Periodic BVPs in ODEs with time singularities

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Abstract: In this paper we show the existence of solutions to a nonlinear singular second order ordinary differential equation,

$$u''(t) = \frac{a}{t}u'(t) + \lambda f(t, u(t), u'(t)),$$

subject to periodic boundary conditions, where $a > 0$ is a given constant, $\lambda > 0$ is a parameter, and the nonlinearity $f(t, x, y)$ satisfies the local Carathéodory conditions on $[0, T] \times \mathbb{R} \times \mathbb{R}$. Here, we study the case that a well-ordered pair of lower and upper functions does not exist and therefore the underlying problem cannot be treated by well-known standard techniques. Instead, we assume the existence of constant lower and upper functions having opposite order. Analytical results are illustrated by means of numerical experiments.

Key words: Singular boundary value problems, periodic boundary conditions, time singularity of the first kind, existence of solutions, lower and upper functions, opposite order, collocation methods

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1 Introduction

Singular periodic boundary value problems for the ordinary differential equation $u''(t) = g(t, u(t), u'(t))$, where $g(t, x, y)$ shows singularities in the phase variable x, have been widely studied for more than 40 years and there exists rich literature on their properties.

In this paper we focus our attention on another type of singular periodic problems, namely on those with singularities in the time variable t. In many applications, cf. [4], [7], [13], and [22], second order singular models, are posed on the interval $(0,T)$ and have the form,

$$u''(t) = \frac{a_1}{t^\alpha}u'(t) + \frac{a_0}{t^{\alpha+1}}u(t) + f(t,u(t),u'(t)), \qquad u''(t) = \frac{a}{t^\alpha}u'(t) + f(t,u(t),u'(t)),$$

where a_1, a_0, a and f are given. We say that for $\alpha = 1$ the problem exhibits a time singularity of the first kind at $t = 0$, while for $\alpha > 1$, the time singularity is essential or of the second kind.

Let $T > 0$, and let us by $L_1[0,T]$ denote the set of functions which are (Lebesgue) integrable on $[0,T]$. The corresponding norm is given by $\|u\|_1 := \int_0^T |u(t)|\mathrm{d}t$. Moreover, let us by $C[0,T]$ and $C^1[0,T]$ denote the sets of functions being continuous on $[0,T]$, and having continuous first derivatives on $[0,T]$, respectively. The maximum norm on $C[0,T]$ is defined as $\|u\|_\infty := \max_{t\in[0,T]}|u(t)|$. We denote by $AC[0,T]$ and $AC^1[0,T]$ the sets of functions which are absolutely continuous on $[0,T]$, and which have absolutely continuous first derivatives on $[0,T]$, respectively. Analogously, $AC^1_{loc}(0,T]$ is the set of functions having absolutely continuous first derivatives on each compact subinterval of $(0,T]$.

In this paper, we investigate the following parameter dependent ordinary differential equation, with a singularity of the first kind,

$$u''(t) = \frac{a}{t}u'(t) + \lambda f(t,u(t),u'(t)), \tag{1}$$

where $a > 0$, $\lambda > 0$, and the function $f(t,x,y)$ is defined for a.e. $t \in [0,T] \subset \mathbb{R}$ and for all $(x,y) \in \mathcal{D} \subset \mathbb{R}\times\mathbb{R}$. Clearly, the above equation is singular at $t = 0$ because of the first term in the right-hand side, which is in general unbounded for $t \to 0$. Moreover, we also alow the function f to be unbounded or bounded but discontinuous for certain values of the time variable $t \in [0,T]$. This form of f is motivated by a variety of initial and boundary value problems known from applications and having nonlinear, discontinuous forcing terms, such as electronic devices which are often driven by square waves or more complicated discontinuous inputs. Typically, such problems are modelled by differential equations where f has jump discontinuities at a discrete set of points in $(0,T)$, cf. [15]. Many other applications, cf. [1]–[2], [4]–[8], [11], [13], [17]–[21] show similar structural difficulties. This motivates to assume the following properties of f in (1):

(A_1) : $f(\cdot,x,y) : [0,T] \to \mathbb{R}$ is measurable for all $(x,y) \in \mathbb{R}^2$ and $f(t,\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is continuous for a.e. $t \in [0,T]$.

(A_2) : For a.e. $t \in [0, T]$ and all $(x, y) \in \mathbb{R}^2$ the estimate

$$|f(t, x, y)| \le g(t)\omega(|y|)$$

holds with positive functions $g \in L_1[0, T]$ and $\omega(y) \in C[0, \infty)$, where ω is nondecreasing.

Definition 1 A function $u : [0, T] \to \mathbb{R}$ is called *a solution of equation* (1) if $u \in AC^1[0, T]$ and

$$u''(t) = \frac{a}{t}u'(t) + \lambda f(t, u(t), u'(t))$$

holds a.e. on $[0, T]$.

In the sequel, we study the differential equation (1) subject to periodic boundary conditions,

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{2}$$

In the literature, there are only very few results for periodic boundary value problems of the form

$$u''(t) = \frac{a}{t}u'(t) + \lambda f(t, u(t), u'(t)), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{3}$$

with a non-integrable singularity in the time variable, see [12], [16], [18], and [19], where existence results for this type of problems are shown using the lower and upper functions technique. All these results are obtained under the assumption that there exists a pair of constant well-ordered lower and upper functions. In this paper we show the existence result for problem (3) in the dual and more difficult case, where the problem has *the opposite-ordered upper and lower functions*, which means that the upper function lies below the lower one. We illustrate this situation with examples to whom the earlier know results do not apply.

The paper is organized as follows: In order to prove that the periodic problem (3) is solvable, we first show in Section 3 that the auxiliary problem,

$$u''(t) = \frac{a}{t}u'(t) + \lambda f(t, u(t), u'(t)), \quad u(0) = u(T), \quad u'(T) = 0, \tag{4}$$

has a solution. The main tool in the proof is the Leray-Schauder degree method, cf. e.g., [6]. Then, applying Lemma 1 from Section 2, we conclude that for solutions u of (4) also $u'(0) = 0$ holds. Hence, the solution u of (4) satisfies (2) and consequently, is a solution of problem (3). We give two examples of boundary value problem (3) which can be analyzed using results developed in this paper and for which on the other hand, a pair of constant well-ordered lower and upper functions does not exist. Finally, in Section 4, we illustrate the theoretical findings by means of numerical experiments.

2 Preliminaries

The following two results are used in the next section in the discussion of the boundary value problem (3).

Lemma 1 *Let* (A_1) *and* (A_2) *hold and let* $\lambda > 0$, $a \neq 0$. *Suppose that* $u \in AC^1_{loc}(0,T]$ *satisfies equation* (1) *a.e. on* $[0,T]$ *and*

$$\sup\{|u(t)| + |u'(t)| : t \in (0,T]\} < \infty.$$

Then $\lim_{t\to 0_+} u'(t) = 0$.

Proof. The proof follows from [20, Corollary 3.5].

Lemma 2 *Let* (A_1) *and* (A_2) *hold and let* $a > 0$. *Then for each* $x, y \in C[0,T]$ *the function*

$$t^a \int_t^T \frac{f(s,x(s),y(s))}{s^a}\,\mathrm{d}s, \quad t \in (0,T]$$

can be extended on the whole interval $[0,T]$ *as a function in* $AC[0,T]$.

