Second Order Periodic Problem with $\phi$-Laplacian and Impulses

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Abstract. Existence principles for the BVP $(\phi(u'))' = f(t,u,u')$, $u(t_i+) = J_i(u(t_i))$, $u'(t_i+) = M_i(u'(t_i))$, $i = 1, 2, \ldots, m$, $u(0) = u(T)$, $u'(0) = u'(T)$ are presented. They are based on the method of lower/upper functions which need not be well-ordered.

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1. Formulation of the problem

Let $m \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$ and $D = \{t_1, t_2, \ldots, t_m\}$. Define $C_D$ (or $C_D^1$) as the sets of functions $u : [0,T] \to \mathbb{R}$,

$$u(t) = \begin{cases} u_{[0]}(t) & \text{if } t \in [0,t_1], \\ u_{[1]}(t) & \text{if } t \in (t_1,t_2), \\ \vdots & \vdots \\ u_{[m]}(t) & \text{if } t \in (t_m,T], \end{cases}$$

where $u_{[i]}$ is continuous on $[t_i, t_{i+1}]$ (or continuously differentiable on $[t_i, t_{i+1}]$) for $i = 0, 1, \ldots, m$. We put $\|u\|_D = \|u\|_\infty + \|u'\|_\infty$, where $\|u\|_\infty = \sup_{t \in [0,T]} |u(t)|$.

Then $C_D$ and $C_D^1$ respectively with the norms $\|\cdot\|_\infty$ and $\|\cdot\|_D$ become Banach spaces. Further, $AC_D$ is the set of functions $u \in C_D$ which are absolutely continuous on each subinterval $(t_i, t_{i+1})$, $i = 0, 1, \ldots, m$.

We consider the problem

\begin{align*}
(1.1) & \quad (\phi(u'(t)))' = f(t,u(t),u'(t)) \quad \text{a.e. on } [0,T], \\
(1.2) & \quad u(t_i+) = J_i(u(t_i)), \quad u'(t_i+) = M_i(u'(t_i)), \quad i = 1, 2, \ldots, m, \\
(1.3) & \quad u(0) = u(T), \quad u'(0) = u'(T),
\end{align*}

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  u'(t_i) = u'(t_i-) = \lim_{t \to t_i^-} u'(t) \quad \text{for } i = 1, 2, \ldots, m + 1, \quad u'(0) = u'(0+) = \lim_{t \to 0^+} u'(t), \]

where \( f \) is an \( L^1 \)-Carathéodory function, functions \( J_i, M_i \) are continuous on \( \mathbb{R} \) and \( \phi \) is an increasing homeomorphism such that \( \phi(0) = 0 \) and \( \phi(\mathbb{R}) = \mathbb{R} \). A typical example of a proper function \( \phi \) is the \( p \)-Laplacian \( \phi_p(y) = |y|^{p-2}y \), where \( p > 1 \).

A solution of the problem (1.1)–(1.3) is a function \( u \in C^1 \) such that \( \phi(u') \in AC \) and (1.1)–(1.3) hold.

A function \( \sigma \in C^1 \) is called a lower function of (1.1)–(1.3) if \( \phi(\sigma') \in AC \) and

\[
  \begin{align*}
  \phi(\sigma'(t))' &\geq f(t, \sigma(t), \sigma'(t)) \quad \text{for a.e. } t \in [0, T], \\
  \sigma(t_i+) = J_i(\sigma(t_i)), \quad \sigma'(t_i+) &\geq M_i(\sigma'(t_i)), \quad i = 1, 2, \ldots, m, \\
  \sigma(0) = \sigma(T), \quad \sigma'(0) &\geq \sigma'(T).
  \end{align*}
\]

Similarly, a function \( \sigma \in C^1 \) with \( \phi(\sigma') \in AC \) is an upper function of (1.1)–(1.3) if it satisfies the relations (1.4) but with reversed inequalities.

