Strictly increasing solutions of a nonlinear singular differential equation arising in hydrodynamics*

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Abstract

The paper investigates singular initial problems

$$(p(t)u')' = p(t)f(u), \quad u(0) = B, \ u'(0) = 0,$$

on the half–line $[0, \infty)$. Here $B < 0$ is a parameter, $p(0) = 0$ and $p'(t) > 0$ on $(0, \infty)$, $f(L) = 0$ for some $L > 0$ and $xf(x) < 0$ if $L_0 < x < L$ and $x \neq 0$. The existence of a strictly increasing solution to the problem for which there exists finite $c > 0$ such that $u(c) = L$ is discussed. This is fundamental for the existence of a strictly increasing solution of the problem having its limit equal to L as $t \to \infty$, which has great importance in applications.

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1 Introduction

Assume that $L > 0$ is a given parameter. A goal of this paper is to prove that for some $B < 0$ there exist $c > 0$ and a solution $u \in C^1([0, c]) \cap C^2((0, c])$ of an initial problem

$$(p(t)u')' = p(t)f(u), \tag{1}$$

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$$u(0) = B, \quad u'(0) = 0 \tag{2}$$

satisfying moreover

$$u(c) = L, \quad u' > 0 \quad \text{on } (0, c]. \tag{3}$$

Definition 1 A solution of (1) – (3) will be called an escape solution of problem (1), (2).

The existence of an escape solution is fundamental in order to get a *homoclinic solution* which plays an important role in hydrodynamics (see [4]-[7], [9]) and which is defined as a function $u \in C^1([0,\infty)) \cap C^2((0,\infty))$ satisfying equation (1) on $(0,\infty)$ and fulfilling

$$u'(0) = 0, \quad u(\infty) = L, \quad u' > 0 \quad \text{on } (0,\infty). \tag{4}$$

See [9] for analytical investigation and [7], [8] for numerical simulations.

Problem (1), (4) can be transformed onto a problem about the existence of a positive solution on the half–line. For $p(t) = t^k$, $k \in \mathbb{N}$ and for $p(t) = t^k$, $k \in (1,\infty)$, such problem was solved by variational methods in [1] and [2], respectively. Related problems were solved e.g. in [3] and [10]. Here we deal with a more general function p and we omit some assumptions for f. In the paper we assume

$$L_0 < 0, \quad f \in Lip([L_0, L]), \quad f(L_0) = f(0) = f(L) = 0, \tag{5}$$

$$xf(x) < 0, \quad x \in (L_0, L) \setminus \{0\}, \tag{6}$$

$$\left\{ \begin{array}{l} \text{there exists } \bar{B} \in (L_0, 0) \text{ such that } F(\bar{B}) = F(L), \\ \text{where } F(x) = -\int_0^x f(z)\,\mathrm{d}z, \ x \in [L_0, L], \end{array} \right. \tag{7}$$

$$p \in C([0,\infty)) \cap C^1((0,\infty)), \ p(0) = 0, \tag{8}$$

$$p' > 0 \quad \text{on } (0,\infty), \ \lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0. \tag{9}$$

Note that equation (1) is singular because $p(0) = 0$. In [13] we have studied problem (1), (2) provided $f \in Lip_{loc}((-\infty, L])$ has a sublinear behaviour near $-\infty$ and has just two zeros in $(-\infty, L]$. In particular, we have assumed

$$f \in Lip_{loc}((-\infty, L)), \ f(0) = f(L) = 0, \ \lim_{x\to-\infty} \frac{|x|}{f(x)} = \infty. \tag{10}$$

Under assumption (6) – (10) we have obtained an escape solution by means of differential and integral inequalities. The lower and upper functions approach has been used in [11] and [12], where the existence of an escape solution has been reached under assumptions (5) – (9) and an additional assumption

$$\left\{ \begin{array}{l} f \in C^1((-\delta, 0)) \text{ and } \lim_{x\to 0-} f'(x) = f'_-(0) < 0, \\ p \in C^2((0,\infty)) \text{ and } \lim_{t\to\infty} \frac{p''(t)}{p(t)} = 0. \end{array} \right. \tag{11}$$

Here, modifying some assertions of [13], we can omit assumption (11) and provide a new existence result for problem (1) – (3). Having an escape solution provided (5) – (9) is assumed, we can use some arguments of [13] and get a homoclinic solution, as well.

