Journal of Difference Equations and Applications,
Vol. 13, No. 5, May 2007, 423–429

Solvability of discrete Dirichlet problem via lower and upper functions method

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(Received 2 May 2006; revised 27 November 2006; in final form 28 November 2006)

We use the lower and upper functions method to prove the existence of a solution of the Dirichlet problem

$$\Delta(p(t)\Delta u(t-1)) + f(t, u(t)) = g(t), \quad t \in [1, T], \quad u(0) = 0, \quad u(T+1) = 0,$$

where $T \in \mathbb{N}$, $[1, T] = \{1, 2, \ldots, T\}$, $p : [1, T+1] \to \mathbb{R}$ is positive and $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is continuous. Provided f fulfils certain sign conditions we get the solvability of the problem for each $g : [1, T] \to \mathbb{R}$.

Keywords: Dirichlet discrete BVP; Lower and upper functions; Brouwer fixed point theorem; Existence

Mathematics Subject Classification 2000: 39A12; 39A10; 39A70

1. Introduction

For fixed $T \in \mathbb{N}$ we define the discrete interval $[1, T] = \{1, 2, \ldots, T\}$. We will study the Dirichlet problem

$$\Delta(p(t)\Delta u(t-1)) + f(t, u(t)) = g(t), \quad t \in [1, T], \tag{1.1}$$

$$u(0) = 0, \quad u(T+1) = 0. \tag{1.2}$$

where

$$\left.\begin{array}{c} p : [1, T+1] \to \mathbb{R} \text{ is positive}, \quad g : [1, T] \to \mathbb{R} \\ f : [1, T] \times \mathbb{R} \to \mathbb{R} \text{ is continuous.} \end{array}\right\} \tag{1.3}$$

Here Δ denotes the forward difference operator with the step size 1, i.e. $\Delta u(t-1) = u(t) - u(t-1)$. Recall that $f(t, x)$ is continuous on $[1, T] \times \mathbb{R}$ if for each $t \in [1, T]$, $f(t, x)$ is a continuous function of x.

DEFINITION 1.1. *By a solution u of problem* (1.1), (1.2) *we mean $u : [0, T+1] \to \mathbb{R}$, u satisfies the difference equation* (1.1) *on $[1, T]$ and the boundary conditions* (1.2).

Discrete boundary value problems arise in the study of solid state physics, chemical reaction, population dynamics and in many other areas, see [1,13,27]. Discrete second order nonlinear boundary value problems have been investigated in several monographs (e.g. [1,5,6,17]) and

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Journal of Difference Equations and Applications
ISSN 1023-6198 print/ISSN 1563-5120 online © 2007 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/10236190601143302

papers (e.g. [8–10,14,16,19–21,23,24,26]). Particularly we can refer to papers [2–4,7,11,12,15,18,22,25], which deal with various difference equations subjected to Dirichlet conditions. Yongjin Li in Ref. [18] used variational approach and proved the existence of a solution of equations (1.1) and (1.2) under the assumptions

$$\exists r > 0 \text{ such that } xf(t,x) \le 0 \text{ for } t \in [1,T] \text{ and } |x| \ge r, \tag{1.4}$$

$$\sum_{t=1}^{T+1} |g(t)|^2 < \frac{m}{2}, \text{ where } m = \min\{p(t) : t \in [1, T+1]\}. \tag{1.5}$$

In this paper, we use a completely different approach based on the lower and upper functions method. By means of this we prove that equation (1.4) yields the solvability of problem (1.1) and (1.2) for each $g[1,T] \to \mathbb{R}$, i.e. that equation (1.5) can be omitted.

2. Green's function

Consider the linear homogeneous equation

$$\Delta(p(t)\Delta u(t-1)) = 0, \quad t \in [1,T], \tag{2.1}$$

where p satisfies equation (1.3). Define

$$P(t) = \sum_{i=1}^{t} \frac{1}{p(i)}, \quad t \in [1, T+1] \text{ and } P(0) = 0, \tag{2.2}$$

and denote

$$M_p = \max\left\{\frac{1}{p(t)} : t \in [1, T+1]\right\} > 0. \tag{2.3}$$

Remark 2.1. The general solution of equation (2.1) has the form $u(t) = c_1 + c_2 P(t)$, $t \in [0, T+1]$, where $c_1, c_2 \in \mathbb{R}$. Therefore equations (1.2) and (1.3) yield $c_1 = c_2 = 0$ and hence problem (2.1) and (2.2) has only the trivial solution.