The aim of this paper is to offer existence principles for problem (1.1)–(1.3) in terms of lower/upper functions. Hence our basic assumption is the existence of lower/upper functions. We will suppose that either

\[
  \text{(1.5) } \sigma_1 \quad \text{and} \quad \sigma_2 \quad \text{are respectively lower and upper functions of (1.1)–(1.3)}
\]

such that \( \sigma_1 \leq \sigma_2 \) on \([0, T]\)

or

\[
  \text{(1.6) } \sigma_1 \quad \text{and} \quad \sigma_2 \quad \text{are respectively lower and upper functions of (1.1)–(1.3)}
\]

such that \( \sigma_1 \not\leq \sigma_2 \) on \([0, T]\), i.e. \( \sigma_1(\tau) > \sigma_2(\tau) \) for some \( \tau \in [0, T] \).

Note that problems with \( \phi \)-Laplacians and impulses have not been studied yet. As concerns problem (1.1), (1.3) (without impulses), there are various results about its solvability. For example the papers [4] and [19] present some results about the existence or multiplicity of periodic solutions of the equation

\[
  (\phi_p(u'))' = f(t, u)
\]

under non resonance conditions imposed on \( f \). The paper [10] presents general existence principles for the vector problem (1.1), (1.3). Using this the authors provide various existence theorems and illustrative examples. The vector case is also considered in [9], [11] and [12]. The existence of periodic solutions of the Liénard type equations with \( p \)-Laplacians has been proved in the resonance case under the Landesman-Lazer conditions in [5] and [6]. Multiplicity results of the Ambrosetti-Prodi type for this problem (with a real parameter) can be found in [8].

The papers which are devoted to the lower/upper functions method for the problem (1.1), (1.3) mostly deal with the condition (1.5), i.e. they assume well-ordered
σ₁/σ₂. We can refer to the papers [1] and [3] which study the problem (1.1), (1.3) under the Nagumo type two-sided growth conditions and to the paper [17] where the second order equation with a φ-Laplacian is considered provided a functional right-hand side of this equation fulfills one-sided growth conditions of the Nagumo type. The significance of the lower/upper functions method is shown in the papers [7] and [18] where this method is used in the investigation of singular periodic problems with a φ-Laplacian. The paper [2] is, to our knowledge, the only one presenting the lower/upper functions method for the problem (1.7), (1.3) with a φ-Laplacian under the assumption that σ₁ ≥ σ₂, i.e., lower/upper functions are in the reverse order. If φ = φ_p the authors get the solvability of (1.7), (1.3) for 1 < p ≤ 2, only. Therefore, the existence principles (Theorems 2.3 and 2.4) which we state here for the impulsive problem (1.1)−(1.3) and the case (1.6) are new even for the non-impulsive problem (1.1), (1.3).

We will work with the following assumptions, where the sets A_i, B(t) ⊂ ℝ, t ∈ [0, T], will be determined later, according to whether (1.5) or (1.6) is assumed:

\begin{align}
(1.8) & \quad \begin{cases} 
  x > \sigma_1(t_i) \implies J_i(x) > J_i(\sigma_1(t_i)) \\
  x < \sigma_2(t_i) \implies J_i(x) < J_i(\sigma_2(t_i))
\end{cases} \quad \text{for } x \in A_i, \ i = 1, 2, \ldots, m; \\
(1.9) & \quad \begin{cases} 
  y \leq \sigma_1'(t_i) \implies M_i(y) \leq M_i(\sigma_1'(t_i)), \\
  y \geq \sigma_2'(t_i) \implies M_i(y) \geq M_i(\sigma_2'(t_i)),
\end{cases} \quad i = 1, 2, \ldots, m; \\
(1.10) & \quad \begin{cases} 
  \text{there is } h \in L_1 \text{ such that } \\
  |f(t, x, y)| \leq h(t) \quad \text{for a.e. } t \in [0, T] \text{ and all } x, y \in \mathbb{R}; \\
\int_0^\infty \frac{d s}{\omega(s)} = \infty \text{ and } |f(t, x, y)| \leq \omega(\phi(|y|)) (|y| + h(t)) \\
  \text{for a.e. } t \in [0, T], \text{ all } x \in B(t) \text{ and } |y| \geq 1,
\end{cases}
\end{align}

