Solvability of Discrete Neumann Boundary Value Problems

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Abstract

In this article we gain solvability to a nonlinear, second-order difference equation with discrete Neumann boundary conditions. Our methods involve new inequalities on the right-hand side of the difference equation and Schaefer's theorem in the finite-dimensional space setting.

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1 Introduction

This paper investigates the following discrete Neumann boundary value problem (BVP)

$$\nabla\Delta y(k) = f(k, y(k), \Delta y(k)), \qquad k = 1, \ldots, n-1; \tag{1.1}$$
$$\Delta y(0) = 0 = \Delta y(n); \tag{1.2}$$

where: f is a continuous, scalar-valued function; $n \geq 2$; and the differences are given by:

$$\Delta y(k) := \begin{cases} y(k+1) - y(k), & \text{for } k = 0, \ldots, n-1, \\ 0, & \text{for } k = n; \end{cases}$$

$$\nabla\Delta y(k) := \begin{cases} y(k+1) - 2y(k) + y(k-1), & \text{for } k = 1, \ldots, n-1, \\ 0, & \text{for } k = 0 \text{ or } k = n. \end{cases}$$

This paper addresses a question of interest regarding the discrete BVP (1.1), (1.2):

- Under what conditions does the discrete BVP (1.1), (1.2) have at least one solution?

Particular significance in the above question lies in the fact strange and interesting distinctions can occur between the theory of differential equations and the theory of difference equations. For example, properties such as existence, uniqueness and multiplicity of solutions may not be shared between the theory of differential equations and the theory of difference equations [1, p.520], even though the right hand side of the equations under consideration may be the same. Moreover, when investigating difference equations, as opposed to differential equations, basic ideas from calculus are not necessarily available to use, such as the intermediate value theorem; the mean value theorem and Rolle's theorem. Thus, new challenges are faced and innovation is required.

The paper is organised as follows.

Section 2 contains the main results of the paper. There, some sufficient conditions are presented, in terms of difference inequalities involving f, such that (1.1), (1.2) will admit at least one solution. The main ideas of the proof involve *a priori* bounds on solutions to a certain family of problems, and also involves Schaefer's Theorem [8, Theorem 4.4.10] in the finite-dimensional space setting.

In Section 3 an example is presented to illustrate how to apply the new theory.

For recent and classical results on difference equations and their comparison with differential equations, including existence, uniqueness and spurious solutions, the reader is referred to: [1]-[7], [9]-[15].

A solution to problem (1.1) is a vector $\mathbf{y} = (y(0), \ldots, y(n)) \in \mathbb{R}^{n+1}$ satisfying (1.1) for $k = 1, \ldots, n-1$.

We will need the following identity in the proof of our main theorem, obtained from the discrete product rule. If $r(t) := [y(t)]^2$, $t \in \mathbb{Z}$ then

$$\nabla\Delta r(t) = 2y(t)\nabla\Delta y(t) + [\Delta y(t)]^2 + [\nabla y(t)]^2. \tag{1.3}$$

2 Main Results

In this section we present and prove the main results of the paper. The main ideas involve new difference inequalities (on f) and Schaefer's Theorem [8, Theorem 4.4.10] in the finite-dimensional space setting.

Theorem 2.1 *Let f be continuous. If there exist non-negative constants α and K such that*

$$|f(t,p,q)-p| \leq \alpha\left[2pf(t,p,q)+q^2\right]+K, \quad \forall (t,p,q) \in \{1,\ldots,n-1\} \times \mathbb{R} \times \mathbb{R}; \tag{2.1}$$

the the discrete BVP (1.1), (1.2) has at least one solution.

