{0\}), \tag{2.7}$$ {phi_3}

and there exists an inverse function ϕ^{-1}, which is continuous and increasing on $\mathbb{R}$. By (1.7), p is positive and increasing on $(0,\infty)$.

Lemma 2.6. *Assume that (1.3)–(1.7), (2.5) and (2.6) hold. Let u be a solution of equation (2.1) and let there exist $b \geq 0$ and $\theta > b$ such that* {lemma2}

$$u(b) \in \left[\bar{B}, 0\right), \quad u'(b) = 0, \quad u(\theta) = 0, \quad u(t) < 0 \text{ for } t \in [b, \theta). \tag{2.8}$$ {23}

Then there exists $a \in (\theta, \infty)$ such that

$$u'(a) = 0, \quad u'(t) > 0 \text{ for } t \in (b, a), \quad u(a) \in (0, L).$$

Proof. Let u be a solution of equation (2.1) on $[0,\infty)$ satisfying (2.8). Then

$$\phi'(u'(t))u''(t) + \frac{p'(t)}{p(t)}\phi(u'(t)) + \tilde{f}(\phi(u(t))) = 0, \quad t \in (0,\infty). \tag{2.9}$$ {25}

By Lemma 2.1 b and by (2.8) we have $u'(t) > 0$ for $t \in (b, \theta]$.

Step 1. We assume that $a > \theta$ satisfying $u'(a) = 0$ does not exist. Then we get

[1] $$u'(t) > 0, \quad t \in (b, \infty), \tag{2.10}$$ {27}

and hence u is increasing on (b,∞). Since $u(\theta) = 0$, the inequality

$$u(t) > 0, \quad t \in (\theta, \infty) \tag{2.11}$$ {28}

holds. Let $(\theta, A) \subset (\theta, \infty)$ be a maximal interval with the property

$$u(t) < L, \quad t \in (\theta, A). \tag{2.12}$$ {29}

Using (1.3), (1.5), (1.6) and (2.7) we obtain $\tilde{f}(\phi(u(t))) > 0$ for $t \in (\theta, A)$. Consequently, equation (2.9) yields

$$u''(t) < 0, \quad t \in (\theta, A), \tag{2.13}$$ {210}

and thus u' is decreasing on (θ, A).

(i) Let $A < \infty$. Then (2.12) implies $u(A) = L$. Multiplying (2.9) by u' and integrating from b to A we get

$$\int_b^A \phi'\left(u'(s)\right) u'(s)u''(s)\,\mathrm{d}s + \int_b^A \frac{p'(s)}{p(s)}\phi\left(u'(s)\right) u'(s)\,\mathrm{d}s + \int_b^A \tilde{f}(\phi(u(s)))u'(s)\,\mathrm{d}s = 0.$$

After substitutions we derive

$$\int_{u'(b)}^{u'(A)} x\phi'(x)\,\mathrm{d}x + \int_b^A \frac{p'(s)}{p(s)}\phi(u'(s))u'(s)\,\mathrm{d}s + \int_{u(b)}^{u(A)} \tilde{f}(\phi(y))\,\mathrm{d}y = 0. \tag{2.14}$$ {211}

Due to (2.8) and (2.10) u' fulfils $u'(b) = 0$ and $u'(A) > 0$. Therefore conditions (1.7) and (2.7) imply

$$\int_{u'(b)}^{u'(A)} x\phi'(x)\,\mathrm{d}x > 0, \quad \int_b^A \frac{p'(s)}{p(s)}\phi(u'(s))u'(s)\,\mathrm{d}s > 0.$$

Using this we derive from (2.14)

$$\int_{u(b)}^{u(A)} \tilde{f}(\phi(y))\,\mathrm{d}y = \int_{u(b)}^{L} \tilde{f}(\phi(y))\,\mathrm{d}y < 0,$$

and hence $\tilde{F}(L) - \tilde{F}(u(b)) < 0$. By Remark 2.3, (2.5) and (2.8) we obtain

$$\tilde{F}(L) < \tilde{F}(u(b)) \leq \tilde{F}\left(\bar{B}\right) = \tilde{F}(L),$$

which is a contradiction.

[1] Původní labely 26 a 26a jsem vyhodil, protože na ně bylo po jednom odkazu hned za rovnicí. Pokud někoho napadne, jak to přeformulovat abychom se zbavili 27, 28, 29 a 27a, 28a, 29a, abychom to mohli hodit na řádek, tak to přepište. Mě nic nenapadá.

(ii) Now we assume that $A = \infty$. Inequalities (2.11) and (2.12) give

$$0 < u(t) < L \text{ for } t \in (\theta, \infty)\,.$$

By (2.10) u is increasing on (θ, ∞) and

$$\lim_{t\to\infty} u(t) = \ell, \tag{2.15}$$ {213}

where $\ell \in (0, L]$. By (2.10) and (2.13) u' is decreasing and positive on (θ, ∞) and $\lim_{t\to\infty} u'(t) \geq 0$. By (2.15) we have

$$\lim_{t\to\infty} u'(t) = 0. \tag{2.16}$$ {214}

Let $\ell = L$. Similarly as before we derive

$$\int_{u'(b)}^{u'(t)} x\phi'(x)\,\mathrm{d}x + \int_b^t \frac{p'(s)}{p(s)}\phi(u'(s))u'(s)\,\mathrm{d}s + \int_{u(b)}^{u(t)} \tilde{f}(\phi(y))\,\mathrm{d}y = 0, \quad t \in (b, \infty)\,.$$

Since the first integral is positive, we have

$$\int_{u(b)}^{u(t)} \tilde{f}(\phi(y))\,\mathrm{d}y < -\int_b^t \frac{p'(s)}{p(s)}\phi(u'(s))u'(s)\,\mathrm{d}s, \quad t \in (b, \infty)\,.$$

This yields

$$\lim_{t\to\infty}\left(\tilde{F}(u(t)) - \tilde{F}(u(b))\right) \leq -\int_b^\infty \frac{p'(s)}{p(s)}\phi(u'(s))u'(s)\,\mathrm{d}s < 0.$$

Using Remark 2.3, (2.5) and (2.8) we deduce

$$\tilde{F}(L) < \tilde{F}(u(b)) \leq \tilde{F}\left(\bar{B}\right) = \tilde{F}(L),$$

which is a contradiction.

Let $\ell \in (0, L)$. For $t \to \infty$ in (2.9) we get by (1.4) and (2.6)

$$\phi'(0) \cdot \lim_{t\to\infty} u''(t) = -\tilde{f}(\phi(\ell)). \tag{2.17}$$ {25n}

Since $-\tilde{f}(\phi(\ell)) \in (-\infty, 0)$, the inequality $\lim_{t\to\infty} u''(t) < 0$ holds, contrary to (2.16).

We have proved that there exists $a > \theta$ such that $u'(a) = 0$.

Step 2. Let $u' > 0$ on $[\theta, a)$. Then $u(a) > 0$ and Lemma 2.1 b) yields $u' > 0$ on (b, a). It remains to prove that $u(a) < L$. Multiplying (2.9) by u' and integrating from b to a we get similarly as before

$$\int_{u(b)}^{u(a)} \tilde{f}(\phi(y))\,\mathrm{d}y < 0, \quad t \in (b, a)\,,$$

and

$$\tilde{F}(u(a)) < \tilde{F}(u(b)) \leq \tilde{F}\left(\bar{B}\right) = \tilde{F}(L).$$

By Remark 2.3, the inequality $u(a) < L$ holds. □

Lemma 2.7. *Assume that (1.3)–(1.7), (2.5) and (2.6) hold. Let u be a solution of equation (2.1) and let there exist $a \geq 0$ and $\theta > a$ such that* {lemma3}

$$u(a) \in (0, L]\,, \quad u'(a) = 0, \quad u(\theta) = 0, \quad u(t) > 0 \textit{ for } t \in [a, \theta)\,. \tag{2.18}$$ {23a}

Then there exists $b \in (\theta, \infty)$ such that

$$u'(b) = 0, \quad u'(t) < 0 \textit{ for } t \in (a, b), \quad u(b) \in \left(\bar{B}, 0\right).$$

Proof. Let u be a solution of equation (2.1) satisfying (2.18). By Lemmas 2.1 a) and 2.2 a) and (2.18), we have $u'(t) < 0$, for $t \in (a, \theta]$.

Step 1. We assume that $b > \theta$ satisfying $u'(b) = 0$ does not exist. Then we get

$$u'(t) < 0, \quad t \in (a, \infty), \tag{2.19}$$ {27a}

and hence u is decreasing on $[a, \infty]$. Since $u(\theta) = 0$, the inequality

$$u(t) < 0, \quad t \in (\theta, \infty) \tag{2.20}$$ {28a}

holds. Let $(\theta, A) \subset (\theta, \infty)$ be a maximal interval with the property

$$u(t) > \bar{B}, \quad t \in (\theta, A). \tag{2.21}$$ {29a}

Using (1.3), (1.5), (1.6) and (2.7) we obtain $\tilde{f}(\phi(u(t))) < 0$ for $t \in (\theta, A)$. Consequently, equation (2.9) yields

$$u''(t) > 0, \quad t \in (\theta, A) \tag{2.22}$$ {210a}

and thus u' is increasing on (θ, A).

(i) Let $A < \infty$. Then (2.21) implies $u(A) = \bar{B}$. We argue similarly as in proof of Lemma 2.6 Step 1 part (i) and we get

$$\tilde{F}\left(\bar{B}\right) < \tilde{F}(u(a)) \le \tilde{F}(L) = \tilde{F}\left(\bar{B}\right),$$

a contradiction.

(ii) Now we assume that $A = \infty$. By (2.20) and (2.21) we have

$$\bar{B} < u(t) < 0 \text{ for } t \in (\theta, \infty).$$

By (2.19) u is decreasing on (θ, ∞) and $\lim_{t\to\infty} u(t) = \ell \in \left[\bar{B}, 0\right)$. Due to (2.19) and (2.22) u' is increasing and negative on (θ, ∞) and $\lim_{t\to\infty} u'(t) \le 0$. By (2.15) we have (2.16).

Similarly as in the proof of Lemma 2.6 Step 1 part (ii) we obtain a contradiction both for $\ell = \bar{B}$ and for $\ell \in \left(\bar{B}, 0\right)$.

We have shown that there exists $b > \theta$ such that $u'(b) = 0$.