\begin{align}
(1.11) & \quad \begin{cases} 
  \text{there are } \omega : [0, \infty) \mapsto (0, \infty) \text{ continuous and } h \in L_1 \text{ such that } \\
  \int_0^{\infty} \frac{d s}{\omega(s)} = \infty \text{ and } |f(t, x, y)| \leq \omega(\phi(|y|)) (|y| + h(t)) \\
  \text{for a.e. } t \in [0, T], \text{ all } x \in B(t) \text{ and } |y| \geq 1,
\end{cases}
\end{align}

\begin{align}
(1.12) & \quad \begin{cases} 
  \text{there are } c_j, d_j \in \mathbb{R}, \ c_j \leq \sigma_k'(t_j) \leq d_j \text{ on } (t_{j-1}, t_j), \ k = 1, 2, \\
  \text{such that } f(t, x, c_j) \leq 0, f(t, x, d_j) \geq 0 \text{ for a.e. } t \in (t_{j-1}, t_j) \\
  \text{and all } x \in B(t), \ j = 1, 2, \ldots, m + 1, \text{ and } c_1 \geq c_{m+1}, d_1 \leq d_{m+1}, \\
  M_i(c_i) \leq c_{i+1}, \ M_i(d_i) \geq d_{i+1}, \ i = 1, 2, \ldots, m.
\end{cases}
\end{align}

2. Main results

Below we formulate our main results:

I. Existence Principles for Well-Ordered Case

2.1 Theorem. Assume that (1.5), (1.8) with A_i = [σ_1(t_i), σ_2(t_i)], i = 1, 2, \ldots, m,
(1.9) and (1.11) with B(t) = [σ_1(t), σ_2(t)] hold. Then the problem (1.1) − (1.3) has
a solution \( u \) satisfying

\[ \sigma_1 \leq u \leq \sigma_2 \text{ on } [0,T]. \]

2.2 Theorem. Assume that (1.5), (1.8) with \( A_i = [\sigma_1(t_i), \sigma_2(t_i)], i = 1, 2, \ldots, m, \)
(1.9) and (1.12) with \( B(t) = [\sigma_1(t), \sigma_2(t)] \) hold.
Then the problem (1.1)–(1.3) has a solution \( u \) satisfying (2.1) and

\[ c_j \leq u(t) \leq d_j \text{ for } t \in (t_{j-1}, t_j], j = 1, 2, \ldots, m + 1. \]

II. Existence principles for non-ordered case

2.3 Theorem. Assume that (1.6), (1.8) with \( A_i = \mathbb{R}, i = 1, 2, \ldots, m, \)
(1.9) and (1.10) hold. Then the problem (1.1)–(1.3) has a solution \( u \) satisfying

\[ |u(t_u)| \leq \max\{|\sigma_1(t_u)|, |\sigma_2(t_u)|\} \text{ for some } t_u \in [0,T]. \]

2.4 Theorem. Assume that (1.6), (1.8) with \( A_i = \mathbb{R}, i = 1, 2, \ldots, m, \)
(1.9) and (1.12) with \( B(t) = \mathbb{R} \) hold. Then the problem (1.1)–(1.3) has a solution \( u \) satisfying
(2.2) and (2.3).

Note that Theorems 2.2 and 2.4 impose no growth restrictions on \( f \). For example, taking \( f(t, x, y) = y(y^{2k}x^{2n}+1) - x^{2n-1} + e(t) \), where \( e \in \mathcal{C}_D, k, n \in \mathbb{N} \), we can check that there are \( c_j \in (-\infty, 0) \) \( d_j \in (0, \infty), j = 1, 2, \ldots, m + 1 \), such that \( c_1 \geq c_{m+1}, d_1 \leq d_{m+1}, f(t, x, c_j) \leq 0 \) and \( f(t, x, d_j) \geq 0 \) for a.e. \( t \in (t_{j-1}, t_j] \) and all \( x \in \mathbb{R}, j = 1, 2, \ldots, m + 1 \).