Proof We consider the following discrete BVP that is equivalent to (1.1), (1.2), namely

$$\nabla\Delta y(k) - y(k) = f(k, y(k), \Delta y(k)) - y(k), \qquad k=1,\ldots,n-1; \tag{2.2}$$

$$\Delta y(0) = 0 = \Delta y(n). \tag{2.3}$$

We will prove that (2.2), (2.3) has at least one solution and thus, so will (1.1), (1.2). We may reformulate (2.2), (2.3) as an equivalent summation equation, namely

$$y(k) = \sum_{i=1}^{n-1} G(t,i)\left[f(i,y(i),\Delta y(i)) - y(i)\right], \quad k=0,\ldots,n,$$

where G is the unique, continuous Green's function associated with the discrete BVP

$$\begin{aligned} &\nabla\Delta y(k) - y(k) = 0, \qquad k=1,\ldots,n-1;\\ &\Delta y(0) = 0 = \Delta y(n). \end{aligned}$$

Introduce the operator (defined componentwise below) $\mathbf{T} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$T_k(\mathbf{y}) = \sum_{i=1}^{n-1} G(k,i)\left[f(i,y(i),\Delta y(i)) - y(i)\right], \quad k=0,\ldots,n, \tag{2.4}$$

Thus, we want to show that there exists at least on $\mathbf{y} \in \mathbb{R}^{n+1}$ such that

$$\mathbf{y} = \mathbf{T}\mathbf{y}. \tag{2.5}$$

To do this, we will use Schaefer's Theorem [8, Theorem 4.4.10] in the finite-dimensional space setting.

Since f and G are continuous, see that $\mathbf{T}$ is a continuous map (and thus compactness of $\mathbf{T}$ in the finite-dimensional space setting is guaranteed).

It remains to show that all possible solutions to

$$\mathbf{y} = \lambda\mathbf{T}\mathbf{y}, \quad \lambda \in [0,1]. \tag{2.6}$$

are bounded *a priori*, with the bound being independent of λ. With this in mind, let $\mathbf{x}$ be a solution to (2.6) and denote

$$G_0 := \max\{|G(k,s)| : \ (k,s) \in [0,n]^2\}.$$

For each $k = 0, \ldots, n$ we have

$$\begin{aligned}
|y(k)| &= \lambda|T_k y(k)| \\
&\leq G_0 \sum_{i=1}^{n-1} \lambda |f(i, y(i), \Delta y(i)) - y(i)| \\
&\leq G_0 \sum_{i=1}^{n-1} \alpha \left[2y(i)\lambda f(i, y(i), \Delta y(i)) + \lambda[\Delta y(i)]^2\right] + \lambda K, \quad \text{from (2.1)} \\
&\leq G_0 \sum_{i=1}^{n-1} \alpha \left[2y(i)\lambda f(i, y(i), \Delta y(i)) + 2(1-\lambda)[y(i)]^2 + [\Delta y(i)]^2 + [\nabla y(i)]^2\right] + K \\
&= G_0 \sum_{i=1}^{n-1} \alpha \left[2y(i)\nabla\Delta y(i) + [\Delta y(i)]^2 + [\nabla y(i)]^2\right] + K \\
&= G_0 \sum_{i=1}^{n-1} \alpha \nabla\Delta r(i) + K, \quad \text{from (1.3)} \\
&= G_0 \left(\alpha[\nabla r(n) - \nabla r(1)] + Kn\right) \\
&= G_0 \left(\alpha\left[(y(n) + y(n-1))\nabla y(n) - (y(1) - y(0))\nabla y(1)\right] + Kn\right) \\
&= G_0 K n, \quad \text{from (1.2)}.
\end{aligned}$$

Hence we see that all solutions to the family (2.6) are bounded *a priori*, with the bound being independent of λ. Schaefer's Theorem applies to $\mathbf{T}$, yielding the existence of at least one fixed point. $\square$

The following corollary easily follows to Theorem 2.1 when f is bounded.

Corollary 2.2 *If $f(t,p,q) - p$ is continuous and bounded on $\{1, \ldots, n-1\} \times \mathbb{R}^2$ then the discrete BVP* (1.1)*,* (1.2) *has at least one solution.*

Proof The proof follows by choosing $\alpha = 0$ and K to be larger than the bound on f. Thus, the conditions of Theorem 2.1 are satisfied and the result follows. $\square$

If the right-hand side of (1.1) does not depend on $\Delta y(k)$ then we obtain the following discrete Neumann BVP

$$\nabla\Delta y(k) = f(k, y(k)), \qquad k = 1, \ldots, n-1; \tag{2.7}$$

$$\Delta y(0) = 0 = \Delta y(n); \tag{2.8}$$

and the following important corollary to Theorem 2.1 follows.