2 Auxiliary initial problem

We work with an auxiliary equation

$$(p(t)u')' = p(t)\tilde{f}(u), \tag{12}$$

where

$$\tilde{f}(x) = \begin{cases} f(x) & x \in [L_0, L], \\ 0 & x \in \mathbb{R} \setminus [L_0, L]. \end{cases} \tag{13}$$

Definition 2 Let $c > 0$. A function $u \in C^1([0,c]) \cap C^2((0,c])$ satisfying (12) on $(0,c]$ and fulfilling (2) is called a solution of problem (12), (2) on $[0,c]$. If u is a solution of problem (12), (2) on $[0,c]$ for all $c > 0$, then it is called a solution of problem (12), (2) on $[0,\infty)$.

Remark 3 Let $c > 0$. A function $u \in C([0,c])$ is a solution of problem (12), (2) on $[0,c]$ if and only if it satisfies

$$u(t) = B + \int_0^t \frac{1}{p(s)} \int_0^s p(\tau)\tilde{f}(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s \tag{14}$$

for $t \in [0,c]$.

Lemma 4 *Let* (5)*,* (6)*,* (8)*,* (9)*,* (13) *be satisfied and* $B \in [L_0, 0]$*. Problem* (12)*,* (2) *has a unique solution on* $[0,\infty)$*. Moreover, for* $B \in (L_0, 0)$ *the solution is strictly increasing on each interval* $[0,b]$ *(*$b > 0$*) on which is negative. If* $B = L_0$ *or* $B = 0$*, the solutions are constant functions.*

Proof. First we will prove the local existence and uniqueness of solution of problem (12), (2). For this purpose we consider the operator $\mathcal{F} : C([0,\eta]) \to C([0,\eta])$ defined by

$$(\mathcal{F}u)(t) = B + \int_0^t \frac{1}{p(s)} \int_0^s p(\tau)\tilde{f}(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s.$$

whose fixed point is the solution of (12), (2) on $[0,\eta]$, $\eta > 0$. From (5), (13) and (9) we deduce that the operator $\mathcal{F}$ is a contraction for sufficiently small $\eta > 0$, and therefore it has the unique fixed point. Due to (5), (8) and (13), this solution can be uniquely extended onto the whole half–line $[0,\infty)$. The rest of the Lemma follows from (5), (6) and (13). □

Remark 5 In what follows, by a solution of problem (12), (2) we mean a solution on $[0,\infty)$.

Lemma 6 *Let* (5)*–*(9)*,* (13) *be satisfied. Let* u *be a solution of problem* (12)*,* (2) *for* $B \in (L_0, 0)$*, let* u *be increasing on* $[0,\infty)$ *and* $u(t) \in [L_0, L]$ *for each* $t \in [0,\infty)$*. Then*

$$\lim_{t\to\infty} u(t) \in \{0, L\} \quad \textit{and} \quad \lim_{t\to\infty} u'(t) = 0.$$

Proof. Let us denote $l = \lim_{t\to\infty} u(t)$. From the assumptions of Lemma 6 we see that $l \in (B, L]$ and from (12), (13) it follows

$$u''(t) = -\frac{p'(t)}{p(t)}u'(t) + f(u(t)), \quad t \in (0,\infty). \tag{15}$$

Multiplying this equality by $u'(t)$ and integrating over interval $(0, t)$ we get from (2) that

$$\frac{u'^2(t)}{2} = -\int_0^t \frac{p'(s)}{p(s)}u'^2(s)\,\mathrm{d}s + \int_0^t f(u(s))u'(s)\,\mathrm{d}s, \quad t \in (0,\infty).$$

According to (7) we have

$$\frac{u'^2(t)}{2} = -\int_0^t \frac{p'(s)}{p(s)}u'^2(s)\,\mathrm{d}s + F(B) - F(u(t)), \quad t \in (0,\infty).$$

From (8), (9) it follows that the right–hand side of the last equality has limit for $t \to \infty$, thus there exists $\lim_{t\to\infty} u'^2(t)$. Since u is increasing on $[0,\infty)$ we have $u'(t) \geq 0$ for each $t \in [0,\infty)$ and consequently there exists nonnegative $\lim_{t\to\infty} u'(t)$. If $\lim_{t\to\infty} u'(t) > 0$ then $\lim_{t\to\infty} u(t) = \infty$, which contradicts the boundedness of the function u. Thus, $\lim_{t\to\infty} u'(t) = 0$. From this fact, (9) and (15) for $t \to \infty$, it follows that

$$\lim_{t\to\infty} u''(t) = f(l).$$

This yields $f(l) = 0$, and thus from (5) and (6) we get $l = 0$ or $l = L$. □

Lemma 7 *Let* (5), (8), (9), (13) *be satisfied. For each $b > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that for any B_1, $B_2 \in [L_0, 0]$*

$$|B_1 - B_2| < \delta \quad \Longrightarrow \quad \Big(|u_1(t) - u_2(t)| + |u_1'(t) - u_2'(t)| < \epsilon, \ t \in [0, b]\Big).$$