3. A fixed point operator

We will transform the problem (1.1)–(1.3) into a fixed point problem in \( \mathcal{C}_D \). First, we borrow some ideas from [10] to get the following two lemmas.

3.1 Lemma. For each \( \ell \in \mathcal{C}_D \) and \( d \in \mathbb{R}, \) the function

\[ \Psi_{\ell,d}: \mathbb{R} \mapsto \mathbb{R}, \quad \Psi_{\ell,d}(a) = d + \int_0^T \phi^{-1}\left(a + \ell(t)\right) \, dt \]

has exactly one zero point \( a(\ell, d) \) in \( \mathbb{R} \).

Proof. Choose \( \ell \in \mathcal{C}_D \) and \( d \in \mathbb{R} \). Since \( \Psi_{\ell,d} \) is continuous, increasing on \( \mathbb{R} \) and \( \Psi_{\ell,d}(\mathbb{R}) = \mathbb{R} \), there is a unique real number \( a(\ell, d) \) such that

\[ \Psi_{\ell,d}(a(\ell, d)) = 0. \]
3.2 Lemma. The mapping \( a : \mathbb{C}_D \times \mathbb{R} \mapsto \mathbb{R} \) defined by (3.1) is continuous and maps bounded sets into bounded sets.  

Proof. (i) Assume that \( A \subset \mathbb{C}_D \times \mathbb{R} \) and \( \gamma \in (0, \infty) \) are such that \( \| \ell \|_{\infty} + |d| \leq \gamma \) for each \( (\ell, d) \in A \) and that there is a sequence \( \{a(\ell_n, d_n)\}_{n=1}^{\infty} \subset a(A) \) such that \( \lim_{n \to \infty} a(\ell_n, d_n) = \infty \) or \( \lim_{n \to \infty} a(\ell_n, d_n) = -\infty \). Let the former possibility occur. Then, by (3.1), we have \( 0 = \lim_{n \to \infty} \Psi_{\ell_n, d_n}(a(\ell_n, d_n)) \geq \lim_{n \to \infty} (-\gamma + T\phi^{-1}(a(\ell_n, d_n) - \gamma)) = \infty \), a contradiction. The latter possibility can be argued similarly.

(ii) Let \( \lim_{n \to \infty}(\ell_n, d_n) = (\ell_0, d_0) \) in \( \mathbb{C}_D \times \mathbb{R} \). By (i) the sequence \( \{a(\ell_n, d_n)\}_{n=1}^{\infty} \) is bounded and hence we can choose a subsequence such that \( \lim_{n \to \infty} a(\ell_{k_n}, d_{k_n}) = a_0 \in \mathbb{R} \). By (3.1), we get

\[
0 = \Psi_{\ell_{k_n}, d_{k_n}}(a(\ell_{k_n}, d_{k_n})) = d_{k_n} + \int_0^T \phi^{-1}(a(\ell_{k_n}, d_{k_n}) + \ell_{k_n}(t)) \, dt,
\]

which, for \( n \to \infty \), yields

\[
0 = d_0 + \int_0^T \phi^{-1}(a_0 + \ell_0(t)) \, dt.
\]

Thus, with respect to Lemma 3.1, we have \( a_0 = a(\ell_0, d_0) = \lim_{n \to \infty} a(\ell_n, d_n) \). \( \square \)

3.3 Lemma. The operator \( \mathcal{N} : \mathbb{C}_D \mapsto \mathbb{C}_D \) given by

\[
(3.2) \quad (\mathcal{N}(x))(t) = \int_0^t f(s, x(s), x'(s)) \, ds + \sum_{i=1}^m [\phi(M_i(x'(t_i))) - \phi(x'(t_i))] \chi(t_i, T)(t)
\]

is absolutely continuous.  

