Asymptotic properties of homoclinic solutions of some singular nonlinear differential equation*

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Dedicated to the memory of Professor Temuri Chanturia

Abstract

We investigate an asymptotic behaviour of homoclinic solutions of the singular differential equation $(p(t)u')' = p(t)f(u)$. Here f is Lipschitz continuous on $\mathbb{R}$ and has at least two zeros 0 and $L > 0$. The function p is continuous on $[0,\infty)$, has a positive continuous derivative on $(0,\infty)$ and $p(0) = 0$.

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1 Introduction

We investigate the differential equation

$$(p(t)u')' = p(t)f(u), \quad t \in (0,\infty), \tag{1}$$

and during the whole paper we assume that f satisfies

$$f \in Lip_{\rm loc}(\mathbb{R}), \quad \exists L \in (0,\infty): \ f(L) = 0, \tag{2}$$

$$\exists L_0 \in [-\infty, 0): \quad xf(x) < 0, \ x \in (L_0, 0) \cup (0, L), \tag{3}$$

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$$\exists \bar{B} \in (L_0, 0) : F(\bar{B}) = F(L), \quad \text{where } F(x) = -\int_0^x f(z)\,\mathrm{d}z, \quad x \in \mathbb{R}, \tag{4}$$

and p fulfils

$$p \in C[0,\infty) \cap C^1(0,\infty), \quad p(0) = 0, \tag{5}$$

$$p'(t) > 0,\ t \in (0,\infty), \quad \lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0. \tag{6}$$

Due to $p(0) = 0$, equation (1) has a singularity at $t = 0$.

Definition 1 A function $u \in C^1[0,\infty) \cap C^2(0,\infty)$ which satisfies equation (1) for all $t \in (0,\infty)$ is called a *solution* of equation (1).

Consider a solution u of equation (1). Since $u \in C^1[0,\infty)$, we have $u(0), u'(0) \in \mathbb{R}$ and the assumption $p(0) = 0$ yields $p(0)u'(0) = 0$. We can find $M > 0$ and $\delta > 0$ such that $|f(u(t))| \le M$ for $t \in (0,\delta)$. Integrating equation (1) and using the fact that p is increasing, we get

$$|u'(t)| = \left| \frac{1}{p(t)} \int_0^t p(s) f(u(s))\,\mathrm{d}s \right| \le \frac{M}{p(t)} \int_0^t p(s)\,\mathrm{d}s \le Mt, \quad t \in (0,\delta).$$

Consequently the condition $u'(0) = 0$ is necessary for each solution u of equation (1). Therefore the set of all solutions of equation (1) forms a one-parameter system of functions u satisfying $u(0) = A$, $A \in \mathbb{R}$.

Definition 2 Let u be a solution of equation (1) and let L be of (2) and (3). Denote $u_{\sup} = \sup\{u(t) : t \in [0,\infty)\}$. If $u_{\sup} = L$ ($u_{\sup} < L$ or $u_{\sup} > L$), then u is called a *homoclinic* solution (a *damped* solution or an *escape* solution).

The existence and properties of these three types of solutions have been investigated in [19]–[23]. In particular, we have proved that if $u(0) \in (0, L)$, than u is a damped solution ([22], Theorem 2.3). Clearly, for $u(0) = 0$ and $u(0) = L$, equation (1) has a unique solution $u \equiv 0$ and $u \equiv L$, respectively.

In this paper we focus our attention on homoclinic solutions. According to the above considerations such solutions have to satisfy the initial conditions

$$u(0) = B, \quad u'(0) = 0, \quad B < 0. \tag{7}$$

Note that if we extend the function $p(t)$ in equation (1) from the half–line onto $\mathbb{R}$ (as an even function), then a homoclinic solution of (1) has the same limit L as $t \to -\infty$ and $t \to \infty$. This is a motivation for Definition 2.

