Singular discrete second order BVPs with $p$-Laplacian

Irena Rachůnková and Lukáš Rachůnek

Department of Mathematics, Palacký University, Tomkova 40, 77900 Olomouc, Czech Republic, e-mail: rachunk@inf.upol.cz

Abstract. We study singular discrete boundary value problems with mixed boundary conditions and with the $p$-Laplacian of the form

$$\Delta \left( \phi_p(\Delta u(t-1)) \right) + f(t,u(t),\Delta u(t-1)) = 0, \quad t \in [1,T+1],$$

$$\Delta u(0) = u(T + 2) = 0,$$

where $[1,T+1] = \{1,2,\ldots,T+1\}$, $T \in \mathbb{N}$, $\phi_p(y) = |y|^{p-2}y$, $p > 1$. We assume that $f$ is continuous on $[1,T+1] \times (0,\infty) \times \mathbb{R}$ and $f(t,x,y)$ has a singularity at $x = 0$. We prove the existence of a positive solution by means of the lower and upper functions method, the Brouwer fixed point theorem and by a convergence of approximate regular problems.

Keywords. Singular discrete BVP, mixed conditions, lower and upper functions, Brouwer fixed point theorem, approximate regular problems

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1 Introduction

Let $T \in \mathbb{N}$ be fixed. We define the discrete interval $[1,T+1] = \{1,2,\ldots,T+1\}$ and consider the second order difference equation with the $p$-Laplacian

$$\Delta \left( \phi_p(\Delta u(t-1)) \right) + f(t,u(t),\Delta u(t-1)) = 0, \quad t \in [1,T+1] \quad (1.1)$$

subjected to the mixed boundary conditions

$$\Delta u(0) = 0, \quad u(T + 2) = 0. \quad (1.2)$$

Here $\Delta$ denotes the forward difference operator with the step size 1, i.e. $\Delta u(t-1) = u(t) - u(t-1)$, and $\phi_p(y) = |y|^{p-2}y$, $p > 1$. We will investigate the solvability of problem (1.1), (1.2).
Definition 1.1 By a solution \( u \) of problem (1.1), (1.2) we mean \( u: [0, T+2] \to \mathbb{R} \), \( u \) satisfies the difference equation (1.1) on \([1, T+1]\) and the boundary conditions (1.2). If \( u(t) > 0 \) for \( t \in [1, T+1] \), we say that \( u \) is a positive solution of problem (1.1), (1.2).

Let \( \mathcal{D} \subset \mathbb{R}^2 \). We say that \( f \) is continuous on \([1, T+1] \times \mathcal{D} \), if \( f(\cdot, x, y) \) is defined on \([1, T+1]\) for each \((x, y) \in \mathcal{D}\) and if \( f(t, \cdot, \cdot) \) is continuous on \( \mathcal{D} \) for each \( t \in [1, T+1] \).

If \( \mathcal{D} = \mathbb{R}^2 \), problem (1.1), (1.2) is called regular. If \( \mathcal{D} \neq \mathbb{R}^2 \) and \( f \) has singularities on \( \partial \mathcal{D} \), then problem (1.1), (1.2) is singular.

Here we will assume that

\[
\begin{align*}
\mathcal{D} &= (0, \infty) \times \mathbb{R}, \ f \text{ is continuous on } [1, T+1] \times \mathcal{D} \\
&\text{and } f \text{ has a singularity at } x = 0, \ i.e. \\
\lim_{x \to 0^+} |f(t,x,y)| &= \infty \text{ for each } t \in [1, T+1] \\
&\text{and for some } y \in \mathbb{R}.
\end{align*}
\]

(1.3)

Discrete second order nonlinear boundary value problems have been investigated in several monographs (e.g. [1], [10], [8], [23]) and papers (e.g. [11], [12], [13], [16], [19], [20], [27], [30]). Most of the above results concern regular problems. Singular discrete problems have received less attention. We refer to [3] and [8] where the solvability of the Dirichlet singular discrete problem was studied. Existence theorems for singular higher order discrete problems can be found in [10]. The paper [19] deals with problem (1.1), (1.2) where \( f \) is regular and has the form \( f(t, x) = a(t) g(x) \). Here, we extend the existence results of [19] onto the singular problem (1.1), (1.2) where \( f \) depends both on \( u \) and on \( \Delta u \).