Proof. The continuity of \( \mathcal{N} \) follows from the continuity of all the mappings involved in the right-hand side of (3.2). Furthermore, let \( \mathcal{H} \subset \mathbb{C}_D \) be bounded. We need to show that the closure \( \overline{\mathcal{N}(\mathcal{H})} \) of \( \mathcal{N}(\mathcal{H}) \) in \( \mathbb{C}_D \) is compact. To this aim, let \( \|x\|_D \leq \gamma < \infty \) for each \( x \in \mathcal{H} \). Then there are \( c \in (0, \infty) \) and \( h \in L^1 \) such that

\[
\sum_{i=1}^m [\phi(M_i(x'(t_i))) - \phi(x'(t_i))] \leq c \quad \text{and} \quad |f(t, x(t), x'(t))| \leq h(t) \quad \text{a.e. on } [0, T]
\]

for all \( x \in \mathcal{H} \). Therefore

\[
(3.3) \quad \|\mathcal{N}(x)\|_{\infty} \leq \|h\|_1 + c \quad \text{for each } x \in \mathcal{H}.
\]

\(^1\)The norm of \( (\ell, d) \in \mathbb{C}_D \times \mathbb{R} \) is defined by \( \|\ell\|_{\infty} + |d| \).

\(^2\)As usual, \( \chi_M \) stands for the characteristic function of the set \( M \subset \mathbb{R} \).
Put \((N_1(x))(t) = \int_0^t f(s, x(s), x'(s)) \, ds\). Then, for \(t_1, t_2 \in [0, T]\), we have
\[
|(N_1(x))(t_2) - (N_1(x))(t_1)| \leq \left| \int_{t_1}^{t_2} h(s) \, ds \right|
\]
wherefrom, by (3.3), we deduce that the functions in \(N_1(H)\) are uniformly bounded and equicontinuous on \([0, T]\). Hence, making use of the Arzelà-Ascoli Theorem in \(C\) (the space of functions continuous on \([0, T]\) with the norm \(\|\cdot\|_\infty\)), we get that each sequence in \(N_1(H)\) contains a subsequence convergent with respect to the norm \(\|\cdot\|_\infty\). This shows that \(N_1(H)\) is compact in \(C_D\). We know that the operator \(N_2 = N - N_1\) is continuous. By (3.3), it maps bounded sets into bounded sets. Moreover, its values are contained in an \(m\)-dimensional subspace of \(C_D\). Thus, \(N_2(H)\) is compact in \(C_D\).

3.4 Theorem. Let \(a : C_D \times R \rightarrow R\) and \(N : C_D \rightarrow C_D\) be respectively defined by (3.1) and (3.2). Furthermore define \(\mathcal{J} : C_D \rightarrow C_D\) by
\[
(\mathcal{J}(x))(t) = \sum_{i=1}^{m} \left[ J_i(x(t_i)) - x(t_i) \right] \chi(t_i, T)(t)
\]
and
\[
(\mathcal{F}(x))(t) = \int_0^t \phi^{-1} \left( a(N(x), (\mathcal{J}(x))(T)) + (N(x))(s) \right) \, ds + x(0) + x'(0) - x'(T) + (\mathcal{J}(x))(t).
\]
Then \(\mathcal{F} : C_D \rightarrow C_D\) is an absolutely continuous operator. Moreover, \(u\) is a solution of the problem (1.1) – (1.3) if and only if \(\mathcal{F}(u) = u\).

Proof. For \(x \in C_D\) and \(t \in [0, T]\), we have
\[
(\mathcal{F}(x))'(t) = \phi^{-1} \left( a(N(x), (\mathcal{J}(x))(T)) + (N(x))(t) \right).
\]
Since the mappings \(a, N\) and \(\mathcal{J}\) included in (3.5) and (3.6) are continuous, it follows that \(\mathcal{F}\) is continuous in \(C_D\).