We have proved in [21], Lemma 3.5, that a solution u of equation (1) is homoclinic if and only if u is strictly increasing and $\lim_{t\to\infty} u(t) = L$. If such homoclinic solution exists, many important physical properties of corresponding models (see below) can be obtained. In particular, equation (1) is a generalization of the equation

$$u'' + \frac{k-1}{t} u' = f(u), \quad t \in (0,\infty), \tag{8}$$

and we can find in [16] that equation (8) with $k > 1$ and special forms of f arises in many areas. For example: In the study of phase transitions of Van der Waals fluids [3], [10], [24], in population genetics, where it serves as a model for the spatial distribution of the genetic composition of a population [8], [9], in the homogeneous nucleation theory [1], in relativistic cosmology for description of particles which can be treated as domains in the universe [18], in the nonlinear field theory, in particular, when describing bubbles generated by scalar fields of the Higgs type in the Minkowski spaces [7]. Numerical simulations of solutions of (8), where f is a polynomial with three zeros have been presented in [6], [14], [17]. Close problems about the existence of positive solutions are investigated in [2], [4], [5].

The main result of this paper is contained in Section 3 in Theorem 12, where we provide an asymptotic formula for homoclinic solutions of equation (1). Let us note that many important results about asymptotic properties of various types of differential equations can be found in the monograph by I. Kiguradze and T. Chanturia [12].

2 Existence of homoclinic solutions

Here we bring theorems about the existence of homoclinic solutions. Remind that assumptions (2)–(6) are common for all these theorems. For a given $B < 0$ we will denote the solution of problem (1), (7) by u_B.

Theorem 3 *Assume that problem* (1), (7) *has an escape solution and let* $\bar{B}$ *be of* (4). *Then there exists* $B^* < \bar{B}$ *such that* u_{B^*} *is a homoclinic solution of problem* (1), (7) *with* $B = B^*$.

Proof. Theorem 2.3 in [22] yields that for any $B \in [\bar{B}, 0)$ there exists a unique solution u_B of problem (1), (7) and u_B is damped. So, if we denote by $\mathcal{M}_d$ the set of all $B < 0$ such that u_B is a damped solution of problem (1), (7), we have $\mathcal{M}_d \neq \emptyset$. Moreover, $\mathcal{M}_d$ is open in $(-\infty, 0)$, due to Theorem 14 in [19]. Further, denote by $\mathcal{M}_e$ the set of all $B < 0$ such that u_B is an escape solution of problem (1), (7). By our assumption, we have $\mathcal{M}_e \neq \emptyset$ and, by Theorem 20 in [19], the set $\mathcal{M}_e$ is open in $(-\infty, 0)$, as well. Therefore the set $\mathcal{M}_h = (-\infty, 0) \setminus (\mathcal{M}_d \cup \mathcal{M}_e)$ is nonempty. Let us choose $B^* \in \mathcal{M}_h$. Then, $B^* < \bar{B}$, and by virtue of Definition 2, the supremum of the solution u_{B^*} on $(0, \infty)$ cannot be less than L and cannot be greater than L. Consequently this supremum is equal to L and u_{B^*} is a homoclinic solution of problem (1), (7) with $B = B^*$. □

Theorem 4 *Assume that* L_0 *of* (3) *satifies*

$$L_0 \in (-\infty, 0), \quad f(L_0) = 0. \tag{9}$$

Then there exists $B^* \in (L_0, \bar{B})$ *such that* u_{B^*} *is a homoclinic solution of problem* (1), (7) *with* $B = B^*$.

Proof. Define

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for} \quad x \le L, \\ 0 & \text{for} \quad x > L, \end{cases}$$

and consider the auxiliary equation

$$(p(t)u')' = p(t)\tilde{f}(u), \quad t \in (0, \infty). \tag{10}$$

By Theorem 10 and Lemma 9 in [20], there exists $B \in (L_0, \bar{B})$ such that u_B is an escape solution of problem (10), (7). If we modify the proof of Theorem 3 working on $(L_0, 0)$ instead of on $(-\infty, 0)$, we get a homoclinic solution u_{B^*} of problem (10), (7) having its starting value B^* in $(L_0, \bar{B})$. Since u_{B^*} is increasing on $(0, \infty)$ (see e.g. Lemma 3.5 in [21]), we have

$$B^* \le u_{B^*}(t) < L, \quad t \in [0, \infty), \tag{11}$$

and u_{B^*} is also a solution of equation (1). □

Theorem 4 assumes that f has the negative finite zero L_0. The following two theorems concern the case that $L_0 = -\infty$ and f is positive on $(-\infty, 0)$. Then a behaviour of f near $-\infty$ plays an important role. Equations with f having sublinear or linear behaviour near $-\infty$ are discussed in the next theorem.

