Nonlinear systems of differential inequalities and solvability of certain boundary value problems

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Summary. In the paper we present some new existence results for nonlinear second order generalized periodic boundary value problems of the form

$$u'' = f(t, u, u'), \quad u(a) = u(b), \quad u'(a) = w(u'(b)).$$

These results are based on the method of lower and upper functions defined as solutions of the system of differential inequalities associated with the problem and their relation to the Leray-Schauder topological degree of the corresponding operator. Our main goal consists in a fairly general definition of these functions as couples from $\mathbb{AC}[a,b] \times \mathbb{BV}[a,b]$. Some conditions ensuring their existence are indicated, as well.

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0 . Introduction

In this paper we give existence theorems for the generalized periodic boundary value problem

$$u'' = f(t, u, u'), \tag{0.1}$$

$$u(a) = u(b), \quad u'(a) = w(u'(b)). \tag{0.2}$$

Using these results (Theorems 4.1 - 4.3) we can get both the existence and multiplicity for solutions of various periodic problems and their generalizations.

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One of such possible applications is shown in Corollary 4.4 which generalizes some results of [3], for other applications see [8] or [9].

The main tool of our arguments is a connection between the existence of lower and upper functions for (0.1), (0.2) (called also lower and upper solutions by some authors) and the Leray-Schauder topological degree of an operator associated with (0.1), (0.2).

The notions of lower and upper functions of the second order boundary value problems have a long history starting in 1931 when G. Scorza Dragoni [10] used them for the Dirichlet problem. So far there have been a lot of definitions introduced. Classically, we understand lower and upper functions as $\mathbb{C}^2$-functions. Differential equations with Carathéodory right hand sides or with singularities involved their generalization, for example as $\mathbb{AC}^1$-functions, $\mathbb{C}^1$-functions having left and right second derivatives or $\mathbb{W}^{2,1}$-functions. The majority of existence results was gained under the ordering assumption that a lower function is less than or equal to an upper one. During the last two decades the extension to non-ordered or reversely ordered lower and upper functions was attained. See [1] and the references mentioned there. Here, we introduce a definition (cf. Definition 1.7) of lower and upper functions of the problem (0.1), (0.2) which generalizes those of [1], [4], [5], [6] or [7] and consider the both cases of their ordering as well as the non-ordered one.

1 . Preliminaries

Throughout the paper we assume:
$-\infty < a < b < \infty$, $w : \mathbb{R} \mapsto \mathbb{R}$ is continuous and nondecreasing and $f : [a,b] \times \mathbb{R}^2 \mapsto \mathbb{R}$ fulfils the Carathéodory conditions on $[a,b] \times \mathbb{R}^2$, i.e. f has the following properties: (i) for each $x \in \mathbb{R}$ and $y \in \mathbb{R}$ the function $f(.,x,y)$ is measurable on $[a,b]$; (ii) for almost every $t \in [a,b]$ the function $f(t,.,.)$ is continuous on $\mathbb{R}^2$; (iii) for each compact set $\mathrm{K} \subset \mathbb{R}^2$ the function $m_{\mathrm{K}}(t) = \sup_{(x,y)\in \mathrm{K}} |f(t,x,y)|$ is Lebesgue integrable on $[a,b]$.

Furthermore, we keep the following notation:
$\mathbb{L}[a,b]$ is the Banach space of Lebesgue integrable functions on $[a,b]$ equipped with the usual norm denoted by $\|.\|_{\mathbb{L}}$. Furthermore, for $k \in \mathbb{N}\cup\{0\}$, $\mathbb{C}^k[a,b]$ and $\mathbb{AC}^k[a,b]$ are the Banach spaces of functions having continuous k-th derivatives on $[a,b]$ and of functions having absolutely continuous k-th derivatives on

$[a,b]$, respectively. As usual, the corresponding norms are defined by

$$\|x\|_{\mathbb{C}^k} = \sum_{i=0}^{k} \max_{t\in[a,b]} |x^{(i)}(t)| \quad \text{and} \quad \|x\|_{\mathbb{AC}^k} = \|x\|_{\mathbb{C}^k} + \|x^{(k+1)}\|_{\mathbb{L}}.$$

The symbols $\mathbb{C}[a,b]$ or $\mathbb{AC}[a,b]$ are used instead of $\mathbb{C}^0[a,b]$ or $\mathbb{AC}^0[a,b]$. Moreover, $\mathbb{BV}[a,b]$ is the set of functions of bounded variation on $[a,b]$. For $u \in \mathbb{BV}[a,b]$, u^{sing} and u^{ac} denote its singular and absolutely continuous parts, respectively. Furthermore, if $u \in \mathbb{BV}[a,b]$, then its one-sided derivatives are denoted by D^+u and D^-u.

$\mathrm{Car}([a,b]\times\mathbb{R}^2)$ is the set of functions satisfying the Carathéodory conditions on $[a,b]\times\mathbb{R}^2$.

Finally, for a given Banach space $\mathbb{X}$ and its subset M, $\mathrm{cl}(M)$ stands for the closure of M and ∂M denotes the boundary of M.

If Ω is an open bounded subset in $\mathbb{C}^1[a,b]$ and the operator $\mathrm{T} : \mathrm{cl}(\Omega) \mapsto \mathbb{C}^1[a,b]$ is compact, then $\deg(\mathrm{I}-\mathrm{T},\Omega)$ denotes the Leray-Schauder topological degree of $\mathrm{I}-\mathrm{T}$ with respect to Ω, where I stands for the identity operator on $\mathbb{C}^1[a,b]$. For a definition and properties of the degree see e.g. [2].

By a *solution of* (0.1),(0.2) we understand a function $u \in \mathbb{AC}^1[a,b]$ satisfying (0.1) for a.e. $t\in[a,b]$ and having the property (0.2).

The following estimate will be helpful later.

1.1. Lemma. *Let a function $m \in \mathbb{L}[a,b]$ and sets $\mathscr{U}(t) \subset \mathbb{R}$, $t\in[a,b]$, be such that $m(t) < 0$ on a subset of $[a,b]$ of a positive measure,*

(1.1) $\quad m(t) < f(t,x,y)$ *for a.e. $t\in[a,b]$ and any* $(x,y)\in\mathscr{U}(t)\times\mathbb{R}$

and

(1.2) $\quad w(y) \ge y$ *for all* $y \in [-\|m\|_{\mathbb{L}}, \|m\|_{\mathbb{L}}]$.

Let u be an arbitrary solution of (0.1), (0.2) *such that $u(t)\in\mathscr{U}(t)$ for all $t\in[a,b]$. Then*

$$\|u'\|_{\mathbb{C}} < \|m\|_{\mathbb{L}}. \tag{1.3}$$

If we suppose $m(t) > 0$ on a subset of $[a,b]$ of a positive measure and

(1.4) $\quad m(t) > f(t,x,y)$ *for a.e. $t\in[a,b]$ and any* $(x,y)\in\mathscr{U}(t)\times\mathbb{R}$

and

(1.5) $\quad w(y) \le y$ *for all* $y \in [-\|m\|_{\mathbb{L}}, \|m\|_{\mathbb{L}}]$

instead of (1.1) *and* (1.2), *then the estimate* (1.3) *remains valid, as well.*

Proof. We shall restrict ourselves only to the proof of the former assertion. The latter can be proved by a similar argument.

Let u be an arbitrary solution of (0.1), (0.2) such that $u(t) \in \mathscr{U}(t)$ for all $t \in [a,b]$ and let (1.1) and (1.2) be fulfilled. Then

$$m(t) < u''(t) \quad \text{for a.e.} \quad t \in [a,b]. \tag{1.6}$$

Certainly, there is $t_0 \in (a,b)$ such that $u'(t_0) = 0$. Hence

$$-\|m\|_{\mathbb{L}} \leq -\int_{t_0}^{t} |m(s)|\mathrm{d}s < u'(t) \quad \text{for} \quad t \in (t_0, b] \tag{1.7}$$

and

$$-\|m\|_{\mathbb{L}} \leq -\int_{t}^{t_0} |m(s)|\mathrm{d}s < -u'(t) \quad \text{for} \quad t \in [a, t_0). \tag{1.8}$$

In particular, with respect to (0.2),

$$w(u'(b)) = u'(a) < \int_{a}^{t_0} |m(s)|\mathrm{d}s \leq \|m\|_{\mathbb{L}}. \tag{1.9}$$

If $u'(b) \geq \|m\|_{\mathbb{L}}$ held, then by (1.2) we would have $w(u'(b)) \geq w(\|m\|_{\mathbb{L}}) \geq \|m\|_{\mathbb{L}}$, a contradiction. This together with (1.7) yields

$$|u'(b)| < \|m\|_{\mathbb{L}}. \tag{1.10}$$

Now, making use of (0.2), (1.2) and (1.7) we obtain for $t \in [a, t_0]$

$$u'(t) \geq u'(a) - \int_{a}^{t} |m(s)|\mathrm{d}s > -\int_{t_0}^{b} |m(s)|\mathrm{d}s - \int_{a}^{t} |m(s)|\mathrm{d}s \geq -\|m\|_{\mathbb{L}}.$$

This together with (1.7) yields

$$-\|m\|_{\mathbb{L}} < u'(t) \quad \text{for all} \quad t \in [a,b]. \tag{1.11}$$

On the other hand, in virtue of (1.9),(1.10) and (1.2) we have for $t \in [t_0, b]$

$$u'(t) < u'(b) + \int_{t}^{b} |m(s)|\mathrm{d}s < \int_{a}^{t_0} |m(s)|\mathrm{d}s + \int_{t}^{b} |m(s)|\mathrm{d}s \leq \|m\|_{\mathbb{L}}.$$

This together with (1.8) and (1.11) completes the proof of (1.3). $\square$

1.2. Remark. If $m(t) \geq 0$ were fulfilled a.e. on $[a,b]$, then in the case that we suppose (1.1) and (1.2), the set of solutions u of (0.1), (0.2) such that $u(t) \in \mathscr{U}(t)$ on $[a,b]$ would be empty. Analogous situation would occur if $m(t) \leq 0$ held a.e. on $[a,b]$ and we supposed (1.4) and (1.5).