The continuous versions of mixed singular problems for differential equations without \( p \)-Laplacian have been investigated e.g. in [6], [9], [15], [21], [24], [26], [28] and for problems with the \( p \)-Laplacian in [7] or [18].

2 Lower and upper function for regular problems

We start our investigation with the equation

\[
\Delta \left( \phi_p(\Delta u(t-1)) \right) + h(t, u(t), \Delta u(t-1)) = 0, \quad t \in [1, T+1],
\]

(2.1)

where \( h \) is continuous on \([1, T+1] \times \mathbb{R}^2\) and we apply the lower and upper functions method for the regular problem (2.1), (1.2).

Definition 2.1 \( \alpha: [0, T+2] \to \mathbb{R} \) is called a lower function of problem (2.1), (1.2) if

\[
\Delta \left( \phi_p(\Delta \alpha(t-1)) \right) + h(t, \alpha(t), \Delta \alpha(t-1)) \geq 0 \text{ for } t \in [1, T+1],
\]

(2.2)
\[ \Delta \alpha(0) \geq 0, \quad \alpha(T + 2) \leq 0. \quad (2.3) \]

\[ \beta: [0, T + 2] \to \mathbb{R} \text{ is called an upper function of problem } (2.1), (1.2) \text{ if} \]

\[ \Delta \left( \phi_p(\Delta \beta(t - 1)) \right) + h(t, \beta(t), \Delta \beta(t - 1)) \leq 0 \text{ for } t \in [1, T + 1], \quad (2.4) \]

\[ \Delta \beta(0) \leq 0, \quad \beta(T + 2) \geq 0. \quad (2.5) \]

**Theorem 2.2** (Lower and upper functions method) Let \( \alpha \) and \( \beta \) be a lower and an upper function, respectively, of (2.1), (1.2) and \( \alpha \leq \beta \) on \([1, T + 1]\). Let \( h \) be continuous on \([1, T + 1] \times \mathbb{R}^2 \) and nonincreasing in its third variable. Then problem (2.1), (1.2) has a solution \( u \) satisfying

\[ \alpha(t) \leq u(t) \leq \beta(t) \text{ for } t \in [0, T + 2]. \quad (2.6) \]

**Proof.** Step 1. For \( t \in [1, T + 1] \), \( x, z \in \mathbb{R} \), define functions

\[ \sigma(t, z) = \begin{cases} 
\beta(t - 1) & \text{if } z > \beta(t - 1) \\
z & \text{if } \alpha(t - 1) \leq z \leq \beta(t - 1) \\
\alpha(t - 1) & \text{if } z < \alpha(t - 1), 
\end{cases} \]

\[ \tilde{h}(t, x, x - z) = \begin{cases} 
h(t, \beta(t), \beta(t) - \sigma(t, z)) - \frac{x - \beta(t)}{x - \beta(t) + 1} & \text{if } x > \beta(t) \\
h(t, x, x - \sigma(t, z)) & \text{if } \alpha(t) \leq x \leq \beta(t) \\
h(t, \alpha(t), \alpha(t) - \sigma(t, z)) + \frac{\alpha(t) - x}{\alpha(t) - x + 1} & \text{if } x < \alpha(t). 
\end{cases} \]

Then \( \tilde{h} \) is continuous on \([1, T + 1] \times \mathbb{R}^2 \) and there exists \( M > 0 \) such that

\[ |\tilde{h}(t, x, y)| \leq M \quad \text{for } t \in [1, T + 1], \quad (x, y) \in \mathbb{R}^2. \quad (2.7) \]

We will study the auxiliary difference equation

\[ \Delta \left( \phi_p(\Delta u(t - 1)) \right) + \tilde{h}(t, u(t), \Delta u(t - 1)) = 0, \quad t \in [1, T + 1], \quad (2.8) \]

and we will prove that problem (2.8), (1.2) has a solution (see Steps 2–3).