Step 2. Let $u' < 0$ on $[\theta, b)$. Then $u(b) < 0$ and Lemma 2.1 a) yields $u' < 0$ on (a, b). It remains to prove that $u(b) > \bar{B}$. We proceed similarly as in the proof of Lemma 2.6 Step 2 and get $\tilde{F}(u(b)) < \tilde{F}\left(\bar{B}\right)$. By Remark 2.3, the inequality $\bar{B} < u(b)$ holds. ☐

Lemma 2.8. *Assume that (1.3)–(1.7) and (2.6) hold. Let u be a solution of equation (2.1) and let there exists $b \ge 0$ such that* {lemma23}

$$u(b) \in (L_0, 0), \quad u'(b) = 0, \quad u(t) < 0 \text{ for } t \in [b, \infty). \tag{2.23}$$ {30}

Then

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0. \tag{2.24}$$ {31}

Proof. By Lemma 2.1 b), $u'(t) > 0$ for $t \in (b, \infty)$. Hence u is increasing on (b, ∞),

$$L_0 < u(t) < 0, \quad t \in (b, \infty) \tag{2.25}$$ {32}

and

$$\lim_{t\to\infty} u(t) =: \ell \in (u(b), 0]. \tag{2.26}$$ {32a}

Multiplying equation (2.9) by u' and integrating it from b to t, we obtain

$$\psi_1(t) + \psi_2(t) + \psi_3(t) = 0, \quad t \in (b, \infty), \tag{2.27}$$ {33}

where

$$\psi_1(t) = \int_{u'(b)}^{u'(t)} x\phi'(x)\,\mathrm{d}x, \quad \psi_2(t) = \int_b^t \frac{p'(s)}{p(s)} \phi(u'(s))u'(s)\,\mathrm{d}s, \quad \psi_3(t) = \int_{u(b)}^{u(t)} \tilde{f}(\phi(x))\,\mathrm{d}x.$$

We have $\psi_3(t) = \tilde{F}(u(t)) - \tilde{F}(u(b))$, where $\tilde{F}$ is defined by (2.5). Since $\tilde{F}(x)$ is decreasing for $x \in (L_0, 0)$ and u is increasing on (b, ∞), $\tilde{F}(u(t))$ is decreasing for $t \in (b, \infty)$ due to (2.25) and $\lim_{t\to\infty} \tilde{F}(u(t)) = \tilde{F}(\ell)$. Therefore

$$\lim_{t\to\infty} \psi_3(t) =: Q_3 \in \left(-\tilde{F}(L_0), 0\right).$$

The positivity of ψ_1 on (b, ∞) yields the inequality $\psi_2(t) < -\psi_3(t)$ for $t \in (b, \infty)$. Since ψ_2 is continuous, increasing and positive on (b, ∞),

$$\lim_{t\to\infty} \psi_2(t) =: Q_2 \in (0, -Q_3].$$

Consequently (2.27) gives

$$\lim_{t\to\infty} \psi_1(t) =: Q_1 \in \left[0, \tilde{F}(L_0)\right).$$

Therefore

$$\lim_{t\to\infty} \Phi(u'(t)) = Q_1,$$

where

$$\Phi(z) := \int_0^z x\phi'(x)\,\mathrm{d}x, \quad z > 0.$$

Φ is positive, continuous and increasing on $(0,\infty)$ and so its inverse Φ^{-1} is positive, continuous and increasing, as well. Thus

$$\lim_{t\to\infty} \Phi^{-1}(\Phi(u'(t))) = \lim_{t\to\infty} u'(t) = \Phi^{-1}(Q_1) \geq 0.$$

According to (2.25),

$$\lim_{t\to\infty} u'(t) = 0. \tag{2.28}$$ {34}

Finally, assume that $\ell \in (u(b), 0)$. Letting $t \to \infty$ in (2.9), we get, by (1.4), (2.6), that (2.17) holds. Since $-\tilde{f}(\phi(\ell)) \in (0,\infty)$, we get $\lim_{t\to\infty} u''(t) > 0$, contrary to (2.16). Therefore $\ell = 0$ due to (2.26). $\square$

Lemma 2.9. *Assume that (1.3)–(1.7) and (2.6) hold. Let u be a solution of equation (2.1) and let there exists $a \geq 0$ such that* {lemma32}

$$u(a) \in (0, L], \quad u'(a) = 0, \quad u(t) > 0 \text{ for } t \in [a,\infty). \tag{2.29}$$ {35}

Then either

$$u(t) = L \quad \text{for } t \in [a,\infty) \tag{2.30}$$ {36}

or (2.24) holds.

Proof. *Step 1.* Let $u(a) \in (0, L)$. We continue analogously as in proof of Lemma 2.8. By Lemma 2.1 a), $u'(t) < 0$ for $t \in (a,\infty)$. Hence

$$0 < u(t) < L, \quad t \in (a,\infty) \tag{2.31}$$ {322}

and

$$\lim_{t\to\infty} u(t) =: \ell \in [0, u(a)). \tag{2.32}$$ {322a}

Multiplying equation (2.9) by u' and integrating it from a to t, we obtain (2.27) with b replaced by a. By Remark 2.3, $\tilde{F}(x)$ is increasing for $x \in (0, L)$ and since u is decreasing on (a,∞), we get $\tilde{F}(u(t))$ is decreasing for $t \in (a,\infty)$ due to (2.31). Consequently $\lim_{t\to\infty} \tilde{F}(u(t)) = \tilde{F}(\ell)$. Let ψ_1, ψ_2 and ψ_3 be defined as in the proof of Lemma 2.8, where b is replaced by a. Then

$$\lim_{t\to\infty} \psi_3(t) = \lim_{t\to\infty} \tilde{F}(u(t)) - \tilde{F}(u(a)) =: Q_3 \in \left(-\tilde{F}(L), 0\right).$$

The positivity of ψ_1 on (a,∞) yields the inequality $\psi_2(t) < -\psi_3(t)$ for $t \in (a,\infty)$. Since ψ_2 is continuous, increasing and positive on (a,∞),

$$\lim_{t\to\infty} \psi_2(t) =: Q_2 \in (0, -Q_3\,].$$

Consequently

$$\lim_{t\to\infty} \psi_1(t) =: Q_1 \in \left[0, \tilde{F}(L)\right).$$

Therefore

$$\lim_{t\to\infty} \Phi(u'(t)) = Q_1,$$

where

$$\Phi(z) := \int_0^z x\phi'(x)\,\mathrm{d}x, \quad z < 0.$$

Φ is positive, continuous and decreasing on $(-\infty, 0)$ and so its inverse Φ^{-1} is positive, continuous and decreasing, as well. Thus

$$\lim_{t\to\infty} \Phi^{-1}(\Phi(u'(t))) = \lim_{t\to\infty} u'(t) = \Phi^{-1}(Q_1) \geq 0.$$

According to (2.31), we have (2.28).

Similarly as in the proof of Lemma 2.8 we derive a contradiction for $\ell \in (0, u(a))$ and hence we get $\ell = 0$ due to (2.32).

Step 2. Let $u(a) = L$. Clearly, function u, defined by (2.30), satisfies equation (2.1) on $[a,\infty)$. Assume that u does not fulfil (2.30). By Lemma 2.2 b) there exists $a_1 \geq a$ such that $0 < u(t) < L$, $u'(t) < 0$, $t \in (a_1,\infty)$, and we can use the arguments from Step 1 to prove (2.24). $\square$

3 A priori estimates

In order to prove the existence and uniqueness of solutions of the auxiliary problem (2.1), (1.2) and of the original problem (1.1), (1.2), a priori estimates derived in this section are needed.

Lemma 3.1. *Assume that (1.3)–(1.7), (2.5) and (2.6) hold. Let u be a solution of problem (2.1), (1.2) with $u_0 \in \big(L_0, \bar{B}\big)$. Let there exist $\theta > 0$, $a > \theta$ such that* {lemma33}

$$u(\theta) = 0, \quad u(t) < 0 \text{ for } t \in [0, \theta), \quad u'(a) = 0, \quad u'(t) > 0 \text{ for } t \in (\theta, a). \tag{3.1}$$ {41}

Then

$$u(a) \in (0, L], \quad u'(t) > 0 \text{ for } t \in (0, a). \tag{3.2}$$ {42}

Proof. From Lemma 2.1 b) and (3.1), we have $u' > 0$ on $(0, a)$. Therefore, $u(a) > 0$. Now, assume that $u(a) > L$. Hence, there exists $a_0 \in (\theta, a)$ such that $u(t) > L$ on $(a_0, a]$. Integrating equation (2.1) over (a_0, a) and using (2.2), we get

$$p(a_0)\phi(u'(a_0)) - p(a)\phi(u'(a)) = \int_{a_0}^{a} p(s)\tilde{f}(\phi(u(s)))\,\mathrm{d}s = 0,$$

and so $p(a_0)\phi(u'(a_0)) = 0$. Thus $u'(a_0) = 0$, contrary to $u' > 0$ on $(0, a)$. We have proved $u(a) \leq L$. $\square$

Lemma 3.2. *Let assumptions (1.3)–(1.7), (2.5) and (2.6) hold. Let u be a solution of problem (2.1), (1.2) with $u_0 \in (L_0, 0) \cup (0, L)$. Then* {lemma34}

$$u_0 \in \big[\bar{B}, 0\big) \cup (0, L) \quad \Rightarrow \quad \bar{B} < u(t) < L, \quad t \in (0, \infty)\,, \tag{3.3}$$ {u1}

$$u_0 \in \big(L_0, \bar{B}\big) \quad \Rightarrow \quad u_0 < u(t), \quad t \in (0, \infty)\,. \tag{3.4}$$ {u2}

Proof. Let $u(0) = u_0 \in (0, L)$. If $u > 0$ on $(0, \infty)$, then, by Lemma 2.1 a), $u' < 0$ on $(0, \infty)$ and (3.3) holds. Assume that there exists $\theta_1 > 0$ such that $u(\theta_1) = 0$, $u(t) > 0$ for $t \in [0, \theta_1)$. According to Lemma 2.7,

$$\exists b \in (\theta_1, \infty)\colon u'(b) = 0, \quad u'(t) < 0 \text{ for } t \in (0, b), \quad u(b) = \big(\bar{B}, 0\big)\,.$$

If $u < 0$ on (b, ∞), then, by Lemma 2.1 b), u is increasing on (b, ∞) and (3.3) is valid. Assume that there exists $\theta_2 > b$ such that $u(\theta_2) = 0$, $u(t) < 0$ for $t \in [b, \theta_2)$. Due to Lemma 2.6,

$$\exists a \in (\theta_2, \infty)\colon u'(a) = 0, \quad u'(t) > 0 \text{ for } t \in (b, a), \quad u(a) = (0, L).$$

Now we use the previous arguments replacing 0 by a.

Let $u(0) = u_0 \in \big[\bar{B}, 0\big)$. We have the same situation as before, where b is replaced by 0. So we argue similarly.