Here u_i is a solution of problem (12), (2) *with $B = B_i$, $i = 1, 2$.*

Proof. Choose $b > 0$, $\epsilon > 0$. Let $K > 0$ be the Lipschitz constant for $\tilde{f}$ on $\mathbb{R}$. Let B_1, $B_2 \in [L_0, 0]$ and u_1, u_2 be corresponding solutions of problem (12), (2) on $[0,\infty)$. From (14), (9) it follows

$$|u_1(t) - u_2(t)| \leq |B_1 - B_2| + Kb\int_0^t |u_1(\tau) - u_2(\tau)|\,\mathrm{d}\tau, \ t \in [0, b].$$

From the Gronwall inequality, we get

$$|u_1(t) - u_2(t)| \leq |B_1 - B_2|e^{Kb^2}, \ t \in [0, b].$$

Similarly

$$\begin{aligned}|u_1'(t) - u_2'(t)| &\leq K\frac{1}{p(t)}\int_0^t p(s)|u_1(s) - u_2(s)|\,\mathrm{d}s \\ &\leq Kb|B_1 - B_2|e^{Kb^2}, \ t \in [0, b].\end{aligned}$$

It suffices to take $\delta > 0$ such that

$$\delta < \frac{\epsilon}{(1+Kb)e^{Kb^2}}. \qquad \square$$

The following lemma is a direct consequence of Lemma 7.

Lemma 8 *Let* (5), (8), (9), (13) *be satisfied. Let* $\{B_n\}_{n=1}^{\infty} \subset [L_0, 0]$, $B_0 \in [L_0, 0]$ *be such that* $\lim_{n\to\infty} B_n = B_0$. *Let* u_n *be the corresponding solution of* (12), (2) *with* $B = B_n$ *for each* $n \in \mathbb{N} \cup \{0\}$. *Then* $\{u_n\}_{n=1}^{\infty}$ *converges to* u_0 *locally uniformly on* $[0, \infty)$.

3 Escape solution

Note that if u is a solution of problem (12), (2) and fulfils condition (3) for some $c > 0$, it is an escape solution of problem (1), (2). Therefore it suffices to search for escape solutions of problem (12), (2).

First we provide some auxiliary conditions which guarantee the existence of an escape solution of problem (12), (2).

Lemma 9 *Let* (5)–(9), (13) *be satisfied. Let* $C \in (L_0, \bar{B})$ *and* $\{B_n\}_{n=1}^{\infty} \subset (L_0, C)$. *Then for each* $n \in \mathbb{N}$
(i) *there exists a solution* u_n *of problem* (12), (2) *with* $B = B_n$,
(ii) *there exists* $b_n > 0$ *such that* $[0, b_n)$ *is the maximal interval on which the solution* u_n *is increasing and its values in this interval are contained in* $[L_0, L]$,
(iii) *there exists* $\gamma_n \in (0, b_n)$ *satisfying* $u_n(\gamma_n) = C$.
If the sequence $\{\gamma_n\}_{n=1}^{\infty}$ *is unbounded, then there exists* $\ell \in \mathbb{N}$ *such that* u_ℓ *is an escape solution of problem* (12), (2).

Proof. In view of Lemma 4 and Lemma 6 solutions u_n of (12), (2) with $B = B_n$ and constants b_n, γ_n exist (b_n can be infinite). Let $\{\gamma_n\}_{n=1}^{\infty}$ be unbounded. Then

$$\lim_{n\to\infty} \gamma_n = \infty, \quad \gamma_n < b_n \quad \text{for all } n \in \mathbb{N} \tag{16}$$