**Step 2.** We denote

\[ E = \{ u: [0, T + 2] \to \mathbb{R}, \quad \Delta u(0) = 0, \quad u(T + 2) = 0 \} \quad (2.9) \]

and define \( \| u \| = \max\{ |u(t)|: t \in [1, T + 1] \} \). Then \( E \) is a Banach space with \( \dim E = T + 1 \). Further we put \( \sum_{i=b}^{a} = 0 \) for each \( a, b \in \mathbb{N} \cup \{ 0 \} \), \( a < b \), and define an operator \( T: E \to E \) by

\[ (Tu)(t) = \sum_{s=t}^{T+1} \phi_q \left( \sum_{i=1}^{s} \tilde{h}(i, u(i), \Delta u(i - 1)) \right), \quad t \in [0, T + 2]. \quad (2.10) \]
Here $\phi_q = \phi_p^{-1}$, $\frac{1}{p} + \frac{1}{q} = 1$.

Since $\phi_q: \mathbb{R} \to \mathbb{R}$ and $\tilde{h}: [1, T + 1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, we see that $\mathcal{T}$ is a continuous operator. Moreover, (2.7) and (2.10) imply that if $r \geq \sum_{s=1}^{T+1} \phi_q(sM)$, then $\mathcal{T}(B(r)) \subset B(r)$, where $B(r) = \{ u \in E: \| u \| < r \}$. Therefore the Brouwer fixed point theorem yields the existence of at least one point $u \in B(r)$ such that $u = \mathcal{T}u$.

**Step 3.** We prove that $u$ is a fixed point of $\mathcal{T}$ if and only if $u$ is a solution of problem (2.8), (1.2).

(i) Assume that $u = \mathcal{T}u$. Then $u \in E$ and so $u$ satisfies (1.2). Further we have

$$\Delta u(t - 1) = u(t) - u(t - 1) = \sum_{s=t}^{T+1} \phi_q \left( \sum_{i=1}^{s} \tilde{h}(i, u(i), \Delta u(i - 1)) \right) - \sum_{s=t-1}^{T+1} \phi_q \left( \sum_{i=1}^{s} \tilde{h}(i, u(i), \Delta u(i - 1)) \right),$$

$$\phi_p(\Delta u(t - 1)) = -\sum_{i=1}^{t} \tilde{h}(i, u(i), \Delta u(i - 1)),$$

$$\Delta \left( \phi_p(\Delta u(t - 1)) \right) = \phi_p(\Delta u(t)) - \phi_p(\Delta u(t - 1)) = -\tilde{h}(t, u(t), \Delta u(t - 1)) \text{ for } t \in [1, T + 1].$$

(ii) Assume that $u$ is a solution of (2.8), (1.2). Then $u \in E$ and $\phi_p(\Delta u(0)) = \phi_p(0) = 0$. Further we have $\Delta \left( \phi_p(\Delta u(0)) \right) = -\tilde{h}(1, u(1), \Delta u(0))$, which yields

$$\phi_p(\Delta u(1)) = -\tilde{h}(1, u(1), \Delta u(0)).$$

Similarly

$$\Delta \left( \phi_p(\Delta u(1)) \right) = -\tilde{h}(2, u(2), \Delta u(1)),$$

and hence

$$\phi_p(\Delta u(2)) = -\tilde{h}(1, u(1), \Delta u(0)) - \tilde{h}(2, u(2), \Delta u(1)).$$

By induction we get

$$\phi_p(\Delta u(t)) = -\sum_{i=1}^{t} \tilde{h}(i, u(i), \Delta u(i - 1))$$

and

$$\Delta u(t) = -\phi_q \left( \sum_{i=1}^{t} \tilde{h}(i, u(i), \Delta u(i - 1)) \right), \quad t \in [1, T + 1]. \quad (2.11)$$
Using (1.2) and (2.11) we get

\[ u(T + 1) = \phi_{q} \left( \sum_{i=1}^{T+1} \tilde{h}(i, u(i), \Delta u(i-1)) \right), \]

\[ u(T) = \phi_{q} \left( \sum_{i=1}^{T} \tilde{h}(i, u(i), \Delta u(i-1)) \right) + \phi_{q} \left( \sum_{i=1}^{T+1} \tilde{h}(i, u(i), \Delta u(i-1)) \right), \]

and by induction we get

\[ u(t) = \sum_{s=t}^{T+1} \phi_{q} \left( \sum_{i=1}^{s} \tilde{h}(i, u(i), \Delta u(i-1)) \right), \quad t \in [0, T + 2]. \]

Note that for \( t = 0 \) and \( t = T + 2 \) we use the equalities \( \sum_{i=1}^{0} = 0 \) and \( \sum_{s=T+2}^{T+1} = 0 \).