Choose an arbitrary bounded set \(H \subset C_D\). We will show that then the set \(\mathcal{F}(H)\) is compact in \(C_D\). Let a sequence \(\{v_n\} \subset \mathcal{F}(H)\) be given. It suffices to show that it contains a subsequence convergent in \(C_D\). Let \(\{x_n\} \subset H\) be such that \(v_n = \mathcal{F}(x_n)\) for \(n \in N\). By Lemma 3.3, there is a subsequence \(\{x_{k_n}\}\) such that \(\{N(x_{k_n})\}\) is convergent in \(C_D\). According to (3.3) and (3.4), there exists \(\gamma \in (0, \infty)\) such that \(\|N(x)\|_\infty + \|(\mathcal{J}(x))(T)\| \leq \gamma\) for all \(x \in H\). Hence, by Lemma 3.2, the sequence \(\{a(N(x_{k_n}), (\mathcal{J}(x_{k_n}))(T))\} \subset R\) is bounded and we can choose a subsequence \(\{x_{l_n}\} \subset \{x_{k_n}\}\) such that \(\{a(N(x_{l_n}), (\mathcal{J}(x_{l_n}))(T)) + N(x_{l_n})\}\) is convergent in \(C_D\). Consequently, \(\{(\mathcal{F}(x_{l_n}))'\}\) and \(\{\mathcal{F}(x_{l_n})\}\) are convergent in \(C_D\), as well. Finally, by a direct computation, we check that (1.1)–(1.3) is equivalent to the problem \(u = \mathcal{F}(u)\).

For more details, see our preprint [15].
4. Proofs of the main results

Sketch of the proof of Theorem 2.1. We can modify the arguments and constructions of [13], where the case $\phi(y) \equiv y$ is considered. By virtue of Theorem 3.4, the problem (1.1)–(1.3) has a solution if and only if the operator $\mathcal{F}$ which is defined by (3.5) has a fixed point. To prove it we argue as follows: (i) we construct an auxiliary operator $\mathcal{F}$ and prove that its Leray-Schauder topological degree is nonzero and consequently $\mathcal{F}$ has a fixed point $u$; (ii) using the method of a priori estimates we show that $u$ is a fixed point of $\mathcal{F}$ satisfying (2.1). Since the realization of these ideas is quite close to the arguments of [13], we skip it. Detailed computation can be found in our preprint [15].

**Proof of Theorem 2.2.** Step 1. Define

\begin{equation}
\beta_j(y) = \begin{cases} 
   c_j & \text{for } y < c_j, \\
   y & \text{for } c_j \leq y \leq d_j, \\
   d_j & \text{for } y > d_j
\end{cases}
\end{equation}

\begin{equation}
\bar{f}(t, x, y) = f(t, x, \beta_j(y)) + \frac{y - \beta_j(y)}{|y - \beta_j(y)| + 1}
\end{equation}

for a.e. $t \in (t_{j-1}, t_j]$, $x, y \in \mathbb{R}$, $j = 1, 2, \ldots, m + 1$;

and

\begin{equation}
\bar{M}_i(y) = M_i(\beta_i(y)) + \frac{y - \beta_i(y)}{|y - \beta_i(y)| + 1}
\end{equation}

for $y \in \mathbb{R}$, $i = 1, 2, \ldots, m$.