Let $u(0) = u_0 \in \big(L_0, \bar{B}\big)$. If $u < 0$ on $(0, \infty)$, then, by Lemma 2.1 b), $u' > 0$ on $(0, \infty)$ and (3.4) is valid. Assume that there exists $\theta_1 > 0$ such that $u(\theta_1) = 0$, $u(t) < 0$ for $t \in [0, \theta_1)$. By Lemma 2.1 b), $u' > 0$ on $(0, \theta_1]$. If $u' > 0$ on (θ_1, ∞), then (3.4) holds. Assume that there exists $a > \theta_1$ such that $u'(a) = 0$, $u'(t) > 0$ for $t \in (\theta_1, a)$. According to Lemma 3.1, (3.2) holds. If $u > 0$ on $[a, \infty)$, (3.4) is valid. Let there exists $\theta_2 > a$ such that $u(\theta_2) = 0$, $u > 0$ on $[a, \theta_2)$. We can apply Lemma 2.7 and argue as before. $\square$

By (2.2), there exists $\tilde{M} > 0$ such that

$$\left|\tilde{f}(\phi(x))\right| \leq \tilde{M}, \quad x \in \mathbb{R}. \tag{3.5}$$ {r2}

Lemma 3.3. *Assume (1.3)–(1.7). Let u be a solution of problem (2.1), (1.2) with $u_0 \in [L_0, L]$. The inequality* {odh}

$$\int_0^{\beta} \frac{p'(t)}{p(t)} |\phi(u'(t))|\,\mathrm{d}t \leq \tilde{M}(\beta - \varphi(\beta)) \tag{3.6}$$ {r3}

is valid for every $\beta > 0$. If moreover (2.5) and (2.6) hold, then there exists $\tilde{c} > 0$ such that

$$|u'(t)| \leq \tilde{c}, \quad t \in [0, \infty), \tag{3.7}$$ {r4}

for every solution u of (2.1), (1.2) with $u_0 \in [L_0, 0) \cup (0, L]$.

Proof. *Step 1.* Let u be solution of (2.1), (1.2) with $u_0 \in [L_0, L]$. Integrating equation (2.1) over $(0,t)$, $t>0$, and using (3.5), we have

$$|\phi(u'(t))| = \left| -\frac{1}{p(t)} \int_0^t p(\tau)\tilde{f}(\phi(u(\tau)))\,\mathrm{d}\tau \right| \le \frac{\tilde{M}}{p(t)} \int_0^t p(\tau)\,\mathrm{d}\tau$$

and

$$\frac{p'(t)}{p(t)}\,|\phi(u'(t))| \le \tilde{M}\frac{p'(t)}{p^2(t)} \int_0^t p(\tau)\,\mathrm{d}\tau.$$

Choose a $\beta > 0$. Integrating this inequality by parts from 0 to β, we get (3.6).

Step 2. Assume moreover that (2.5) and (2.6) hold. Denote

$$\Psi_1(z) := \int_0^z x\phi'(x)\,\mathrm{d}x; \quad \Psi_2(z) := \int_0^z x\phi'(-x)\,\mathrm{d}x; \quad z \in [0,\infty).$$

Clearly, Ψ_1, Ψ_2 are positive, continuous and increasing on $(0,\infty)$. Put

$$\tilde{c} = \max\left\{\Psi_1^{-1}\left(\tilde{F}(L_0)\right), \Psi_2^{-1}\left(\tilde{F}(L)\right)\right\}, \qquad (3.8) \qquad \{\mathtt{t2}\}$$

where $\tilde{F}$ is defined in (2.5).

Let $u(0) = u_0 \in (L_0, 0)$, $u'(0) = 0$ and let u be a solution of equation (2.1). Then (2.9) holds.

1. Assume that $u < 0$ on $[0,\infty)$. By Lemma 2.1 b) $u' > 0$ on $(0,\infty)$, and by Lemma 2.8 $\lim_{t\to\infty} u'(t) = 0$. Therefore there exists $\xi \in (0,\infty)$ such that

$$\max_{t\in[0,\infty)} |u'(t)| = u'(\xi) > 0, \quad u(\xi) \in (u_0, 0). \qquad (3.9) \qquad \{\mathtt{t4}\}$$

Multiplying (2.9) by u' and integrating over $[0,\xi]$ we get

$$\int_{u'(0)}^{u'(\xi)} x\phi'(x)\,\mathrm{d}x + \int_0^\xi \frac{p'(t)}{p(t)}\phi\left(u'(t)\right)u'(t)\,\mathrm{d}t + \int_{u(0)}^{u(\xi)} \tilde{f}\left(\phi(x)\right)\,\mathrm{d}x = 0. \qquad (3.10) \qquad \{\mathtt{t5}\}$$

Since the second integral in (3.10) is positive, (3.9) and (3.10) yield

$$\Psi_1\left(u'(\xi)\right) < \tilde{F}(u_0) - \tilde{F}\left(u(\xi)\right) < \tilde{F}(u_0) < \tilde{F}(L_0).$$

Therefore

$$0 < u'(\xi) < \Psi_1^{-1}\left(\tilde{F}(L_0)\right). \qquad (3.11) \qquad \{\mathtt{t6}\}$$

Due to (3.8) and (3.9) estimate (3.7) is proved.

2. Assume that $\theta \in (0,\infty)$ is such that $u < 0$ on $[0,\theta)$, $u(\theta) = 0$. Then by Lemma 2.1 b), $u' > 0$ on $(0,\theta]$. Let $a > \theta$ be such that $u' > 0$ on (θ, a), $u'(a) = 0$. On interval (θ, a) we have $u > 0$, $u' > 0$ and by (1.3), (1.6), (1.7), (2.7) and (2.9) we get $u'' < 0$ on $[\theta, a)$. Therefore u' is decreasing on $[\theta, a)$ and there exists $\xi \in (0,\theta)$ such that

$$\max_{t\in[0,a)} |u'(t)| = u'(\xi) > 0, \quad u(\xi) \in (u_0, 0). \qquad (3.12) \qquad \{\mathtt{t7}\}$$

Analogously as in part 1 we get (3.11) and if $a = \infty$ then estimate (3.7) is proved.

3. Assume that $a < \infty$ in (3.12). We have $u'(a) = 0$ and by Lemma 2.6 and Lemma 3.1 we deduce that $u(a) \in (0, L]$. Let $u > 0$ on $[a,\infty)$. Then Lemma 2.9 gives $\lim_{t\to\infty} u'(t) = 0$ and hence there exists $\eta \in (a,\infty)$ such that

$$\max_{t\in[a,\infty)} |u'(t)| = -u'(\eta) > 0, \quad u(\eta) \in (0, u(a)). \qquad (3.13) \qquad \{\mathtt{t8}\}$$

Multiplying (2.9) by u' and integrating over $[a,\eta]$ we get

$$\int_{u'(a)}^{|u'(\eta)|} x\phi'(-x)\,\mathrm{d}x + \int_a^\eta \frac{p'(t)}{p(t)}\phi\left(u'(t)\right)u'(t)\,\mathrm{d}t + \int_{u(a)}^{u(\eta)} \tilde{f}\left(\phi(x)\right)\,\mathrm{d}x = 0. \qquad (3.14) \qquad \{\mathtt{t9}\}$$

Since the second integral in (3.10) is positive, (3.9) and (3.10) yield

$$\Psi_2\left(|u'(\eta)|\right) < \tilde{F}(u(a)) - \tilde{F}\left(u(\eta)\right) < \tilde{F}(L).$$

Then

$$0 < |u'(\eta)| < \Psi_2^{-1}\left(\tilde{F}(L)\right). \qquad (3.15) \qquad \{\mathtt{t10}\}$$

Using (3.11), (3.12), (3.13) and (3.15) we obtain (3.7) due to (3.8).

4. Assume, that there exists $\chi \in (a, \infty)$ which is the next zero of u. Summarized, we have $u(a) \in (0, L]$, $u'(a) = 0$, $u(\chi) = 0$, $u > 0$ on $[a, \chi)$. By Lemma 2.7 there exists $b \in (\chi, \infty)$ such that $u'(b) = 0$, $u' < 0$ on (a, b), $u(b) \in \left(\bar{B}, 0\right)$ and by (2.9) we have $u'' > 0$ on $[\chi, b)$. Consequently there exists $\eta \in (a, \chi)$ such that
$$\max_{t\in[a,b]} |u'(t)| = -u'(\eta) > 0, \quad u(\eta) \in (0, u(a)) \,. \tag{3.16}$$ {t11}

   Similarly as in part 3 we get (3.15) and (3.7).

5. Since $u(b) < 0$ we continue repeating the argument of parts 1 - 3 with b on place of 0 and the arguments of part 4 writing $\tilde{b}$ instead of b. After finite or infinite number of steps we obtain (3.7).

If $u_0 \in (0, L)$, we can argue similarly. □

4 Existence and continuous dependence of solutions on initial values

The existence of solutions of the auxiliary problem (2.1), (1.2) is proved in Theorem 4.1 by means of the Schauder Fixed Point Theorem. Moreover, the question about continuous dependence of solutions on initial values is discussed in Theorems 4.3, 4.6, 4.8. These results are extended to the existence and uniqueness of damped solutions of the original problem (1.1), (1.2) on grounds of a priori estimates derived in Section 5.

For the following investigation, we introduce a function φ
$$\varphi(t) := \frac{1}{p(t)} \int_0^t p(s)\,\mathrm{d}s, \quad t \in (0, \infty), \quad \varphi(0) = 0.$$
This function is continuous on $[0, \infty)$ and satisfies
$$0 < \varphi(t) \le t, \quad t \in (0, \infty), \qquad \lim_{t\to 0+} \varphi(t) = 0. \tag{4.1}$$ {p_4}

Theorem 4.1 **(Existence of solutions of problem (2.1), (1.2)).** *Assume (1.3)–(1.7). Then, for each $u_0 \in [L_0, L]$, there exists a solution u of problem (2.1), (1.2).* {exi}

Proof. Clearly, for $u_0 = L_0$, $u_0 = 0$ and $u_0 = L$ there exists a solution by Remark 1.4. Assume that $u_0 \in (L_0, 0) \cup (0, L)$. Integrating equation (2.1), we get the equivalent form of problem (2.1), (1.2)
$$u(t) = u_0 + \int_0^t \phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\tilde{f}(\phi(u(\tau)))\,\mathrm{d}\tau\right)\mathrm{d}s, \quad t \in [0, \infty)\,. \tag{4.2}$$ {34a}

Choose a $\beta > 0$, consider the Banach space $C\,[0, \beta]$ with the maximum norm and define an operator $\mathcal{F}\colon C\,[0, \beta] \to C\,[0, \beta]$,
$$(\mathcal{F}u)(t) = u_0 + \int_0^t \phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\tilde{f}(\phi(u(\tau)))\,\mathrm{d}\tau\right)\mathrm{d}s.$$
Put $\Lambda = \max\{|L_0|, L\}$ and consider the ball $\mathcal{B}\,(0, R) = \left\{u \in C\,[0, \beta] : \|u\|_{C[0,\beta]} \le R\right\}$, where $R = \Lambda + \beta\phi^{-1}\left(\tilde{M}\beta\right)$ and $\tilde{M}$ is from (3.5). Since ϕ is increasing on $\mathbb{R}$, ϕ^{-1} is also increasing on $\mathbb{R}$ and, by (4.1), $\phi^{-1}\left(\tilde{M}\varphi(t)\right) \le \phi^{-1}\left(\tilde{M}\beta\right)$, $t \in [0, \beta]$. The norm of $\mathcal{F}u$ can be estimated as follows
$$\begin{aligned}\|\mathcal{F}u\|_{C[0,\beta]} &= \max_{t\in[0,\beta]}\left|u_0 + \int_0^t \phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\tilde{f}(\phi(u(\tau)))\,\mathrm{d}\tau\right)\mathrm{d}s\right| \le \Lambda + \int_0^t \left|\phi^{-1}\left(\tilde{M}\varphi(s)\right)\right|\mathrm{d}s \\ &\le \Lambda + \int_0^t \phi^{-1}\left(\tilde{M}\beta\right)\mathrm{d}s \le \Lambda + \beta\phi^{-1}\left(\tilde{M}\beta\right) = R,\end{aligned}$$
which yields that $\mathcal{F}$ maps $\mathcal{B}\,(0, R)$ on itself.