Theorem 5 *Assume that $f(x) > 0$ for $x \in (-\infty, 0)$ and*

$$0 \le \limsup_{x \to -\infty} \frac{f(x)}{|x|} < \infty. \tag{12}$$

Then there exists $B^* < \bar{B}$ such that u_{B^*} is a homoclinic solution of problem (1), (7) *with $B = B^*$.*

Proof. In the linear case, that is if we assume

$$0 < \limsup_{x \to -\infty} \frac{f(x)}{|x|} < \infty,$$

the assertion follows from Theorem 5.1 in [21]. Consider the sublinear case, when we work with the condition

$$\limsup_{x \to -\infty} \frac{f(x)}{|x|} = 0.$$

Assumption $f > 0$ on $(-\infty, 0)$ gives

$$\lim_{x \to -\infty} \frac{f(x)}{|x|} = 0,$$

and Theorem 19 in [19] guarantees the existence of $B < \bar{B}$ such that u_B is an escape solution of problem (10), (7) . Theorem 3 and estimate (11) yield $B^* < \bar{B}$ such that u_{B^*} is a homoclinic solution of problem (1), (7) with $B = B^*$.
□

Theorem 6 *Assume that $f(x) > 0$ for $x \in (-\infty, 0)$ and that there exists $k \geq 2$ such that*

$$\lim_{t\to 0+} \frac{p'(t)}{t^{k-2}} \in (0, \infty). \tag{13}$$

Further, let $r \in (1, \frac{k+2}{k-2})$ be such that f fulfils

$$\lim_{x\to -\infty} \frac{f(x)}{|x|^r} \in (0, \infty). \tag{14}$$

Then there exists $B^* < \bar{B}$ such that u_{B^*} is a homoclinic solution of problem (1), (7) with $B = B^*$.

Proof. Theorem 2.10 in [23] guarantees the existence of $B < \bar{B}$ such that u_B is an escape solution of problem (10), (7) . Theorem 3 and estimate (11) yield $B^* < \bar{B}$ such that u_{B^*} is a homoclinic solution of problem (1), (7) with $B = B^*$. □

Theorem 6 discusses a superlinear behaviour of f near $-\infty$. Note that if $k = 2$, we can take any $r \in (0, \infty)$. Last existence theorem imposes an additional assumption on p only.

Theorem 7 *Assume that p satifies*

$$\int_0^1 \frac{\mathrm{d}s}{p(s)} < \infty. \tag{15}$$

Then there exists $B^* < \bar{B}$ such that u_{B^*} is a homoclinic solution of problem (1), (7) with $B = B^*$.

Proof. Using Theorem 18 in [19] instead of Theorem 2.10 in [23], we argue as in the proof of Theorem 6. □

In the next section, the generalized Matell's theorem which can be found as Theorem 6.5 in the monograph by I. Kiguradze [11] will be useful. For our purpose we provide its following special case.

Consider an interval $J \subset \mathbb{R}$. We write $AC(J)$ for the set of functions absolutely continuous on J and $AC_{loc}(J)$ for the set of functions belonging to $AC(I)$ for each compact interval $I \subset J$. Choose $T > 0$ and a function matrix $A(t) = (a_{i,j}(t))_{i,j\leq 2}$ which is defined on (T, ∞). Denote by $\lambda(t)$ and $\mu(t)$ eigenvalues of $A(t)$, $t \in (T, \infty)$. Further, suppose that

$$\lambda = \lim_{t\to\infty} \lambda(t) \quad \text{and} \quad \mu = \lim_{t\to\infty} \mu(t)$$

are different eigenvalues of the matrix $A = \lim_{t\to\infty} A(t)$ and let $\mathbf{l}$ and $\mathbf{m}$ be eigenvectors of A corresponding to λ and μ, respectively.