Now, consider the auxiliary problem

\begin{equation}
(\phi(u'(t)))' = \bar{f}(t, u(t), u'(t)) \quad \text{a.e. on } [0, T];
\end{equation}

\begin{equation}
u(t_i+) = J_i(u(t_i)), \quad u'(t_i+) = \bar{M}_i(u'(t_i)), \quad i = 1, 2, \ldots, m,
\end{equation}

\begin{equation}
u(0) = u(T), \quad \beta_i(u'(0)) = u'(T).
\end{equation}

We see that $\bar{f}$ and $\bar{M}_i$ have the same properties as $f$ and $M_i$. In particular, $\bar{f}$ satisfies (1.11) with $\omega(s) \equiv 1$, $\bar{M}_i$ fulfills (1.9) and $\sigma_1/\sigma_2$ are lower/upper functions for (4.4)–(4.6). Since we work with (4.6) instead of (1.3), we have to replace the expression $x(0) + x'(0) - x'(T)$ in (3.5) by $x(0) + \beta_1(x'(0)) - x'(T)$. Then we get the existence of a solution $u$ of (4.4)–(4.6) satisfying (2.1) in the same way as in the proof of Theorem 2.1 for (1.1)–(1.3).

**Step 2.** Having the solution $u$ of (4.4)–(4.6), it remains to show that (2.2) is true.

(i) Let $j \in \{1, 2, \ldots, m + 1\}$ and $\xi \in [t_{j-1}, t_j)$ be such that

\begin{equation}
sup\{u'(t) : t \in [0, T]\} = u' (\xi+) > d_j.
\end{equation}
Then there is $\delta > 0$ such that $(\xi, \xi + \delta) \subset (t_{j-1}, t_j)$ and $u' > d_j$ on $(\xi, \xi + \delta)$. By (1.12),

$$(\phi(u'(t)))' = f(t, u(t), d_j) + \frac{u'(t) - d_j}{u'(t) - d_j + 1} > 0 \text{ for a.e. } t \in (\xi, \xi + \delta),$$

i.e. $\phi(u'(t)) > \phi(u'(\xi +))$ and so $u'(t) > u'(\xi +)$ for each $t \in (\xi, \xi + \delta)$, which contradicts (4.7).

(ii) Assume that

$$(4.8) \quad \sup\{u'(t) : t \in [0, T]\} = u'(t_j) > d_j \quad \text{for some } t_j \in D.$$  

If $j = m + 1$, i.e. $u'(T) > d_{m+1}$, then, by (1.12), we have also $u'(T) > d_1$. Since (4.1) and (4.6) imply $u'(T) \leq d_1$, we get a contradiction.

If $j < m + 1$, then

$$\bar{M}_j(u'(t_j)) = M_j(d_j) + \frac{u'(t) - d_j}{u'(t) - d_j + 1} > M_j(d_j) \geq d_{j+1},$$

so $u'(t_{j+}) > d_{j+1}$. By part (i) we know that $u'(t) = d_{j+1}$ cannot achieve a positive maximum inside $(t_j, t_{j+1})$. Consequently, we have $u'(t_{j+1}) > d_{j+1}$. Repeating this procedure we get $u'(T) > d_{m+1}$ and a contradiction as before.

We have proved that $u'(t) \leq d_j$ on $(t_{j-1}, t_j]$, $j = 1, 2, \ldots, m + 1$. The remaining inequalities in (2.2) can be derived analogously. Finally, since $u$ fulfills (2.2), $u$ is a solution of (1.1)–(1.3).

**Sketch of the proof of Theorem 2.3.** We borrow ideas of [14], where non-ordered lower/upper functions to periodic impulsive problem without $\phi$-Laplacian ($\phi(y) = y$) have been studied. Here, we define the operator $\mathcal{F}$ by (3.5). Then, according to $\mathcal{F}$, we construct auxiliary operators and compute their Leray–Schauder degrees by a similar procedure as in [14]. For this we need a priori estimates of solutions of corresponding auxiliary problems. Now we consider problems with $\phi$-Laplacians but the basic evaluation of estimates of $\phi(u')$ are similar to those of $u'$ in [14] and hence we omit their computation here. For details see our preprint [16].