Furthermore, we can see that provided $w(y) \equiv y$ (i.e. the boundary conditions (0.2) reduce to the periodic ones), we get (1.3) under the assumption (1.1) as well as under (1.4).

The equation (0.1) may be rewritten as the system of two equations of the first order

$$x' = y, \quad y' = f(t,x,y).$$

Generalization of the notions of lower and upper functions for systems of differential equations of the first order leads to the following concepts of "coupled" lower and upper functions which will be suitable for our purposes.

1.3. Definition. Functions $(\sigma_1, \rho_1) \in \mathbb{AC}[a,b] \times \mathbb{BV}[a,b]$ are said to be *lower functions of the equation* (0.1) (on $[a,b]$), if the singular part ρ_1^{sing} of ρ_1 is nondecreasing on $[a,b]$ and the following system of differential inequalities is satisfied:

$$\sigma_1'(t) = \rho_1(t) \qquad \text{a.e. on } [a,b], \tag{1.12}$$

$$\rho_1'(t) \geq f(t, \sigma_1(t), \rho_1(t)) \quad \text{a.e. on } [a,b]. \tag{1.13}$$

1.4. Definition. Functions $(\sigma_2, \rho_2) \in \mathbb{AC}[a,b] \times \mathbb{BV}[a,b]$ are said to be *upper functions of the equation* (0.1) (on $[a,b]$), if the singular part ρ_2^{sing} of ρ_2 is nonincreasing on $[a,b]$ and the following system of differential inequalities is satisfied:

$$\sigma_2'(t) = \rho_2(t) \qquad \text{a.e. on } [a,b], \tag{1.14}$$

$$\rho_2'(t) \leq f(t, \sigma_2(t), \rho_2(t)) \quad \text{a.e. on } [a,b]. \tag{1.15}$$

1.5. Remark. If (σ_1, ρ_1) and (σ_2, ρ_2) are respectively lower and upper functions to the given equation, then the monotonicity properties of the singular parts of the functions ρ_i $(i = 1,2)$ yield the relations

$$\begin{aligned} &\rho_1(t+) - \rho_1(t) \geq 0 \text{ and } \rho_2(t+) - \rho_2(t) \leq 0 \text{ for all } t \in [a,b) \\ &\rho_1(s) - \rho_1(s-) \geq 0 \text{ and } \rho_2(s) - \rho_2(s-) \leq 0 \text{ for all } s \in (a,b]. \end{aligned} \tag{1.16}$$

1.6. Remark. Obviously, if (σ_1, ρ_1) are lower functions of the equation (0.1), then $\sigma_1'(t) = \rho_1(t)$ for any point t of continuity of ρ_1 in (a, b), while the relations $\mathrm{D}^+\sigma_1(t) = \rho_1(t+)$ and $\mathrm{D}^-\sigma_1(s) = \rho_1(s-)$ are satisfied for any $t \in [a, b)$ and $s \in (a, b]$. Analogous relations are true for upper functions (σ_2, ρ_2) of (0.1), of course. On the other hand, for a given $i \in \{1, 2\}$, $\sigma_i''(t)$ need not be defined even for any $t \in [a, b]$ where $\mathrm{D}^+\sigma_i(t) = \mathrm{D}^-\sigma_i(t) = \sigma_i'(t)$ and thus $\sigma_i''(t)$ need not be defined for any $t \in [a, b]$, which generalizes the notion of $\mathbb{W}^{2.1}$-lower and -upper functions introduced in [1]. Other definitions which generalize the notions of $\mathbb{W}^{2.1}$-lower and -upper functions, but not so suitable for our purposes, were given by Ch. Fabry and P. Habets in [4]. Recently, it was shown by I. Vrkoč in [11] that our Definitions 1.3 and 1.4 are equivalent to those from [4].

1.7. Definition. Lower functions (σ_1, ρ_1) of (0.1) which satisfy

$$\sigma_1(a) = \sigma_1(b) \quad \text{and} \quad \rho_1(a+) \geq w(\rho_1(b-)) \tag{1.17}$$

are called *lower functions of the problem* (0.1), (0.2).

Upper functions (σ_2, ρ_2) of (0.1) which satisfy

$$\sigma_2(a) = \sigma_2(b) \quad \text{and} \quad \rho_2(a+) \leq w(\rho_2(b-)) \tag{1.18}$$

are called *upper functions of the problem* (0.1), (0.2).

1.8. Remark. If $f(t, r_1, 0) \leq 0$ a.e. on $[a, b]$ and $w(0) \leq 0$, then $(r_1, 0)$ are lower functions of (0.1), (0.2) and, similarly, if $f(t, r_2, 0) \geq 0$ a.e. on $[a, b]$ and $w(0) \geq 0$, then $(r_2, 0)$ are upper functions of (0.1), (0.2). On the other hand, it is easy to see that if $f(t, \sigma(t), \rho(t)) > 0$ a.e. on $[a, b]$ and w fulfils(1.2), then (σ, ρ) could not be lower functions of (0.1), (0.2). Analogously, $f(t, \sigma(t), \rho(t)) < 0$ a.e. on $[a, b]$ with (1.5) can be true for no upper functions (σ, ρ) of (0.1), (0.2).

Let us denote

$$\mathrm{L} : x \in \mathbb{AC}^1[a, b] \mapsto (x'' - x, x(a) - x(b), x'(a)) \in \mathbb{L}[a, b] \times \mathbb{R}^2 \tag{1.19}$$

and

$$\mathrm{F} : x \in \mathbb{C}^1[a, b] \mapsto \mathrm{F}x \in \mathbb{L}[a, b] \times \mathbb{R}^2, \tag{1.20}$$

where

$$(\mathrm{F}x)(t) = (f(t, x(t), x'(t)) - x(t), 0, w(x'(b))) \quad \text{a.e. on} \quad [a, b].$$

Then L is a linear bounded operator and the operator F is continuous.

After a careful computation we can check that if we put

(1.21)
$$\Gamma_0(t,s)=\begin{cases} \dfrac{\left(e^{2a-s-t}+e^{t-s}\right)\left(e^{2s}-e^{2b}\right)}{2\,\left(e^a-e^b\right)^2} & \text{if } \ t<s,\\[2ex] -\dfrac{\left(e^{2b-s-t}-e^{t-s}\right)\left(e^{2s}+e^{2a}\right)-2\,e^{a+b}\left(e^{t-s}-e^{s-t}\right)}{2\,\left(e^a-e^b\right)^2} & \text{if } \ t>s \end{cases}$$

and

(1.22)
$$\Gamma_1(t)=-\frac{e^{2a+b-t}+e^{b+t}}{\left(e^b-e^a\right)^2} \quad \text{and} \quad \Gamma_2(t)=-\frac{e^{a+b-t}+e^t}{e^b-e^a} \text{ on } [a,b],$$

then

$$\max_{t,s\in[a,b]}|\Gamma_0(t,s)|+\sup{}_{t,s\in[a,b]}\left|\frac{\partial\Gamma_0(t,s)}{\partial t}\right|<\infty,$$
$$\max_{t\in[a,b]}\Big(|\Gamma_1(t)|+|\Gamma_1'(t)|\Big)+\max_{t\in[a,b]}\Big(|\Gamma_2(t)|+|\Gamma_2'(t)|\Big)<\infty$$

and for any $(y,r_1,r_2)\in\mathbb{L}[a,b]\times\mathbb{R}^2$ the unique solution of the linear boundary value problem

$$x''-x=y(t), \quad x(a)-x(b)=r_1, \quad x'(a)=r_2$$

is defined by

$$x(t)=\int_a^b\Gamma_0(t,s)y(s)\mathrm{d}s+\Gamma_1(t)r_1+\Gamma_2(t)r_2 \text{ on } \ [a,b].$$

Furthermore, the operator L^+ defined by

(1.23)
$$\mathrm{L}^+:(y,r_1,r_2)\in\mathbb{L}[a,b]\times\mathbb{R}^2\mapsto\mathrm{L}^+(y,r_1,r_2)\in\mathbb{C}^1[a,b],$$

where

$$(\mathrm{L}^+(y,r_1,r_2))(t)=\int_a^b\Gamma_0(t,s)y(s)\mathrm{d}s+\Gamma_1(t)r_1+\Gamma_2(t)r_2 \text{ on } \ [a,b],$$

is linear and bounded and the operator $\mathrm{L}^+\mathrm{F}:\mathbb{C}^1[a,b]\mapsto\mathbb{C}^1[a,b]$ is compact. The problem (0.1),(0.2) is equivalent to the operator equation

$$\big(\mathrm{I}-\mathrm{L}^+\mathrm{F}\big)x=0$$

and if for some open bounded set $\Omega \subset \mathbb{C}^1[a,b]$ the relation

$$\deg(\mathrm{I} - \mathrm{L}^{+}\mathrm{F}, \Omega) \neq 0 \tag{1.24}$$

is true, then the problem (0.1), (0.2) possesses at least one solution in Ω.

