MULTIPLEMENT RESULTS FOR FOUR-POINT BOUNDARY VALUE PROBLEMS

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1. INTRODUCTION

Let \( R = (\infty, +\infty) \), \( I = [a, b] \), \( -\infty < a < c < d < b < +\infty \), \( f: I \times \mathbb{R}^2 \to \mathbb{R} \) be continuous functions. This paper proves existence and multiplicity results of Ambrosetti–Prodi type for the four-point resonance problem

\[
\begin{align*}
    u'' + f(t, u, u') &= s, \\
    u(a) &= u(c), \quad u(d) = u(b),
\end{align*}
\]

where \( s \) is a real parameter.

Our results have been motivated by similar ones concerning the number of solutions of periodic problems for first and second order differential equations [1, 3]. Our method of proof is close to that of [1]. It is based on the use of strict upper and lower solutions and on coincidence topological degree arguments.

This four-point problem can be understood as an approximation of the Neumann problem, where derivatives at the points \( a, b \) are replaced by differences.

We write \( C^k(I) \) for the space of real valued \( C^k \)-functions \( u \) on \( I \) with the norm

\[
    \|u\|_k = \sum_{i=0}^{k} \max\{|u^{(i)}(t)| : t \in I\}.
\]

We recall that \( \sigma_1, \sigma_2 \in C^2(I) \) are lower and upper solutions for (1.1), (1.2), respectively, if

\[
    [\sigma_i^t + f(t, \sigma_i, \sigma_i') - s](-1)^i \leq 0 \quad \text{for each } t \in I, \tag{1.3}
\]

\[
    [\sigma_i(a) - \sigma_i(c) + 1]^i \geq 0, \quad [\sigma_i(d) - \sigma_i(b) + 1]^i \leq 0, \quad i \in \{1, 2\}. \tag{1.4}
\]

Similarly, \( \sigma_1, \sigma_2 \in C^2(I) \) are strict lower and upper solutions for (1.1), (1.2), respectively, if

\[
    [\sigma_i^t + f(t, \sigma_i, \sigma_i') - s](-1)^i < 0 \quad \text{for each } t \in I, \tag{1.5}
\]

\[
    \sigma_i(a) = \sigma_i(c), \quad \sigma_i(d) = \sigma_i(b), \quad i \in \{1, 2\}. \tag{1.6}
\]

A continuous function \( \omega: (0, +\infty) \to (\epsilon, +\infty) \), with \( \epsilon > 0 \), will be called a Nagumo function, if

\[
    \int_{0}^{+\infty} \frac{z \, dz}{\omega(z)} = +\infty. \tag{1.7}
\]
We say that \( f: I \times \mathbb{R}^2 \to \mathbb{R} \) satisfies the Bernstein–Nagumo conditions, if for any \( r \in (0, +\infty) \) there exists a Nagumo function \( \omega_i \) such that

\[
f(t, x, y) \sgn y \geq -\omega_i(|y|) \quad \text{on } I \times [-r, r] \times \mathbb{R}
\]

and

\[
f(t, x, y) \sgn y \leq \omega_i(|y|) \quad \text{on } [a, c] \times [-r, r] \times \mathbb{R}.
\]

In what follows

\[
D(-r_1) = \{ x \in C^2(I) : x(t) > -r_1 \text{ for each } t \in I\},
\]

\[
D(r_1) = \{ x \in C^2(I) : x(t) < r_1 \text{ for each } t \in I\},
\]

where \( r_1 \in (0, +\infty) \).

2. AUXILIARY RESULTS

We shall need some lemmas whose proofs follow the approach proposed in [5]. Let us consider the equation

\[
u'' = g(t, u, u')
\]

where \( g \in C^0(I \times \mathbb{R}^2) \).

**Lemma 1.** Let \( \sigma_1 \) be a lower solution and \( \sigma_2 \) an upper solution of (2.1), (1.2) with \( \sigma_1(t) \leq \sigma_2(t) \) for each \( t \in I \). Further, let there exist \( k \in (0, +\infty) \) such that for each \( t \in I, x, y \in \mathbb{R}, \) where \( \sigma_1(t) \leq x \leq \sigma_2(t) \), the inequality

\[ |g(t, x, y)| \leq k \]

is fulfilled.