Theorem 8 [*11*] *Assume that*

$$a_{i,j} \in AC_{loc}(T,\infty), \qquad \left| \int_T^\infty a'_{i,j}(t)\,\mathrm{d}t \right| < \infty, \quad i,j = 1,2, \tag{16}$$

and that there exists $c_0 > 0$ *such that*

$$\int_s^t Re(\lambda(\tau) - \mu(\tau))\,\mathrm{d}\tau \le c_0, \quad T \le s < t, \tag{17}$$

or

$$\int_T^\infty Re(\lambda(\tau) - \mu(\tau))\,\mathrm{d}\tau = \infty, \int_s^t Re(\lambda(\tau) - \mu(\tau))\,\mathrm{d}\tau \ge -c_0, \quad T \le s < t. \tag{18}$$

Then the differential system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) \tag{19}$$

has a fundamental system of solutions $\mathbf{x}(t)$, $\mathbf{y}(t)$ *such that*

$$\lim_{t\to\infty} \mathbf{x}(t)\mathrm{e}^{-\int_T^t \lambda(\tau)\,\mathrm{d}\tau} = \mathbf{l}, \quad \lim_{t\to\infty} \mathbf{y}(t)\mathrm{e}^{-\int_T^t \mu(\tau)\,\mathrm{d}\tau} = \mathbf{m}. \tag{20}$$

3 Asymptotic behaviour of homoclinic solutions

In this section we assume that $B < \bar{B}$ is such that the corresponding solution u of initial problem (1), (7) is homoclinic. Hence u fulfils

$$u(0) = B, \quad u'(0) = 0, \quad u'(t) > 0, \ t \in (0,\infty), \quad \lim_{t\to\infty} u(t) = L. \tag{21}$$

Moreover, due to (1),

$$u''(t) + \frac{p'(t)}{p(t)} u'(t) = f(u(t)), \quad t > 0, \tag{22}$$

and, by multiplication and integration over $[0,t]$,

$$\frac{u'^2(t)}{2} + \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s = F(u(0)) - F(u(t)), \quad t > 0. \tag{23}$$

Therefore

$$0 \le \lim_{t\to\infty} \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s \le F(B) - F(L) < \infty,$$

and hence there exists

$$\lim_{t\to\infty} \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s.$$

Consequently, according to (23), $\lim_{t\to\infty} u'^2(t)$ exists, as well. Since u is bounded on $[0,\infty)$, we get

$$\lim_{t\to\infty} u'^2(t) = \lim_{t\to\infty} u'(t) = 0. \tag{24}$$

In order to derive an asymptotic formula for u we need to characterize a behaviour of p in ∞ and a behaviour of f near L more precisely. In particular we put

$$h(x) := \frac{f(x)}{x-L}, \quad x < L,$$

and we will work with the following assumptions:

$$\exists c, \eta > 0 : \quad h \in C^1[L-\eta, L], \quad \lim_{x \to L-} h(x) = h(L) = c, \tag{25}$$

$$p' \in AC_{loc}(0,\infty), \quad \exists n \geq 2 : \quad \lim_{t\to\infty} \frac{p'(t)}{t^{n-2}} \in (0,\infty). \tag{26}$$

For simplicity transform L to the origin by the substitution

$$z(t) = L - u(t), \quad t \in [0,\infty), \tag{27}$$

and put

$$g(y) = -f(L-y), \quad y > 0. \tag{28}$$

Then the function z given by (27) is a solution of the equation

$$(p(t)z')' = p(t)g(z), \quad t \in (0,\infty), \tag{29}$$

and satisfies

$$z(0) = L + |B|, \quad z'(0) = 0, \quad z'(t) < 0, \ t \in (0,\infty), \tag{30}$$

$$\lim_{t\to\infty} z(t) = 0, \quad \lim_{t\to\infty} z'(t) = 0. \tag{31}$$

Lemma 9 *Assume that condition* (25) *holds and let z be given by* (27)*. Then there exists $T > 0$ such that*

$$|z'(t)| > \sqrt{\frac{c}{2}}z(t), \quad t \geq T. \tag{32}$$

Proof. According to (29) the function z fulfils

$$z''(t) = -\frac{p'(t)}{p(t)}z'(t) + g(z(t)), \quad t \in (0,\infty). \tag{33}$$