2 . Strict lower and upper functions and topological degree

The following definition is motivated by the similar one used in [1] for the periodic problem $x'' = f(t,x)$, $x(a) = x(b), x'(a) = x'(b)$.

2.1. Definition. Lower functions (σ_1, ρ_1) of (0.1), (0.2) such that σ_1 is not a solution of this problem are called *strict lower functions of* (0.1), (0.2) if there exists $\varepsilon > 0$ such that

$$\begin{aligned} &\rho_1'(t) \geq f(t,x,y) \quad \text{for a.e.} \quad t \in [a,b] \\ &\qquad \text{and all} \quad (x,y) \in [\sigma_1(t), \sigma_1(t)+\varepsilon] \times [\rho_1(t)-\varepsilon, \rho_1(t)+\varepsilon]. \end{aligned} \tag{2.1}$$

Analogously, upper functions (σ_2, ρ_2) of (0.1), (0.2) are said to be strict upper functions of (0.1), (0.2) if σ_2 is not a solution of this problem and there exists $\varepsilon > 0$ such that

$$\begin{aligned} &\rho_2'(t) \leq f(t,x,y) \quad \text{for a.e.} \quad t \in [a,b] \\ &\qquad \text{and all} \quad (x,y) \in [\sigma_2(t)-\varepsilon, \sigma_2(t)] \times [\rho_2(t)-\varepsilon, \rho_2(t)+\varepsilon]. \end{aligned} \tag{2.2}$$

In this section we want to prove theorems giving sufficient conditions for (1.24) in terms of strict lower and upper functions of (0.1),(0.2). We shall need the following two lemmas.

2.2. Lemma. *Let* (σ_1, ρ_1) *and* (σ_2, ρ_2) *be respectively strict lower and upper functions of the problem* (0.1), (0.2) *such that*

$$\sigma_1(t) < \sigma_2(t) \ \textit{on} \ [a,b]. \tag{2.3}$$

Then for any solution u *of* (0.1), (0.2) *fulfilling*

$$\sigma_1(t) \leq u(t) \leq \sigma_2(t) \ \textit{on} \ [a,b] \tag{2.4}$$

we have $\sigma_1(t) < u(t) < \sigma_2(t)$ *on* $[a,b]$.

Proof. i) Suppose

$$u(t_0) - \sigma_2(t_0) = \max_{t\in[a,b]} \Big(u(t) - \sigma_2(t)\Big) = 0 \quad \text{and} \quad t_0 \in (a,b). \tag{2.5}$$

In particular, $u'(t_0) - \rho_2(t_0-) \geq 0 \geq u'(t_0) - \rho_2(t_0+)$ and thus, with respect to (1.16),

$$u'(t_0) = \lim_{t\to t_0} \rho_2(t) = \rho_2(t_0). \tag{2.6}$$

Hence, if $\varepsilon > 0$ is such that (2.2) is true, then there is $\delta \in (0, b - t_0]$ such that the relations $\sigma_2(t) - \varepsilon \leq u(t) \leq \sigma_2(t)$ and $\rho_2(t) - \varepsilon \leq u'(t) \leq \rho_2(t) + \varepsilon$ are satisfied for all $t \in [t_0 - \delta, t_0 + \delta]$ and consequently, making use of (2.2), (2.6) and the monotonicity of ρ_2^{sing}, we get for any $t \in [t_0, t_0 + \delta]$

$$\begin{aligned} 0 &\leq \int_{t_0}^t \big(f(s, u(s), u'(s)) - \rho_2'(s)\big)\,\mathrm{d}s = \int_{t_0}^t \big(u''(t) - \rho_2'(s)\big)\,\mathrm{d}s \\ &= u'(t) - \rho_2^{\text{ac}}(t) - u'(t_0) + \rho_2^{\text{ac}}(t_0) = u'(t) - \rho_2(t) + \rho_2^{\text{sing}}(t) - \rho_2^{\text{sing}}(t_0) \\ &\leq u'(t) - \rho_2(t). \end{aligned} \tag{2.7}$$

By (1.16), (2.7) and (2.4) we have

$$0 \geq u(t) - \sigma_2(t) = \int_{t_0}^t \Big(u'(s) - \rho_2(s)\Big)\,\mathrm{d}s \geq 0 \text{ on } \ [t_0, t_0 + \delta],$$

i.e. $u(t) = \sigma_2(t)$ on $[t_0, t_0 + \delta]$.

Let us put $t^* = \sup \Big\{\tau \in [t_0, b] \,:\, u(t) = \sigma_2(t) \text{ on } [t_0, \tau]\Big\}$. Then $t^* \geq t_0 + \delta$, $u(t^*) = \sigma_2(t^*)$ and $u'(t^*) = \rho_2(t^*-)$. Let us assume that $t^* < b$. Then, by (1.16), we have $u'(t^*) \geq \rho_2(t^*+)$. If $u'(t^*) > \rho_2(t^*+)$ were valid, then $0 = u(t_0) - \sigma_2(t_0) = u(t^*) - \sigma_2(t^*)$ could not be the maximum value of $u(t) - \sigma_2(t)$ on $[a, b]$ and this would contradict the assumption (2.5). Thus, $u'(t^*) = \rho_2(t^*+)$. Repeating the above considerations with t^* in place of t_0, we would obtain further that there is $\delta^* \in (0, b - t^*]$ such that $u(t) = \sigma_2(t)$ on $[t^*, t^* + \delta^*]$, a contradiction with the definition of t^*. It means that $t^* = b$ and $u(t) = \sigma_2(t)$ on $[t_0, b]$. Similarly, we could prove that $u(t) = \sigma_2(t)$ on $[a, t_0]$, i.e. $u(t) = \sigma_2(t)$ on $[a, b]$. This contradicts our assumption that σ_2 is not a solution of the problem (0.1), (0.2) on $[a, b]$, i.e. $u(t) < \sigma_2(t)$ on (a, b).

ii) Suppose

$$0 = u(b) - \sigma_2(b) = u(a) - \sigma_2(a) = \max_{t\in[a,b]} \Big(u(t) - \sigma_2(t)\Big). \tag{2.8}$$

This is possible only if $u'(a) \le \rho_2(a+)$ and $u'(b) \ge \rho_2(b-)$. On the other hand, by (0.2) and (1.18) we have $0 \ge u'(a) - \rho_2(a+) \ge w(u'(b)) - w(\rho_2(b-)) \ge 0$ and hence

$$u'(a) = \rho_2(a+). \tag{2.9}$$

Similarly as in part i) of the proof, we can deduce from the relations (2.8) and (2.9) that $u(t) \equiv \sigma_2(t)$ on $[a,b]$. This being impossible by Definition 2.1, we conclude that $u(t) < \sigma_2(t)$ on $[a,b]$.

iii) Similarly we can show that under our assumptions the relation $u(t) > \sigma_1(t)$ is true for all $t \in [a,b]$, as well. $\square$

2.3. Lemma. *Let* (σ_1,ρ_1) *and* (σ_2,ρ_2) *be respectively strict lower and upper functions of* (0.1), (0.2) *such that* (2.3) *is true. Let us put*

$$\widetilde{f}(t,x,y) = \begin{cases} f(t,\sigma_1(t),y) - \sigma_1(t) & \text{if } x < \sigma_1(t), \\ f(t,x,y) - x & \text{if } \sigma_1(t) \le x \le \sigma_2(t), \\ f(t,\sigma_2(t),y) - \sigma_2(t) & \text{if } \sigma_2(t) < x. \end{cases} \tag{2.10}$$

Then $\widetilde{f} \in \mathrm{Car}([a,b]\times\mathbb{R}^2)$ *and for any solution* u *of the problem*

$$u'' - u = \widetilde{f}(t,u,u'), \quad (0.2) \tag{2.11}$$

the relations (2.4) *are satisfied.*

Proof. In view of (2.10), we have $\widetilde{f} \in \mathrm{Car}([a,b]\times\mathbb{R}^2)$. Let u be an arbitrary solution of the problem (2.11) and let

$$u(t_0) - \sigma_2(t_0) = \max_{t\in[a,b]} \Big(u(t) - \sigma_2(t)\Big) > 0. \tag{2.12}$$