Let us prove that $\mathcal{F}$ is compact on $\mathcal{B}\,(0, R)$. Choose a sequence $\{u_n\} \subset C\,[0, \beta]$ such that $\lim_{n\to\infty} \|u_n - u\|_{C[0,\beta]} = 0$. We have
$$(\mathcal{F}u_n)(t) - (\mathcal{F}u)(t) = \int_0^t \left(\phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\tilde{f}(\phi(u_n(\tau)))\,\mathrm{d}\tau\right) - \phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\tilde{f}(\phi(u(\tau)))\,\mathrm{d}\tau\right)\right)\mathrm{d}s.$$

Since $\tilde{f}(\phi)$ is continuous on $[0,\beta]$, we get

$$\lim_{n\to\infty}\left\|\tilde{f}(\phi(u_n))-\tilde{f}(\phi(u))\right\|_{C[0,\beta]}=0.$$

Put

$$\begin{aligned}
A_n(t)&=-\frac{1}{p(t)}\int_0^t p(\tau)\tilde{f}(\phi(u_n(\tau)))\,\mathrm{d}\tau,\\
A(t)&=-\frac{1}{p(t)}\int_0^t p(\tau)\tilde{f}(\phi(u(\tau)))\,\mathrm{d}\tau,\quad t\in(0,\beta],\quad A_n(0)=A(0)=0,\quad n\in\mathbb{N}.
\end{aligned}$$

Then, for a fixed $n\in\mathbb{N}$,

$$|A_n(t)-A(t)|=\left|\frac{1}{p(t)}\int_0^t p(\tau)\left(\tilde{f}(\phi(u(\tau)))-\tilde{f}(\phi(u_n(\tau)))\,\mathrm{d}\tau\right)\right|,\quad t\in(0,\beta]$$

and, by (4.1) and (3.5), $\lim_{t\to0^+}|A_n(t)-A(t)|=0$. Therefore $A_n-A\in C[0,\beta]$ and

$$\|A_n-A\|_{C[0,\beta]}\le\left\|\tilde{f}(\phi(u_n))-\tilde{f}(\phi(u))\right\|_{C[0,\beta]}\beta,\quad n\in\mathbb{N}.$$

This implies that $\lim_{n\to\infty}\|A_n-A\|_{C[0,\beta]}=0$. Using the continuity of ϕ^{-1} on $\mathbb{R}$, we have $\lim_{n\to\infty}\left\|\phi^{-1}(A_n)-\phi^{-1}(A)\right\|_{C[0,\beta]}=0$. Therefore

$$\begin{aligned}
\lim_{n\to\infty}\|\mathcal{F}u_n-\mathcal{F}u\|_{C[0,\beta]}&=\lim_{n\to\infty}\left\|\int_0^t\left(\phi^{-1}(A_n(s))-\phi^{-1}(A(s))\right)\mathrm{d}s\right\|_{C[0,\beta]}\\
&\le\beta\lim_{n\to\infty}\left\|\phi^{-1}(A_n)-\phi^{-1}(A)\right\|_{C[0,\beta]}=0,
\end{aligned}$$

that is the operator $\mathcal{F}$ is continuous.

Choose an arbitrary $\varepsilon>0$ and put $\delta=\frac{\varepsilon}{\phi^{-1}(\tilde{M}\beta)}$. Then, for $t_1,t_2\in[0,\beta]$ and for $u\in\mathcal{B}(0,R)$, we have

$$\begin{aligned}
|t_1-t_2|<\delta\Rightarrow|(\mathcal{F}u)(t_1)-(\mathcal{F}u)(t_2)|&=\left|\int_{t_2}^{t_1}\phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\tilde{f}(\phi(u(\tau)))\,\mathrm{d}\tau\right)\mathrm{d}s\right|\le\left|\int_{t_2}^{t_1}\phi^{-1}\left(\tilde{M}\varphi(s)\right)\mathrm{d}s\right|\\
&\le\left|\int_{t_2}^{t_1}\phi^{-1}\left(\tilde{M}\beta\right)\mathrm{d}s\right|=\phi^{-1}\left(\tilde{M}\beta\right)|t_1-t_2|<\phi^{-1}\left(\tilde{M}\beta\right)\delta=\varepsilon.
\end{aligned}$$

Hence, functions in $\mathcal{F}(\mathcal{B}(0,R))$ are equicontinuous, and, by the Arzelà–Ascoli theorem, the set $\mathcal{F}(\mathcal{B}(0,R))$ is relatively compact. Consequently, the operator $\mathcal{F}$ is compact on $\mathcal{B}(0,R)$.

The Schauder fixed point theorem yields a fixed point $u^\star$ of $\mathcal{F}$ in $\mathcal{B}(0,R)$. Therefore,

$$u^\star(t)=u_0+\int_0^t\phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\tilde{f}(\phi(u^\star(\tau)))\,\mathrm{d}\tau\right)\mathrm{d}s.$$

Hence, $u^\star(0)=u_0$,

$$(p(t)\phi((u^\star)'(t)))'=-p(t)\tilde{f}\left(\phi(u^\star(t))\right),\quad t\in[0,\beta].$$

Further,

$$|(u^\star)'(t)|=\left|\phi^{-1}\left(-\frac{1}{p(t)}\int_0^t p(s)\tilde{f}(\phi(u^\star(s)))\,\mathrm{d}s\right)\right|\le\phi^{-1}\left(\tilde{M}\varphi(t)\right),\quad t\in[0,\beta].$$

Thus, by (4.1), $\lim_{t\to0^+}\phi^{-1}\left(\tilde{M}\varphi(t)\right)=\phi^{-1}(0)=0$ and therefore $\lim_{t\to0^+}(u^\star)'(t)=0=(u^\star)'(0)$. According to (2.2), $\tilde{f}(\phi(u^\star(t)))$ is bounded on $[0,\infty)$ and hence, by Theorem 11.5 in [11], $u^\star$ can be extended to interval $[0,\infty)$ as a solution of equation (2.1). □

Example 4.2. Consider $\phi\colon\mathbb{R}\to\mathbb{R}$ given by one of the next formulas {mex1}

$$\phi(x)=|x|^\alpha\operatorname{sgn}x,\quad\alpha\ge1,\tag{4.3}$$ {exphi}

$$\phi(x)=\left(x^4+2x^2\right)\operatorname{sgn}x,\tag{4.4}$$ {exphi1}

$$\phi(x)=\sinh x=\frac{e^x-e^{-x}}{2},\tag{4.5}$$ {exphi2}

$$\phi(x)=\operatorname{arg\,sinh}x=\ln\left(x+\sqrt{x^2+1}\right),\tag{4.6}$$ {exphi3}

$$\phi(x)=\ln(|x|+1)\operatorname{sgn}x,\tag{4.7}$$ {exphi4}

$$\phi(x)=((|x|+1)^\alpha-1)\operatorname{sgn}x.\tag{4.8}$$ {exphi5}

Assume that $\phi(L) < -\phi(L_0)$ and put

$$p(t) = t^\beta, \quad t \in [0, \infty), \ \beta > 0,$$
$$f(x) = k|x|^\gamma \operatorname{sgn} x (x - \phi(L_0))(\phi(L) - x), \quad x \in [\phi(L_0), \phi(L)], \ \gamma > 0, \ k > 0.$$

Then the functions p, ϕ and f fulfil all assumptions of Theorem 4.1. In particular $\phi \in \mathrm{Lip}_{loc}(\mathbb{R})$ for each ϕ given by (4.3)–(4.8). Therefore the auxiliary problem (2.1), (1.2) has a solution for every $u_0 \in [L_0, L]$.

Further we examine the uniqueness of solutions of the auxiliary problem (2.1), (1.2). Our arguments will be based on a continuous dependence on initial values expressed in Theorem 4.3, Theorem 4.6 and Theorem 4.8.

Assumption (1.3) implies that $\phi \in \mathrm{Lip}_{loc}(\mathbb{R})$. This need not be true for ϕ^{-1} as we have shown in Introduction for $\phi(x) = |x|^\alpha \operatorname{sgn} x$, $\alpha > 1$. The next theorem discusses the special case when both ϕ and ϕ^{-1} are locally Lipschitz continuous.

Theorem 4.3 (Uniqueness and continuous dependence on initial values I). *Assume (1.3)–(1.7) and* {invlip}

$$f \in \mathrm{Lip}\,[\phi(L_0), \phi(L)], \tag{4.9}$$ {cdlip}
$$\phi^{-1} \in \mathrm{Lip}_{loc}(\mathbb{R}). \tag{4.10}$$ {liplip}

Let u_i be a solution of problem (2.1), (1.2) with $u_0 = B_i \in [L_0, L]$, $i = 1, 2$. Then, for each $\beta > 0$, there exists $K > 0$ such that

$$\|u_1 - u_2\|_{C^1[0,\beta]} \le K|B_1 - B_2|. \tag{4.11}$$ {KK}

Furthermore, any solution of problem (2.1), (1.2) with $u_0 \in [L_0, L]$ is unique on $[0, \infty)$.

Proof. Let $i \in 1, 2$ and let u_i be a solution of problem (2.1), (1.2) with $u_0 = B_i$. By integrating (2.1) over $(0, t)$, we obtain

$$\phi(u_i'(t)) = A_i(t), \qquad u_i(t) = B_i + \int_0^t \phi^{-1}(A_i(s))\, \mathrm{d}s, \ t \in [0, \infty), \tag{4.12}$$ {conIII}

where

$$A_i(s) = -\frac{1}{p(s)} \int_0^s p(\tau) \tilde{f}(\phi(u_i(\tau)))\, \mathrm{d}\tau.$$

Choose $\beta > 0$. Since $u_i, \phi(u_i') \in C[0, \beta]$, there exist $m, M \in \mathbb{R}$ such that

$$m \le u_i(t) \le M, \quad m \le \phi(u_i'(t)) \le M, \quad \text{for } t \in [0, \beta], \ i = 1, 2.$$

According to (1.3), (4.9) and (4.10) there exist positive constants $\Lambda_f, \Lambda_\phi, \Lambda_{inv}$ satisfying

$$|f(x_1) - f(x_2)| \le \Lambda_f |x_1 - x_2|, \quad x_1, x_2 \in [\phi(L_0), \phi(L)],$$
$$|\phi(x_1) - \phi(x_2)| \le \Lambda_\phi |x_1 - x_2|, \quad x_1, x_2 \in [m, M],$$
$$\left|\phi^{-1}(x_1) - \phi^{-1}(x_2)\right| \le \Lambda_{inv} |x_1 - x_2|, \quad x_1, x_2 \in [m, M].$$