Define the Lyapunov function V by

$$V(t) = \frac{z'^2(t)}{2} + G(z(t)), \tag{34}$$

where

$$G(x) = -\int_0^x g(s)\,\mathrm{d}s.$$

Then, due to (3), (4) and $B < \bar{B}$, the function G fulfils

$$G(L + |B|) = -\int_0^{L+|B|} g(s)\,\mathrm{d}s = \int_B^L f(s)\,\mathrm{d}s = F(B) - F(L) > 0.$$

Thus $V(0) = G(L + |B|) > 0$. Further, using (33), we have

$$V'(t) = z'(t)z''(t) - g(z(t))z'(t) = -\frac{p'(t)}{p(t)}z'^2(t) < 0, \quad t > 0.$$

So, V is decreasing on $(0, \infty)$ and, by (31), (34), we get $\lim_{t\to\infty} V(t) = 0$. Consequently $V(t) > 0$ for $t \in [0, \infty)$ which yields

$$\frac{z'^2(t)}{2} > -G(z(t)), \quad t > 0. \tag{35}$$

Let $y = L - x$. Then, using (25) and (28), we deduce that

$$-\lim_{y\to 0+} \frac{G(y)}{y^2} = \lim_{y\to 0+} \frac{g(y)}{2y} = \frac{1}{2} \lim_{x\to L-} \frac{f(x)}{x - L} = \frac{c}{2}.$$

Hence, by (31), there exists $T > 0$ such that

$$-\frac{G(z(t))}{z^2(t)} > \frac{c}{4}, \quad t \geq T.$$

This together with (35) leads to

$$\frac{z'^2(t)}{2} > \frac{c}{4}z^2(t), \quad t \geq T.$$

Consequently we get (32). □

Lemma 10 *Assume that condition* (25) *holds and let z and g be given by* (27) *and* (28), *respectively. Then*

$$\int_1^\infty \left| \frac{g(z(\tau))}{z(\tau)} - c \right| \mathrm{d}\tau < \infty. \tag{36}$$

Proof. Let us put

$$\tilde{h}(y) = \frac{g(y)}{y}, \quad y > 0. \tag{37}$$

By (25) and (28), we have

$$h(L - y) = \tilde{h}(y),\ y > 0, \quad \tilde{h} \in C^1[0, \eta], \quad \lim_{y\to 0+} \tilde{h}(y) = \tilde{h}(0) = c, \tag{38}$$

and there exists $M_0 \in (0, \infty)$ such that

$$\left| \frac{\mathrm{d}\tilde{h}(y)}{\mathrm{d}y} \right| \leq M_0, \quad y \in [0, \eta].$$

The Mean Value Theorem guarantees the existence of $\theta \in (0,1)$ such that

$$\tilde{h}(y) = c + y\frac{\mathrm{d}\tilde{h}(\theta y)}{\mathrm{d}y}, \quad y \in (0, \eta].$$

By (31), there exists $T \geq 1$ such that $0 < z(t) \leq \eta$ for $t \geq T$ and hence, according to (37),

$$\left| \frac{g(z(t))}{z(t)} - c \right| \leq M_0 z(t), \quad t \geq T. \tag{39}$$