Proof. Let us choose $x, y \in C[0,T]$ and let

$$r(t) := t^a \int_t^T \frac{f(s,x(s),y(s))}{s^a}\,\mathrm{d}s, \quad t \in (0,T].$$

Then

$$r'(t) = at^{a-1}\int_t^T \frac{f(s,x(s),y(s))}{s^a}\,\mathrm{d}s - f(t,x(t),y(t)) \ \text{ for a.e. } t \in [0,T].$$

Let $\varepsilon \in (0,T)$. Since, by (A_2),

$$|f(t,x(t),y(t))| \le g(t)\omega(|y(t)|) \le g(t)\omega(\|y\|_\infty) \ \text{ for a.e. } t \in [0,T]$$

holds, we integrate by parts and obtain

$$\begin{aligned}
\int_\varepsilon^T \left| t^{a-1}\int_t^T \frac{f(s,x(s),y(s))}{s^a}\,\mathrm{d}s\right| \mathrm{d}t &\le \omega(\|y\|_\infty)\int_\varepsilon^T t^{a-1}\left(\int_t^T \frac{g(s)}{s^a}\,\mathrm{d}s\right)\mathrm{d}t \\
&= \frac{\omega(\|y\|_\infty)}{a}\left(\int_\varepsilon^T g(s)\,\mathrm{d}s - \varepsilon^a \int_\varepsilon^T \frac{g(s)}{s^a}\,\mathrm{d}s\right) \\
&< \frac{\omega(\|y\|_\infty)}{a}\int_\varepsilon^T g(s)\,\mathrm{d}s.
\end{aligned} \tag{5}$$

Hence,

$$\begin{aligned}
\int_\varepsilon^T |r'(t)|\,\mathrm{d}t \ &\le \ a\int_\varepsilon^T \left| t^{a-1}\int_t^T \frac{f(s,x(s),y(s))}{s^a}\,\mathrm{d}s\right| \mathrm{d}t + \int_\varepsilon^T |f(t,x(t),y(t))|\,\mathrm{d}t \\
&\le \ 2\omega(\|y\|_\infty)\int_\varepsilon^T g(t)\,\mathrm{d}t.
\end{aligned}$$

Taking the limit $\varepsilon \to 0^+$, we have

$$\int_0^T |r'(t)|\,\mathrm{d}t \le 2\omega(\|y\|_\infty) \int_0^T g(t)\,\mathrm{d}t.$$

Consequently, $r' \in L_1[0,T]$ and therefore, there exists a finite limit $\lim_{t\to 0^+} r(t) =: c$. For $r(0) := c$ we now conclude that $r \in AC[0,T]$. $\square$

Remark 1 Taking the limit $\varepsilon \to 0^+$ in (5) implies

$$\int_0^T \left| t^{a-1} \int_t^T \frac{f(s,x(s),y(s))}{s^a}\,\mathrm{d}s \right| \mathrm{d}t \le \frac{\omega(\|y\|_\infty)}{a} \int_0^T g(s)\,\mathrm{d}s$$

for all $x, y \in C[0,T]$, and

$$\int_0^T t^{a-1} \left(\int_t^T \frac{g(s)}{s^a}\,\mathrm{d}s \right) \mathrm{d}t \le \frac{1}{a} \int_0^T g(t)\,\mathrm{d}t.$$

3 Analytical investigations

The main analytical result for the periodic problem (3) is given in the following theorem.

Theorem 1 *Let $a > 0$. Let conditions (A_1) and (A_2) hold. Suppose that there exist $A, B \in \mathbb{R}$, such that $A < B$ and*

$$f(t,x,y) > 0 \quad \text{for a.e. } t \in [0,T] \ \text{ and all } x \le A, \ \ y \in \mathbb{R}, \tag{6}$$

$$f(t,x,y) < 0 \quad \text{for a.e. } t \in [0,T] \ \text{ and all } x \ge B, \ \ y \in \mathbb{R}. \tag{7}$$

Let

$$\lambda^* = \int_0^\infty \frac{\mathrm{d}s}{\omega(s)} \cdot \left(\int_0^T g(t)\,\mathrm{d}t \right)^{-1}.$$

Then problem (3) has a solution for each $\lambda \in (0, \lambda^*)$.

Proof. Let us choose $\lambda \in (0, \lambda^*)$. Also, let $G(z) := \int_0^z (1/\omega(s))\,\mathrm{d}s$ for $z \in [0,\infty)$. Then G is increasing on $[0,\infty)$ and $\lim_{z\to\infty} G(z) > \lambda \int_0^T g(t)\,\mathrm{d}t$. Hence, the inverse function G^{-1} of G is defined on an interval $[0, L)$, where $L > \lambda \int_0^T g(t)\,\mathrm{d}t$. Let $\varepsilon \in \left(0, L - \lambda \int_0^T g(t)\,\mathrm{d}t\right)$ and define $S := G^{-1}\left(\lambda \int_0^T g(t)\,\mathrm{d}t + \varepsilon\right)$. Let

$$\Omega = \{(x,c) \in C^1[0,T] \times \mathbb{R} : \|x\|_\infty < ST, \|x'\|_\infty < S, |c| < ST + \max\{|A|,|B|\}\}.$$

Then Ω is an open, bounded and symmetric with respect to $(0,0)$ subset of the Banach space $C^1[0,T]\times\mathbb{R}$ equipped with the norm $\|(x,c)\|_\Omega = \|x\|_\infty + \|x'\|_\infty + |c|$.

Keeping in mind that for each $x, y \in C[0,T]$ the function r,

$$r(t) = t^a \int_t^T \frac{f(s,x(s),y(s))}{s^a}\,\mathrm{d}s, \quad t \in (0,T],$$

can be extended on $[0,T]$ in such a way that $r \in AC[0,T]$, cf. Lemma 2, we define an operator $\mathcal{Q} : [0,1]\times\overline{\Omega} \to C^1[0,T]\times\mathbb{R}$ as

$$\mathcal{Q}(\mu,x,c)(t) = (\mathcal{Q}_1(\mu,x,c)(t), \mathcal{Q}_2(\mu,x,c)),$$

where

$$\mathcal{Q}_1(\mu,x,c)(t) = -\mu\lambda \int_0^t \left(s^a \int_s^T \frac{f(\xi,x(\xi)+c,x'(\xi))}{\xi^a}\,\mathrm{d}\xi\right)\mathrm{d}s,$$

$$\mathcal{Q}_2(\mu,x,c) = (2-\mu)c - \mu\lambda \int_0^T \left(s^a \int_s^T \frac{f(\xi,x(\xi)+c,x'(\xi))}{\xi^a}\,\mathrm{d}\xi\right)\mathrm{d}s.$$

Suppose that $(y,b) \in \overline{\Omega}$ is a fixed point of $\mathcal{Q}(1,\cdot,\cdot)$, that is, $\mathcal{Q}(1,y,b) = (y,b)$. Then

$$y(t) = -\lambda \int_0^t \left(s^a \int_s^T \frac{f(\xi,y(\xi)+b,y'(\xi))}{\xi^a}\,\mathrm{d}\xi\right)\mathrm{d}s, \quad t \in [0,T], \tag{8}$$

and

$$\int_0^T \left(s^a \int_s^T \frac{f(\xi,y(\xi)+b,y'(\xi))}{\xi^a}\,\mathrm{d}\xi\right)\mathrm{d}s = 0. \tag{9}$$

With $u(t) := y(t)+b$ for $t \in [0,T]$, it follows from (8) and (9),

$$u(t) = b - \lambda \int_0^t \left(s^a \int_s^T \frac{f(\xi,u(\xi),u'(\xi))}{\xi^a}\,\mathrm{d}\xi\right)\mathrm{d}s, \quad t \in [0,T],$$

and

$$\int_0^T \left(s^a \int_s^T \frac{f(\xi,u(\xi),u'(\xi))}{\xi^a}\,\mathrm{d}\xi\right)\mathrm{d}s = 0.$$

Hence, u is a solution of (1), subject to $u(0) = u(T) = b$ and $u'(T) = 0$.