By (0.2) and (1.18) it suffices to consider the cases $t_0 \in (a,b)$ and $t_0 = a$. If $t_0 \in (a,b)$, then similarly as in the proof of Lemma 2.2 we obtain that $\lim_{t\to t_0} \rho_2(t) = \rho_2(t_0) = u'(t_0)$. If $t_0 = a$, then like in the second part of the proof of Lemma 2.2 we get $u'(a) = \rho_2(a+)$. In particular, in both cases, if

$\varepsilon > 0$ is such that (2.2) is satisfied, then there is $\delta \in (0, b - t_0]$ such that $u'(t) \in [\rho_2(t) - \varepsilon, \rho_2(t) + \varepsilon]$ and $u(t) > \sigma_2(t)$ on $[t_0, t_0 + \delta]$. Hence, owing to (2.10) we have

$$\begin{aligned} u''(t) - \rho_2'(t) &= f(t, \sigma_2(t), u'(t)) + u(t) - \sigma_2(t) - \rho_2'(t) \\ &> f(t, \sigma_2(t), u'(t)) - \rho_2'(t) \geq 0 \quad \text{a.e. on} \ \ [t_0, t_0 + \delta] \end{aligned}$$

and like in (2.7) for $t \in (t_0, t_0 + \delta]$ we obtain

$$0 < \int_{t_0}^{t} \Big(u''(s) - \rho_2'(s)\Big) \mathrm{d}s \leq u'(t) - \rho_2(t).$$

Consequently,

$$0 < \int_{t_0}^{t} \Big(u'(s) - \rho_2(s)\Big) \mathrm{d}s \leq \Big(u(t) - \sigma_2(t)\Big) - \Big(u(t_0) - \sigma_2(t_0)\Big) \ \text{on} \ \ (t_0, t_0 + \delta].$$

As this contradicts the assumption (2.12), it follows that $u(t) \leq \sigma_2(t)$ on $[a, b]$. Similarly we could show that $\sigma_1(t) \leq u(t)$ on $[a, b]$. ☐

2.4. Theorem. *Let* (σ_1, ρ_1) *and* (σ_2, ρ_2) *be respectively strict lower and upper functions of* (0.1), (0.2) *satisfying* (2.3). *Further, let us assume that either* (1.1) *and* (1.2) *or* (1.4) *and* (1.5) *are satisfied with* $m \in \mathbb{L}[a, b]$ *and* $\mathscr{U}(t) = [\sigma_1(t), \sigma_2(t)]$ *for* $t \in [a, b]$. *Let us denote*

(2.13)
$$\Omega_1 = \Big\{x \in \mathbb{C}^1[a, b] \ : \ \sigma_1(t) < x(t) < \sigma_2(t) \ \textit{and} \ \ \|x'\|_{\mathbb{C}} < \|m\|_{\mathbb{L}} \ \textit{on} \ \ [a, b]\Big\}$$

and let the operators L^+ *and* F *be given by (1.23) and (1.20), respectively. Then*

$$\deg(\mathrm{I} - \mathrm{L}^+\mathrm{F}, \Omega_1) = 1.$$

Proof. Assume (1.1) and (1.2) and for some $c \in (0, \infty)$ put

$$g(t, x, y) = \begin{cases} f(t, x, -c) & \text{if} \ \ y < -c, \\ f(t, x, y) & \text{if} \ \ |y| \leq c, \\ f(t, x, c) & \text{if} \ \ y > c \end{cases}$$

and

$$\widetilde{w}(y) = \begin{cases} w(-c) & \text{if } y < -c, \\ w(y) & \text{if } |y| \le c, \\ w(c) & \text{if } y > c. \end{cases}$$

Let $\widetilde{f}$ be given by (2.10), where we put g instead of f and choose $c > \|m\|_{\mathbb{L}}$ such that (σ_1, ρ_1) and (σ_2, ρ_2) are strict lower and upper functions of

$$u'' = g(t, u, u'), \quad u(a) - u(b) = 0, \quad u'(a) = \widetilde{w}(u'(b)). \tag{2.14}$$

Now consider the parameter system of boundary value problems

(2.15)
$$u'' - u = \lambda \widetilde{f}(t, u, u'), \quad u(a) - u(b) = 0, \quad u'(a) = \lambda \widetilde{w}(u'(b)), \quad \lambda \in [0,1].$$

Defining for $x \in \mathbb{C}^1[a,b]$ and for a.e. $t \in [a,b]$

$$(\widetilde{\mathrm{F}}x)(t) = (\widetilde{f}(t, x(t), x'(t)), 0, \widetilde{w}(x'(b))),$$

we get a continuous operator $\widetilde{\mathrm{F}} : \mathbb{C}^1[a,b] \mapsto \mathbb{L}[a,b] \times \mathbb{R}^2$ and the system (2.15) can be rewritten as the parameter system of operator equations

$$u - \lambda \mathrm{L}^+ \widetilde{\mathrm{F}} u = 0, \quad \lambda \in [0,1].$$

For $\lambda \in [0,1]$, a function $u \in \mathbb{C}^1[a,b]$ is a solution to (2.15) if and only if it satisfies the relation

$$u(t) = \lambda \left(\int_a^b \Gamma_0(t,s) \widetilde{f}(s, u(s), u'(s)) \mathrm{d}s + \Gamma_2(t) \widetilde{w}(u'(b)) \right) \text{ on } [a,b],$$

where Γ_0 and Γ_2 are defined by (1.21) and (1.22). Therefore there is $r \in (0, \infty)$ such that

$$\Omega_1 \subset \mathscr{K}(r) = \left\{ x \in \mathbb{C}^1[a,b] \,:\, \|x\|_{\mathbb{C}^1} < r \right\}$$

and for any $\lambda \in [0,1]$ any solution u to (2.15) belongs to $\mathscr{K}(r)$. Thus, the operator $\mathrm{I} - \lambda \mathrm{L}^+ \widetilde{\mathrm{F}}$ is a homotopy on $\mathscr{K}(r) \times [0,1]$ and

$$\deg \left(\mathrm{I} - \mathrm{L}^+ \widetilde{\mathrm{F}}, \, \mathscr{K}(r) \right) = \deg \left(\mathrm{I}, \, \mathscr{K}(r) \right) = 1.$$

Now, let $\lambda = 1$ and let u be an arbitrary solution of the corresponding problem (2.15). We can apply Lemma 2.3 and get (2.4). Hence u is a solution of (2.14).

Since $g(t,x,y) > m(t)$ for a.e. $t \in [a,b]$ and all $(x,y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}$, $\sigma_1(t) \le u(t) \le \sigma_2(t)$ on $[a,b]$ and $\widetilde{w}(y) = w(y)$ for $y \in [-\|m\|_{\mathbb{L}}, \|m\|_{\mathbb{L}}]$, we can use Lemma 1.1 and get $\|u'\|_{\mathbb{C}} < \|m\|_{\mathbb{L}} < c$. It follows that u is a solution of (0.1), (0.2). Consequently, we can make use of Lemma 2.2 to show that $\sigma_1(t) < u(t) < \sigma_2(t)$ on $[a,b]$.

To summarize, for $\lambda = 1$ and for any solution u of (2.15) we have $u \in \Omega_1$. Since $\widetilde{\mathrm{F}} = \mathrm{F}$ on $\mathrm{cl}(\Omega_1)$, this means that

$$\begin{aligned}&\deg\big(\mathrm{I} - \mathrm{L}^{+}\mathrm{F},\, \Omega_1\big)\\ &\qquad = \deg\big(\mathrm{I} - \mathrm{L}^{+}\widetilde{\mathrm{F}},\, \Omega_1\big) = \deg\big(\mathrm{I} - \mathrm{L}^{+}\widetilde{\mathrm{F}},\, \mathscr{K}(r)\big) = 1.\end{aligned}$$

The case that (1.4) and (1.5) are satisfied instead of (1.1) and (1.2) could be treated in a similar way. □

Now, we prove an analogous theorem provided σ_1, σ_2 are ordered in the opposite way, i.e.

$$\sigma_2(t) < \sigma_1(t) \text{ for all } t \in [a,b]. \tag{2.16}$$

2.5. Theorem. *Let (σ_1, ρ_1) and (σ_2, ρ_2) be respectively strict lower and upper functions of (0.1),(0.2) satisfying (2.16). Further, let us assume that either (1.1) and (1.2) or (1.4) and (1.5) are satisfied with $m \in \mathbb{L}[a,b]$ and $\mathscr{U}(t) \equiv \mathbb{R}$. Let $A \in \mathbb{R}$ be such that $\|\sigma_1\|_{\mathbb{C}} + \|\sigma_2\|_{\mathbb{C}} + (b-a)\|m\|_{\mathbb{L}} \le A$ and let*

$$\begin{aligned}\Omega_2 = \Big\{&x \in \mathbb{C}^1[a,b] \,:\, \|x\|_{\mathbb{C}} < A,\ \|x'\|_C < \|m\|_{\mathbb{L}}\\ &\textit{and there exists } t_x \in [a,b] \textit{ such that } \sigma_2(t_x) < x(t_x) < \sigma_1(t_x)\Big\}.\end{aligned}$$