Then problem (2.1), (1.2) has a solution \( u \) fulfilling

\[
\sigma_1(t) \leq u(t) \leq \sigma_2(t) \quad \text{for each } t \in I.
\]

**Proof.** Similarly, to the proof of [5, lemma 6], we put

\[
w_i(t, x, y) = (-1)^i m(x - \sigma_i)[g(t, \sigma_i, \sigma_i') - g(t, \sigma_i, y) + (-1)^i r_0 / m], \quad i = 1, 2
\]

\[
g_m(t, x, y) = \begin{cases} 
g(t, \sigma_1, \omega_1) - r_0 / m & \text{for } x \leq \sigma_1(t) - 1/m \\
g(t, \sigma_1, \omega_1) + w_1 & \text{for } \sigma_1(t) - 1/m < x < \sigma_1(t) \\
g(t, x, y) & \text{for } \sigma_1(t) \leq x \leq \sigma_2(t) \\
g(t, \sigma_2, \omega_2) & \text{for } \sigma_2(t) < x < \sigma_2(t) + 1/m \\
g(t, \sigma_2, \omega_2) + r_0 / m & \text{for } x \geq \sigma_2(t) + 1/m,
\end{cases}
\]

where \( m \) is a natural number and \( (t, x, y) \in I \times \mathbb{R}^2 \), and consider the equation

\[
u'' = (1/m)u + g_m(t, u, u').
\]

By the Fredholm nonlinear alternative theorem, problem (2.3), (1.2) has a solution \( u_m \), because \( g_m \) is bounded and the linear problem corresponding to (2.3), (1.2) has only the trivial solution.

Similarly to [5, lemma 6], it can be checked that

\[
\sigma_1(t) = 1/m \leq u_m(t) \leq \sigma_2(t) + 1/m
\]

for each \( t \in I \) and any natural \( m \). This implies, by (1.2), (2.3), that the sequences \( (u_m)_m \) and \( (u'_m)_m \) are uniformly bounded and equi-continuous on \( I \) and thus, by the Arzelà–Ascoli
theorem, we conclude that \((u_m)\) contains a subsequence converging in \(C^1(I)\). Writing, for every \(m\), equations (2.3) in integral forms, it is easily seen that the limit of that subsequence is a solution of (2.1), (1.2) and satisfies (2.2). The proof is complete.

**Lemma 2** (on a priori estimate). Let, for \(r \in (0, +\infty)\), \(\omega\), be a Nagumo function. Then there exists a number \(\rho = \rho(r, \omega_r)\) such that for any function \(u \in C^2(I)\) the conditions

\[
\|u\|_0 \leq r, \quad u(a) = u(c), \quad u(b) = u(d),
\]

\[
u^\circ \operatorname{sgn} u' \leq \omega_r(|u'|) \quad \text{for each } t \in I,
\]

and

\[
u^\circ \operatorname{sgn} u' \geq -\omega_r(|u'|) \quad \text{for each } t \in [a, c]
\]

imply the estimate

\[
\|u'\|_0 < \rho.
\]

**Proof.** In view of (2.4) we can choose \(a_1 \in (a, c)\) such that \(u'(a_1) = 0\). From (1.7) it follows that there exists \(\rho \in (r, +\infty)\) such that

\[
\int_0^\rho \frac{z \, dz}{\omega_r(z)} > 2r.
\]

Now, let us suppose that there exists \(t_0 \in (a_1, b)\) such that

\[
u'(t_0) \geq \rho.
\]

Let \([\alpha, \beta] \subset [a, b]\) be the maximal interval containing \(t_0\) with \(u'(t) \geq 0\) for \(t \in [\alpha, \beta]\). Let \(t^* \in (\alpha, \beta)\) be such point that \(u'(t^*) = c_1 = \max\{|u'(t): \alpha \leq t \leq \beta|\}\). Then, from (2.5), it follows

\[
\int_0^{c_1} \frac{z \, dz}{\omega(z)} \leq 2r,
\]

which implies, by (2.8), \(c_1 < \rho\). The latter inequality contradicts (2.9). Similarly, supposing that there exists \(t_0 \in (a_1, b)\) with

\[
u'(t_0) \leq -\rho
\]

and choosing the maximal interval \([\alpha, \beta] \subset [a_1, b]\) such that \(t_0 \in (\alpha, \beta)\) and \(u'(t) \leq 0\) on \([\alpha, \beta]\), we can get the same contradiction. Finally, if we suppose that \(t_0\) satisfying (2.9) or (2.10) can be chosen in \([a, a_1]\), then using (2.6) instead of (2.5) we obtain a contradiction by the same arguments as above. Therefore \(\|u'\|_0 < \rho\) and the lemma is proved.