Now $u \in AC^1[0,T]$ follows, since $f(t,u(t),u'(t)) \in L_1[0,T]$ and since, by Remark 1, for $x(t) = u(t)$, $y(t) = u'(t)$,

$$\frac{u'(t)}{t} = -\lambda t^{a-1} \int_t^T \frac{f(s,u(s),u'(s))}{s^a}\,\mathrm{d}s \in L_1[0,T]$$

holds. Furthermore, $u'(0) = 0$ by Lemma 1. Consequently, u is a solution of problem (3).

In order to prove that the operator $\mathcal{Q}(1,\cdot,\cdot)$ has a fixed point, we use the Leray-Schauder degree method, which means that a fixed point of the operator $\mathcal{Q}(1,\cdot,\cdot)$ exists if

$$\deg\left(\mathcal{I}-\mathcal{Q}(1,\cdot,\cdot),\Omega,0\right)\neq 0, \tag{10}$$

where 'deg' is the Leray-Schauder degree and $\mathcal{I}$ is the identical operator on the space $C^1[0,T]\times\mathbb{R}$. By the homotopy property, relation (10) holds if

(i) $\mathcal{Q}$ is a compact operator,
(ii) $\deg\left(\mathcal{I}-\mathcal{Q}(0,\cdot,\cdot),\Omega,0\right)\neq 0$,
(iii) $\mathcal{Q}(\mu,x,c)\neq(x,c)$ for $\mu\in[0,1]$ and $(x,c)\in\partial\Omega$.

We first show (i). The operator $\mathcal{Q}$ is continuous. To see this let $\{(\mu_n,x_n,c_n)\}\subset[0,1]\times\overline{\Omega}$ be a convergent sequence and let $\lim_{n\to\infty}(\mu_n,x_n,c_n)=(\mu,x,c)$. For $t\in[0,T]$ and $n\in\mathbb{N}$ the relation

$$\begin{aligned}
&|\mathcal{Q}_1'(\mu_n,x_n,c_n)(t)-\mathcal{Q}_1'(\mu,x,c)(t)|\\
&\quad=\lambda\left|\mu_n t^a\int_t^T\frac{f(s,x_n(s)+c_n,x_n'(s))}{s^a}\,\mathrm{d}s-\mu t^a\int_t^T\frac{f(s,x(s)+c,x'(s))}{s^a}\,\mathrm{d}s\right|\\
&\quad\leq\mu_n\lambda t^a\int_t^T\frac{|f(s,x_n(s)+c_n,x_n'(s))-f(s,x(s)+c,x'(s))|}{s^a}\,\mathrm{d}s\\
&\qquad+|\mu_n-\mu|\lambda t^a\int_t^T\frac{|f(s,x(s)+c,x'(s))|}{s^a}\,\mathrm{d}s\\
&\quad\leq\lambda\int_0^T|f(s,x_n(s)+c_n,x_n'(s))-f(s,x(s)+c,x'(s))|\,\mathrm{d}s\\
&\qquad+|\mu_n-\mu|\lambda\omega(S)\int_0^T g(s)\,\mathrm{d}s
\end{aligned}$$

holds. Here, $\mathcal{Q}_1'(\mu,x,c)(t)=\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q}_1(\mu,x,c)(t)$. Then

$$\begin{aligned}
&|\mathcal{Q}_1(\mu_n,x_n,c_n)(t)-\mathcal{Q}_1(\mu,x,c)(t)|=\left|\int_0^t[\mathcal{Q}_1'(\mu_n,x_n,c_n)(s)-\mathcal{Q}_1'(\mu,x,c)(s)]\,\mathrm{d}s\right|\\
&\quad\leq T\lambda\int_0^T|f(s,x_n(s)+c_n,x_n'(s))-f(s,x(s)+c,x'(s))|\,\mathrm{d}s\\
&\qquad+|\mu_n-\mu|\lambda T\omega(S)\int_0^T g(s)\,\mathrm{d}s.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&|\mathcal{Q}_2(\mu_n,x_n,c_n)-\mathcal{Q}_2(\mu,x,c)|\\
&\quad=\left|(2-\mu_n)c_n-\mu_n\lambda\int_0^T\left(s^a\int_s^T\frac{f(\xi,x_n(\xi)+c_n,x_n'(\xi))}{\xi^a}\,\mathrm{d}\xi\right)\mathrm{d}s\right.\\
&\qquad\left.-(2-\mu)c+\mu\lambda\int_0^T\left(s^a\int_s^T\frac{f(\xi,x(\xi)+c,x'(\xi))}{\xi^a}\,\mathrm{d}\xi\right)\mathrm{d}s\right|
\end{aligned}$$

$$\le T\lambda \int_0^T |f(s, x_n(s) + c_n, x_n'(s)) - f(s, x(s) + c, x'(s))| \,\mathrm{d}s$$
$$+ |(2-\mu_n)c_n - (2-\mu)c| + |\mu_n - \mu|\lambda T\omega(S) \int_0^T g(s)\,\mathrm{d}s.$$

Hence,

$$\begin{aligned} \|\mathcal{Q}(\mu_n, x_n, c_n) - \mathcal{Q}(\mu, x, c)\|_\Omega &\le |(2-\mu_n)c_n - (2-\mu)c| \\ &+ (2T+1)\lambda \int_0^T |f(s, x_n(s) + c_n, x_n'(s)) - f(s, x(s) + c, x'(s))| \,\mathrm{d}s \\ &+ |\mu_n - \mu|\lambda(2T+1)\omega(S) \int_0^T g(s)\,\mathrm{d}s. \end{aligned} \tag{11}$$

Since $\lim_{n\to\infty} f(t, x_n(t)+c_n, x_n'(t)) = f(t, x(t)+c, x'(t))$, $|f(t, x_n(t)+c_n, x_n'(t))| \le g(t)\omega(S)$ for a.e. $t \in [0,T]$ and all $n \in \mathbb{N}$, the Lebesgue dominated convergence theorem implies

$$\lim_{n\to\infty} \int_0^T |f(t, x_n(t) + c_n, x_n'(t)) - f(t, x(t) + c, x'(t))| \,\mathrm{d}t = 0.$$

With $n \to \infty$, (11) yields $\lim_{n\to\infty} \|\mathcal{Q}(\mu_n, x_n, c_n) - \mathcal{Q}(\mu, x, c)\|_\Omega = 0$. Hence, $\mathcal{Q}$ is a continuous operator.