Then

$$\deg(\mathrm{I} - \mathrm{L}^{+}\mathrm{F}, \Omega_2) = -1. \tag{2.17}$$

Proof. Put $\widetilde{A} = A + (b-a)$. Assume (1.1) and (1.2) and consider an auxiliary equation

$$u'' = g(t,u,u'), \tag{2.18}$$

where

$$g(t,x,y)=\begin{cases} f(t,x,y)+|m(t)| & \text{if } x\ge \widetilde{A}+1,\\ f(t,x,y)+(x-\widetilde{A})|m(t)| & \text{if } \widetilde{A}<x<\widetilde{A}+1,\\ f(t,x,y) & \text{if } -\widetilde{A}\le x\le \widetilde{A},\\ f(t,x,y)+(\widetilde{A}+x)[f(t,x,y)+|m(t)|] & \text{if } -\widetilde{A}-1<x<-\widetilde{A},\\ -|m(t)| & \text{if } x\le -\widetilde{A}-1. \end{cases}$$

We have $g\in \mathrm{Car}([a,b]\times\mathbb{R}^2)$ and

$$(2.19)\quad g(t,x,y)>-(|m(t)|+1)$$
$$\text{for a.e. } \ t\in[a,b] \ \text{ and all } \ (x,y)\in[-(\widetilde{A}+2),(\widetilde{A}+2)]\times\mathbb{R}.$$

The couples of functions (σ_1,ρ_1) and (σ_2,ρ_2) are respectively strict lower and upper functions to the problem (2.18), (0.2). Furthermore, in virtue of the assumption (1.1), also $(\sigma_3,\rho_3)=(-(\widetilde{A}+2),0)$ and $(\sigma_4,\rho_4)=(\widetilde{A}+2,0)$ are respectively strict lower and upper functions to the problem (2.18), (0.2) which are "well-ordered", i.e. $\sigma_3(t)<\sigma_4(t)$ on $[a,b]$. Let us define sets

$$\Omega \ =\Big\{x\in\mathbb{C}^1[a,b]\,:\,\|x\|_{\mathbb{C}}<\widetilde{A}+2,\|x'\|_{\mathbb{C}}<\|m\|_{\mathbb{L}}+1\Big\},$$
$$\Delta_1=\Big\{x\in\Omega\,:\,\sigma_1(t)<x(t) \text{ on } \ [a,b]\Big\}$$

and

$$\Delta_2=\Big\{x\in\Omega\,:\,x(t)<\sigma_2(t) \text{ on } \ [a,b]\Big\},$$

and an operator

$$\mathrm{G}\ :\ x\in\mathbb{C}^1[a,b]\mapsto \mathrm{G}x\in\mathbb{L}[a,b]\times\mathbb{R}^2,$$

where

$$(\mathrm{G}x)(t)=(g(t,x(t),x'(t)),0,w(x'(b))) \text{ a.e. on } \ [a,b].$$

Owing to Theorem 2.4 we have

$$\deg(\mathrm{I}-\mathrm{L}^+\mathrm{G},\Omega)=\deg(\mathrm{I}-\mathrm{L}^+\mathrm{G},\Delta_1)=\deg(\mathrm{I}-\mathrm{L}^+\mathrm{G},\Delta_2)=1.$$

Let us denote $\Delta=\Omega\setminus\mathrm{cl}(\Delta_1\cup\Delta_2)$. Then

$$\Delta=\Big\{x\in\Omega\,:\ \text{ there is } \ t_x\in[a,b] \text{ such that } \ \sigma_2(t_x)<x(t_x)<\sigma_1(t_x)\Big\}$$

and by the additivity of the degree we have

$$
\begin{aligned}
&\deg(\mathrm{I}-\mathrm{L}^{+}\mathrm{G},\Delta)\\
&\qquad=\deg(\mathrm{I}-\mathrm{L}^{+}\mathrm{G},\Omega)-\deg(\mathrm{I}-\mathrm{L}^{+}\mathrm{G},\Delta_1)-\deg(\mathrm{I}-\mathrm{L}^{+}\mathrm{G},\Delta_2)=-1.
\end{aligned}
$$

Let u be a solution to (2.18), (0.2) and let $u\in\Delta$. Then there is $t_u\in(a,b)$ such that $\sigma_2(t_u)\le u(t_u)\le\sigma_1(t_u)$. Consequently, for any $t\in[a,b]$ we have

$$
|u(t)|=\left|u(t_u)+\int_{t_u}^{t}u'(s)\mathrm{d}s\right|\le\|\sigma_1\|_{\mathbb{C}}+\|\sigma_2\|_{\mathbb{C}}+(b-a)\|u'\|_{\mathbb{C}},\tag{2.20}
$$

wherefrom by (2.19) and Lemma 1.1 the relation $\|u\|_{\mathbb{C}}<\widetilde{A}$ follows. Therefore u is a solution of (0.1), (0.2) and using Lemma 1.1 and (2.20) once more we get $\|u'\|_{\mathbb{C}}<\|m\|_{\mathbb{L}}$ and $\|u\|_{\mathbb{C}}<A$, i.e. $u\in\Omega_2$. Consequently, the excision property of the degree yields

$$
\deg(\mathrm{I}-\mathrm{L}^{+}\mathrm{G},\Omega_2)=-1,
$$

wherefrom, since $\mathrm{G}=\mathrm{F}$ on $\mathrm{cl}(\Omega_2)$, we obtain (2.17).

In the case that (1.4) and (1.5) are satisfied instead of (1.1) and (1.2) we can argue similarly. □

The case

$$
\text{there are } r \text{ and } s\in[a,b] \text{ such that } \sigma_1(r)<\sigma_2(r) \text{ and } \sigma_2(s)<\sigma_1(s)\tag{2.21}
$$

is treated by the following theorem.

2.6. Theorem. *Let (σ_1,ρ_1) and (σ_2,ρ_2) be respectively strict lower and upper functions of* (0.1), (0.2) *satisfying* (2.21). *Further, let us assume that either* (1.1) *and* (1.2) *or* (1.4) *and* (1.5) *are satisfied with $m\in\mathbb{L}[a,b]$ and $\mathscr{U}(t)\equiv\mathbb{R}$. Let $A\in\mathbb{R}$ be such that $\|\sigma_1\|_{\mathbb{C}}+\|\sigma_2\|_{\mathbb{C}}+(b-a)\|m\|_{\mathbb{L}}\le A$ and let*

$$
\begin{aligned}
\Omega_3=\Big\{x\in\mathbb{C}^1[a,b]\,:\,&\|x\|_{\mathbb{C}}<A,\ \|x'\|_{C}<\|m\|_{\mathbb{L}}\ \text{ and there exist}\\
&r_x,s_x\in[a,b]\ \text{such that}\ \sigma_1(r_x)>x(r_x)\ \text{and}\ \sigma_2(s_x)<x(s_x)\Big\}.
\end{aligned}
$$

Then

$$
\deg(\mathrm{I}-\mathrm{L}^{+}\mathrm{F},\Omega_3)=-1.
$$

Proof. Let $g, \mathrm{G}, \widetilde{A}, \Delta_1, \Delta_2$ and Ω have the same meaning as in the proof of Theorem 2.5. Taking into account that in the case (2.21), $\Omega \setminus \mathrm{cl}(\Delta_1 \cup \Delta_2)$ is the set of all $x \in \Omega$ for which there exist r_x and $s_x \in [a,b]$ such that $\sigma_1(r_x) > x(r_x)$ and $\sigma_2(s_x) < x(s_x)$, it is easy to see that the proof of this theorem can be completed by an argument analogous to that used in the proof of Theorem 2.5. □

3 . Lower and upper functions and topological degree

In this section we give proper modifications of the results described in the previous section to the case of lower and upper functions which need not be strict.

