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MULTIPLICITY RESULTS FOR FOUR-POINT BOUNDARY VALUE PROBLEMS

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1. INTRODUCTION

LET $R = (-\infty, +\infty)$, $I = [a, b]$, $-\infty < a < c \leq d < b < +\infty$, $f: I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous functions. This paper proves existence and multiplicity results of Ambrosetti–Prodi type for the four-point resonance problem

$$u'' + f(t, u, u') = s, \tag{1.1}$$

$$u(a) = u(c), \qquad u(d) = u(b), \tag{1.2}$$

where s is a real parameter.

Our results have been motivated by similar ones concerning the number of solutions of periodic problems for first and second order differential equations [1, 3]. Our method of proof is close to that of [1]. It is based on the use of strict upper and lower solutions and on coincidence topological degree arguments.

This four-point problem can be understood as an approximation of the Neumann problem, where derivatives at the points a, b are replaced by differences.

We write $C^k(I)$ for the space of real valued C^k-functions u on I with the norm

$$\|u\|_k = \sum_{i=0}^{k} \max\{|u^{(i)}(t)| : t \in I\}.$$

We recall that $\sigma_1, \sigma_2 \in C^2(I)$ are lower and upper solutions for (1.1), (1.2), respectively, if

$$[\sigma_i'' + f(t, \sigma_i, \sigma_i') - s](-1)^i \leq 0 \qquad \text{for each } t \in I, \tag{1.3}$$

$$[\sigma_i(a) - \sigma_i(c)](-1)^i \geq 0, \qquad [\sigma_i(d) - \sigma_i(b)](-1)^i \leq 0, \qquad i \in \{1, 2\}. \tag{1.4}$$

Similarly, $\sigma_1, \sigma_2 \in C^2(I)$ are strict lower and upper solutions for (1.1), (1.2), respectively, if

$$[\sigma_i'' + f(t, \sigma_i, \sigma_i') - s](-1)^i < 0 \qquad \text{for each } t \in I, \tag{1.5}$$

$$\sigma_i(a) = \sigma_i(c), \qquad \sigma_i(d) = \sigma_i(b), \qquad i \in \{1, 2\}. \tag{1.6}$$

A continuous function $\omega: (0, +\infty) \to (\varepsilon, +\infty)$, with $\varepsilon > 0$, will be called a Nagumo function, if

$$\int_0^{+\infty} \frac{z\,dz}{\omega(z)} = +\infty. \tag{1.7}$$

We say that $f: I \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Bernstein–Nagumo conditions, if for any $r \in (0, +\infty)$ there exists a Nagumo function ω_r such that

$$f(t, x, y) \operatorname{sgn} y \geq -\omega_r(|y|) \qquad \text{on } I \times [-r, r] \times \mathbb{R} \tag{1.8}$$

and

$$f(t, x, y) \operatorname{sgn} y \leq \omega_r(|y|) \qquad \text{on } [a, c] \times [-r, r] \times \mathbb{R}. \tag{1.9}$$

In what follows

$$\begin{aligned} D(-r_1) &= \{x \in C^2(I) : x(t) > -r_1 \text{ for each } t \in I\}, \\ D(r_1) &= \{x \in C^2(I) : x(t) < r_1 \text{ for each } t \in I\}, \end{aligned} \tag{1.10}$$

where $r_1 \in (0, +\infty)$.

2. AUXILIARY RESULTS

We shall need some lemmas whose proofs follow the approach proposed in [5]. Let us consider the equation

$$u'' = g(t, u, u') \tag{2.1}$$

where $g \in C^0(I \times R^2)$.

LEMMA 1. Let σ_1 be a lower solution and σ_2 an upper solution of (2.1), (1.2) with $\sigma_1(t) \leq \sigma_2(t)$ for each $t \in I$. Further, let there exist $k \in (0, +\infty)$ such that for each $t \in I$, $x, y \in \mathbb{R}$, where $\sigma_1(t) \leq x \leq \sigma_2(t)$, the inequality

$$|g(t, x, y)| \leq k$$

is fulfilled.

Then problem (2.1), (1.2) has a solution u fulfilling

$$\sigma_1(t) \leq u(t) \leq \sigma_2(t) \qquad \text{for each } t \in I. \tag{2.2}$$