From the estimates ($\mu \in [0,1]$, $t \in [0,T]$ and $(x,c) \in \overline{\Omega}$),

$$\begin{aligned} |\mathcal{Q}_1'(\mu, x, c)(t)| &= \mu\lambda \left| t^a \int_t^T \frac{f(s, x(s) + c, x'(s))}{s^a} \,\mathrm{d}s \right| \le \omega(S)\lambda \int_0^T g(s)\,\mathrm{d}s, \\ |\mathcal{Q}_1(\mu, x, c)(t)| &= \left| \int_0^t \mathcal{Q}_1'(\mu, x, c)(s)\,\mathrm{d}s \right| \le T\omega(S)\lambda \int_0^T g(s)\,\mathrm{d}s, \end{aligned}$$

$$\begin{aligned} |\mathcal{Q}_2(\mu, x, c)| &= \left| (2-\mu)c + \mu\lambda \int_0^T \left(s^a \int_s^T \frac{f(\xi, x(\xi) + c, x'(\xi))}{\xi^a} \,\mathrm{d}\xi \right) \mathrm{d}s \right| \\ &\le 2(ST + \max\{|A|, |B|\}) + \omega(S)\lambda \int_0^T \left(\int_s^T g(\xi)\,\mathrm{d}\xi \right) \mathrm{d}s \\ &\le 2(ST + \max\{|A|, |B|\}) + T\omega(S)\lambda \int_0^T g(s)\,\mathrm{d}s, \end{aligned}$$

it follows that the set $\mathcal{Q}([0,T] \times \overline{\Omega})$ is bounded in $C^1[0,T] \times \mathbb{R}$. Moreover, for a.e. $t \in [0,T]$ and all $\mu \in [0,1]$, $(x,c) \in \overline{\Omega}$, the relation

$$|\mathcal{Q}_1''(\mu, x, c)(t)| = \mu\lambda \left| a t^{a-1} \int_t^T \frac{f(s, x(s) + c, x'(s))}{s^a} \,\mathrm{d}s - f(t, x(t) + c, x'(t)) \right| \le \rho(t)$$

holds, where

$$\rho(t) := \omega(S)\lambda \left(a t^{a-1} \int_t^T \frac{g(s)}{s^a} \mathrm{d}s + g(t) \right).$$

By (A_2) and Remark 1, $\rho \in L_1[0,T]$, and consequently, the set $\{\mathcal{Q}_1'(\mu,x,c) : \mu \in [0,1], (x,c) \in \overline{\Omega}\}$ is equicontinuous on $[0,T]$. Hence, the Arzelà-Ascoli theorem and the Bolzano-Weierstrass theorem guarantee that the set $\mathcal{Q}([0,T]\times\overline{\Omega})$ is relatively compact in $C^1[0,T]\times\mathbb{R}$. Consequently, $\mathcal{Q}$ is a compact operator and (i) follows.

Since $\mathcal{Q}(0,x,c)(t) = (0,2c)$ is an odd operator, property (ii) follows from the Borsuk antipodal theorem.

It remains to verify property (iii). Let $\mathcal{Q}(\mu,x,c) = (x,c)$ for some $\mu \in [0,1]$ and $(x,c) \in \overline{\Omega}$. Then

$$x(t) = -\mu\lambda \int_0^t \Big(s^a \int_s^T \frac{f(\xi, x(\xi)+c, x'(\xi))}{\xi^a}\,\mathrm{d}\xi\Big)\,\mathrm{d}s, \quad t \in [0,T],$$

and

$$(1-\mu)c = \mu\lambda \int_0^T \Big(s^a \int_s^T \frac{f(\xi, x(\xi)+c, x'(\xi))}{\xi^a}\,\mathrm{d}\xi\Big)\,\mathrm{d}s. \tag{12}$$

Therefore,

$$x'(t) = -\mu\lambda t^a \int_t^T \frac{f(s, x(s)+c, x'(s))}{s^a}\,\mathrm{d}s, \quad t \in [0,T],$$

and thus, by (A_2),

$$|x'(t)| \le \lambda t^a \int_t^T \frac{g(s)\omega(|x'(s)|)}{s^a}\,\mathrm{d}s \le \lambda \int_t^T g(s)\omega(|x'(s)|)\,\mathrm{d}s, \quad t \in [0,T].$$

Hence,

$$|x'(T-t)| \le \lambda \int_0^t g(T-s)\omega(|x'(T-s)|)\,\mathrm{d}s, \quad t \in [0,T],$$

and by the Bihari inequality, cf. [3],

$$|x'(T-t)| \le G^{-1}\Big(\lambda \int_0^t g(T-s)\,\mathrm{d}s\Big) \le G^{-1}\Big(\lambda \int_0^T g(s)\,\mathrm{d}s\Big) < S, \quad t \in [0,T].$$

Consequently, $\|x'\|_\infty < S$ and from $x(0)=0$, $|x(t)| = |\int_0^t x'(s)\,\mathrm{d}s|$ we have

$$\|x\|_\infty < ST, \quad \|x'\|_\infty < S. \tag{13}$$

In the final part of the proof we distinguish between three cases, $\mu = 0$, $\mu \in (0,1)$, and $\mu = 1$.
Case 1. Let $\mu = 0$. Then, by (12), $c = 0$.
Case 2. Let $\mu \in (0,1)$. If $c = ST + \max\{|A|,|B|\}$, then $c \ge ST + B$. Hence, $x(t) + c > -ST + c \ge B$ for $t \in [0,T]$, which gives $f(t, x(t)+c, x'(t)) < 0$ for a.e. $t \in [0,T]$ by (7). Consequently,

$$\int_0^T \Big(s^a \int_s^T \frac{f(\xi, x(\xi)+c, x'(\xi))}{\xi^a}\,\mathrm{d}\xi\Big)\mathrm{d}s < 0,$$

contradicting (12) since $c > 0$. If $c = -ST - \max\{|A|, |B|\}$, then $c \leq -ST + A$, and so $x(t) + c < ST + c \leq A$ for $t \in [0, T]$. Then, by (6), $f(t, x(t) + c, x'(t)) > 0$ for a.e. $t \in [0, T]$. Hence,

$$\int_0^T \Big(s^a \int_s^T \frac{f(\xi, x(\xi) + c, x'(\xi))}{\xi^a}\,\mathrm{d}\xi\Big)\mathrm{d}s > 0,$$

again contradicting (12) since $c < 0$.
Case 3. Let $\mu = 1$. Then, by (12),

$$\int_0^T \Big(s^a \int_s^T \frac{f(\xi, x(\xi) + c, x'(\xi))}{\xi^a}\,\mathrm{d}\xi\Big)\mathrm{d}s = 0. \tag{14}$$

If $c = ST + \max\{|A|, |B|\}$, then $x(t) + c \geq B$ for $t \in [0, T]$, and so, by (7), $f(t, x(t) + c, x'(t)) < 0$ for a.e. $t \in [0, T]$, which contradicts (14). Analogously, if $c = -ST - \max\{|A|, |B|\}$, then $f(t, x(t) + c, x'(t)) > 0$ for a.e. $t \in [0, T]$, which once more contradicts (14).

We summarize: it holds

$$|c| < ST + \max\{|A|, |B|\} \tag{15}$$

and hence, it follows from (13) and (15) that $(x, c) \notin \partial\Omega$. This means that (iii) holds, which completes the proof. □

The following examples have been specified for the numerical simulations in Section 4.

