Solvability of discrete Dirichlet problem via lower and upper functions method

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Abstract. We use the lower and upper functions method to prove the existence of a solution of the Dirichlet problem

$$\Delta(p(t)\Delta u(t-1)) + f(t,u(t)) = g(t), \quad t \in [1,T],$$

$$u(0) = 0, \quad u(T+1) = 0,$$

where $T \in \mathbb{N}$, $[1,T] = \{1,2,\ldots,T\}$, $p\colon [1,T+1] \to \mathbb{R}$ is positive and $f\colon [1,T]\times\mathbb{R} \to \mathbb{R}$ is continuous. Provided f fulfils certain sign conditions we get the solvability of the problem for each $g\colon [1,T] \to \mathbb{R}$.

Keywords. Dirichlet discrete BVP, lower and upper functions, Brouwer fixed point theorem, existence

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1 Introduction

For fixed $T \in \mathbb{N}$ we define the discrete interval $[1,T] = \{1,2,\ldots,T\}$. We will study the Dirichlet problem

$$\Delta(p(t)\Delta u(t-1)) + f(t,u(t)) = g(t), \quad t \in [1,T], \tag{1.1}$$

$$u(0) = 0, \quad u(T+1) = 0. \tag{1.2}$$

where

$$\left.\begin{array}{c} p\colon [1,T+1] \to \mathbb{R} \text{ is positive}, \quad g\colon [1,T] \to \mathbb{R} \\ f\colon [1,T]\times\mathbb{R} \to \mathbb{R} \text{ is continuous.} \end{array}\right\} \tag{1.3}$$

Here Δ denotes the forward difference operator with the step size 1, i.e. $\Delta u(t-1) = u(t) - u(t-1)$. Recall that $f(t,x)$ is continuous on $[1,T]\times\mathbb{R}$ if for each $t \in [1,T]$, $f(t,x)$ is a continuous function of x.

Definition 1.1 By *a solution* u of problem (1.1), (1.2) we mean $u\colon [0,T+1] \to \mathbb{R}$, u satisfies the difference equation (1.1) on $[1,T]$ and the boundary conditions (1.2).

Discrete boundary value problems arise in the study of solid state physics, chemical reaction, population dynamics and in many other areas, see [1], [13], [27]. Discrete second order nonlinear boundary value problems have been investigated in several monographs (e.g. [1], [6], [5], [17]) and papers (e.g. [8], [9], [10], [14], [16], [19], [20], [21], [23], [24], [26]). Particularly we can refer to papers [2], [3], [4], [7], [11], [12], [15], [18], [22], [25], which deal with various difference equations subjected to Dirichlet conditions. Yongjin Li in [18] used variational approach and proved the existence of a solution of (1.1), (1.2) under the assumptions

$$\exists r > 0 \quad \text{such that} \quad xf(t,x) \le 0 \quad \text{for } t \in [1,T] \text{ and } |x| \ge r, \tag{1.4}$$

$$\sum_{t=1}^{T+1} |g(t)|^2 < \frac{m}{2}, \quad \text{where} \quad m = \min\{p(t)\colon t \in [1,T+1]\}. \tag{1.5}$$

In this paper we use a completely different approach based on the lower and upper functions method. By means of this we prove that (1.4) yields the solvability of problem (1.1), (1.2) for each $g\colon [1,T] \to \mathbb{R}$, i.e. that (1.5) can be omitted.

2 Green's function

Consider the linear homogeneous equation

$$\Delta(p(t)\Delta u(t-1)) = 0, \quad t \in [1,T], \tag{2.1}$$

where p satisfies (1.3). Define

$$P(t) = \sum_{i=1}^{t} \frac{1}{p(i)}, \quad t \in [1,T+1] \quad \text{and} \quad P(0) = 0, \tag{2.2}$$

and denote

$$M_p = \max\left\{\frac{1}{p(t)}\colon t \in [1,T+1]\right\} > 0. \tag{2.3}$$

Remark 2.1 The general solution of (2.1) has the form $u(t) = c_1 + c_2P(t)$, $t \in [0,T+1]$, where $c_1, c_2 \in \mathbb{R}$. Therefore (1.2) and (1.3) yield $c_1 = c_2 = 0$ and hence problem (2.1), (1.2) has only the trivial solution.