(otherwise we take a subsequence). Assume on the contrary that for any $n \in \mathbb{N}$, u_n is not an escape solution. Choose $n \in \mathbb{N}$. If $b_n = \infty$, we write $u_n(b_n) = \lim_{t\to\infty} u_n(t)$ and $u_n'(b_n) = \lim_{t\to\infty} u_n'(t)$. If b_n is finite, then

$$u_n(b_n) \in [0, L] \quad \text{and} \quad u_n'(b_n) = 0. \tag{17}$$

In view of Lemma 6, relations (17) are valid for $b_n = \infty$, as well. Due to (17), (2) and (ii) there exists $\bar{\gamma}_n \in [\gamma_n, b_n)$ satisfying

$$u_n'(\bar{\gamma}_n) = \max\{u_n'(t) : t \in [\gamma_n, b_n)\} \tag{18}$$

By (ii), (12) and (13), u_n satisfies equation

$$u_n''(t) + \frac{p'(t)}{p(t)} u_n'(t) = f(u_n(t)), \quad t \in (0, b_n),$$

Integrating it over $[0,t]$ we get

$$\frac{u_n'^2(t)}{2} + F(u_n(t)) = F(B_n) - \int_0^t \frac{p'(s)}{p(s)} u_n'^2(s)\,\mathrm{d}s,\ t \in (0, b_n). \tag{19}$$

Put

$$E_n(t) = \frac{u_n'^2(t)}{2} + F(u_n(t)),\ t \in (0, b_n). \tag{20}$$

Then, by (19),

$$\frac{\mathrm{d}E_n(t)}{\mathrm{d}t} = -\frac{p'(t)}{p(t)} u_n'^2(t) < 0,\ t \in (0, b_n). \tag{21}$$

We see that E_n is decreasing. From (6) and (7) we get that F is increasing on $[0, L]$ and consequently by (17) and (20) we have

$$E_n(\gamma_n) > F(u_n(\gamma_n)) = F(C),\ E_n(b_n) = F(u_n(b_n)) \le F(L). \tag{22}$$

Integrating (21) over (γ_n, b_n) and using (18), we obtain

$$E_n(\gamma_n) - E_n(b_n) = \int_{\gamma_n}^{b_n} \frac{p'(t)}{p(t)} u_n'^2(t)\,\mathrm{d}t \le u_n'(\bar{\gamma}_n)(L - C)K_n,$$

where

$$K_n = \sup\left\{\frac{p'(t)}{p(t)} :\ t \in [\gamma_n, b_n)\right\} \in (0, \infty).$$

Further, by (22),

$$F(C) < E_n(\gamma_n) \le F(L) + u_n'(\bar{\gamma}_n)(L - C)K_n, \tag{23}$$

and

$$\frac{F(C) - F(L)}{L - C} \cdot \frac{1}{K_n} < u_n'(\bar{\gamma}_n).$$

Conditions (9) and (16) yield $\lim_{n\to\infty} K_n = 0$, which implies

$$\lim_{n\to\infty} u_n'(\bar{\gamma}_n) = \infty. \tag{24}$$

By (20) and (23),

$$\frac{u_n'^2(\bar{\gamma}_n)}{2} \le E_n(\bar{\gamma}_n) \le E_n(\gamma_n) \le F(L) + u_n'(\bar{\gamma}_n)(L - C)K_n,$$

and consequently

$$u_n'(\bar{\gamma}_n)\left(\frac{1}{2}u_n'(\bar{\gamma}_n) - (L - C)K_n\right) \le F(L) < \infty,\ n \in \mathbb{N},$$

which contradicts (24). Therefore at least one escape solution of (12), (2) with $B < \bar{B}$ must exist. □

The main result is contained in the next theorem.

Theorem 10 *Let* (5)–(9) *be satisfied. Then there exists at least one escape solution of problem* (1), (2).

Proof. Let us take $C \in (L_0, 0)$ and a sequence $\{B_n\} \subset (L_0, C)$ such that $\lim_{n\to\infty} B_n = L_0$. We consider the corresponding sequence $\{u_n\}$ of solutions of problem (12), (2) with $B = B_n$, and sequence $\{\gamma_n\}$ from Lemma 9. Lemma 8 and Lemma 4 yield that $\{u_n\}$ converges to $u \equiv L_0$ locally uniformly. Consequently, we can find subsequence $\{\gamma_{k_n}\} \subset (0, \infty)$ such that

$$\lim_{n\to\infty} \gamma_{k_n} = \infty.$$

Therefore, by Lemma 9 there exists at least one escape solution. □

Example 11 Let us assume problem (1), (2) with

$$f(x) = A(x - L_0)x(x - L), \quad x \in \mathbb{R},$$

such that $A > 0$, $L_0 < 0 < L$ and $|L_0| > |L|$ and

$$p(t) = t^k, \quad t \in [0, \infty),$$

where $k \in \mathbb{N}$. We can check, that the assumptions (5)–(9) are satisfied and by Theorem 10 the problem (1), (2) has an escape solution. Special case of this problem has arisen in hydrodynamics, see for example [9].

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