Example 1 Here, we consider the boundary value problem (3) of the form

$$u''(t) = \frac{a}{t}u'(t) + \lambda\left(\frac{t}{3} - \frac{(1 + u'(t)^2)}{4\sqrt{t}}\arctan u(t)\right), \;\; u(0) = u(1),\; u'(0) = u'(1). \tag{16}$$

The function

$$f(t, x, y) := \frac{t}{3} - \frac{1}{4\sqrt{t}}(1 + y^2)\arctan x$$

satisfies assumptions (A_1) and (A_2) on $[0, 1] \times \mathbb{R}^2$ with $g(t) = \frac{t}{3} + \frac{\pi}{8\sqrt{t}}$ and $\omega(s) = 1 + s^2$. It follows from the relations

$$\begin{aligned} f(t, x, y) \;&>\; -\frac{1}{4\sqrt{t}}(1 + y^2)\arctan x \geq 0, \quad t \in (0, 1],\;\; x \leq 0,\;\; y \in \mathbb{R}, \\ f(t, x, y) \;&<\; \frac{1}{3} - \frac{1}{4}\arctan x \leq 0, \quad t \in (0, 1],\;\; x \geq \tan\left(\frac{4}{3}\right),\;\; y \in \mathbb{R}, \end{aligned}$$

that inequalities (6) and (7) hold for $A=0$ and $B=\tan\left(\frac{4}{3}\right)$. From

$$\int_0^\infty \frac{\mathrm{d}s}{\omega(s)} = \frac{\pi}{2}, \quad \int_0^1 g(t)\,\mathrm{d}t = \frac{1}{6}+\frac{\pi}{4},$$

we have

$$\lambda^* = \int_0^\infty \frac{\mathrm{d}s}{\omega(s)} \cdot \left(\int_0^1 g(t)\,\mathrm{d}t\right)^{-1} = \frac{6\pi}{2+3\pi}.$$

Consequently, by Theorem 1, problem (16) has a solution for any $\lambda \in (0, 6\pi/(2+3\pi))$, and $a>0$. For the numerical experiments we choose $\lambda = 1$.

We now show that for this example we cannot apply results based on the existence of a well-ordered pair of constant lower and upper functions because they do not exist. Let us assume that problem (16) has such pair of constant functions $\widetilde{A}, \widetilde{B} \in \mathbb{R}$. Then $\widetilde{A} < \widetilde{B}$ and

$$f(t, \widetilde{A}, 0) \le 0, \quad f(t, \widetilde{B}, 0) \ge 0, \tag{17}$$

for all $t \in (0,1]$. From (16) we conclude

$$f(t,\widetilde{A},0) = \frac{t}{3} - \frac{1}{4\sqrt{t}}\arctan\widetilde{A} > \frac{t}{3} - \frac{1}{4\sqrt{t}}\arctan\widetilde{B} = f(t,\widetilde{B},0)$$

for all $t \in (0,1]$ which contradicts (17).

Example 2 Let $\mu \in (0,1)$. Consider the boundary value problem (3), where the function

$$f(t,x,y) = h(t,x,y) - \frac{1}{\sqrt[\beta]{t}}\frac{x}{\sqrt{1+x^2}}(1+|y|)^\alpha \tag{18}$$

depends on parameters $\beta \in (1,\infty)$ and $\alpha \in (0,\infty)$. Here, $h \in C([0,1]\times\mathbb{R}^2)$ and $|h(t,x,y)| \le \mu$ on $[0,1]\times\mathbb{R}^2$. Then f satisfies assumptions (A_1) and (A_2) on $[0,1]\times\mathbb{R}^2$ with $g(t) = \mu + \frac{1}{\sqrt[\beta]{t}}$, $\omega(s) = (1+s)^\alpha$. For $A \le -\mu/\sqrt{1-\mu^2}$ and $B \ge \mu/\sqrt{1-\mu^2}$,

$$\begin{aligned} f(t,x,y) &> -\mu + \frac{|A|}{\sqrt{1+A^2}} \ge 0, \quad t\in(0,1],\ x \le A,\ y\in\mathbb{R}, \\ f(t,x,y) &< \mu - \frac{B}{\sqrt{1+B^2}} \le 0, \quad t\in(0,1],\ x\ge B,\ y \in \mathbb{R}. \end{aligned}$$

Therefore, inequalities (6) and (7) are satisfied. Since

$$\int_0^\infty \frac{\mathrm{d}s}{\omega(s)} = \begin{cases} \infty, & \alpha \le 1, \\ \dfrac{1}{\alpha-1}, & \alpha > 1, \end{cases} \quad \int_0^1 g(t)\,\mathrm{d}t = \mu + \frac{\beta}{\beta-1},$$

we have

$$\lambda^* = \int_0^\infty \frac{\mathrm{d}s}{\omega(s)} \cdot \left(\int_0^1 g(t)\,\mathrm{d}t\right)^{-1} = \begin{cases} \infty, & \alpha \le 1, \\ \dfrac{\beta - 1}{(\alpha-1)(\beta+\mu(\beta-1))}, & \alpha > 1. \end{cases}$$

Consequently, by Theorem 1, problem (3) has a solution for any $a > 0$, f given by (18) and $\lambda \in (0, \lambda^*)$.

Finally, we show that also for this example a well-ordered pair of constant lower and upper functions does not exist. Let us assume that problem (3) with $T = 1$ and f given by (18) has such a pair of constant functions $\widetilde{A}, \widetilde{B} \in \mathbb{R}$. Then $\widetilde{A} < \widetilde{B}$ and (17) holds. By (18), we have for all $t \in (0, 1]$

$$\begin{aligned} f(t, \widetilde{B}, 0) - f(t, \widetilde{A}, 0) &= h(t, \widetilde{B}, 0) - h(t, \widetilde{A}, 0) + \frac{1}{\sqrt[\beta]{t}} \left(\frac{\widetilde{A}}{\sqrt{1+\widetilde{A}^2}} - \frac{\widetilde{B}}{\sqrt{1+\widetilde{B}^2}} \right) \\ &\le 2\mu + \frac{1}{\sqrt[\beta]{t}}\, \widetilde{C}, \end{aligned}$$

where

$$\widetilde{C} = \frac{\widetilde{A}}{\sqrt{1+\widetilde{A}^2}} - \frac{\widetilde{B}}{\sqrt{1+\widetilde{B}^2}} < 0.$$

Therefore, there exists $\delta > 0$ such that for $t \in (0, \delta)$,

$$f(t, \widetilde{B}, 0) < f(t, \widetilde{A}, 0)$$

which again contradicts (17).

For $\alpha = \beta = 3$, $\mu = \frac{5}{6}$, we have $\lambda^* = \frac{3}{14}$. For the numerical simulation we choose $\lambda = 1/5$ and

$$f(t, x, y) = \frac{5}{6}\sin(5t) - \frac{1}{\sqrt[3]{t}} \frac{x}{\sqrt{1+x^2}} \left(1 + |y|\right)^3, \tag{19}$$

where $(t, x, y) \in [0, 1] \times \mathbb{R}^2$. Consequently, the boundary value problem (3) with the above data reads,

$$u''(t) = \frac{a}{t}u'(t) + \frac{1}{6}\sin(5t) - \frac{1}{5\sqrt[3]{t}} \frac{u(t)\left(1 + |u'(t)|\right)^3}{\sqrt{1+u^2(t)}}, \quad u(0) = u(1),\ u'(0) = u'(1). \tag{20}$$

4 Numerical Simulation

To illustrate the analytical results discussed in the previous section, we solved numerically Examples 1 and 2 using a MATLAB™ software package `bvpsuite` designed to solve boundary value problems in ordinary differential equations and differential algebraic equations. The solver routine is based on a class of collocation methods whose orders may vary from 2 to 8. The code also provides the asymptotically correct estimate for the global error of the numerical approximation. To enhance the efficiency of the method, a mesh adaptation strategy is implemented, which attempts to choose grids related to the solution behavior, in such a way that the tolerance is satisfied with the least possible effort. Error estimate procedure and the mesh adaptation work dependably provided that the solution of the problem and its global error are appropriately smooth[1]. The code and the manual can be downloaded from `http://www.math.tuwien.ac.at/∼ewa`. For further information see [14]. This software was already used for the approximation of numerous singular boundary value problems important for applications, see e.g. [4], [7], [13], [17].