Lemma 2.2 *Let p satisfy* (1.3). *Then the Green's function of problem* (2.1), (1.2) *has the form*

$$G(t,s) = -\begin{cases} \dfrac{P(s)}{P(T+1)}(P(T+1) - P(t)) & \text{if} \quad 0 \le s \le t \le T+1 \\[2ex] \dfrac{P(t)}{P(T+1)}(P(T+1) - P(s)) & \text{if} \quad 0 \le t \le s \le T+1. \end{cases} \tag{2.4}$$

Proof. The proof can be done similarly as in [17], Example 6.12. □

Due to (2.2), (2.3) and (2.4) we see that

$$G(0,s)=0, \quad G(T+1,s)=0 \quad \text{for } s\in[0,T+1], \tag{2.5}$$

$$-TM_p < G(t,s) < 0 \quad \text{for } t,s\in[1,T]. \tag{2.6}$$

Further we have

$$\Delta G(t-1,s) = \frac{1}{p(t)P(T+1)} \begin{cases} P(s) & \text{for} \quad s+1\le t \\ P(s)-P(T+1) & \text{for} \quad t\le s \end{cases}$$

and

$$\Delta(p(t)\Delta G(t-1,s)) = \begin{cases} 0 & \text{for} \quad t\le s+1 \text{ and } t\ge s+1 \\ 1 & \text{for} \quad t=s. \end{cases}$$

Therefore, according to Remark 2.1 and Lemma 2.2, we get the following lemma for the nonhomogeneous linear equation

$$\Delta(p(t)\Delta u(t-1)) = g(t), \quad t\in[1,T], \tag{2.7}$$

where p and q satisfy (1.3).

Lemma 2.3 *Problem* (2.7), (1.2) *has the unique solution of the form*

$$u_0(t) = \sum_{s=1}^{T} G(t,s)g(s), \quad t\in[0,T+1]. \tag{2.8}$$

3 Lower and upper functions

Lower and upper functions are important tools for the investigation of solvability of boundary value problems. Here we bring their definition for problem (1.1), (1.2).

Definition 3.1 $\alpha\colon[0,T+1]\to\mathbb{R}$ is called *a lower function* of problem (1.1), (1.2) if

$$\Delta(p(t)\Delta\alpha(t-1)) + f(t,\alpha(t)) \ge g(t) \quad \text{for } t\in[1,T], \tag{3.1}$$

$$\alpha(0)\le 0, \quad \alpha(T+1)\le 0. \tag{3.2}$$

$\beta\colon[0,T+1]\to\mathbb{R}$ is called *an upper function* of problem (1.1), (1.2) if

$$\Delta(p(t)\Delta\beta(t-1)) + f(t,\beta(t)) \le g(t) \quad \text{for } t\in[1,T], \tag{3.3}$$

$$\beta(0)\ge 0, \quad \beta(T+1)\ge 0. \tag{3.4}$$

Theorem 3.2 (Lower and upper functions method) *Assume* (1.3). *Let* α *and* β *be a lower and an upper function of* (1.1), (1.2) *and* $\alpha \le \beta$ *on* $[1,T]$. *Then problem* (1.1), (1.2) *has a solution* u *satisfying*

$$\alpha(t) \le u(t) \le \beta(t) \quad \textit{for } t \in [0, T+1]. \tag{3.5}$$

Theorem 3.2 is a slight modification of Theorem 9.7 in [17], where $p(t) \equiv 1$. However for the reader's convenience we will prove Theorem 3.2 here.

Proof. *Step* 1. For $t \in [1,T]$, $x \in \mathbb{R}$, define function

$$\tilde{f}(t,x) = \begin{cases} f(t,\beta(t)) - \dfrac{x-\beta(t)}{x-\beta(t)+1} & \text{if} \quad x > \beta(t) \\ f(t,x) & \text{if} \quad \alpha(t) \le x \le \beta(t) \\ f(t,\alpha(t)) + \dfrac{\alpha(t)-x}{\alpha(t)-x+1} & \text{if} \quad x < \alpha(t). \end{cases}$$

Since $\tilde{f}$ is continuous on $[1,T] \times \mathbb{R}$, there exists $M > 0$ such that

$$|\tilde{f}(t,x)| \le M \quad \text{for } t \in [1,T],\ x \in \mathbb{R}. \tag{3.6}$$

We will study the auxiliary difference equation

$$\Delta(p(t)\Delta u(t-1)) + \tilde{f}(t,u(t)) = g(t), \quad t \in [1,T], \tag{3.7}$$

and we will prove that problem (3.7), (1.2) has a solution (see Steps 2–3).