**Step 4.** We prove that the solution \( u \) of (2.8), (1.2) satisfies (2.6). Put \( v(t) = u(t) - \beta(t) \) for \( t \in [0, T + 2] \) and assume that \( \max\{v(t): t \in [0, T + 2]\} = v(\ell) > 0 \). Conditions (1.2) and (2.5) imply \( \ell \in [1, T + 1] \). Thus we have \( v(\ell + 1) \leq v(\ell), \) \( v(\ell - 1) \leq v(\ell) \), and consequently \( \Delta u(\ell) \leq \Delta \beta(\ell), \) \( \Delta u(\ell - 1) \geq \Delta \beta(\ell - 1) \). This leads to \( \phi_{p}(\Delta u(\ell)) \leq \phi_{p}(\Delta \beta(\ell)), \) \( \phi_{p}(\Delta u(\ell - 1)) \geq \phi_{p}(\Delta \beta(\ell - 1)) \) and

\[ \Delta \left( \phi_{p}(\Delta u(\ell - 1)) \right) \leq \Delta \left( \phi_{p}(\Delta \beta(\ell - 1)) \right). \quad (2.12) \]

On the other hand, since \( h \) is nonincreasing in its third variable, we get by (2.8)

\[ \Delta \left( \phi_{p}(\Delta u(\ell - 1)) \right) - \Delta \left( \phi_{p}(\Delta \beta(\ell - 1)) \right) = \]

\[ = -\tilde{h}(\ell, u(\ell), \Delta u(\ell - 1)) - \Delta \left( \phi_{p}(\Delta \beta(\ell - 1)) \right) = \]

\[ = -h(\ell, \beta(\ell), \beta(\ell) - \sigma(\ell, u(\ell - 1))) + \frac{v(\ell)}{2v(\ell) + 1} - \Delta \left( \phi_{p}(\Delta \beta(\ell - 1)) \right) \geq \]

\[ \geq -h(\ell, \beta(\ell), \Delta \beta(\ell - 1)) + \frac{v(\ell)}{v(\ell) + 1} - \Delta \left( \phi_{p}(\Delta \beta(\ell - 1)) \right) \geq \]

\[ \geq \frac{v(\ell)}{v(\ell) + 1} > 0, \]

which contradicts (2.12). So, we have proved \( u(t) \leq \beta(t) \) for \( t \in [0, T + 2] \). The inequality \( \alpha(t) \leq u(t) \) for \( t \in [0, T + 2] \) can be proved similarly. Therefore \( u \) satisfies (2.6) and hence \( u \) is a solution of problem (2.1), (1.2). \( \square \)
3 Main result and example

The next theorem provides sufficient conditions for the solvability of the singular problem (1.1), (1.2). The proof is based on the construction of a sequence of approximating auxiliary regular problems and on the lower and upper functions method from Theorem 2.2.

**Theorem 3.1** Assume (1.3) and let the following conditions hold:

there exists $c \in (0, \infty)$ such that $f(t, c, 0) \leq 0$ for $t \in [1, T + 1], \quad (3.1)$

$f$ is nonincreasing in $y$ for $t \in [1, T + 1], x \in (0, c], \quad (3.2)$

$$
\lim_{x \to 0^+} f(t, x, y) = \infty \text{ for } t \in [1, T + 1], y \in [-c, c].
$$