4.1 Example 1

We consider the boundary value problem

$$u''(t) = \frac{a}{t}u'(t) + \frac{t}{3} - \frac{(1+u'(t))^2}{4\sqrt{t}} \arctan u(t), \quad u(0) = u(1),\ u'(0) = u'(1) \tag{21}$$

with $a = 0.4, 0.5, 0.7, 0.9, 1, 2,$ and 5.

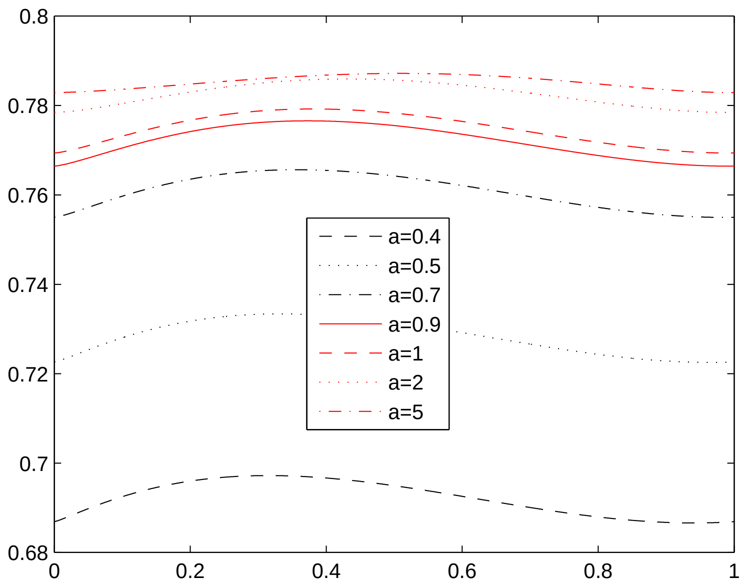

Figure 1: Example 1: Numerical solutions for different values of a

[1]The required smoothness of higher derivatives is related to the order of the used collocation method.

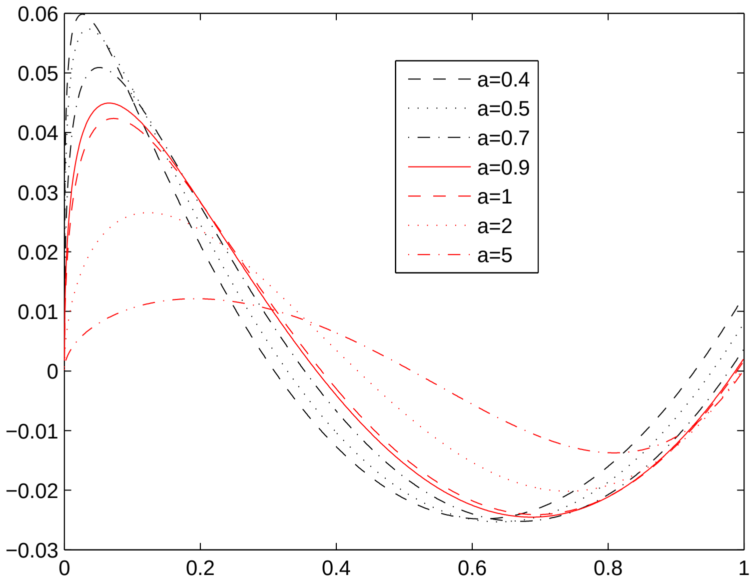

Figure 2: Example 1: First derivative for different values of a

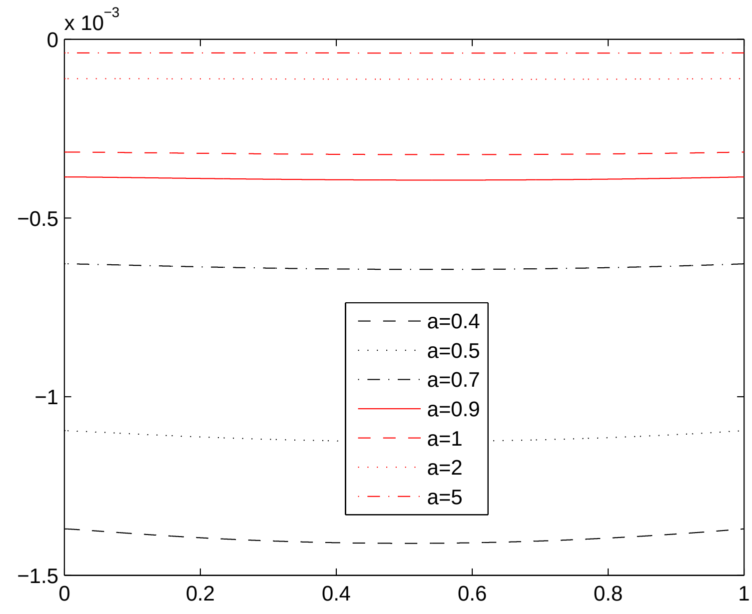

Figure 3: Example 1: Global errors for different values of a

Numerical results shown in Figures 1 to 3 have been obtained using 4 uniform collocation points (convergence order 4 uniformly in t) and the uniform mesh with 100 subintervals on $[0, 1]$. Although the curves in Figure 1 look smooth and fully harmless, the problem is very difficult to solve and the accuracy of the approximation varies from 10^{-3} to 10^{-5} depending on a. The larger a the more accurate he results. Roughly speaking, for smooth problems with solutions whose higher derivatives are moderate in size, one would expect the global error to be $O(h^4) = O(10^{-8})$ with a moderate error constant. This is not the case here. When we look at the first derivatives in Figure 2, we immediately see that the second (and higher) derivatives are large, especially for small values of a. The loss of accuracy can most probably be attributed to the large higher derivatives. Therefore, although the theory predicts that any analytical solution u satisfies $u'(0) = u'(1) = 0$, the last relation does not seem to be very accurately reflected by the numerical solution, at least not for small values of a.

Let us first, in Figures 4 and 5 look closer at the values of the first derivatives in the regions close to $t = 0$ and $t = 1$, respectively. We can see that indeed the larger the value of a the smaller the values of the derivative at the interval ends.