Denote $\rho(t) := \max\{|u_1(s) - u_2(s)| : s \in [0, t]\}$, $t \in [0, \beta]$. Then, by (4.1),

$$\begin{aligned} |A_1(s) - A_2(s)| &\le \frac{1}{p(s)} \int_0^s p(\tau) \left|\tilde{f}(\phi(u_1(\tau))) - \tilde{f}(\phi(u_2(\tau)))\right| \mathrm{d}\tau \\ &\le \Lambda_f \Lambda_\phi \frac{1}{p(s)} \int_0^s p(\tau) |u_1(\tau) - u_2(\tau)|\, \mathrm{d}\tau \le \Lambda_f \Lambda_\phi \rho(s) \beta, \end{aligned}$$

and by virtue of (4.12)

$$\begin{aligned} \rho(t) &\le |B_1 - B_2| + \int_0^t \left|\phi^{-1}(A_1(s)) - \phi^{-1}(A_2(s))\right| \mathrm{d}s \le |B_1 - B_2| + \Lambda_{inv} \int_0^t |A_1(s) - A_2(s)|\, \mathrm{d}s \\ &\le |B_1 - B_2| + \Lambda_f \Lambda_\phi \Lambda_{inv} \beta \int_0^t \rho(s)\, \mathrm{d}s, \quad t \in [0, \beta]. \end{aligned}$$

The Gronwall lemma yields

$$\rho(t) \le |B_1 - B_2| e^{L\beta^2}, \quad t \in [0, \beta], \tag{4.13}$$ {conIII_rh

where $L := \Lambda_f \Lambda_\phi \Lambda_{inv}$. Similarly, from (4.12) it follows

$$|u_1'(t) - u_2'(t)| \le \left|\phi^{-1}(A_1(t)) - \phi^{-1}(A_2(t))\right| \le \Lambda_{inv} |A_1(t) - A_2(t)| \le L \rho(t) \beta, \quad t \in [0, \beta].$$

Applying (4.13), we get

$$\max\{|u_1'(t)-u_2'(t)|: t\in[0,\beta]\}\le |B_1-B_2|L\beta e^{L\beta^2}.$$

Consequently,

$$\|u_1-u_2\|_{C^1[0,\beta]}\le |B_1-B_2|(1+L\beta)e^{L\beta^2},$$

that is (4.11) holds for

$$K:=(1+L\beta)e^{L\beta^2}.$$

Clearly, if $B_1=B_2$, we have $u_1=u_2$ on each $[0,\beta]\subset\mathbb{R}$ and the uniqueness for problem (2.1), (1.2) on $[0,\infty)$ follows. □

Remark 4.4. If also (2.5) and (2.6) are fulfilled, we can use (3.7) and get universal estimates for $\phi(u_i')$ and u_i. This is the case that K in (4.11) does not depend on choice of u_1, u_2. {unire}

Example 4.5. In order to apply Theorem 4.3 we need both ϕ and ϕ^{-1} from $\mathrm{Lip}_{loc}(\mathbb{R})$. Let us check the functions ϕ in Example 4.2 from this point of view:

$$\begin{array}{lll}
\phi(x)=|x|^\alpha\operatorname{sgn}x,\quad \alpha\ge 1 & \Rightarrow \phi^{-1}(x)=|x|^{\frac{1}{\alpha}}\operatorname{sgn}x & \notin \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})\\
\phi(x)=\left(x^4+2x^2\right)\operatorname{sgn}x, & \Rightarrow \phi^{-1}(x)=\sqrt{\sqrt{|x|+1}-1} & \notin \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})\\
\phi(x)=\sinh x=\dfrac{e^x-e^{-x}}{2}, & \Rightarrow \phi^{-1}(x)=\operatorname{arg\,sinh}x & \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})\\
\phi(x)=\operatorname{arg\,sinh}x=\ln\left(x+\sqrt{x^2+1}\right) & \Rightarrow \phi^{-1}(x)=\sinh x & \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})\\
\phi(x)=\ln(|x|+1)\operatorname{sgn}x & \Rightarrow \phi^{-1}(x)=\left(e^{|x|-1}\right)\operatorname{sgn}x & \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})\\
\phi(x)=((|x|+1)^\alpha-1)\operatorname{sgn}x,\quad \alpha\in(0,1) & \Rightarrow \phi^{-1}(x)=\left((|x|+1)^{\frac{1}{\alpha}}-1\right)\operatorname{sgn}x & \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}).
\end{array}$$

Consider p, f from Example 4.2 with $\gamma\ge 1$ and ϕ given by one of the formulas (4.5)–(4.8). Then all assumptions of Theorem 4.3 are fulfilled and problem (2.1), (1.2) has a unique solution for $u_0\in[L_0,L]$.

Note that if $\gamma\in(0,1)$, then f is not Lipschitz continuous on a neighbourhood of 0, that is (4.9) is not valid. Similarly, in the case that ϕ is given by (4.3) or (4.4), then ϕ^{-1} is not Lipschitz continuous on a neighbourhood of 0 and hence (4.10) falls.

In next two theorems we show assumptions under which solutions of problem (2.1), (1.2) continuously depend on their initial values in the case that ϕ^{-1} is not locally Lipschitz continuous.

Theorem 4.6 (Continuous dependence on initial values II). *Assume (1.3)–(1.7), (2.5), (2.6), (4.9) and* {th28a}

$$\limsup_{x\to 0^-}\left(-x\left(\phi^{-1}\right)'(x)\right)<\infty,\quad \phi' \text{ is nonincreasing on } (-\infty,0). \tag{4.14}$$ {cdII10}

Let B_1, B_2 satisfy

$$B_1\in(2\varepsilon,L-2\varepsilon),\quad |B_1-B_2|<\varepsilon$$

for some $\varepsilon>0$. Let u_i be a solution of problem (2.1), (1.2) with $u_0=B_i$, $i=1,2$. Then for each $\beta>0$ where

$$u_i'<0 \text{ on } (0,\beta],\ i=1,2,$$

there exists $K\in(0,\infty)$ such that

$$\|u_1-u_2\|_{C^1[0,\beta]}\le K|B_1-B_2|.$$

Proof. Let u_i be a solution of problem (2.1), (1.2) with $u_0=B_i$, $i=1,2$. Then by integrating (2.1) over $(0,t)$, $t\in(0,\infty)$, we obtain

$$\begin{aligned}
\phi(u_i'(t))&=-\frac{1}{p(t)}\int_0^t p(s)f(\phi(u_i(s)))\,\mathrm{d}s=:A_i(t)\\
u_i(t)&=B_i+\int_0^t\phi^{-1}(A_i(s))\,\mathrm{d}s,\quad i=1,2.
\end{aligned}$$

Therefore

$$|u_1(t)-u_2(t)|\le|B_1-B_2|+\int_0^t\left|\phi^{-1}(A_1(s))-\phi^{-1}(A_2(s))\right|\mathrm{d}s,\quad t\in(0,\infty).$$

In order to obtain the required estimate, we restrict our consideration on a small interval $[0,\delta]$ for a suitably chose $\delta>0$ in Step 1. Then we prolongate the result on $[0,\beta]$ in Step 2.

Step 1. Assumptions (1.3)–(1.6), (4.9), (4.14) yield the existence of positive constants $\Lambda_f, \Lambda_\phi, K_1, K_2$ such that

$$\begin{aligned} &|f(y_1)-f(y_2)| \le \Lambda_f |y_1-y_2|, \quad y_1,y_2 \in [\phi(L_0),\phi(L)],\\ &|\phi(x_1)-\phi(x_2)| \le \Lambda_\phi |x_1-x_2|, \quad x_1,x_2\in[L_0,L],\\ &K_1 = \min\{f(\phi(x))\colon x\in[B_1-2\varepsilon, B_1+2\varepsilon]\}, \end{aligned} \tag{4.15}$$ {K1}

$$0 < -x\left(\phi^{-1}\right)'(x) \le K_2, \quad x\in[-1,0). \tag{4.16}$$ {72a}

By Lemma 3.3, there exists $\tilde c>0$ such that $|u_i'|\le \tilde c, i=1,2$ on $[0,\infty)$. Let us choose δ such that

$$0<\delta \le \min\left\{\frac{\varepsilon}{\tilde c}, \frac{1}{K_1}, \frac{K_1}{2K_2\Lambda_f\Lambda_\phi}\right\}. \tag{4.17}$$ {cdIIdelta

Then we get for $t\in[0,\delta]$

$$|B_1-u_1(t)| = |u_1(0)-u_1(t)| \le \tilde c\delta \le \varepsilon.$$

This yields $u_1(t)\in[B_1-\varepsilon, B_1+\varepsilon]$ for $t\in[0,\delta]$. Moreover,

$$|B_1-u_2(t)| \le |B_1-B_2| + |u_2(0)-u_2(t)| \le \varepsilon+\delta\tilde c \le 2\varepsilon,$$

thus $u_2(t)\in[B_1-2\varepsilon, B_1+2\varepsilon]$ holds for $t\in[0,\delta]$. Consequently,

$$f(\phi(u_i)(t)) \ge K_1, \quad t\in[0,\delta], \quad i=1,2.$$

Therefore

$$A_i(s) = -\int_0^s \frac{p(\tau)}{p(s)} f(\phi(u_i(\tau)))\,\mathrm{d}\tau \le -K_1\int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau.$$

Moreover,

$$|A_1(s)-A_2(s)| \le \int_0^s \frac{p(\tau)}{p(s)} |f(\phi(u_1(\tau))) - f(\phi(u_2(\tau)))|\,\mathrm{d}\tau \le \Lambda_f\Lambda_\phi \|u_1-u_2\|_{C[0,\delta]} \int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau.$$

Let $s\in(0,\delta]$ be fixed. By the Mean Value Theorem there exists $A^\star(s)$ between $A_1(s)$ and $A_2(s)$ such that

$$\left|\phi^{-1}(A_1(s)) - \phi^{-1}(A_2(s))\right| \le \left(\phi^{-1}\right)'(A^\star(s))\,|A_1(s)-A_2(s)|.$$

Since $(\phi^{-1})'$ is a nondecreasing function on $(-\infty,0)$, we get

$$\begin{aligned} \left|\phi^{-1}(A_1(s)) - \phi^{-1}(A_2(s))\right| &\le (\phi^{-1})'\left(-K_1\int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau\right)|A_1(s)-A_2(s)|\\ &\le (\phi^{-1})'\left(-K_1\int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau\right)\frac{\Lambda_f\Lambda_\phi\|u_1-u_2\|_{C[0,\delta]}}{K_1} K_1\int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau. \end{aligned}$$

By (4.17)

$$0 < K_1\int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau \le K_1\delta \le 1,$$

and hence we use (4.16) and get

$$\left|\phi^{-1}(A_1(s)) - \phi^{-1}(A_2(s))\right| \le \frac{K_2}{K_1}\Lambda_f\Lambda_\phi\|u_1-u_2\|_{C[0,\delta]}.$$

Consequently, by (4.17)

$$\begin{aligned} |u_1(t)-u_2(t)| &\le |B_1-B_2| + \int_0^t \frac{K_2}{K_1}\Lambda_f\Lambda_\phi\|u_1-u_2\|_{C[0,\delta]}\,\mathrm{d}s\\ &\le |B_1-B_2| + \delta\frac{K_2}{K_1}\Lambda_f\Lambda_\phi\|u_1-u_2\|_{C[0,\delta]}\\ &\le |B_1-B_2| + \frac{1}{2}\|u_1-u_2\|_{C[0,\delta]}. \end{aligned}$$

This yields

$$\|u_1 - u_2\|_{C[0,\delta]} \le 2|B_1 - B_2|. \tag{4.18}$$ {cdIIdelta

Furthermore,

$$|u_1'(t) - u_2'(t)| = \left|\phi^{-1}(A_1(t)) - \phi^{-1}(A_2(t))\right| \le \frac{K_2}{K_1}\Lambda_f\Lambda_\phi\|u_1 - u_2\|_{C[0,\delta]}.$$

Hence

$$\|u_1' - u_2'\|_{C[0,\delta]} \le K_3|B_1 - B_2|, \tag{4.19}$$ {cdIIdelta

where $K_3 := 2\frac{K_2}{K_1}\Lambda_f\Lambda_\phi$. Finally

$$\|u_1 - u_2\|_{C^1[0,\delta]} \le K_{S1}|B_1 - B_2|,$$

with $K_{S1} := 2\left(\frac{K_2}{K_1}\Lambda_f\Lambda_\phi + 1\right)$.