**Lemma 3.** Let \(s\) be a real number. Assume that the function \(f\) in equation (1.1) satisfies the Bernstein–Nagumo conditions. Further let \(\sigma_1\) and \(\sigma_2\) be lower and upper solutions of problem (1.1), (1.2), respectively, with \(\sigma_1(t) \leq \sigma_2(t)\) for each \(t \in I\).

Then problem (1.1), (1.2) has at least one solution \(u\) satisfying (2.2).

**Proof.** Let

\[
r_i = \max\{|\sigma_1(t)| + |\sigma_2(t)|: t \in I\}, \quad i = 0, 1.
\]

Then for \(r_0\) there exists a Nagumo function \(\omega_r\) such that \(f\) satisfies (1.8) and (1.9) where \(r = r_0\). Put \(\tilde{\omega}(z) = |s| + \omega_{r_0}(z), \, z \in (0, +\infty)\). We can easily verify that \(\tilde{\omega}\) is also a Nagumo function.
Further, let $\rho = \rho(r_0, \omega)$ be the number found by lemma 2. Put $\mu = \rho + r_0 + r_1$ and

$$
\chi(\mu, z) = \begin{cases} 
1 & \text{for } |z| \leq \mu \\
2 - z/\mu & \text{for } \mu < |z| < 2\mu \\
0 & \text{for } |z| \geq 2\mu,
\end{cases}
$$

$$
\tilde{f}(t, x, y) = \chi(\mu, |x| + |y|)f(t, x, y).
$$

Next, for fixed real $s$ consider the equation

$$
u^n + \tilde{f}(t, u, u') = s.
$$

Since $\|\sigma_i\| < \mu$, $i = 1, 2$, we can see that $\sigma_1$ is a lower solution and $\sigma_2$ an upper solution for (2.12), (1.2). Moreover

$$
|s - \tilde{f}(t, x, y)| \leq k + |s| \quad \text{on } I \times \mathbb{R}^2,
$$

where $k = \max|f(t, x, y)| : t \in I, |x| + |y| \leq 2\mu$. Thus, by lemma 1, problem (2.12), (1.2) has a solution $u$ satisfying (2.2). Therefore $u$ fulfills (2.4). Further, by (2.11) and the first part of the proof, $u$ satisfies (2.5) and (2.6) where $r = r_0$ and $\omega = \omega$. So, applying lemma 2 we get estimate (2.7). Therefore $\|u\|_1 < \mu$ and $u$ is also a solution of problem (1.1), (1.2). The lemma is proved.

3. EXISTENCE RESULTS

THEOREM 1. Let $f \in C^0(I \times \mathbb{R}^2)$ satisfy the Bernstein–Nagumo conditions and let there exist numbers $r_1 > 0$ and $s_1$ such that for all $t \in I$

$$
f(t, -r_1, 0) > s_1 > f(t, 0, 0). \tag{3.1}
$$

Then there exists $s_0 < s_1$ (with the possibility $s_0 = -\infty$) such that

(a) for $s < s_0$, (1.1), (1.2) has no solution in $D(-r_1)$,

(b) for $s \in (s_0, s_1)$, (1.1), (1.2) has at least one solution $u \in D(-r_1)$. [For $D(-r_1)$ see (1.10).]

The proof of theorem 1 follows the approach proposed in [1], for periodic solutions. However condition (3.1) is weaker than the corresponding one in [1], where it is assumed

$$
f(t, x, 0) > s_1 > f(t, 0, 0) \quad \text{for all } t \in \mathbb{R} \text{ and all } x \leq -r_1. \tag{3.2}
$$