Proof. Similarly, to the proof of [5, lemma 6], we put

$$w_i(t, x, y) = (-1)^i m(x - \sigma_i)[g(t, \sigma_i, \sigma_i') - g(t, \sigma_i, y) + (-1)^i r_0/m], \qquad i = 1, 2,$$

$$g_m(t, x, y) = \begin{cases} g(t, \sigma_1, \sigma_1') - r_0/m & \text{for } x \leq \sigma_1(t) - 1/m \\ g(t, \sigma_1, y) + w_1 & \text{for } \sigma_1(t) - 1/m < x < \sigma_1(t) \\ g(t, x, y) & \text{for } \sigma_1(t) \leq x \leq \sigma_2(t) \\ g(t, \sigma_2, y) + w_2 & \text{for } \sigma_2(t) < x < \sigma_2(t) + 1/m \\ g(t, \sigma_2, \sigma_2') + r_0/m & \text{for } x \geq \sigma_2(t) + 1/m, \end{cases}$$

where m is a natural number and $(t, x, y) \in I \times R^2$, and consider the equation

$$u'' = (1/m)u + g_m(t, u, u'). \tag{2.3}$$

By the Fredholm nonlinear alternative theorem, problem (2.3), (1.2) has a solution u_m, because g_m is bounded and the linear problem corresponding to (2.3), (1.2) has only the trivial solution.

Similarly to [5, lemma 6], it can be checked that

$$\sigma_1(t) = 1/m \leq u_m(t) \leq \sigma_2(t) + 1/m$$

for each $t \in I$ and any natural m. This implies, by (1.2), (2.3), that the sequences $(u_m)_1^\infty$ and $(u_m')_1^\infty$ are uniformly bounded and equi-continuous on I and thus, by the Arzelo–Ascoli theorem, we conclude that $(u_m)_1^\infty$ contains a subsequence converging in $C^1(I)$. Writing, for every m, equations (2.3) in integral forms, it is easily seen that the limit of that subsequence is a solution of (2.1), (1.2) and satisfies (2.2). The proof is complete.

LEMMA 2 (on *a priori* estimate). Let, for $r \in (0, +\infty)$, ω_r be a Nagumo function. Then there exists a number $\rho = \rho(r, \omega_r)$ such that for any function $u \in C^2(I)$ the conditions

$$\|u\|_0 \le r, \qquad u(a) = u(c), \qquad u(b) = u(d), \tag{2.4}$$

$$u'' \operatorname{sgn} u' \le \omega_r(|u'|) \qquad \text{for each } t \in I, \tag{2.5}$$

and

$$u'' \operatorname{sgn} u' \ge -\omega_r(|u'|) \qquad \text{for each } t \in [a, c] \tag{2.6}$$

imply the estimate

$$\|u'\|_0 < \rho. \tag{2.7}$$

Proof. In view of (2.4) we can choose $a_1 \in (a, c)$ such that $u'(a_1) = 0$. From (1.7) it follows that there exists $\rho \in (r, +\infty)$ such that

$$\int_0^\rho \frac{z\,dz}{\omega_r(z)} > 2r. \tag{2.8}$$

Now, let us suppose that there exists $t_0 \in (a_1, b]$ such that

$$u'(t_0) \ge \rho. \tag{2.9}$$

Let $[\alpha, \beta] \subset [a, b]$ be the maximal interval containing t_0 with $u'(t) \ge 0$ for $t \in [\alpha, \beta]$. Let $t^* \in (\alpha, \beta]$ be such point that $u'(t^*) = c_1 = \max\{u'(t): \alpha \le t \le \beta\}$. Then, from (2.5), it follows

$$\int_0^{c_1} \frac{z\,dz}{\omega(z)} \le 2r,$$

which implies, by (2.8), $c_1 < \rho$. The latter inequality contradicts (2.9). Similarly, supposing that there exists $t_0 \in (a_1, b]$ with

$$u'(t_0) \le -\rho \tag{2.10}$$

and choosing the maximal interval $[\alpha, \beta] \subset [a_1, b]$ such that $t_0 \in (\alpha, \beta]$ and $u'(t) \le 0$ on $[\alpha, \beta]$, we can get the same contradiction. Finally, if we suppose that t_0 satisfying (2.9) or (2.10) can be chosen in $[a, a_1)$, then using (2.6) instead of (2.5) we obtain a contradiction by the same arguments as above. Therefore $\|u'\|_0 < \rho$ and lemma is proved.

LEMMA 3. Let s be a real number. Assume that the function f in equation (1.1) satisfies the Bernstein–Nagumo conditions. Further let σ_1 and σ_2 be lower and upper solutions of problem (1.1), (1.2), respectively, with $\sigma_1(t) \le \sigma_2(t)$ for each $t \in I$.

Then problem (1.1), (1.2) has at least one solution u satisfying (2.2).

Proof. Let

$$r_i = \max\{|\sigma_1^{(i)}(t)| + |\sigma_2^{(i)}(t)| : t \in I\}, \qquad i = 0, 1.$$

Then for r_0 there exists a Nagumo function ω_{r_0} such that f satisfies (1.8) and (1.9) where $r = r_0$. Put $\tilde{\omega}(z) = |s| + \omega_{r_0}(z)$, $z \in (0, +\infty)$. We can easily verify that $\tilde{\omega}$ is also a Nagumo function.