Step 2. In this step, we extend the continuous dependence on initial values from $[0,\delta]$ to $[0,\beta]$, where $u_i'(t) < 0$ for $t \in (0,\beta]$, $i = 1,2$. To this aim, we denote

$$\nu_i = \max\{u_i'(t) \colon t \in [\delta,\beta]\} < 0, \quad m_1 = \max\{\nu_1,\nu_2\}, \quad m = \min\{-\tilde{c}, L_0\}.$$

Moreover, (1.3) yields the existence of positive Lipschitz constants $\Lambda_m, \Lambda_{\phi^{-1}}$ such that

$$\begin{aligned}
&|\phi(x_1) - \phi(x_2)| \le \Lambda_m|x_1 - x_2|; \quad x_1, x_2 \in [m, L],\\
&\left|\phi^{-1}(y_1) - \phi^{-1}(y_2)\right| \le \Lambda_{\phi^{-1}}|y_1 - y_2|; \quad y_1, y_2 \in [\phi(-\tilde{c}), \phi(m_1)].
\end{aligned}$$

By integrating (2.1) over $[\delta,t]$, $t \in [\delta,\beta]$, we get

$$\phi(u_i'(t)) = \frac{p(\delta)}{p(t)}\phi(u_i'(\delta)) - \frac{1}{p(t)}\int_\delta^t p(s)\tilde{f}(\phi(u_i(s)))\,\mathrm{d}s.$$

Let us denote

$$\begin{aligned}
\tilde{A}_i(t) &:= -\int_\delta^t \frac{p(s)}{p(t)}\tilde{f}(\phi(u_i(s)))\,\mathrm{d}s,\\
x_i(t) &:= \frac{p(\delta)}{p(t)}\phi(u_i'(\delta)) + \tilde{A}_i(t) = \phi(u_i'(t)), \quad t \in [\delta,\beta].
\end{aligned}$$

Then

$$u_i'(t) = \phi^{-1}(x_i(t)), \quad t \in [\delta,\beta]. \tag{4.20}$$ {cdIIder}

Since $-\tilde{c} \le u_i'(t) \le m_1$, then

$$x_i(t) \in [\phi(-\tilde{c}), \phi(m_1)], \quad t \in [\delta,\beta].$$

Integrating (4.20) from δ to t, $t \in [\delta,\beta]$, we get

$$u_i(t) = u_i(\delta) + \int_\delta^t \phi^{-1}(x_i(s))\,\mathrm{d}s.$$

By (4.18) we obtain for $t \in [\delta,\beta]$

$$\begin{aligned}
|u_1(t) - u_2(t)| &\le |u_1(\delta) - u_2(\delta)| + \int_\delta^t \left|\phi^{-1}(x_1(s)) - \phi^{-1}(x_2(s))\right|\mathrm{d}s\\
&\le 2|B_1 - B_2| + \Lambda_{\phi^{-1}}\int_\delta^t |x_1(s) - x_2(s)|\,\mathrm{d}s.
\end{aligned}$$

Further by (1.7), (4.19) we obtain for $s \in [\delta,\beta]$

$$\begin{aligned}
|x_1(s) - x_2(s)| &\le \frac{p(\delta)}{p(s)}|\phi(u_1'(\delta)) - \phi(u_2'(\delta))| + \left|\tilde{A}_1(s) - \tilde{A}_2(s)\right|\\
&\le \Lambda_m|u_1'(\delta) - u_2'(\delta)| + \int_\delta^s \left|\tilde{f}(\phi(u_1(\tau))) - \tilde{f}(\phi(u_2(\tau)))\right|\mathrm{d}\tau\\
&\le \Lambda_m K_3|B_1 - B_2| + \Lambda_f\Lambda_m\int_\delta^s |u_1(\tau) - u_2(\tau)|\,\mathrm{d}\tau.
\end{aligned}$$

Therefore,

$$\begin{aligned}|u_1(t)-u_2(t)| &\le 2|B_1-B_2| + \Lambda_{\phi^{-1}}\Lambda_m K_3\beta|B_1-B_2| + \Lambda_{\phi^{-1}}\Lambda_f\Lambda_m\int_\delta^t\int_\delta^s |u_1(\tau)-u_2(\tau)|\,\mathrm{d}\tau\,\mathrm{d}s\\ &\le K_4|B_1-B_2| + K_5\int_\delta^t |u_1(\tau)-u_2(\tau)|\,\mathrm{d}\tau, \quad t\in[\delta,\beta],\end{aligned}$$

where $K_4 = 2+\Lambda_{\phi^{-1}}\Lambda_m K_3\beta$, $K_5 = \Lambda_{\phi^{-1}}\Lambda_f\Lambda_m\beta$. Next we set for $t\in(\delta,\beta]$

$$\rho(t) = \max\{|u_1(s)-u_2(s)|\colon s\in[\delta,t]\}.$$

Then

$$\rho(t) \le K_4|B_1-B_2| + K_5\int_\delta^t \rho(\tau)\,\mathrm{d}\tau.$$

The Gronwall Lemma yields that

$$\begin{gathered}\rho(t)\le K_4|B_1-B_2|e^{K_5\beta}, \quad t\in[\delta,\beta]\\ \|u_2-u_2\|_{C[\delta,\beta]} \le K_6|B_1-B_2|, \text{ where } K_6 = K_4e^{K_5\beta}.\end{gathered}$$

Finally,

$$\begin{aligned}|u_1'(t)-u_2'(t)| &\le \left|\phi^{-1}(x_1(t))-\phi^{-1}(x_2(t))\right| \le \Lambda_{\phi^{-1}}|x_1(t)-x_2(t)|\\ &\le \Lambda_{\phi^{-1}}\Lambda_m K_3|B_1-B_2| + \Lambda_{\phi^{-1}}\Lambda_f\Lambda_m\beta\|u_1-u_2\|_{C[\delta,\beta]} \le K_7|B_1-B_2|,\end{aligned}$$

where $K_7 = \Lambda_{\phi^{-1}}\Lambda_m K_3 + \Lambda_{\phi^{-1}}\Lambda_f\Lambda_m\beta K_6$. Hence

$$\|u_1'-u_2'\|_{C[\delta,\beta]} \le K_7|B_1-B_2|$$

and

$$\|u_1-u_2\|_{C^1[\delta,\beta]} \le K_{S2}|B_1-B_2|$$

with $K_{S2} = K_6+K_7$

Finally, there exists $K = K_{S1}+K_{S2}$ such that

$$\|u_1-u_2\|_{C^1[0,\beta]} \le K|B_1-B_2|.$$

This completes the proof. □

Remark 4.7. The approach developed in the proof of Theorem 4.6 cannot be used for $B_1 = L$ because then the positive constant K_1 in (4.15) which is crucial in the proof does not exists. {pozn45}

Theorem 4.8 (Continuous dependence on initial values III). *Assume (1.3)–(1.7), (2.5), (2.6), (4.9) and* {th28}

$$\limsup_{x\to0^+}\left(x\left(\phi^{-1}\right)'(x)\right) < \infty, \quad \phi' \textit{ is nondecreasing on } (0,\infty). \tag{4.21}$$ {cd10}

Let B_1, B_2 satisfy

$$B_1\in(L_0+2\varepsilon,-2\varepsilon), \quad |B_1-B_2|<\varepsilon$$

for some $\varepsilon>0$. Let u_i be a solution of problem (2.1), (1.2) with $u_0 = B_i$, $i=1,2$. Then for each $\beta>0$ where

$$u_i' > 0 \textit{ on } (0,\beta], \ i=1,2,$$

there exists $K\in(0,\infty)$ such that

$$\|u_1-u_2\|_{C^1[0,\beta]} \le K|B_1-B_2|.$$

Proof. The proof of this theorem we proceed similarly as in the proof of the Theorem 4.6. In *Step 1.* we make following changes:

$$K_1 = \min\{|f(\phi(x))|:\ x\in[B_1-2\varepsilon, B_1+2\varepsilon]\};$$

by (4.21) there exists positive constant K_2 such that

$$0 < x\left(\phi^{-1}\right)'(x) \le K_2, \quad x\in(0,1];$$

$$-f(\phi(u_i)(t)) = |f(\phi(u_i)(t))| \geq K_1, \quad t \in [0,\delta], \quad i = 1,2;$$
$$A_i(s) = -\int_0^s \frac{p(\tau)}{p(s)} f(\phi(u_i(\tau)))\,\mathrm{d}\tau \geq K_1 \int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau;$$

$(\phi^{-1})'$ is nonincreasing on $(0,\infty)$ and we get

$$\left|\phi^{-1}(A_1(s)) - \phi^{-1}(A_2(s))\right| \leq (\phi^{-1})'\left(K_1 \int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau\right) |A_1(s) - A_2(s)|$$
$$\leq (\phi^{-1})'\left(K_1 \int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau\right) \frac{\Lambda_f \Lambda_\phi \|u_1 - u_2\|_{C[0,\delta]}}{K_1} K_1 \int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau.$$

In *Step 2.* we have $u_i'(t) > 0$ for $t \in (0,\beta]$, $i = 1,2$ and we denote

$$\nu_i = \min\{u_i'(t) \colon t \in [\delta,\beta]\} > 0, \quad m_0 = \min\{\nu_1,\nu_2\}, \quad M = \max\{\tilde{c}, L\}.$$

By (1.3) there exists positive Lipschitz constants $\Lambda_m, \Lambda_{\phi^{-1}}$ such that

$$|\phi(x_1) - \phi(x_2)| \leq \Lambda_m |x_1 - x_2|; \quad x_1, x_2 \in [L_0, M],$$
$$\left|\phi^{-1}(y_1) - \phi^{-1}(y_2)\right| \leq \Lambda_{\phi^{-1}} |y_1 - y_2|; \quad y_1, y_2 \in [\phi(m_0), \phi(\tilde{c})]\,.$$

We have $m_0 \leq u_i'(t) \leq \tilde{c}$ and

$$x_i(t) \in [\phi(m_0), \phi(\tilde{c})]\,, \quad t \in [\delta,\beta]$$

Futrther we argue as in the proof of Theorem 4.6. □

5 Existence and uniqueness of damped solutions of problem (1.1), (1.2)

{mainsecti

Main results of the present paper are formulated in this section. The existence of damped solutions is proved in Theorem 5.2 and the uniqueness is derived in Theorem 5.5. Both results hold not only for the auxiliary problem (2.1), (1.2), but above all for the original problem (1.1), (1.2). Immediately from Theorem 4.1 and Lemma 3.2, we obtain

Theorem 5.1 (**Existence of damped solutions of problem** (2.1)**,** (1.2)**).** *Assume* (1.3)*–*(1.7)*,* (2.5) *and* (2.6)*. Then, for each $u_0 \in \left[\bar{B}, L\right)$, problem* (2.1)*,* (1.2) *has a solution. Every solution of problem* (2.1)*,* (1.2) *with $u_0 \in \left[\bar{B}, L\right)$ is damped.* {dam1}

Let u be a solution of problem (2.1), (1.2) with $u_0 \in \left[\bar{B}, L\right)$. According to (2.2) and (3.3), we have $f(\phi(u(t))) = \tilde{f}(\phi(u(t)))$ for $t \in [0,\infty)$. Hence we can formulate the previous theorem also for the original problem (1.1), (1.2).