LEMMA 2.2. *Let p satisfy equation* (1.3). *Then the Green's function of problem* (2.1) *and* (1.2) *has the form*

$$G(t,s) = -\begin{cases} \dfrac{P(s)}{P(T+1)}(P(T+1) - P(t)) \ \textit{if} \ \ 0 \le s \le t \le T+1 \\ \dfrac{P(t)}{P(T+1)}(P(T+1) - P(s)) \ \textit{if} \ \ 0 \le t \le s \le T+1. \end{cases} \tag{2.4}$$

Proof. The proof can be done similarly as in Ref. [17], Example 6.12. □

Due to equations (2.2)–(2.4) we see that

$$G(0,s) = 0, \quad G(T+1, s) = 0 \text{ for } s \in [0, T+1], \tag{2.5}$$

$$-TM_p < G(t,s) < 0 \text{ for } t, s \in [1,T]. \tag{2.6}$$

Further, we have

$$\Delta G(t-1,s) = \frac{1}{p(t)P(T+1)} \begin{cases} P(s) & \text{for } s+1 \leq t \\ P(s) - P(T+1) & \text{for } t \leq s \end{cases}$$

and

$$\Delta(p(t)\Delta G(t-1,s)) = \begin{cases} 0 & \text{for } t \leq s+1 \text{ and } t \geq s+1 \\ 1 & \text{for } t = s. \end{cases}$$

Therefore, according to Remark 2.1 and Lemma 2.2, we get the following lemma for the non-homogeneous linear equation

$$\Delta(p(t)\Delta u(t-1)) = g(t), \quad t \in [1,T], \tag{2.7}$$

where p and q satisfy equation (1.3).

LEMMA 2.3. *Problem* (2.7) *and* (1.2) *has the unique solution of the form*

$$u_0(t) = \sum_{s=1}^{T} G(t,s)g(s), \quad t \in [0,T+1]. \tag{2.8}$$

3. Lower and upper functions

Lower and upper functions are important tools for the investigation of solvability of boundary value problems. Here, we bring their definition for problem (1.1) and (1.2).

DEFINITION 3.1. $\alpha : [0,T+1] \to \mathbb{R}$ *is called* a lower function *of problem* (1.1) *and* (1.2) *if*

$$\Delta(p(t)\Delta\alpha(t-1)) + f(t,\alpha(t)) \geq g(t) \ \textit{for} \ t \in [1,T], \tag{3.1}$$

$$\alpha(0) \leq 0, \quad \alpha(T+1) \leq 0. \tag{3.2}$$

$\beta : [0,T+1] \to \mathbb{R}$ *is called* an upper function *of problem* (1.1) *and* (1.2) *if*

$$\Delta(p(t)\Delta\beta(t-1)) + f(t,\beta(t)) \leq g(t) \ \textit{for} \ t \in [1,T], \tag{3.3}$$

$$\beta(0) \geq 0, \quad \beta(T+1) \geq 0. \tag{3.4}$$

THEOREM 3.2 (LOWER AND UPPER FUNCTIONS METHOD). *Assume equation* (1.3). *Let* α *and* β *be a lower and an upper function of equations* (1.1), (1.2) *and* $\alpha \leq \beta$ *on* [1,*T*]. *Then problem equations* (1.1) *and* (1.2) *has a solution u satisfying*

$$\alpha(t) \leq u(t) \leq \beta(t) \ \textit{for} \ t \in [0,T+1]. \tag{3.5}$$

Theorem 3.2 is a slight modification of Theorem 9.7 in Ref. [17], where $p(t) \equiv 1$. However for the reader's convenience we will prove Theorem 3.2 here.

Proof. *Step* 1. For $t \in [1,T]$, $x \in \mathbb{R}$, define function

$$\tilde{f}(t,x) = \begin{cases} f(t,\beta(t)) - \frac{x-\beta(t)}{x-\beta(t)+1} & \text{if } x > \beta(t) \\ f(t,x) & \text{if } \alpha(t) \le x \le \beta(t) \\ f(t,\alpha(t) + \frac{\alpha(t)-x}{\alpha(t)-x+1} & \text{if } x < \alpha(t). \end{cases}$$

Since $\tilde{f}$ is continuous on $[1,T] \times \mathbb{R}$, there exists $M > 0$ such that

$$|\tilde{f}(t,x)| \le M \text{ for } t \in [1,T], \quad x \in \mathbb{R}. \tag{3.6}$$

We will study the auxiliary difference equation

$$\Delta(p(t)\Delta u(t-1)) + \tilde{f}(t,u(t)) = g(t), \quad t \in [1,T], \tag{3.7}$$

and we will prove that problem (3.7) and (1.2) has a solution (see Steps 2 and 3).