3.1. Lemma. *Let the assumptions of Theorem* 2.4 *be fulfilled but with* (σ_1, ρ_1) *and* (σ_2, ρ_2) *not necessarily strict. For a.e.* $t \in [a,b]$ *and any* $\zeta \in [0,1]$ *let us put*

$$\omega_1(t,\zeta) = \sup_{z \in \mathbb{R},\, |\rho_1(t) - z| \le \zeta} |f(t, \sigma_1(t), \rho_1(t)) - f(t, \sigma_1(t), z)|, \tag{3.1}$$

$$\omega_2(t,\zeta) = \sup_{z \in \mathbb{R},\, |\rho_2(t) - z| \le \zeta} |f(t, \sigma_2(t), \rho_2(t)) - f(t, \sigma_2(t), z)| \tag{3.2}$$

Furthermore, let us define

(3.3)

$$h(t,x,y) = \begin{cases} f(t,\sigma_1(t),y) \quad -\sigma_1(t) - \omega_1(t, \dfrac{\sigma_1(t) - x}{\sigma_1(t) - x + 1}) & \text{if } x < \sigma_1(t), \\ f(t,x,y) \quad -x & \text{if } x \in [\sigma_1(t), \sigma_2(t)], \\ f(t,\sigma_2(t),y) \quad -\sigma_2(t) + \omega_2(t, \dfrac{x - \sigma_2(t)}{x - \sigma_2(t) + 1}) & \text{if } x > \sigma_2(t) \end{cases}$$

and

$$\widetilde{w}(y) = \begin{cases} w(-\|m\|_{\mathbb{L}}) + y + \|m\|_{\mathbb{L}} & \text{for } y < -\|m\|_{\mathbb{L}}, \\ w(y) & \text{for } |y| \le \|m\|_{\mathbb{L}}, \\ w(\|m\|_{\mathbb{L}}) + y - \|m\|_{\mathbb{L}} & \text{for } y > \|m\|_{\mathbb{L}}. \end{cases}$$

Then $h \in \mathrm{Car}([a,b] \times \mathbb{R}^2)$ *and for any solution* u *of the problem*

$$u'' - u = h(t,u,u'), \quad u(a) = u(b), \; u'(a) = \widetilde{w}(u'(b)) \tag{3.4}$$

the relations (2.4) *and* (1.3) *are true.*

Proof. The functions $\omega_i : [a,b] \times [0,1] \mapsto \mathbb{R}^+$ $(i = 1,2)$ given by (3.1) and (3.2) are nondecreasing in the second variable and belong to the class $\mathrm{Car}([a,b] \times [0,1])$. Hence $h \in \mathrm{Car}([a,b] \times \mathbb{R}^2)$ as well. Let u be an arbitrary solution of (3.4) and suppose

$$u(t_0) - \sigma_2(t_0) = \max_{t \in [a,b]} \Big(u(t) - \sigma_2(t) \Big) > 0.$$

In virtue of (0.2) and (1.18) it suffices to consider the cases $a < t_0 < b$ and $t_0 = a$. As in the proof of Lemma 2.3 we have

$$u'(t_0) = \lim_{t \to t_0} \rho_2(t) = \rho_2(t_0)$$

in the former case and $u'(a) = \rho_2(a+)$ in the latter. Making use of the continuity of σ_2, u and u' we conclude that in both cases there are $\delta > 0$ and $\eta \in (0,1)$ such that for all $t \in [t_0, t_0 + \delta]$ we have

$$|\rho_2(t) - u'(t)| < \eta < \frac{u(t) - \sigma_2(t)}{u(t) - \sigma_2(t) + 1} < u(t) - \sigma_2(t)$$

and, with respect to (3.2),

$$\begin{aligned} &|f(t, \sigma_2(t), \rho_2(t)) - f(t, \sigma_2(t), u'(t))| \\ &\qquad \le \omega_2(t, |\rho_2(t) - u'(t)|) \le \omega_2(t, \frac{u(t) - \sigma_2(t)}{u(t) - \sigma_2(t) + 1}). \end{aligned}$$

Consequently, by means of (1.15), for any $t \in [t_0, t_0 + \delta]$ we get

$$\begin{aligned} &u''(t) - \rho_2'(t) \\ &\quad = u(t) + f(t, \sigma_2(t), u'(t)) + \omega_2(t, \frac{u(t) - \sigma_2(t)}{u(t) - \sigma_2(t) + 1}) - \sigma_2(t) - \rho_2'(t) \\ &\quad > \eta + f(t, \sigma_2(t), \rho_2(t)) - \rho_2'(t) > 0. \end{aligned}$$

Like in the proof of Lemma 2.3 this yields a contradiction with the assumption that $u(t_0) - \sigma_2(t_0)$ is the maximal value of $u(t) - \sigma_2(t)$ on $[a,b]$. Thus, the relation $u(t) \le \sigma_2(t)$ is true on $[a,b]$. Similarly we can show that $\sigma_1(t) \le u(t)$ on $[a,b]$ as well, i.e. u satisfies (2.4). Therefore u is a solution of (0.1) on $[a,b]$. Moreover, $\widetilde{w}$ satisfies (1.2) or (1.5) for all $y \in \mathbb{R}$. Hence by Lemma 1.1 we get (1.3). $\square$

3.2. Lemma. *Let the assumptions of Lemma* 3.1 *be fulfilled. Then for any* $\mu > 0$ *the couples* $(\sigma_1 - \mu, \rho_1)$ *and* $(\sigma_2 + \mu, \rho_2)$ *are respectively strict lower and upper functions to the problem* (3.4).

Proof. Let (σ_1, ρ_1) and (σ_2, ρ_2) be respectively lower and upper functions to the problem (0.1), (0.2) such that (2.3) is true. Let an arbitrary $\mu > 0$ be given and let us define

$$\widetilde{\sigma}_2(t) = \sigma_2(t) + \mu \text{ on } [a,b].$$

Obviously, the couple $(\widetilde{\sigma}_2, \rho_2)$ satisfies the boundary conditions (1.18). Further, making use of (1.15) and (3.2), we get for a.e. $t \in [a,b]$

$$\begin{aligned}\widetilde{\sigma}_2(t) + h(t, \widetilde{\sigma}_2(t), \rho_2(t)) &= \mu + f(t, \sigma_2(t), \rho_2(t)) + \omega_2(t, \frac{\mu}{\mu+1}) \\ &> f(t, \sigma_2(t), \rho_2(t)) \geq \rho_2'(t).\end{aligned}$$

This means that $(\widetilde{\sigma}_2, \rho_2)$ are upper functions to (3.4) and $\widetilde{\sigma}_2$ is not a solution of (3.4).

Now, let us put $\varepsilon = \frac{\frac{\mu}{2}}{\frac{\mu}{2}+1}$. Since $\varepsilon < \frac{\mu}{2}$, for any $t \in [a,b]$ and any couple $(x,y) \in \mathbb{R}^2$ such that

$$(3.5) \qquad |x - \widetilde{\sigma}_2(t)| < \varepsilon \quad \text{and} \quad |y - \rho_2(t)| < \varepsilon$$

we obtain $x - \sigma_2(t) > \frac{\mu}{2}$ and $|y - \rho_2(t)| < \dfrac{x - \sigma_2(t)}{x - \sigma_2(t) + 1}$ and hence also

$$\omega_2(t, |y - \rho_2(t)|) \leq \omega_2(t, \frac{x - \sigma_2(t)}{x - \sigma_2(t) + 1}).$$

Consequently, for a.e. $t \in [a,b]$ and all $(x,y) \in \mathbb{R}^2$ fulfilling (3.5) we can compute

$$\begin{aligned}&x + h(t,x,y) \\ &\quad \geq x - \sigma_2(t) + \omega_2(t, \frac{x - \sigma_2(t)}{x - \sigma_2(t) + 1}) - \omega_2(t, |y - \rho_2(t)|) + f(t, \sigma_2(t), \rho_2(t)) \\ &\quad > f(t, \sigma_2(t), \rho_2(t)) \geq \rho_2'(t),\end{aligned}$$

i.e. the functions $(\widetilde{\sigma}_2, \rho_2)$ are strict upper functions to the problem (3.4). Analogously we could show that for any $\mu > 0$ the functions $(\sigma_1 - \mu, \rho_1)$ are strict lower functions of (3.4). $\square$

3.3. Theorem. *Let the assumptions of Theorem* 2.4 *be fulfilled, but with* (σ_1, ρ_1) *and* (σ_2, ρ_2) *not necessarily strict. Then either the problem* (0.1), (0.2) *has a solution which belongs to* $\partial\Omega_1$ *or*

$$\deg(\mathrm{I} - \mathrm{L}^+\mathrm{F}, \Omega_1) = 1. \tag{3.6}$$

Proof. Let (σ_1, ρ_1) and (σ_2, ρ_2) be respectively lower and upper functions to the problem (0.1), (0.2) fulfilling the relation (2.3). Let us choose an arbitrary $\mu > 0$. By Lemma 3.2 the couples $(\sigma_1 - \mu, \rho_1)$ and $(\sigma_2 + \mu, \rho_2)$ are respectively strict lower and upper functions to the modified problem (3.4). It means that by Theorem 2.4

$$\deg(\mathrm{I} - \mathrm{L}^+\mathrm{H}, \Omega_\mu) = 1,$$

where $\mathrm{H}: \, x \in \mathbb{C}^1[a,b] \mapsto \mathrm{H}x \in \mathbb{L}[a,b] \times \mathbb{R}^2$,

$$(\mathrm{H}x)(t) = (h(t, x(t), x'(t)), 0, \widetilde{w}(x'(b))) \text{ a.e. in } [a,b],$$

$$\Omega_\mu = \left\{ x \in \mathbb{C}^1[a,b] : \sigma_1(t) - \mu < x(t) < \sigma_2(t) + \mu \text{ on } [a,b] \text{ and } \|x'\|_{\mathbb{C}} < \|\widetilde{m}\|_{\mathbb{L}} \right\}$$

and either $\widetilde{m}(t) = m(t) - \mu - \omega_1(t,1)$ or $\widetilde{m}(t) = m(t) + \mu + \omega_2(t,1)$ (according to whether we assume (1.1), (1.2) or (1.4), (1.5)). On the other hand, by Lemma 3.1 the problem (3.4) does not possess any solution in $\Omega_\mu \setminus \mathrm{cl}(\Omega_1)$. Moreover, $\mathrm{H} = \mathrm{F}$ on $\mathrm{cl}(\Omega_1)$ and so if the problem (0.1), (0.2) has no solution belonging to $\partial\Omega_1$, the modified problem (3.4) has no solution belonging to $\partial\Omega_1$, either. Therefore, by the excision property of the degree we have (3.6). □

In the case that σ_1 and σ_2 fulfil the relation (2.16) or (2.21), making use of Theorem 3.3 we can modify the proofs of Theorems 2.5 and 2.6 in such a way that we get the following assertions.