Step 2. We define the space

$$E = \{u\colon [0,T+1] \to \mathbb{R},\ u(0) = 0,\ u(T+1) = 0\}$$

with the norm $\|u\| = \max\{|u(t)|\colon t \in [1,T]\}$. Then E is a Banach space with $\dim E = T$. Further we define an operator $\mathcal{F}\colon E \to E$ by

$$(\mathcal{F}u)(t) = \sum_{s=1}^{T} G(t,s)\Big(g(s) - \tilde{f}(s,u(s))\Big), \quad t \in [0,T+1]. \tag{3.8}$$

Due to (1.3), $\mathcal{F}$ is a continuous operator. Denote $B(r) = \{u \in E\colon \|u\| < r\}$ and

$$M_g = \max\{|g(t)|\colon t \in [1,T]\}. \tag{3.9}$$

Let us choose $r^* \ge T^2 M_p(M_g + M)$, where M_p and M are given by (2.6) and (3.6), respectively. Then by (2.5) and (3.8) we get $\mathcal{F}\left(\overline{B(r^*)}\right) \subset \overline{B(r^*)}$. Therefore the Brouwer fixed point theorem yields the existence of at least one point $u \in \overline{B(r^*)}$

such that $u = \mathcal{F}u$. According to Lemma 2.3 we see that if u is a fixed point of $\mathcal{F}$, then u satisfies (3.7) and (1.2).

Step 3. We prove that the solution u of (3.7), (1.2) satisfies (1.1). Put $v(t) = \alpha(t) - u(t)$ for $t \in [0, T+1]$ and assume that $\max\{v(t)\colon t \in [0, T+1]\} = v(\ell) > 0$. Conditions (1.2) and (3.2) imply $\ell \in [1, T]$. Thus we have $v(\ell+1) \le v(\ell)$, $v(\ell-1) \le v(\ell)$, and consequently $\Delta\alpha(\ell) \le \Delta u(\ell)$, $\Delta\alpha(\ell-1) \ge \Delta u(\ell-1)$. This leads to $p(\ell+1)\Delta\alpha(\ell) \le p(\ell+1)\Delta u(\ell)$, $p(\ell)\Delta\alpha(\ell-1) \ge p(\ell)\Delta u(\ell-1)$ and

$$\Delta(p(\ell)\Delta u(\ell-1)) \ge \Delta(p(\ell)\Delta\alpha(\ell-1)). \tag{3.10}$$

On the other hand we get by (3.1) and (3.7)

$$\Delta(p(\ell)\Delta\alpha(\ell-1)) - \Delta(p(\ell)\Delta u(\ell-1)) =$$

$$= \Delta(p(\ell)\Delta\alpha(\ell-1)) - \Big(g(\ell) - \tilde{f}(\ell, u(\ell))\Big) =$$

$$= \Delta(p(\ell)\Delta\alpha(\ell-1)) + f(\ell, \alpha(\ell)) + \frac{v(\ell)}{v(\ell)+1} - g(\ell) \ge$$

$$\ge \frac{v(\ell)}{v(\ell)+1} > 0,$$

which contradicts (3.10). So, we have proved $\alpha(t) \le u(t)$ for $t \in [0, T+1]$. The inequality $u(t) \le \beta(t)$ for $t \in [0, T+1]$ can be proved similarly. Therefore u satisfies (3.5) and hence u is a solution of problem (1.1), (1.2). $\square$

4 Main results

Our main result is contained in the next theorem, which provides sufficient conditions for the solvability of problem (1.1), (1.2). The proof is based on the lower and upper functions method from Theorem 3.2.