Proof. Put

$$
h(t, x, y) = \begin{cases} 
f(t, x, y) & \text{for } x \geq -r_1 \\
f(t, -r_1, y) & \text{for } x < -r_1
\end{cases}
$$

and consider the equation

$$
u^n + h(t, u, u') = s. \tag{3.4}
$$

We can see that $h$ satisfies the Bernstein–Nagumo conditions. Let $s^* = \max[h(t, 0, 0) : t \in I]$. Then for $s = s^*$, 0 is an upper solution and $-r_1$ is a lower solution for (3.4), (1.2). Then, by lemma 3, problem (3.4), (1.2) has a solution $u^*$ with $-r_1 \leq u^*(t) \leq 0$ on $I$. By (3.3), $u^*$ is a solution for (1.1), (1.2) as well.
Now, suppose first that (3.4), (1.2) has a solution \( u \) for some \( s \leq s_1 \) and show that \( u \in D(-r_1) \). Let, on the contrary, \( \min \{u(t) : t \in I\} = u(t_0) \leq -r_1 \). Then, by (1.2), \( u'(t_0) = 0, u''(t_0) \geq 0 \). On the other hand, from (3.1), (3.3), it follows that

\[
u''(t_0) = s - h(t_0, u(t_0), 0) = s - f(t_0, -r_1, 0) \leq s_1 - f(t_0, -r_1, 0) < 0,\]

a contradiction.

Next, let us show that if problem (3.4), (1.2) has a solution \( \bar{u} \) for \( s = \bar{s} < s_1 \), then it has at least one solution for each \( s \in [\bar{s}, s_1] \). From the above considerations it follows that \( \bar{u} \in D(-r_1) \).

Further, \( \bar{u}'' + h(t, \bar{u}, \bar{u}') = \bar{s} \leq s \) and so \( \bar{u} \) is an upper solution for (3.4), (1.2), where \( s \in [\bar{s}, s_1] \).

Similarly, since \( h(t, -r_1, 0) > s_1 \geq s, -r_1 \) is a lower solution for (3.4), (1.2), where \( s \in [\bar{s}, s_1] \).

Hence, we can use lemma 3 again to get that (3.4), (1.2) has at least one solution in \( D(-r_1) \) provided \( s \in [\bar{s}, s_1] \). From the latter it is a solution for (1.1), (1.2) as well.

Finally, taking \( s_0 = \inf \{s \in \mathbb{R} : (1.1), (1.2) \text{ has at least one solution in } D(-r_1)\} \), we have \( s_0 \leq s_* < s_1 \) and from the above considerations (a) and (b) follow. The theorem is proved.

**Theorem 2.** Let \( f \in C^0(I \times \mathbb{R}^2) \) satisfy the Bernstein–Nagumo conditions and let there exist numbers \( r_1 > 0 \) and \( s_1 \) such that for all \( t \in I \)

\[
f(t, 0, 0) > s_1 > f(t, r_1, 0).
\]

Then there exists \( s_0 > s_1 \) (with the possibility \( s_0 = +\infty \)) such that

(a) for \( s > s_0 \), (1.1), (1.2) has no solution in \( \overline{D(-r_1)} \),

(b) for \( s \in [s_1, s_0] \), (1.1), (1.2) has at least one solution in \( D(r_1) \).

**Proof.** Theorem 2 can be obtained from theorem 1 if \( f \) is replaced by \( -f \) and \( x \) by \( -x \).

4. **MULTICLICITY RESULTS**

**Theorem 3.** Let \( f \in C^0(I \times \mathbb{R}^2) \) satisfy the Bernstein–Nagumo conditions and let there exist \( r_1, r_2 \in (0, +\infty), s_1 \in \mathbb{R} \) such that for all \( t \in I \) the inequality (3.1) is fulfilled and for all \( s \leq s_1 \) any solution \( u_s \) of (1.1), (1.2) belonging to \( D(-r_1) \) satisfies

\[
u_s(t) < r_2 \quad \text{for each } t \in I.
\]

Then the number \( s_0 \) in theorem 1 is finite and

(a) for \( s < s_0 \), problem (1.1), (1.2) has no solution in \( \overline{D(-r_1)} \),

(b) for \( s = s_0 \), problem (1.1), (1.2) has at least one solution in \( \overline{D(-r_1)} \),

(c) for \( s \in [s_0, s_1] \), problem (1.1), (1.2) has at least two solutions in \( D(-r_1) \).

A similar theorem for a periodic problem is proved in [1], where the stronger condition (3.2) is assumed instead of (3.1) and moreover the function \( f(\cdot, \cdot, 0) \) is required to be bounded below.

Theorem 3 is valid not only for problem (1.1), (1.2) but also for Neumann and periodic problems.