**Proof of Theorem 2.4.** First, we will prove the following a priori estimate:

**Claim.** There exist $a_j \in (0, \infty)$, $j = 1, 2, \ldots, m + 1$, such that for each function $u \in C^1_0$ satisfying (1.2), (1.3), (2.2) and (2.3), the estimates

$$(4.9) \quad |u(t)| \leq a_j \quad \text{for } t \in (t_{j-1}, t_j], \ j = 1, 2, \ldots, m + 1$$

are valid.
Indeed, let \( u \) satisfy the assumptions of Claim and let
\[
\rho_0 = \max\{\|\sigma_1\|_{\infty}, \|\sigma_2\|_{\infty}\} \quad \text{and} \quad \gamma_i = \max\{|c_i|, |d_i|\}, \quad i = 1, 2, \ldots, m + 1.
\]

(i) If \( t_u \in [0, t_1] \), then \( |u(t)| \leq \gamma_1 t_1 + \rho_0 \) for \( t \in [0, t_1] \). Put \( a^0_1 = \gamma_1 t_1 + \rho_0 \) and \( b^0_1 = \max\{|J_1(x)| : x \in [-a^0_1, a^0_1]\} \). Then \( |u(t)| \leq \gamma_2(t_2 - t_1) + b^0_1 \) for \( t \in (t_1, t_2] \).

Further, put \( a^0_2 = \gamma_2(t_2 - t_1) + b^0_1 \) and \( b^0_2 = \max\{|J_2(x)| : x \in [-a^0_1, a^0_1]\} \). Then \( |u(t)| \leq \gamma_3(t_3 - t_2) + b^0_2 \) for \( t \in (t_2, t_3] \). By induction we get that \( |u(t)| \leq a^i_1 \) for \( t \in (t_{i-1}, t_i] \), where \( a^i_1 = \gamma_i(t_{i+1} - t_i) + \max\{|J_i(x)| : x \in [-a^i_1, a^i_1]\}, \quad i = 1, 2, \ldots, m \).

(ii) If \( t_u \in (t_j, t_{j+1}] \) for some \( j \in \{1, 2, \ldots, m\} \), we get similarly as in (i) that \( |u(t)| \leq a^j_1 \) for \( t \in (t_{j-1}, t_j] \), \( i \in 1, 2, \ldots, m + 1 \), where \( a^j_1 = \gamma_j(t_{j+1} - t_j) + \rho_0 \), \( a^{i+1}_{j+1} = \gamma_{i+1}(t_{i+1} - t_i) + \max\{|J_i(x)| : x \in [-a^j_1, a^j_1]\}, \quad i = 1, 2, \ldots, j - 1, j + 1, \ldots, m \), \( a^j_1 = \gamma_j t_1 + a^j_{m+1} \).

Setting
\[
a_j = \max\{\rho_0, a^j_0, a^j_1, \ldots, a^j_m\} \quad \text{for} \quad j = 1, 2, \ldots, m + 1,
\]
we complete the proof of Claim.

Now, take \( \beta_j \) by (4.1) and for \( a_j \) of Claim put
\[
\alpha_j(x) = \begin{cases} 
-a_j & \text{for} \ x < -a_j, \\
\quad x & \text{for} \ -a_j \leq x \leq a_j, \\
\quad a_j & \text{for} \ x > a_j
\end{cases}
\]
and
\[
\bar{f}(t, x, y) = f(t, \alpha_j(x), \beta_j(y)) + \frac{y - \beta_j(y)}{|y - \beta_j(y)| + 1}
\]
for a.e. \( t \in (t_{j-1}, t_j] \), all \( x, y \in \mathbb{R}, \quad j = 1, 2, \ldots, m + 1 \).

Finally, define \( M_j \) by (4.3). We see that all assumptions of Theorem 2.3 are satisfied for the problem (4.4)-(4.6) and consequently it has a solution \( u \) satisfying (2.3). As in the proof of Theorem 2.2, Step 2, we get that \( u \) fulfills (2.2). Hence \( u \) satisfies (1.2), (1.3) and, by Claim, also (4.8). Therefore, \( u \) is a solution of (1.1)-(1.3).

References


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