Theorem 4.1 *Assume that* (1.3) *and* (1.4) *hold. Then problem* (1.1), (1.2) *has at least one solution.*

Proof. By Lemma 2.3, problem (2.7), (1.2) has the unique solution u_0 given by (2.8). Using (3.9), (2.5) and (2.6) we have

$$|u_0(t)| \le T^2 M_p M_g \quad \text{for } t \in [0, T+1].$$

Choose $A, B \in \mathbb{R}$ such that

$$A \le -T^2 M_p M_g - r, \quad B \ge T^2 M_p M_g + r$$

and define functions

$$\alpha(t) = u_0(t) + A, \quad \beta(t) = u_0(t) + B, \quad t \in [0, T+1].$$

Then $\alpha(t) \le -r$, $\beta(t) \ge r$ for $t \in [0, T+1]$. This implies that α and β satisfy (3.2) and (3.4) respectively. Moreover, by (1.4),

$$\Delta(p(t)\Delta\alpha(t-1)) + f(t, \alpha(t)) = \Delta(p(t)\Delta u_0(t-1)) + f(t, \alpha(t)) \ge$$

$$\ge \Delta(p(t)\Delta u_0(t-1)) = g(t) \quad \text{for } t \in [1, T].$$

Similarly

$$\Delta(p(t)\Delta\beta(t-1)) + f(t, \beta(t)) \le g(t) \quad \text{for } t \in [1, T].$$

Therefore α and β are a lower and an upper function of (1.1), (1.2), respectively, and $\alpha \le \beta$ on $[1, T]$. Hence Theorem 3.2 guarantees the existence of at least one solution u of (1.1), (1.2) satisfying (3.5). $\square$

Example. Assume $k \in \mathbb{N}$, $c \in \mathbb{R}$, $a(t)\colon [1, T] \to \mathbb{R}$, $b(t)\colon [1, T] \to (-\infty, 0)$ and consider the equation

$$\Delta(t^3\Delta u(t-1)) + a(t) + b(t)u^{2k-1}(t) = ct^2\mathrm{e}^t, \quad t \in [1, T]. \tag{4.1}$$

By Theorem 4.1, problem (4.1), (1.2) has a solution.

Corollary 4.2 *Assume that* (1.3) *holds. Let*

$$g(t) < 0, \quad f(t, 0) \ge 0 \quad \textit{for } t \in [1, T], \tag{4.2}$$

$$\exists r > 0 \quad \textit{such that} \quad f(t, x) \le 0 \quad \textit{for } t \in [1, T] \textit{ and } x \ge r. \tag{4.3}$$

Then problem (1.1)*,* (1.2) *has a solution u such that*

$$u(t) > 0 \quad \textit{for } t \in [1, T]. \tag{4.4}$$

Proof. Condition (4.2) implies that if we put $\alpha(t) \equiv 0$, we have

$$\Delta(p(t)\Delta\alpha(t-1)) + f(t, \alpha(t)) = f(t, 0) \ge 0 > g(t), \quad t \in [1, T].$$

If $\beta(t)$ is the same as in the proof of Theorem 4.1, we get a solution u of problem (1.1), (1.2) such that

$$0 \le u(t) \le \beta(t) \quad \text{for } t \in [0, T+1].$$

Since $f(t, 0) > g(t)$ on $[1, T]$, we obtain (4.4). $\square$

Corollary 4.3 *Assume that* (1.3) *holds. Let*

$$g(t) > 0, \quad f(t,0) \le 0 \quad \textit{for } t \in [1,T], \tag{4.5}$$

$$\exists r > 0 \quad \textit{such that} \quad f(t,x) \ge 0 \quad \textit{for } t \in [1,T] \textit{ and } x \le -r. \tag{4.6}$$

Then problem (1.1), (1.2) *has a solution* u *such that*

$$u(t) < 0 \quad \textit{for } t \in [1,T]. \tag{4.7}$$

Proof. We argue similarly as in the proof of Corollary 4.2 putting $\alpha(t)$ as in the proof of Theorem 4.1 and $\beta(t) \equiv 0$. □

Example. If $a(t) \ge 0$ on $[1,T]$ and $c < 0$, then by Corollary 4.2, problem (4.1), (1.2) has a solution, which is positive on $[1,T]$.

If $a(t) \le 0$ on $[1,T]$ and $c > 0$, then by Corollary 4.3, problem (4.1), (1.2) has a solution, which is negative on $[1,T]$.

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