Theorem 5.2 (**Existence of damped solutions of problem** (1.1)**,** (1.2)**).** *Assume* (1.3)*–*(1.7)*,* (2.6) *and* {dam2}

$$\exists \bar{B} \in (L_0, 0) : \int_0^{\bar{B}} f(\phi(z))\,\mathrm{d}z = \int_0^L f(\phi(z))\,\mathrm{d}z.$$

Then, for each $u_0 \in \left[\bar{B}, L\right)$, problem (1.1)*,* (1.2) *has a solution. Every solution of problem* (1.1)*,* (1.2) *with $u_0 \in \left[\bar{B}, L\right)$ is damped.*

Example 5.3. Problem (1.1), (1.2) with p, f and ϕ from Example 4.2 has for each $u_0 \in \left[\bar{B}, L\right]$ a damped solution.

Remark 5.4. By Theorem 5.2, we can get homoclinic or escape solutions only if $u_0 \in \left(L_0, \bar{B}\right)$. {hoes}

If $\phi^{-1} \notin \mathrm{Lip}_{loc}(\mathbb{R})$, we derive results about the uniqueness by means of Theorems 4.6 and 4.8. Since the next uniqueness result concerns damped solutions, it can be formulated directly for the original problem (1.1), (1.2).

Theorem 5.5 (**Uniqueness of damped solutions**)**.** *Assume* (1.3)*–*(1.7)*,* (2.5)*,* (2.6)*,* (4.9)*,* (4.14) *and* (4.21)*. Let u be a damped solution of problem* (1.1)*,* (1.2) *with $u_0 \in (L_0, 0) \cup (0, L)$. Then u is a unique solution of this problem.* {uni1}

Proof. Assume that u is a damped solution of the auxiliary problem (2.1), (1.2) and that there exists another solution v of problem (2.1), (1.2). Definition 1.3 yields

$$u(t) < L, \qquad t \in [0, \infty). \tag{5.1}$$ {6.1}

By Lemma 3.2, we have

$$L_0 < u(t), \quad L_0 < v(t), \quad t \in (0, \infty]\,. \tag{5.2}$$ {334}

Step 1. Let $u_0 \in (L_0, 0)$.

(i) According to Lemma 2.1 b), there exists $\beta > 0$ such that

$$u'(t) > 0, \quad v'(t) > 0, \qquad t \in (0, \beta]. \tag{5.3}$$ {6.2}

Put

$$a = \sup\{\beta > 0 \colon (5.3) \text{ holds}\},$$
$$\rho(t) = u(t) - v(t), \qquad t \in [0, \infty).$$

Since $u' > 0$, $v' > 0$ on $(0, a)$ and $B_1 := u_0 = v(0) =: B_2$, Theorem 4.8 yields

$$\rho(t) = 0, \qquad t \in [0, a). \tag{5.4}$$ {6.3}

If $a = \infty$, then

$$u(t) = v(t), \qquad t \in [0, \infty). \tag{5.5}$$ {6.4}

Consequently, by (5.1) and (5.2), u is a unique solution of problem (1.1), (1.2).
Let $a < \infty$. Since u, $v \in C^1[0, \infty)$, we get, by (5.4),

$$\left.\begin{aligned} \lim_{t\to a^-} \rho(t) = \rho(a) = u(a) - v(a) = 0, \\ \lim_{t\to a^-} \rho'(t) = \rho'(a) = u'(a) - v'(a) = 0. \end{aligned}\right\} \tag{5.6}$$ {6.41}

Therefore $u'(a) = v'(a)$.

(ii) According to the definition of number a, we have $u'(a) = v'(a) = 0$. By (5.1) and Lemma 2.6 or Lemma 3.1, $u(a) = v(a) \in (0, L)$. Due to Lemma 2.1 a), there exists $\gamma > a$ such that

$$u'(t) < 0, \quad v'(t) < 0, \qquad t \in (a, \gamma]. \tag{5.7}$$ {6.5}

Put $b = \sup\{\gamma > a \colon (5.7) \text{ holds}\}$. Since $u' < 0$, $v' < 0$ on (a, b) and $u(a) = v(a) \in (0, L)$, by Theorem 4.6 (working with a, γ, $u(a)$ and $v(a)$ instead of 0, β, B_1 and B_2 respectively), we get

$$\rho(t) = 0, \qquad t \in [a, b). \tag{5.8}$$ {6.6}

If $b = \infty$, then (5.5) holds and, by (5.1), (5.2), u is a unique solution of problem (1.1), (1.2).
Let $b < \infty$. Since u, $v \in C^1[0, \infty)$, (5.8) yields

$$\lim_{t\to b^-} \rho(t) = \rho(b) = u(b) - v(b) = 0,$$
$$\lim_{t\to b^-} \rho'(t) = \rho'(b) = u'(b) - v'(b) = 0.$$

Hence $u'(b) = v'(b)$ and, due to the definition of b, $u'(b) = v'(b) = 0$. Lemma 2.7 implies $u(b) = v(b) \in \big(\bar{B}, 0\big)$. Repeating the arguments in parts (i) and (ii), we get that u is a unique solution of problem (1.1), (1.2).

Step 2. Let $u_0 \in (0, L)$. We have the same situation as in part (ii) of Step 1, where a is repleced by 0, and so we argue similarly. □

6 Uniqueness of homoclinic and escape solutions

In this section we discuss homoclinic and escape solutions and hence, by Remark 5.4, we take $u_0 \in \big(L_0, \bar{B}\big)$.

Theorem 6.1 (Nonexistence of singular homoclinic solutions). *Assume (1.3)–(1.7), (4.9) and (4.10). Then each homoclinic solution of problem (1.1), (1.2) with $u_0 \in \big(L_0, \bar{B}\big)$ is regular.* {neex}

Proof. Let u be a singular homoclinic solution of problem (1.1), (1.2) with

$$u_0 \in \left(L_0, \bar{B}\right). \tag{6.1}$$ {h1}

Then, by Definition 1.3, there exists $t_0 > 0$ such that

$$u(t_0) = L, \qquad u'(t_0) = 0, \tag{6.2}$$ {h2}

and

$$u(t) < L, \qquad t \in [0, t_0). \tag{6.3}$$ {h3}

Using the substitution

$$s = t_0 - t, \quad q(s) = p(t), \quad v(s) = u(t), \quad t \in \left[\frac{t_0}{2}, t_0\right],$$

we transform the terminal value problem (1.1), (6.2) on $\left[\frac{t_0}{2}, t_0\right]$ to the initial value problem

$$-(q(s)\phi(-v'(s)))' + q(s)\tilde{f}(\phi(v(s))) = 0, \qquad s \in \left[0, \frac{t_0}{2}\right], \tag{6.4}$$ {h4}

$$v(0) = L, \qquad v'(0) = 0. \tag{6.5}$$ {h5}

By Theorem 4.3, the only possible function satisfying (6.4) and (6.5) is the constant function $v(s) = L$ for $s \in \left[0, \frac{t_0}{2}\right]$. Therefore $u(t) = L$ for $t \in \left[\frac{t_0}{2}, t_0\right]$, which contradicts (6.3). □

Theorem 6.1 discusses the case where $\phi^{-1} \in \mathrm{Lip}_{loc}(\mathbb{R})$. Now we will study the case where condition (4.10) falls, that is $\phi^{-1} \notin \mathrm{Lip}_{loc}(\mathbb{R})$. Then both regular and singular homoclinic solutions may exist and, according to Remark 4.7, we are able to prove the uniqueness just for regular ones.

Theorem 6.2 (Uniqueness of regular homoclinic solutions). *Assume* (1.3)–(1.7), (2.5), (2.6), (4.9), (4.14) *and* (4.21). *Let u be a regular homoclinic solution of problem* (1.1), (1.2) *with $u_0 \in (L_0, 0) \cup (0, L)$. Then u is a unique solution of this problem.* {uni11}

Proof. Assume that u is a regular homoclinic solution of the auxiliary problem (2.1), (1.2) and that there exists another solution v of problem (2.1), (1.2). Since, by Definition 1.3 and Lemma 3.2, the inequalities (5.1) and (5.2) hold, we can argue as in the proof of Theorem 5.5. □

Lemma 6.3 (Homoclinic solution is increasing). *Assume* (1.3)–(1.7), (2.5), (2.6). *Let u be a regular homoclinic solution of problem* (1.1), (1.2) *with $u_0 \in (L_0, \bar{B})$. Then* {homros}

$$\lim_{t\to\infty} u(t) = L, \quad u'(t) > 0 \text{ for } t \in (0, \infty). \tag{6.6}$$ {hom}

Moreover

$$\lim_{t\to\infty} u'(t) = 0. \tag{6.7}$$ {limprime0

Proof. Let u be a regular homoclinic solution of problem (1.1), (1.2) with $u_0 \in (L_0, \bar{B})$. Thus, by Definition 1.3, $u_{sup} = L$.

Step 1. By Lemma 2.1 b) there exists $\theta_0 > 0$ such that $u(\theta_0) = 0$, $u(t) < 0$ for $t \in (0, \theta_0)$ and $u'(t) > 0$ for $t \in (0, \theta_0]$. Assume on contrary with (6.6), that $a_1 > \theta_0$ is the first zero of u'. Since u is a regular homoclinic solution $u(a_1)$ belongs to $(0, L)$. If $u > 0$ on $[a_1, \infty)$, then by Lemma 2.1 a), u is decreasing which contradicts $u_{sup} = L$. Therefore, there exists $\theta_1 > a_1$ such that $u(\theta_1) = 0$, $u'(t) < 0$ for $t \in (a_1, \theta_1]$. Hence we have

$$u(a_1) \in (0, L), \;\; u'(a_1) = 0, \;\; u'(t) > 0, \;\; t \in (0, a_1). \tag{6.8}$$ {a1}

By Lemma 2.7 there exists $b_1 > \theta_1$ such that

$$u(b_1) \in (\bar{B}, 0), \;\; u'(b_1) = 0, \;\; u'(t) < 0, \;\; t \in [\theta_1, b_1).$$

Since, $u_{sup} = L$, there exists $\theta_2 > b_1$, such that $u(\theta_2) = 0$, $u'(t) > 0$ for $t \in (b_1, \theta_2]$. By Lemma 2.6 there exists $a_2 > \theta_2$ such that

$$u(a_2) \in (0, L), \;\; u'(a_2) = 0, \;\; u'(t) > 0, \;\; t \in (b_1, a_2).$$

Repeating this procedure, we obtain a sequence of zeros $\{\theta_n\}_{n=0}^{\infty}$ of u and a sequence of local maxima $\{u(a_n)\}_{n=1}^{\infty}$ of u.