Using (2), (28) and $z > 0$ on $[1, T]$, we can find $M_1 \in (0, \infty)$ that

$$\int_1^T \left| \frac{g(z(\tau))}{z(\tau)} - c \right| \mathrm{d}\tau \leq M_1,$$

and, without loss of generality we may assume that T is chosen in such a way, that (32) is valid, as well. Therefore, using (32) and (39), we get

$$\int_1^t \left| \frac{g(z(\tau))}{z(\tau)} - c \right| \mathrm{d}\tau \leq M_1 + M_0 \int_T^t z(\tau)\,\mathrm{d}\tau < M_1 + \sqrt{\frac{2}{c}} M_0 \int_T^t |z'(\tau)|\,\mathrm{d}\tau =$$

$$= M_1 - \sqrt{\frac{2}{c}} M_0 \int_T^t z'(\tau)\,\mathrm{d}\tau = M_1 + \sqrt{2c} M_0 (z(T) - z(t)), \quad t \geq T.$$

Letting $t \to \infty$ and using (31), we get (36). □

Lemma 11 *Assume that condition* (26) *holds. Then*

$$\int_1^\infty \left(\frac{p'(\tau)}{p(\tau)} \right)^2 \mathrm{d}\tau < \infty. \tag{40}$$

Proof. Condition (26) implies that there exists $c_0 \in (0, \infty)$ such that

$$\lim_{t \to \infty} \frac{p'(t)}{t^{n-2}} = c_0, \quad \lim_{t \to \infty} \frac{p(t)}{t^{n-1}} = \frac{c_0}{n-1}.$$

Therefore

$$\lim_{t \to \infty} t^2 \left(\frac{p'(t)}{p(t)} \right)^2 = (n-1)^2.$$

Hence, we can find $T \geq 1$ such that

$$\left(\frac{p'(t)}{p(t)} \right)^2 < \frac{n^2}{t^2}, \quad t \geq T, \tag{41}$$

and, due to (5) and (6), we can find $M_3 \in (0, \infty)$ such that

$$\int_1^T \left(\frac{p'(\tau)}{p(\tau)} \right)^2 \mathrm{d}\tau \leq M_3.$$

Consequently,

$$\int_1^t \left(\frac{p'(\tau)}{p(\tau)}\right)^2 \mathrm{d}\tau < M_3 + n^2 \int_T^t \frac{\mathrm{d}\tau}{\tau^2} = n^2\left(\frac{1}{T} - \frac{1}{t}\right), \quad t \ge T.$$

Letting $t \to \infty$, we get (40). $\square$

The main result about asymptotic behaviour of homoclinic solutions is contained in the next theorem.

Theorem 12 *Assume that* (25) *and* (26) *hold. Let* $B < \bar{B}$ *be such that the corresponding solution* u *of initial problem* (1), (7) *is homoclinic. Then* u *fulfils*

$$\lim_{t\to\infty} (L - u(t))\mathrm{e}^{\sqrt{c}t}\sqrt{p(t)} \in (0, \infty). \tag{42}$$

Remark 13 A similar asymptotic formula for positive solutions of equation (8), where $k > 1$ and $f(x) = x - |x|^r \text{sign } x$, $r > 1$, has been derived in [13], Theorem 6.1.

Proof. Step 1. *Construction of an auxiliary linear differential system.* Consider the function z given by (27). According to (29), z satisfies

$$z'' + \frac{p'(t)}{p(t)}z' = \frac{g(z(t))}{z(t)}z(t), \quad t \in (0, \infty). \tag{43}$$

Having this z, we introduce the linear differential equation

$$v'' + \frac{p'(t)}{p(t)}v' = \frac{g(z(t))}{z(t)}v, \tag{44}$$

and the corresponding linear differential system

$$x_1' = x_2, \quad x_2' = \frac{g(z(t))}{z(t)}x_1 - \frac{p'(t)}{p(t)}x_2. \tag{45}$$

Denote

$$A(t) = (a_{i,j}(t))_{i,j\le 2} = \begin{pmatrix} 0 & 1 \\ \frac{g(z(t))}{z(t)} & -\frac{p'(t)}{p(t)} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}.$$

By (6), (31), (37) and (38),

$$A = \lim_{t\to\infty} A(t).$$

Eigenvalues of A are numbers $\lambda = \sqrt{c}$ and $\mu = -\sqrt{c}$, eigenvectors of A are $\mathbf{l} = (1, \sqrt{c})$ and $\mathbf{m} = (1, -\sqrt{c})$, respectively. Denote

$$D(t) = \left(\frac{p'(t)}{2p(t)}\right)^2 + \frac{g(z(t))}{z(t)}, \quad t \in (0, \infty). \tag{46}$$