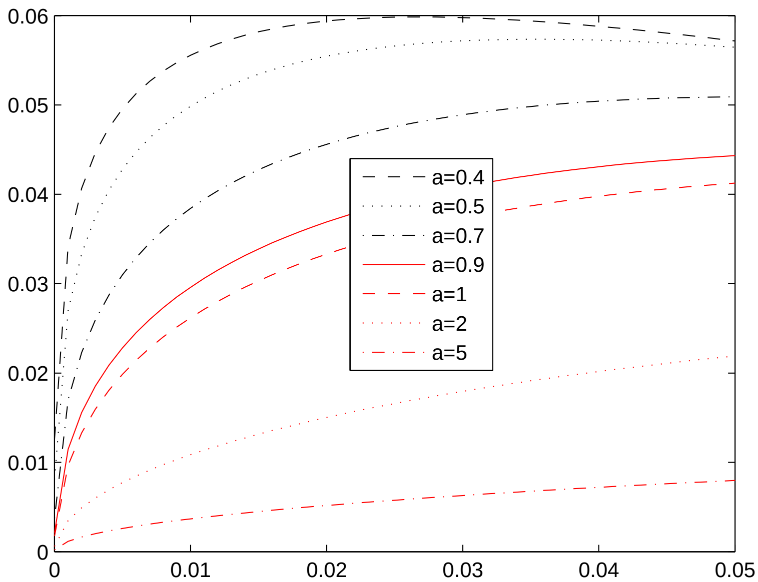

Figure 4: Example 1: First derivative in the vicinity of $t = 0$

a	$u(0) = u(1)$	$u'(0) = u'(1)$	$u''(0)$	maximal error in u
0.4	0.6663581	$1.022613 \cdot 10^{-2}$	$6.047415 \cdot 10^{1}$	$1.42 \cdot 10^{-3}$
0.5	0.7062183	$6.203709 \cdot 10^{-3}$	$4.973534 \cdot 10^{1}$	$1.13 \cdot 10^{-3}$
0.7	0.7455672	$2.709670 \cdot 10^{-3}$	$3.147486 \cdot 10^{1}$	$6.44 \cdot 10^{-4}$
0.9	0.7606680	$1.525785 \cdot 10^{-3}$	$2.065479 \cdot 10^{1}$	$3.94 \cdot 10^{-4}$
1	0.7646459	$1.232963 \cdot 10^{-3}$	$1.722183 \cdot 10^{1}$	$3.23 \cdot 10^{-4}$
2	0.7767797	$4.198916 \cdot 10^{-4}$	$5.939505 \cdot 10^{0}$	$1.12 \cdot 10^{-4}$
5	0.7823136	$1.439668 \cdot 10^{-4}$	$1.947079 \cdot 10^{0}$	$3.86 \cdot 10^{-5}$

Table 1: Example 1: Numerical approximations for the values $u(0)$, $u'(0)$, $u''(0)$, $u'(1)$, and the maximal global error in u

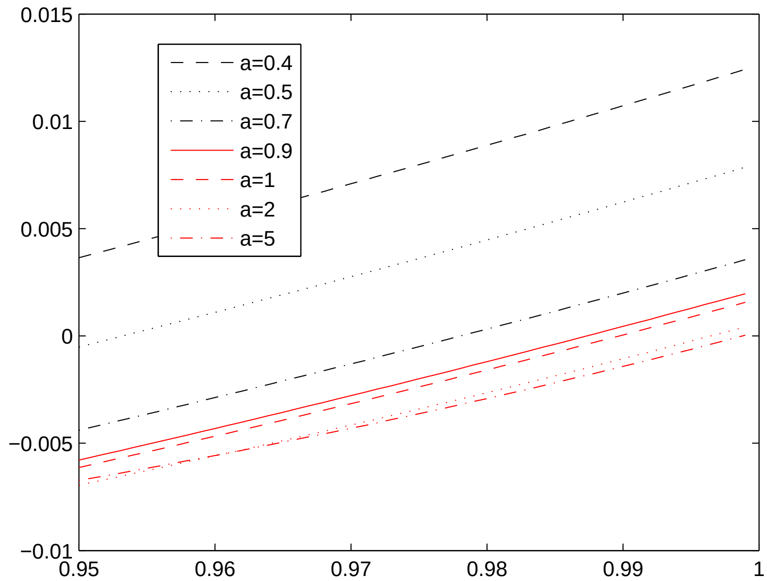

Figure 5: Example 1: First derivative in the vicinity of $t = 1$

Finally, in Table 1, we specify the values $u(0)$, $u'(0)$, $u''(0)$, $u'(1)$, and the maximal global error in u. Indeed the values of the second derivative at $t = 0$ are large for small values of a. This fact can be supported by looking at the differential equation in (21). For a large, the first term in the right-hand side becomes dominant and the solution smoothness can be related to the linear equation of the form

$$y'(t) = \frac{a}{t} y(t),$$

were $y(t) = ct^a$. This immediately explains why the higher derivatives of the solution are smoother for large values of a and consequently, why such solutions are easier to approximate.

The above explanations are by no means strict analysis. They only make clear that the relatively large numerical values for $u'(0) = u'(1)$ are not contradicting the theory which predicts that $u'(0) = u'(1) = 0$ holds. Within given accuracy, the numerical values can be regarded as a reasonable approximation for the correct values of $u'(0) = u'(1)$, especially since for small a the analytical solution is very unsmooth.

4.2 Example 2

We now consider the second example, where we have to solve the differential equation

$$u''(t) = \frac{a}{t}u'(t) + \frac{1}{6}\sin(5t) - \frac{1}{5\sqrt[3]{t}} \frac{u(t)(1 + |u'(t)|)^3}{\sqrt{1 + u(t)^2}}$$

subject to boundary conditions $u(0) = u(1)$, $u'(0) = u'(1)$ and $a = 0.4$, 0.5, $0.7, 0.9, 1, 2$ and 5. Figures 6 to 10 correspond to Figures 1 to 5, respectively, and Table 2 corresponds to Table 1.
Here, as we can see, the problem data is considerably smoother and the approximation accuracy is very good, especially for large a. The number of collocation points was again 4 with 100 subintervals in $[0, 1]$.