We prove that the sequence $\{u(a_n)\}_{n=1}^{\infty}$ is nonincreasing. Choose $n \in \mathbb{N}$. Multiplying equation (1.1) by u'/p, then integrating from a_n to a_{n+1} we obtain

$$\int_{a_n}^{a_{n+1}} \phi'(u'(t))u''(t)u'(t)\,\mathrm{d}t + \int_{a_n}^{a_{n+1}} \frac{p'(t)}{p(t)}\phi(u'(t))u'(t)\,\mathrm{d}t + \int_{a_n}^{a_{n+1}} f(\phi(u(t)))u'(t)\,\mathrm{d}t = 0.$$

The first integral is equal zero since $u'(a_n) = u'(a_{n+1}) = 0$. The second integral is nonnegative due to (1.7) and (2.7). Therefore,

$$0 \geq \int_{a_n}^{a_{n+1}} f(\phi(u(t)))u'(t)\,\mathrm{d}t = \int_{u(a_n)}^{u(a_{n+1})} f(\phi(y))\,\mathrm{d}y = F(u(a_{n+1})) - F(u(a_n)).$$

Since F is increasing function, we get

$$u(a_n) \geq u(a_{n+1}).$$

The sequence $\{u(a_n)\}_{n=1}^{\infty}$ is nonincreasing, because n is chosen arbitrarily. Thus $u_{sup} < L$, which cannot be fulfilled because u is homoclinic solution. This contradiction yields that

$$u'(t) > 0,\ t \in (0, \infty).$$

Since $u_{sup} = L$, then $\lim_{t\to\infty} u(t) = L$.

Step 2. Since, $u > 0$ on (θ_0, ∞) we have $f(\phi(u)) > 0$ on (θ_0, ∞). From (1.1) we obtain that

$$0 > (p(t)\phi(u'(t)))' = p'(t)\phi(u'(t)) + p(t)\left(\phi(u'(t))\right)',\ t \in (\theta_0, \infty).$$

Since p, p', u' and $\phi(u')$ are positive on $(0, \infty)$, we get that $\phi(u')$ is decreasing on (θ_0, ∞). On the other hand ϕ is an increasing function . Therefore u' is a decreasing function on (θ_0, ∞). Since $u' > 0$ on $(0, \infty)$, there exists a nonnegative limit

$$\lim_{t\to\infty} u'(t) =: K \geq 0.$$

If $K > 0$, then

$$K(t - \theta_0) \leq \int_{\theta_0}^{t} u'(s)\,\mathrm{d}s = u(t) - u(\theta_0) = u(t).$$

The limit as t tends to infinity yields,

$$L = \lim_{t\to\infty} u(t) \geq \lim_{t\to\infty} K(t - \theta_0) = \infty,$$

a contradiction. Therefore (6.7) holds. □

Since assumptions (1.6) are imposed to f on the interval $[\phi(L_0), \phi(L)]$ and we have no information about a behaviour of f out of this interval, we formulate results concerning escape solutions for the auxiliary problem (2.1), (1.2).

Lemma 6.4 (Escape solution is increasing). *Assume that* (1.3)–(1.7), (2.5) *and* (2.6) *hold. Let u be an escape solution of problem* (2.1), (1.2) *with $u_0 \in (L_0, \bar{B})$. Then* {lemma41}

$$u'(t) > 0, \quad t \in (0, \infty). \tag{6.9}$$ {unik}

Proof. Let u be an escape solution of problem (2.1), (1.2) with $u_0 \in (L_0, \bar{B})$. Thus, by Definition 1.3, $u_{sup} > L$. Then there exists a point $c \in (0, \infty)$ such that $u(c) = L$, $u'(c) \geq 0$ and $u(t) < L$ for $t \in [0, c)$. First we exclude the case $u'(c) = 0$. Lemma 2.2 yields that if $u'(c) = 0$ then either u has a zero point $u(\theta) = 0$, $u(t) \leq L$, $t \in [c, \theta]$ or u is positive and nonincreasing on $[c, \infty)$. The later case is in contradiction with u being an escape solution. Therefore, such zero point $\theta > c$ must exist. Applying Lemma 2.1 a), b) and Lemma 2.7 and repeating the arguments as in Step1 in the proof of Lemma 6.3, we get that u has a nonincreasing sequence $\{u(a_n)\}_{n=1}^{\infty}$ of its local maxima. Thus $u(t) \leq L$ for $t \geq 0$ on contrary that u is an escape solution. Therefore $u'(c) > 0$.

Let $c_1 > c$ be such that $u'(c_1) = 0$ and $u(t) > L$, $u'(t) > 0$ for $t \in (c, c_1)$. Integrating (2.1) over $[c, c_1]$ we get, due to (1.3), (1.4), (1.7) and (2.2),

$$\phi(u'(c_1)) = \frac{p(c)\phi(u'(c))}{p(c_1)} > 0,$$

contrary to $u'(c_1) = 0$. We have proved $u'(t) > 0$ for $t > c$.

Further, we prove that $u'(t) > 0$ for $t \in (0, \theta_0]$. Since $u_0 \in (L_0, 0)$, Lemma 2.1 b) yields that there exists $\theta_0 > 0$ such that $u(\theta_0) = 0$, $u(t) < 0$ for $t \in (0, \theta_0)$, $u'(t) > 0$ for $t \in (0, \theta_0]$.

It remains to prove that $u'(t) > 0$ for $t \in (\theta_0, c)$. Assume on the contrary that there exists $a_1 \in (\theta_0, c)$ such that 6.8 holds. We derive a contradiction as in Step 1 in the proof of Lemma 6.3. To summarize, $u'(t) > 0$ for $t > 0$. □

Theorem 6.5 (**Uniqueness of escape solutions**). *Assume* (1.3)–(1.7), (2.5), (2.6), (4.9) *and* (4.21). *Let u be an escape solution of problem* (2.1), (1.2) *with $u_0 \in \left(L_0, \bar{B}\right)$. Then u is a unique solution of this problem.* {uni2}

Proof. Let u be an escape solutions of problem (2.1), (1.2). According to Lemma 6.4, (6.9) holds. Consider that v is another solution of problem (2.1), (1.2). Assume that there exists $c > 0$ such that $v'(c) = 0$. By Lemma 2.1 b), there exists $\theta > 0$ such that $v(\theta) = 0$, $v'(t) > 0$ for $t \in (0, \theta]$. Therefore $c > \theta$ and there exists $a \in (\theta, c]$ such that $v'(a) = 0$, $v'(t) > 0$ for $t \in (0, a)$. Put

$$\rho(t) = u(t) - v(t), \qquad t \in [0, \infty).$$

Let $a < \infty$. Since $u' > 0$, $v' > 0$ on $(0, a)$, Theorem 4.8, where $u_0 = B_1 = B_2$, gives

$$\rho(t) = 0, \quad \rho'(t) = 0, \quad t \in [0, a). \tag{6.10}$$ {342}

Since u, $v \in C^1[0, \infty)$, we get that (5.6) holds. Thus $u'(a) = v'(a)$. According to the definition of number a, we have $u'(a) = v'(a) = 0$, which contradicts (6.9). Therefore $a = \infty$ and, by (6.10), u is a unique solution of problem (2.1), (1.2). □

Example 6.6. Consider p, f from Example 4.2 with $\gamma \geq 1$ and ϕ given by (4.3). Then

$$\phi^{-1}(x) = |x|^{\frac{1}{\alpha}} \operatorname{sgn} x, \qquad \left(\phi^{-1}\right)'(x) = \frac{1}{\alpha}|x|^{\frac{1}{\alpha}-1}, \qquad \lim_{x\to 0} x \left(\phi^{-1}\right)'(x) = \frac{1}{\alpha} \lim_{x\to 0} x|x|^{\frac{1}{\alpha}-1} = 0 \in \mathbb{R},$$

$$\phi'(x) = \alpha|x|^{\alpha-1}, \qquad \phi''(x) = \frac{\alpha(\alpha-1)|x|^{\alpha-1}}{x} \begin{cases} \leq 0 & \text{for } x < 0, \\ \geq 0 & \text{for } x > 0. \end{cases}$$

Hence ϕ' is nonincreasing on $(-\infty, 0)$, nondecreasing on $(0, \infty)$ and conditions (4.14) and (4.21) hold.

If ϕ is given by (4.4), then

$$\phi^{-1}(x) = \sqrt{\sqrt{|x|+1} - 1}, \qquad \left(\phi^{-1}\right)'(x) = \frac{\operatorname{sgn} x}{4\sqrt{\sqrt{|x|+1} - 1}\,\sqrt{|x|+1}},$$

$$\lim_{x\to 0} x \left(\phi^{-1}\right)'(x) = \lim_{x\to 0} \frac{|x|}{4\sqrt{\sqrt{|x|+1} - 1}\,\sqrt{|x|+1}} = 0 \in \mathbb{R},$$

$$\phi'(x) = 4\left(x^3 + x\right) \operatorname{sgn} x, \qquad \phi''(x) = 4\left(3x^2 + 1\right) \operatorname{sgn} x \begin{cases} < 0 & \text{for } x < 0, \\ > 0 & \text{for } x > 0. \end{cases}$$

Therefore ϕ' is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. Function ϕ satisfies conditions (1.3), (1.4), (4.14) and (4.21).

In both cases all assumptions of Theorem 4.1, Theorem 5.5 and Theorem 6.5 are fulfilled. Therefore problem (2.1), (1.2) has for $u_0 \in [L_0, L]$ a solution u. If $u_0 \in (L_0, L) \cup (0, L)$ and $u < L$ on $[0, \infty)$, then u is a solution of the original problem (1.1), (1.2) and it is a unique solution of this problem. If $u_0 \in \left(L_0, \bar{B}\right)$ and u is an escape solution of problem (2.1), (1.2), then u is a unique solution of this problem.

Remark 6.7. Theorem 6.1 does not cover equations having a ϕ-Laplacian in the form (4.3) or (4.4) because such ϕ-Laplacian does not fulfil condition (4.10). Therefore to find conditions which guarantee that singular homoclinic solutions do not exist while $\phi^{-1} \notin \mathrm{Lip}_{loc}(\mathbb{R})$ is an open problem and we plan to solve it in our next paper where we also will discuss the existence and asymptotic properties of regular homoclinic and escape solutions. {open}

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