Then eigenvalues of $A(t)$ have the form

$$\lambda(t) = -\frac{p'(t)}{2p(t)} + \sqrt{D(t)}, \quad \mu(t) = -\frac{p'(t)}{2p(t)} - \sqrt{D(t)}, \quad t \in (0,\infty). \tag{47}$$

We see that

$$\lim_{t\to\infty} \lambda(t) = \lambda, \quad \lim_{t\to\infty} \mu(t) = \mu.$$

Step 2. *Verification of the assumptions of Theorem* 8. Due to (31) and (38), we can find $T \geq 1$ such that

$$0 < z(t) \leq \eta, \quad D(t) > 0, \quad t \in (T,\infty). \tag{48}$$

Therefore, by (37) and (38),

$$a_{21}(t) = \frac{g(z(t))}{z(t)} \in AC_{loc}(T,\infty),$$

and so

$$\left| \int_T^\infty \left(\frac{g(z(t))}{z(t)} \right)' \mathrm{d}t \right| = \left| \lim_{t\to\infty} \frac{g(z(t))}{z(t)} - \frac{g(z(T))}{z(T)} \right| = \left| c - \frac{g(z(T))}{z(T)} \right| < \infty.$$

Further, by (26), $a_{22}(t) = -\frac{p'(t)}{p(t)} \in AC_{loc}(T,\infty)$. Hence, due to (6),

$$\left| \int_T^\infty \left(\frac{p'(t)}{p(t)} \right)' \mathrm{d}t \right| = \left| \lim_{t\to\infty} \frac{p'(t)}{p(t)} - \frac{p'(T)}{p(T)} \right| = \frac{p'(T)}{p(T)} < \infty.$$

Since $a_{11}(t) \equiv 0$ and $a_{12}(t) \equiv 1$, we see that (16) is satisfied. Using (47) we get $Re(\lambda(t) - \mu(t)) = 2\sqrt{D(t)} > 0$ for $t \in (T,\infty)$. Since $\lim_{t\to\infty} \sqrt{D(t)} = \sqrt{c} > 0$, we have

$$\int_T^\infty Re(\lambda(\tau) - \mu(\tau))\,\mathrm{d}\tau = \infty, \quad \int_s^t Re(\lambda(\tau) - \mu(\tau))\,\mathrm{d}\tau > 0, \quad T \leq s < t.$$

Consequently (18) is valid.

Step 3. *Application of Theorem* 8. By Theorem 8 there exists a fundamental system $\mathbf{x}(t) = (x_1(t), x_2(t))$, $\mathbf{y}(t) = (y_1(t), y_2(t))$ of solutions of (45) such that (20) is valid. Hence

$$\lim_{t\to\infty} x_1(t) \mathrm{e}^{-\int_T^t \lambda(\tau)\,\mathrm{d}\tau} = 1, \quad \lim_{t\to\infty} y_1(t) \mathrm{e}^{-\int_T^t \mu(\tau)\,\mathrm{d}\tau} = 1. \tag{49}$$

Using (47) we get for $t \geq T$

$$\exp\left(-\int_T^t \lambda(\tau)\,\mathrm{d}\tau \right) = \exp\left(\int_T^t \left(\frac{p'(\tau)}{2p(\tau)} - \sqrt{D(\tau)} \right) \mathrm{d}\tau \right)$$

$$= \exp\left(\frac{1}{2}\ln\frac{p(t)}{p(T)}\right)\exp\left(-\int_T^t \sqrt{D(\tau)}\,\mathrm{d}\tau\right) = \sqrt{\frac{p(t)}{p(T)}}\exp\left(-\int_T^t \sqrt{D(\tau)}\,\mathrm{d}\tau\right),$$

and

$$\exp\left(-\int_T^t \mu(\tau)\,\mathrm{d}\tau\right) = \exp\left(\int_T^t \left(\frac{p'(\tau)}{2p(\tau)} + \sqrt{D(\tau)}\right)\,\mathrm{d}\tau\right)$$

$$= \exp\left(\frac{1}{2}\ln\frac{p(t)}{p(T)}\right)\exp\left(\int_T^t \sqrt{D(\tau)}\,\mathrm{d}\tau\right) = \sqrt{\frac{p(t)}{p(T)}}\exp\left(\int_T^t \sqrt{D(\tau)}\,\mathrm{d}\tau\right).$$