3.4. Theorem. *Let the assumptions of Theorem* 2.5 *be fulfilled, but with* (σ_1, ρ_1) *and* (σ_2, ρ_2) *not necessarily strict. Then either the problem* (0.1), (0.2) *has a solution which belongs to* $\partial\Omega_2$ *or*

$$\deg(\mathrm{I} - \mathrm{L}^+\mathrm{F}, \Omega_2) = -1.$$

Proof. Let (σ_1, ρ_1) and (σ_2, ρ_2) be respectively lower and upper functions to the problem (0.1), (0.2) and let m, A, $\widetilde{A}$, g, G, (σ_3, ρ_3), (σ_4, ρ_4), Ω, Δ_1, Δ_2 and Δ have the same meaning as in the proof of Theorem 2.5. The couples (σ_1, ρ_1)

and (σ_2, ρ_2) are respectively lower and upper functions to the problem (2.18), (0.2) which need not be strict now. By Theorem 2.4 we have again

$$\deg(\mathrm{I} - \mathrm{L}^+\mathrm{G}, \Omega) = 1.$$

Let u be a solution of (0.1), (0.2) such that $u \in \partial\Omega_2$. Then u is also a solution to (2.18), (0.2). Moreover, as in the proof of Theorem 2.5, making use of (2.20) and of Lemma 1.1 we can show that

$$\|u\|_{\mathbb{C}} < A \quad \text{and} \quad \|u'\|_{\mathbb{C}} < \|m\|_{\mathbb{L}}. \tag{3.7}$$

Thus, there exist $i \in \{1, 2\}$ and $t_u \in [a, b]$ such that

$$u(t_u) = \sigma_i(t_u), \tag{3.8}$$

i.e. $u \in \partial\Delta_i$.

On the other hand, let u be a solution of (2.18), (0.2) such that $u \in \partial\Delta_1 \cup \partial\Delta_2$. By Lemma 2.2 we have $-(\widetilde{A} + 2) < u(t) < \widetilde{A} + 2$ on $[a, b]$. Furthermore, by (2.19) and Lemma 1.1 we get $\|u'\|_{\mathbb{C}} < \|m\|_{\mathbb{L}} + 1$. As in the proof of Theorem 2.5 this implies by (2.20) that $\|u\|_{\mathbb{C}} < \widetilde{A}$, i.e. u is a solution of (0.1), (0.2). Now, using (2.20) and Lemma 1.1 once more we obtain again (3.7) and (3.8), i.e. $u \in \partial\Omega_2$.

To summarize, (0.1), (0.2) possesses a solution belonging to $\partial\Omega_2$ if and only if (2.18), (0.2) possesses a solution belonging to $\partial\Delta_1 \cup \partial\Delta_2$.

Consequently, if the problem (0.1), (0.2) possesses no solution u such that $u \in \partial\Omega_2$, then making use of Theorem 3.3 we get

$$\deg(\mathrm{I} - \mathrm{L}^+\mathrm{G}, \Delta_1) = 1 \quad \text{and} \quad \deg(\mathrm{I} - \mathrm{L}^+\mathrm{G}, \Delta_2) = 1.$$

Finally, by the same argument as in the proof of Theorem 2.5 we can show that any solution $u \in \Delta$ of the problem (2.18), (0.2) belongs to Ω_2. Therefore

$$\begin{aligned}&\deg(\mathrm{I} - \mathrm{L}^+\mathrm{G}, \Omega_2)\\ &\qquad = \deg(\mathrm{I} - \mathrm{L}^+\mathrm{G}, \Omega) - \deg(\mathrm{I} - \mathrm{L}^+\mathrm{G}, \Delta_1) - \deg(\mathrm{I} - \mathrm{L}^+\mathrm{G}, \Delta_2) = -1\end{aligned}$$

and taking into account that $\mathrm{F} = \mathrm{G}$ on $\mathrm{cl}(\Omega_2)$, we complete the proof. □

3.5. Theorem. *Let the assumptions of Theorem* 2.6 *be fulfilled, but with* (σ_1, ρ_1) *and* (σ_2, ρ_2) *not necessarily strict. Then either the problem* (0.1), (0.2) *has a solution which belongs to* $\partial\Omega_3$ *or*

$$\deg(\mathrm{I} - \mathrm{L}^+\mathrm{F}, \Omega_3) = -1.$$

Proof follows from Theorem 3.3 by a modification of the proof of Theorem 2.6 similar to that used in the proof of Theorem 3.4. □

4 . Existence theorems

Theorems 3.3 - 3.5 give directly existence results for our problem (0.1), (0.2). Similarly as in [7] (cf. Theorem 6) it is possible to show the existence of a solution to this problem even in the cases that the strict inequalities (2.3) and (2.16) are replaced by the non-strict ones.

4.1. Theorem. *Let the assumptions of Theorem* 2.4 *be satisfied but with* (σ_1, ρ_1) *and* (σ_2, ρ_2) *not necessarily strict and instead of* (2.3) *let us assume*

$$\sigma_1(t) \le \sigma_2(t) \ \text{ on } \ [a,b]. \tag{4.1}$$

Then the problem (0.1), (0.2) *possesses a solution* u *such that* $u \in \mathrm{cl}(\Omega_1)$ *(with* Ω_1 *given by* (2.13).

Proof. Consider an auxiliary problem

$$u'' = \widetilde{f}(t,u,u'), \ (0.2), \tag{4.2}$$

where $\widetilde{f}$ is for a.e. $t \in [a,b]$ and any $y \in \mathbb{R}$ given by

$$\widetilde{f}(t,x,y) = \begin{cases} f(t,x,y) & \text{if } \ x \le \sigma_2(t), \\ f(t,\sigma_2(t),y) & \text{if } \ x > \sigma_2(t). \end{cases}$$

Clearly, (σ_1, ρ_1) are lower functions to (4.2). Now, let an arbitrary $k \in \mathbb{N}$ be given. The functions $(\sigma_2 + \frac{1}{k}, \rho_2)$ are then upper functions to (4.2) and by Theorem 3.3 the problem (4.2) possesses a solution x_k such that

$$x_k(t) \in [\sigma_1(t), \sigma_2(t) + \tfrac{1}{k}] \quad \text{on} \quad [a,b] \quad \text{and} \quad \|x_k'\|_{\mathbb{C}} \le \|m\|_{\mathbb{L}}.$$

Using the Arzelà-Ascoli theorem and the Lebesgue Dominated Convergence Theorem for the sequence $\{x_k\}$ we get a solution $x \in \mathrm{cl}(\Omega_1)$ of (0.1), (0.2) as a $\mathbb{C}^1$-limit of a proper subsequence of $\{x_k\}$. $\square$

4.2. Theorem. *Let the assumptions of Theorem* 2.5 *be satisfied but with* (σ_1, ρ_1) *and* (σ_2, ρ_2) *not necessarily strict and instead of* (2.16) *let us assume*

$$\sigma_2(t) \le \sigma_1(t) \ \text{ on } \ [a,b]. \tag{4.3}$$

Then the problem (0.1), (0.2) *possesses a solution* u *such that* $u \in \mathrm{cl}(\Omega_2)$ *(with* Ω_2 *given in Theorem* 2.5*).*

Proof. For any $k \in \mathbb{N}$, a.e. $t \in [a,b]$ and any $x, y \in \mathbb{R}$ put

$$g_k(t,x,y) = k\big(f(t,\sigma_2(t),y) - f(t,x,y)\big)\big(x - (\sigma_2(t) - \tfrac{1}{k})\big)$$

and

$$\widetilde{f}_k(t,x,y) = \begin{cases} f(t,x,y) & \text{if } \ x < \sigma_2(t) - \frac{2}{k}, \\ f(t,\sigma_2(t),y) + g_k(t,x,y) & \text{if } \ x \in [\sigma_2 - \frac{2}{k}, \sigma_2(t) - \frac{1}{k}), \\ f(t,\sigma_2(t),y) & \text{if } \ x \in [\sigma_2 - \frac{1}{k}, \sigma_2(t)), \\ f(t,x,y) & \text{if } \ x \ge \sigma_2(t). \end{cases}$$