Corollary 2.3 *Let f be continuous. If there exist non-negative constants α and K such that*

$$(2.9) \qquad |f(t,p) - p| \leq 2\alpha p f(t,p) + K, \quad \forall (t,p) \in \{1,\ldots,n-1\} \times \mathbb{R};$$

the the discrete BVP (2.7), (2.8) *has at least one solution.*

Proof If (2.9) holds then it is easy to see that

$$|f(t,p) - p| \leq \alpha[2pf(t,p) + q^2] + K, \quad \forall (t,p,q) \in \{1,\ldots,n-1\} \times \mathbb{R} \times \mathbb{R};$$

and so the conditions of Theorem 2.1 hold with the result following from there. □

3 An Example

In this section an example is discussed to highlight how to apply the theory of Section 2.

Example 3.1 Consider the discrete Neumann BVP (2.7), (2.8) where f is given by

$$f(t,p) = p^5 + p + t, \quad t = 1,\ldots,9,$$

(and $n = 10$). We claim that for the above f, the discrete BVP (2.7), (2.8) has at least one solution.

Proof We want to show that there exist non-negative constants α and K such that (2.9) holds.

See that, for $(t,p) \in \{1,\ldots,9\} \times \mathbb{R}$ we have

$$|f(t,p) - p| \leq |p^5| + 9.$$

For α and K to be chosen below, for $(t,p) \in \{1,\ldots,9\} \times \mathbb{R}$ consider

$$\begin{aligned} 2\alpha p f(t,p) + K &= 2\alpha[p^6 + p^2 + pt] + K \\ &= (p^6 + 1) + [p^2 + pt + 49], \quad \text{for } \alpha = 1/2, K = 50 \\ &\geq (|p^5|) + [9] \geq |f(t,p) - p| \end{aligned}$$

and the result follows from Corollary 2.3. □

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Note. I. The condition (2.9) implies that **there exist constant lower and upper functions** (solutions) to problem (2.7), (2.8) . But this implies that there exists at least one solution between them! Let us prove the existence of constant lower and upper functions:

1. Let $p > 0$. Then (2.9) yields

$$f(t,p) - p \geq -2\alpha p f(t,p) - K,$$

$$f(t,p)(1+2\alpha p) \ge p-K,$$

$$f(t,p) \ge \frac{p-K}{1+2\alpha p} > 0 \quad \text{for each } p > K \ge 0.$$

2. Let $p<0$. Then (2.9) yields

$$f(t,p)-p \le 2\alpha p f(t,p)+K,$$

$$f(t,p)(1-2\alpha p) \le p+K,$$

$$f(t,p) \le \frac{p+K}{1-2\alpha p} < 0 \quad \text{for each} \quad p < -K \le 0.$$

Therefore we can find $r>0$ such that

$$f(t,-r)<0, \quad f(t,r)>0 \quad \text{for } t\in\{1,\dots,n-1\}. \tag{3.10}$$

So, the constant functions $\sigma_1(t)=-r$, $\sigma_2(t)=r$ are lower and upper functions of problem (2.7), (2.8) and if we prove lower and upper functions method for Neumann problem, we get **better result** than Cor.2.3. **Fortunately it is not the case** of Theorem 2.1 - see II below.

II. We can prove that condition (2.1) implies the existence of constant lower and upper functions to problem (1.1), (1.2). Similarly as in I, we can find $r>0$ such that

$$f(t,-r,0)<0, \quad f(t,r,0)>0 \quad \text{for } t\in\{1,\dots,n-1\}. \tag{3.11}$$

So, the constant functions $\sigma_1(t)=-r$, $\sigma_2(t)=r$ are lower and upper functions of problem (1.1), (1.2). If we have fixed step (here it is 1) and (3.11) holds, then for the solvability of (1.1), (1.2) the **monotonicity** of $f(t,p,q)$ in q is sufficient. We could prove it for Neumann problem (in some our further paper) similarly as we did in our second paper for Dirichler problem.

III. Therefore it is very important to add an example of $f(t,p,q)$ which satisfies (2.1) and which **is not monotonous in** q, because in such a way we show that the existence result based on the condition (2.1) is not completely contained in the lower and upper functions method. I hope that f in Example 3.2 has this needed properties.

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