Further,

$$\int_T^t \sqrt{D(\tau)}\,\mathrm{d}\tau = E_0(t) + \sqrt{c}(t-T),$$

where

$$E_0(t) = \int_T^t \frac{D(\tau)-c}{\sqrt{D(\tau)}+\sqrt{c}}\,\mathrm{d}\tau, \quad t \geq T. \tag{50}$$

Hence,

$$\exp\left(-\int_T^t \lambda(\tau)\,\mathrm{d}\tau\right) = \sqrt{\frac{p(t)}{p(T)}}\mathrm{e}^{-E_0(t)}\mathrm{e}^{-\sqrt{c}(t-T)}, \quad t \geq T, \tag{51}$$

$$\exp\left(-\int_T^t \mu(\tau)\,\mathrm{d}\tau\right) = \sqrt{\frac{p(t)}{p(T)}}\mathrm{e}^{E_0(t)}\mathrm{e}^{\sqrt{c}(t-T)}, \quad t \geq T \tag{52}$$

Using (36), (40), (46), we can find $K_0 \in (0,\infty)$ such that for $t \geq T$,

$$\int_T^t \left|\frac{D(\tau)-c}{\sqrt{D(\tau)}+\sqrt{c}}\right|\mathrm{d}\tau \leq \frac{1}{\sqrt{c}}\left(\int_T^t \left(\frac{p'(\tau)}{2p(\tau)}\right)^2 \mathrm{d}\tau + \int_T^t \left|\frac{g(z(\tau))}{z(\tau)} - c\right|\mathrm{d}\tau\right) \leq K_0.$$

Consequently, due to (50),

$$\lim_{t\to\infty} E_0(t) = E_0 \in \mathbb{R}.$$

Therefore (49), (51) and (52) imply

$$1 = \lim_{t\to\infty} x_1(t)\sqrt{\frac{p(t)}{p(T)}}\mathrm{e}^{-E_0}\mathrm{e}^{-\sqrt{c}(t-T)}, \quad 1 = \lim_{t\to\infty} y_1(t)\sqrt{\frac{p(t)}{p(T)}}\mathrm{e}^{E_0}\mathrm{e}^{\sqrt{c}(t-T)}.$$

Since, by (26),

$$\lim_{t\to\infty}\sqrt{p(t)}\mathrm{e}^{-\sqrt{c}t} = \lim_{t\to\infty}\sqrt{\frac{p(t)}{t^{n-1}}}t^{(n-1)/2}\mathrm{e}^{-\sqrt{c}t} = 0, \quad \lim_{t\to\infty}\sqrt{p(t)}\mathrm{e}^{\sqrt{c}t} = \infty,$$

we obtain

$$\lim_{t\to\infty} x_1(t) = \infty, \quad \lim_{t\to\infty} y_1(t) = 0. \tag{53}$$

Step 4. *Asymptotic formula.* According to (43), z is also a solution of (44). Therefore there are $c_1, c_2 \in \mathbb{R}$ such that $z(t) = c_1 x_1(t) + c_2 y_1(t)$, $t \in (0, \infty)$. Having in mind (30), (31), (49) and (53), we get $c_1 = 0$, $c_2 y_1(t) > 0$ on $(0, \infty)$, and $c_2 \in (0, \infty)$. Consequently, $z(t) = c_2 y_1(t)$ and

$$1 = \lim_{t\to\infty} \frac{1}{c_2} z(t) \sqrt{\frac{p(t)}{p(T)}} \mathrm{e}^{E_0} \mathrm{e}^{\sqrt{c}(t-T)},$$

which together with (27) yields (42). □

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