Then problem (1.1), (1.2) has a solution $u$ satisfying

$$
0 < u(t) \leq c \text{ for } t \in [0, T + 1]. \quad (3.4)
$$

**Proof.** Step 1. For $k \in \mathbb{N}$, $t \in [1, T + 1]$, $(x, y) \in \mathbb{R}^2$ define

$$
f_k(t, x, y) = \begin{cases} 
  f(t, |x|, y) & \text{if } |x| \geq \frac{1}{k}, \\
  f(t, \frac{1}{k}, y) & \text{if } |x| < \frac{1}{k}.
\end{cases}
$$

Then $f_k$ is continuous on $[1, T + 1] \times \mathbb{R}^2$ and nonincreasing in $y$ for $t \in [1, T + 1]$, $x \in [-c, c]$. Assumption (3.3) implies the existence of $k_0 \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq k_0$

$$
f_k(t, 0, 0) = f(\frac{t}{k}, 0) > 0 \text{ for } t \in [1, T + 1].
$$

Consider an auxiliary sequence of equations

$$
\Delta \left( \phi_p(\Delta u(t - 1)) \right) + f_k(t, u(t), \Delta u(t - 1)) = 0, \quad t \in [1, T + 1], \quad (3.5)
$$

$k \in \mathbb{N}$, $k \geq k_0$. Put $\alpha(t) = 0$, $\beta(t) = c$ for $t \in [0, T + 2]$. Then $\alpha$ and $\beta$ are a lower and an upper function of each problem (3.5), (1.2) and $\alpha(t) < \beta(t)$ for $t \in [1, T + 1]$. By Theorem 2.2, there exists a solution $u_k$ of problem (3.5), (1.2) satisfying

$$
0 \leq u_k(t) \leq c \text{ for } t \in [0, T + 2], k \in \mathbb{N}, k \geq k_0.
$$

Consequently

$$
|\Delta u_k(t)| \leq c \text{ for } t \in [0, T + 1], k \in \mathbb{N}, k \geq k_0. \quad (3.7)
$$

Step 2. Let $k \in \mathbb{N}$, $k \geq k_0$. Since $u_k$ satisfies (3.5), we get by (2.11)

$$
\Delta u_k(t) = \phi_q \left( -\sum_{i=1}^{t} f_k(i, u_k(i), \Delta u_k(i - 1)) \right), \quad t \in [1, T + 1]. \quad (3.8)
$$
By (3.3) there exists $\varepsilon_1 \in \left(0, \frac{1}{k_0}\right)$ such that if $k \geq \frac{1}{\varepsilon_1}$, then
\[ f_k(1, x, y) > \phi_p(c) \quad x \in (0, \varepsilon_1], \ y \in [-c, c]. \] (3.9)
Assume that $k \geq \frac{1}{\varepsilon_1}$ and $u_k(1) < \varepsilon_1$. Then, by (3.8) and (3.9), we get
\[ \Delta u_k(1) = \phi_q\left(-f_k(1, u_k(1), \Delta u_k(0))\right) < \phi_q(-\phi_p(c)) = -c, \]
which contradicts (3.7). Therefore
\[ u_k(1) \geq \varepsilon_1 \quad \text{for each} \ k \in \mathbb{N}, \ k \geq \frac{1}{\varepsilon_1}. \] (3.10)

Denote
\[ m_1 = \max\{|f_k(1, x, y)|: x \in [\varepsilon_1, c], \ y \in [-c, c]\}. \] By (3.3) there exists $\varepsilon_2 \in (0, \varepsilon_1]$ such that if $k \geq \frac{1}{\varepsilon_2}$, then
\[ f_k(2, x, y) > \phi_p(c) + m_1 \quad \text{for} \ x \in (0, \varepsilon_2], \ y \in [-c, c]. \] (3.11)
Assume that $k \geq \frac{1}{\varepsilon_2}$ and $u_k(2) < \varepsilon_2$. Then, by (3.8), (3.10) and (3.11), we get
\[ \Delta u_k(2) = \phi_q\left(-f_k(1, u_k(1), \Delta u_k(0)) - f_k(2, u_k(2), \Delta u_k(1))\right) < \]
\[ < \phi_q\left(m_1 - f_k(2, u_k(2), \Delta u_k(1))\right) < \phi_q(-\phi_p(c)) = -c, \]
which contradicts (3.7). Therefore
\[ u_k(2) \geq \varepsilon_2 \quad \text{for each} \ k \in \mathbb{N}, \ k \geq \frac{1}{\varepsilon_2}. \]