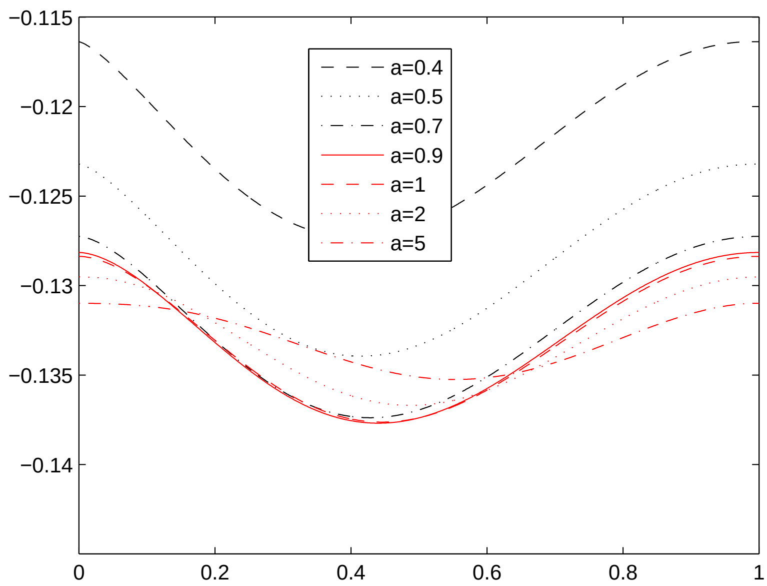

Figure 6: Example 2: Numerical solutions for different values of a

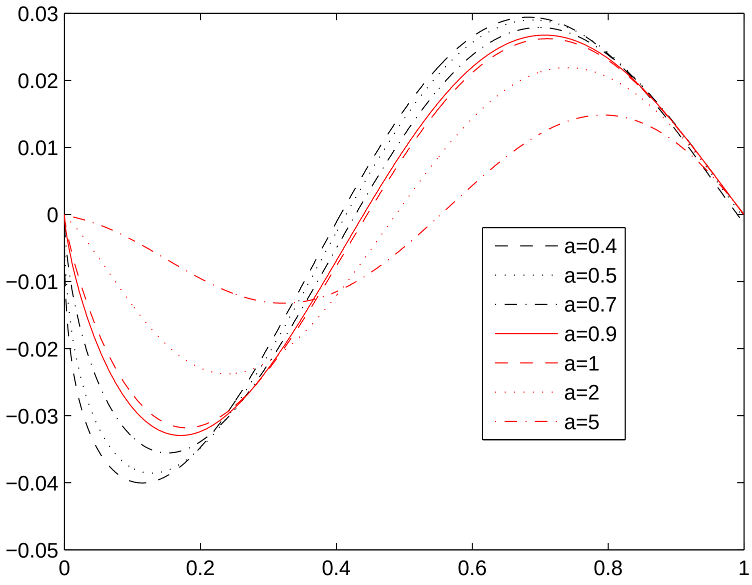

Figure 7: Example 2: First derivative for different values of a

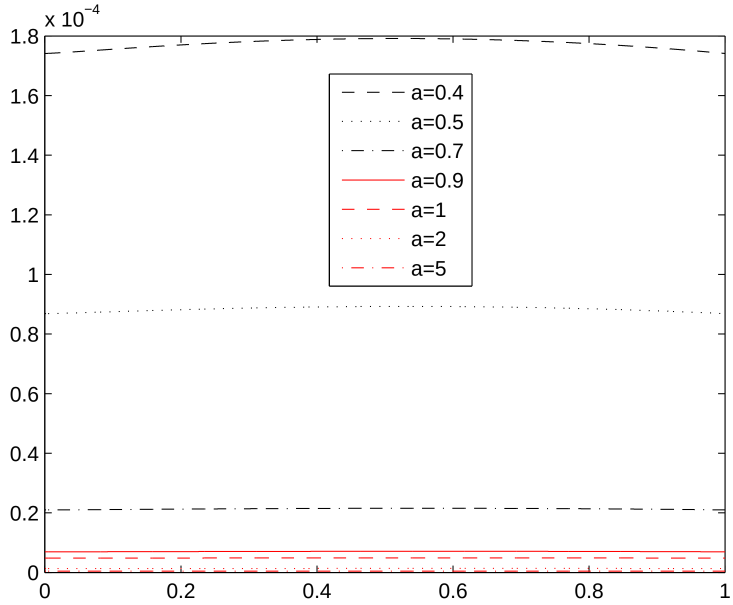

Figure 8: Example 2: Global errors for different values of a

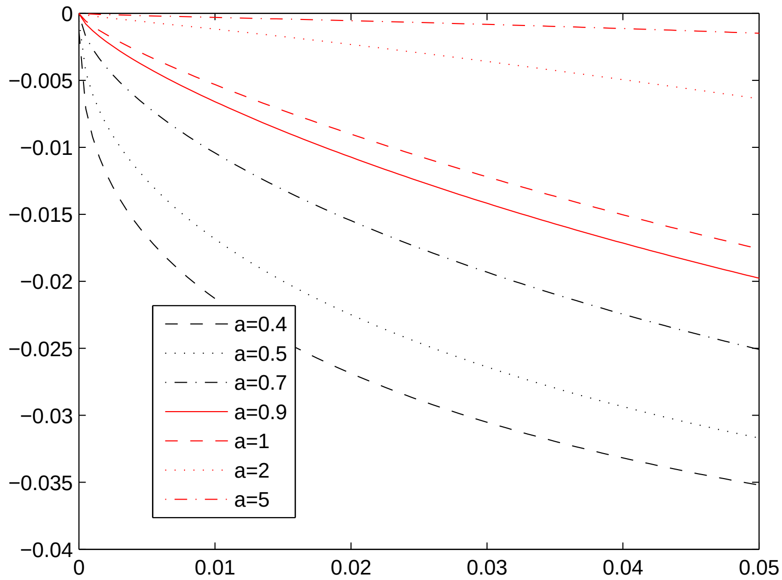

Figure 9: Example 1: First derivative in the vicinity of $t = 0$

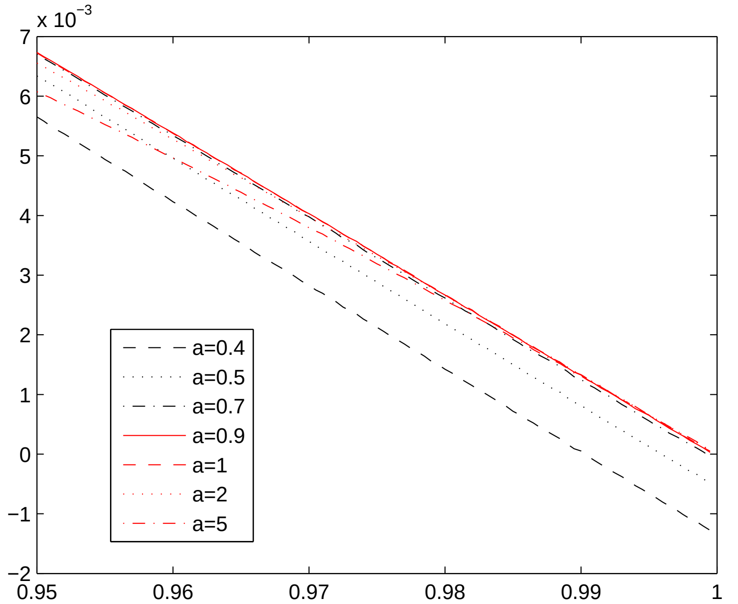

Figure 10: Example 1: First derivative in the vicinity of $t = 1$

a	$u(0) = u(1)$	$u'(0) = u'(1)$	$u''(0) = u''(1)$	maximal error in u
0.4	-0.1163804	$-1.344652 \cdot 10^{-3}$	$-3.445983 \cdot 10^{1}$	$1.80 \cdot 10^{-4}$
0.5	-0.1232172	$-5.484997 \cdot 10^{-4}$	$-2.016584 \cdot 10^{1}$	$8.95 \cdot 10^{-5}$
0.7	-0.1272538	$-1.059086 \cdot 10^{-4}$	$-6.832530 \cdot 10^{0}$	$2.16 \cdot 10^{-5}$
0.9	-0.1281526	$-3.320831 \cdot 10^{-5}$	$-2.734550 \cdot 10^{0}$	$7.10 \cdot 10^{-6}$
1	-0.1283698	$-2.308953 \cdot 10^{-5}$	$-1.909115 \cdot 10^{0}$	$4.92 \cdot 10^{-6}$
2	-0.1295174	$-6.364872 \cdot 10^{-6}$	$-4.225124 \cdot 10^{-1}$	$1.35 \cdot 10^{-6}$
5	-0.1309907	$-2.168133 \cdot 10^{-6}$	$-1.271241 \cdot 10^{-1}$	$4.60 \cdot 10^{-6}$

Table 2: Example 2: Numerical approximations for the values $u(0)$, $u'(0)$, $u''(0)$, $u'(1)$, and the maximal global error in u

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