The couples (σ_1, ρ_1) and $(\sigma_2 - \frac{1}{k}, \rho_2)$ are then respectively lower and upper functions to

$$u'' = \widetilde{f}_k(t,u,u'), (0.2) \tag{4.4}$$

and satisfy (2.16). It is easy to verify that for any $k \in \mathbb{N}$ the function $\widetilde{f}_k$ satisfies the assumptions for f of Theorem 2.5 with the same $m \in \mathbb{L}[a,b]$. Thus by Theorem 3.4 for any $k \in \mathbb{N}$ there are a solution x_k to the problem (4.4) and a point $s_k \in [a,b]$ such that

$$\|x_k\|_{\mathbb{C}} \le A + \tfrac{1}{k}, \quad \|x_k'\|_{\mathbb{C}} \le \|m\|_{\mathbb{L}} \quad \text{and} \quad \sigma_2(s_k) - \tfrac{1}{k} \le x_k(s_k) \le \sigma_1(s_k),$$

where A has the same meaning as in Theorem 2.5. Using the compactness of the interval $[a,b]$ and the Arzelà-Ascoli theorem we get the existence of a subsequence $\{x_{k_\ell}\}$ in $\{x_k\}$, $s^* \in [a,b]$ and $x \in \mathbb{C}^1[a,b]$ such that

$$\lim_{\ell\to\infty} \|x_{k_\ell} - x\|_{\mathbb{C}^1} = 0 \quad \text{and} \quad \lim_{\ell\to\infty} s_{k_\ell} = s^*.$$

Obviously, $x \in \mathrm{cl}(\Omega_2)$ and by virtue of the Lebesgue Dominated Convergence Theorem, x is a solution of (0.1), (0.2). □

4.3. Theorem. *Let the assumptions of Theorem* 2.6 *be satisfied but with* (σ_1, ρ_1) *and* (σ_2, ρ_2) *not necessarily strict. Then the problem* (0.1), (0.2) *possesses a solution* u *such that* $u \in \mathrm{cl}(\Omega_3)$ *(with* Ω_3 *given in Theorem* 2.6*).*

Proof. If σ_1 and σ_2 satisfy neither (4.1) nor (4.2), they fulfil (2.21) and hence by Theorem 3.5 we have a solution $u \in \mathrm{cl}(\Omega_3)$ to (0.1), (0.2). □

4.4. Corollary. *Let* $z_1,\ z_2 \in \mathbb{C}[a,b]$,

$$m_1 = \max_{t\in[a,b]} z_1(t) < m_2 = \min_{t\in[a,b]} z_2(t) \tag{4.5}$$

and let for a.e. $t \in [a,b]$ *and all* $x, y \in \mathbb{R}$

$$f(t,x,y) < 0 \quad \text{if} \;\; x \in (z_1(t), z_2(t)) \tag{4.6}$$

and

$$f(t,x,y) > 0 \quad \text{if} \;\; x < z_1(t) \;\; \text{or} \;\; x > z_2(t). \tag{4.7}$$

Further, let (1.1) *be satisfied with* $m \in \mathbb{L}[a,b]$ *and* $\mathscr{U}(t) = [z_1(t), z_2(t)]$, $t \in [a,b]$. *Then*

(i) there are at least two different solutions u *and* v *to the periodic boundary value problem*

$$u'' = f(t,u,u'), \quad u(a) = u(b), \quad u'(a) = u'(b) \tag{4.8}$$

such that

$$v(t_v) \le m_1 \;\; \text{for some} \;\; t_v \in [a,b] \tag{4.9}$$

and

$$\max\{m_2, v(t)\} \le u(t) \;\; \text{on} \;\; [a,b]; \tag{4.10}$$

(ii) if we suppose in addition that for any compact $K \subset [m_2,\infty) \times \mathbb{R}$ *there is a nonnegative function* $h_k \in \mathbb{L}[a,b]$ *such that*

$$\begin{aligned} &f(t,x_1,y_1) - f(t,x_2,y_2) > -h_k(t)|y_1 - y_2| \\ &\quad \text{for a.e.} \;\; t \in [a,b] \text{ and all } \;(x_1,y_1),(x_2,y_2) \in K \text{ such that } \; x_1 > x_2, \end{aligned} \tag{4.11}$$

then u *is the only solution of* (4.8) *bounded below by* m_2.

Proof. (i) Without any loss of generality we may assume that $m(t) \le 0$ a.e. on $[a,b]$, i.e. we have

$$f(t,x,y) \ge m(t) \quad \text{for a.e.} \;\; t \in [a,b] \text{ and all } \;(x,y) \in \mathbb{R}^2.$$

Furthermore, by (4.5) there are r_1, r_2, such that

$$r_1 < \min_{t\in[a,b]} z_1(t) \le m_1 < m_2 \le \max_{t\in[a,b]} z_2(t) < r_2.$$

According to (4.6) the couples $(m_1, 0)$ and $(m_2, 0)$ are lower functions of (4.8) and by (4.7) the couples $(r_1, 0)$ and $(r_2, 0)$ are upper functions of (4.8). Hence, by Theorems 4.1 and 4.2 there are solutions v and v_1 of (4.8) such that

$$r_1 < v(t_v) < m_1 \text{ for some } \; t_v \in (a,b) \text{ and } \; m_2 \le v_1(t) \le r_2 \; \text{ for all } \; t \in [a,b].$$

Suppose that v and v_1 are not ordered on $[a,b]$, i.e. there is s_v such that $v_1(s_v) < v(s_v)$, and set

$$\sigma_1(t) = \max\{v(t), v_1(t)\} \quad \text{for } \; t \in [a,b]. \tag{4.12}$$

Then $\sigma_1 \in \mathbb{AC}[a,b]$, $\sigma_1' \in \mathbb{BV}[a,b]$, σ_1 is not a solution of (4.8) but the functions (σ_1, σ_1') are lower functions of (4.8). According to (4.7) we can find a number $r^* > \|\sigma_1\|_{\mathbb{C}}$ such that $(r^*, 0)$ are upper functions of (4.8). This implies the existence of a solution u of (4.8) satisfying $\sigma_1(t) \le u(t) \le r^*$ on $[a,b]$. Provided v and v_1 are ordered, we set $u = v_1$.

(ii) Suppose (4.11) and let $u_1 \ne u$ be a solution of (4.8) such that $m_2 \le u_1(t)$ on $[a,b]$. Set $z(t) = u_1(t) - u(t)$ and choose a compact K such that $(u(t), u'(t)) \in K$ and $(u_1(t), u_1'(t)) \in K$ for all $t \in [a,b]$. We can assume that $\max_{t\in[a,b]} z(t) = z(t_0) > 0$ and $z'(t_0) = 0$ for some $t_0 \in [a,b)$. Then there exists $t^* > t_0$ such that $z'(t^*) \le 0$ and $z(t) > 0$ on $[t_0, t^*]$. Now, (4.11) implies

$$z''(t) > -h_k(t)|z'(t)| = -\big(h_k(t)\,\mathrm{sgn}(z'(t))\big)z'(t) \quad \text{for a.e. } \; t \in [t_0, t^*].$$

Thus,

$$\Big(z'(t)\exp\Big(\int_{t_0}^{t}\big(h_k(s)\,\mathrm{sgn}(z'(s))\big)\,\mathrm{d}s\Big)\Big)' > 0 \text{ on } \; [t_0, t^*]$$

and

$$z'(t^*)\exp\Big(\int_{t_0}^{t^*}\big(h_k(s)\,\mathrm{sgn}(z'(s))\big)\,\mathrm{d}s\Big) > z'(t_0) = 0,$$

a contradiction. □

4.5. Remark. Provided z_i is a constant function for some $i \in \{1,2\}$, it is a solution of (4.8). In this case we can set $v(t) = z_i(t)$. If z_1 is not constant, then there exists $s_v \in [a,b]$ such that $v(s_v) > z_1(s_v)$. Similarly, if z_2 is not constant, we get $u(t_u) < z_2(t_u)$ for some $t_u \in [a,b]$. These observations follow from the fact that any solution of (4.8) cannot have all its values outside $(z_1(t), z_2(t))$.

4.6. Remark. In the case that $f(t,x,y) \equiv g(t,x)$ the assertion (i) of Corollary 4.4 is fulfilled under the assumptions (4.5), (4.6) and (4.7). Thus our Corollary 4.4 generalizes Theorem 4.7 from [3]. Further, the assertion (ii) of Corollary 4.4 is true provided g is increasing in x on $[m_2, \infty)$ for a.e. $t \in [a,b]$.

4.7. Remark. The lower and upper functions method which is described in this section (cf. Theorems 4.1 - 4.3 and Corollary 4.4) can be used for singular boundary value problems, as well. For multiplicity results for periodic boundary value problems which were obtained by this method, see [8].

4.8. Remark. Conditions ensuring the existence of constant lower and upper functions of the problem (0.1), (0.2) were mentioned in Remark 1.8. In the proof of Corollary 4.4 we constructed nonconstant lower functions whose first component was the maximum of two solutions of the problem (4.8) (cf. (4.12)). In general, it is not easy to find conditions which guarantee the existence of nonconstant lower and upper functions. One of the possibilities is shown in [9] where they are constructed as solutions of linear boundary value problems for generalized linear differential equations.

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