We continue similarly for $t = 3, \ldots, T$ and get $0 < \varepsilon_T \leq \varepsilon_{T-1} \leq \cdots \leq \varepsilon_1$ such that
\[ u_k(t) \geq \varepsilon_i \quad \text{for} \ t \in [1, T], \ k \in \mathbb{N}, \ k \geq \frac{1}{\varepsilon_T}. \] (3.12)
If we denote
\[ m_i = \max\{|f_k(i, x, y)|: x \in [\varepsilon_i, c], \ y \in [-c, c]\}, \quad i \in [1, T] \]
then by virtue of (3.3) there exists $\varepsilon_{T+1} \in (0, \varepsilon_T]$ such that if $k \geq \frac{1}{\varepsilon_{T+1}}$, then
\[ f_k(T+1, x, y) > \phi_p(c) + \sum_{i=1}^{T} m_i \quad \text{for} \ x \in (0, \varepsilon_{T+1}], \ y \in [-c, c]. \] (3.13)
Assume that $k \geq \frac{1}{\varepsilon_{T+1}}$ and $u_k(T+1) < \varepsilon_{T+1}$. Then, by (3.8), (3.12) and (3.13), we get
\[ \Delta u_k(T+1) = \]
\[= \phi_q \left( - \sum_{i=1}^{T} f_k(i, u_k(i), \Delta u_k(i-1)) - f_k(T+1, u_k(T+1), \Delta u_k(T)) \right) <\]

\[< \phi_q \left( \sum_{i=1}^{T} m_i - f_k(T+1, u_k(T+1), \Delta u_k(T)) \right) < \phi_q(-\phi_p(c)) = -c,\]

which contradicts (3.7). Therefore, if we put \(\varepsilon = \varepsilon_{T+1}\), we get

\[0 < \varepsilon \leq u_k(t) \leq c \quad \text{for} \quad t \in [0, T + 1], \quad k \in \mathbb{N}, \quad k \geq \frac{1}{\varepsilon}, \quad (3.14)\]

Since \(u_k\) satisfies (3.14) and (1.2) we can choose a subsequence \(\{u_{k_n}\} \subset \{u_k\}\) such that \(\lim_{n \to \infty} u_{k_n}(t) = u(t), \quad t \in [0, T + 2]\), where \(u \in E\) (see (2.9)). Moreover, (3.8) yields for each sufficiently large \(n \in \mathbb{N}\)

\[\Delta u_{k_n}(t) = -\phi_q \left( \sum_{i=1}^{t} f(i, u_{k_n}(i), \Delta u_{k_n}(i-1)) \right), \quad t \in [1, T + 1].\]

Letting \(n \to \infty\) and using the continuity of \(\phi_q\) on \(\mathbb{R}\) and \(f\) on \([1, T + 1] \times D_\varepsilon\), where \(D_\varepsilon = [\varepsilon, \infty) \times \mathbb{R}\), we get

\[\Delta u(t) = -\phi_q \left( \sum_{i=1}^{t} f(i, u(i), \Delta u(i-1)) \right), \quad t \in [1, T + 1],\]

from which the equality

\[\Delta \phi_p(\Delta u(t-1)) = -f(t, u(t), \Delta u(t-1)), \quad t \in [1, T + 1]\]

follows. Therefore \(u\) is a solution of (1.1) and, by (3.14), \(u\) satisfies (3.4). The theorem is proved. \(\square\)

**Example.** Let \(T \in \mathbb{N}, \alpha \in [0, \infty), c, \beta \in (0, \infty), p \in (1, \infty), a: [1, T + 1] \to \mathbb{R}\). By Theorem 3.1 the problem

\[\Delta \left( \phi_p(\Delta u(t-1)) \right) + (a(t) + (u(t))^\alpha + (u(t))^{-\beta})(c - u(t)) - (\Delta u(t-1))^3,\]

\[t \in [1, T + 1], \quad \Delta u(0) = u(T + 2) = 0\]

has a solution \(u\) satisfying (3.4).

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