Step 2. We define the space

$$E = \{u : [0, T+1] \to \mathbb{R}, u(0) = 0, u(T+1) = 0\}$$

with the norm $\|u\| = \max\{|u(t)| : t \in [1,T]\}$. Then E is a Banach space with $\dim E = T$. Further, we define an operator $\mathcal{F} : E \to E$ by:

$$(\mathcal{F}u)(t) = \sum_{s=1}^{T} G(t,s)(g(s) - \tilde{f}(s,u(s))), \quad t \in [0, T+1]. \tag{3.8}$$

Due to equation (1.3), $\mathcal{F}$ is a continuous operator. Denote $B(r) = \{u \in E : \|u\| < r\}$ and

$$M_g = \max\{|g(t)| : t \in [1,T]\}. \tag{3.9}$$

Let us choose $r^* \ge T^2 M_p(M_g + M)$, where M_p and M are given by equations (2.6) and (3.6), respectively. Then by equations (2.5) and (3.8) we get $\mathcal{F}(\overline{B(r^*)}) \subset \overline{B(r^*)}$. Therefore, the Brouwer fixed point theorem yields the existence of at least one point $u \in \overline{B(r^*)}$ such that $u = \mathcal{F}u$. According to Lemma 2.3 we see that if u is a fixed point of $\mathcal{F}$, then u satisfies equations (3.7) and (1.2).

Step 3. We prove that the solution u of equations (3.7) and (1.2) satisfies equation (1.1). Put $v(t) = \alpha(t) - u(t)$ for $t \in [0, T+1]$ and assume that $\max\{v(t) : t \in [0, T+1]\} = v(\ell) > 0$. Conditions (1.2) and (3.2) imply $\ell \in [1,T]$. Thus, we have $v(\ell+1) \le v(\ell)$, $v(\ell-1) \le v(\ell)$, and consequently $\Delta\alpha(\ell) \le \Delta u(\ell)$, $\Delta\alpha(\ell-1) \ge \Delta u(\ell-1)$. This leads to $p(\ell+1)\Delta\alpha(\ell) \le p(\ell+1)\Delta u(\ell)$, $p(\ell)\Delta\alpha(\ell-1) \ge p(\ell)\Delta u(\ell-1)$ and

$$\Delta(p(\ell)\Delta u(\ell-1)) \ge \Delta(p(\ell)\Delta\alpha(\ell-1)). \tag{3.10}$$

On the other hand, we get by equations (3.1) and (3.7)

$$\begin{aligned} \Delta(p(\ell)\Delta\alpha(\ell-1)) - \Delta(p(\ell)\Delta u(\ell-1)) &= \Delta(p(\ell)\Delta\alpha(\ell-1)) - (g(\ell) - \tilde{f}(\ell,u(\ell))) \\ &= \Delta(p(\ell)\Delta\alpha(\ell-1)) + f(\ell,\alpha(\ell)) \\ &\quad + \frac{v(\ell)}{v(\ell)+1} - g(\ell) \\ &\ge \frac{v(\ell)}{v(\ell)+1} > 0, \end{aligned}$$

which contradicts equation (3.10). So, we have proved $\alpha(t) \leq u(t)$ for $t \in [0, T+1]$. The inequality $u(t) \leq \beta(t)$ for $t \in [0, T+1]$ can be proved similarly. Therefore, u satisfies equation (3.5) and hence u is a solution of problems (1.1) and (1.2). □

4. Main results

Our main result is contained in the next theorem, which provides sufficient conditions for the solvability of problems (1.1) and (1.2). The proof is based on the lower and upper functions method from Theorem 3.2.

THEOREM 4.1. *Assume that equations* (1.3) *and* (1.4) *hold. Then problem* (1.1) *and* (1.2) *has at least one solution.*

Proof. By Lemma 2.3, problem (2.7) and (1.2) has the unique solution u_0 given by equation (2.8). Using equations (3.9), (2.5) and (2.6) we have

$$|u_0(t)| \leq T^2 M_p M_g \quad \text{for} \quad t \in [0, T+1].$$

Choose $A, B \in \mathbb{R}$ such that

$$A \leq -T^2 M_p M_g - r, \quad B \geq T^2 M_p M_g + r$$

and define functions

$$\alpha(t) = u_0(t) + A, \quad \beta(t) = u_0(t) + B, \quad t \in [0, T+1].$$

Then $\alpha(t) \leq -r$, $\beta(t) \geq r$ for $t \in [0, T+1]$. This implies that α and β satisfy equations (3.2) and (3.4), respectively. Moreover, by equation (1.4),

$$\Delta(p(t)\Delta\alpha(t-1)) + f(t, \alpha(t)) = \Delta(p(t)\Delta u_0(t-1)) + f(t, \alpha(t)) \geq \Delta(p(t)\Delta u_0(t-1))$$
$$= g(t) \quad \text{for} \quad t \in [1, T].$$

Similarly

$$\Delta(p(t)\Delta\beta(t-1)) + f(t, \beta(t)) \leq g(t) \quad \text{for} \quad t \in [1, T].$$

Therefore, α and β are a lower and an upper function of equations (1.1) and (1.2), respectively, and $\alpha \leq \beta$ on $[1, T]$. Hence, Theorem 3.2 guarantees the existence of at least one solution u of equations (1.1) and (1.2) satisfying equation (3.5). □

Example. Assume $k \in \mathbb{N}$, $c \in \mathbb{R}$, $a(t) : [1, T] \to \mathbb{R}$, $b(t) : [1, T] \to (-\infty, 0)$ and consider the equation

$$\Delta(t^3 \Delta u(t-1)) + a(t) + b(t)u^{2k-1}(t) = ct^2 \mathrm{e}^t, \quad t \in [1, T]. \tag{4.1}$$

By Theorem 4.1, problem (4.1) and (1.2) has a solution.

Corollary 4.2. *Assume that equation* (1.3) *holds. Let*

$$g(t) < 0, \quad f(t,0) \geq 0 \;\; for \;\; t \in [1,T], \tag{4.2}$$

$$\exists r > 0 \;\; such\, that \;\; f(t,x) \leq 0 \;\; for \;\; t \in [1,T] \;\; and \;\; x \geq r. \tag{4.3}$$

Then problem (1.1) *and* (1.2) *has a solution u such that*

$$u(t) > 0 \;\; for \;\; t \in [1,T]. \tag{4.4}$$

Proof. Condition (4.2) implies that if we put $\alpha(t) \equiv 0$, we have

$$\Delta(p(t)\Delta\alpha(t-1)) + f(t,\alpha(t)) = f(t,0) \geq 0 > g(t), \quad t \in [1,T].$$

If $\beta(t)$ is the same as in the proof of Theorem 4.1, we get a solution u of problem (1.1), (1.2) such that

$$0 \leq u(t) \leq \beta(t) \;\; \text{for} \;\; t \in [0, T+1].$$

Since $f(t,0) > g(t)$ on $[1,T]$, we obtain equation (4.4). □

Corollary 4.3. *Assume that equation* (1.3) *holds. Let*

$$g(t) > 0, \quad f(t,0) \leq 0 \;\; for \;\; t \in [1,T], \tag{4.5}$$

$$\exists r > 0 \;\; such\, that \;\; f(t,x) \geq 0 \;\; for \;\; t \in [1,T] \;\; and \;\; x \leq -r. \tag{4.6}$$

Then problem (1.1) *and* (1.2) *has a solution u such that:*

$$u(t) < 0 \;\; for \;\; t \in [1,T]. \tag{4.7}$$

Proof. We argue similarly as in the proof of Corollary 4.2 putting $\alpha(t)$ as in the proof of Theorem 4.1 and $\beta(t) \equiv 0$. □

Example. If $a(t) \geq 0$ on $[1,T]$ and $c < 0$, then by Corollary 4.2, problem (4.1) and (1.2) has a solution, which is positive on $[1,T]$.

If $a(t) \leq 0$ on $[1,T]$ and $c > 0$, then by Corollary 4.3, problem (4.1) and (1.2) has a solution, which is negative on $[1,T]$.

Acknowledgements

Supported by the Council of Czech Government MSM 6198959214.

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