**Proof.** Let us consider the equation (3.4) where \( h \) satisfies (3.3). Then \( h \) fulfills the Bernstein–Nagumo conditions and, according to the proof of theorem 1, each solution of problem (3.4), (1.2) belongs to \( D(-r_1) \) provided \( s \leq s_1 \).
Now, proving theorem 3, we shall need several auxiliary propositions.

**Proposition 1.** There exist numbers \( \sigma, M, \sigma < s_1 < M \), such that for any \( s \leq s_1 \) and any solution \( u_s \) of (3.4), (1.2)

\[
\sigma \leq h(t, u_s, 0) \leq M \quad \text{for each } t \in I. \tag{4.2}
\]

**Proof of proposition 1.** Let \( s \leq s_1 \). Then, by (4.1), any solution \( u_s \) of (3.4), (1.2) fulfills

\[
-r_1 < u_s(t) < r_2 \quad \text{for each } t \in I. \tag{4.3}
\]

Therefore we can put

\[
\sigma = \min\{h(t, x, 0) : t \in I, x \in [-r_1, r_2]\} \quad \text{and} \quad M = \max\{h(t, x, 0) : t \in I, x \in [-r_1, r_2]\}.
\]

From (3.1), (3.3) it follows that

\[
\sigma < s_1 < M. \tag{4.4}
\]

**Proposition 2.** There exists \( s_0 \in [\sigma, s_1] \) such that for \( s < s_0 \), problem (3.4), (1.2) has no solution and for \( s \in (s_0, s_1] \) it has at least one solution in \( D(-r_1) \).

**Proof of proposition 2.** Suppose on the contrary that for \( s < \sigma \) problem (3.4), (1.2) has a solution. Then, by (4.3), \( \min\{u(t) : t \in I\} = u(t_0) \in (-r_1, r_2) \), \( u'(t_0) = 0 \), \( u''(t_0) \geq 0 \). On the other hand, by (4.2), \( u''(t_0) < 0 \), which is impossible. Hence there exists \( s_0 \geq \sigma \) such that (3.4), (1.2) has no solution for \( s < s_0 \). By (4.4) and theorem 1 we can deduce \( s_0 < s_1 \) and (3.4), (1.2) has at least one solution in \( D(-r_1) \) for each \( s \in (s_0, s_1] \).

From now on, let \( \xi \in (s_0, s_1) \) be fixed and \( \bar{u} \) denote a solution of (3.4), (1.2) for \( s = \xi \). Then \( \bar{u} \in D(-r_1) \). Further, let for \( t \in I, x, y \in \mathbb{R} \)

\[
\alpha(x) = \begin{cases} 
-r_1 & \text{for } x < -r_1 \\
0 & \text{for } -r_1 \leq x \leq \bar{u}(t) \\
\bar{u}(t) & \text{for } x > \bar{u}(t)
\end{cases}
\]

and

\[
g(t, x, y) = f(t, \alpha(x), y) - x + \alpha(x). \tag{4.5}
\]

We shall consider the equation

\[
u'' + g(t, u, u') = s. \tag{4.6}
\]

**Proposition 3.** For each \( s \in (\xi, s_1] \) any solution \( u \) of problem (4.6), (1.2) satisfies

\[-r_1 < u < \bar{u} \quad \text{on } I. \]

**Proof of proposition 3.** Let \( u \) be a solution of (4.6), (1.2) where \( s \in (\xi, s_1] \). Suppose that for some \( t \in I \) \( u(t) \geq \bar{u}(t) \). Then there exists \( t_0 \in (a, b) \) such that \( u(t_0) \geq \bar{u}(t_0) \), \( u'(t_0) = \bar{u}'(t_0) \), \( u''(t_0) \leq \bar{u}''(t_0) \). But from (4.5) we can get \( u''(t_0) > \bar{u}''(t_0) \), which is a contradiction. The inequality \(-r_1 < u \) can be proved by similar arguments.
Now, for an arbitrary fixed $s \leq s_1$, let us consider the class of equations

$$u'' - (1 - \lambda)u + \lambda[g(t, u, u') - s] = 0, \quad (4.7\lambda)$$

where a real parameter $\lambda$ varies from 0 to 1.

**Proposition 4.** There exist positive numbers $R$, $\rho$ such that for any $s \in [s_0, s_1]$ and any parameter $\lambda \in [0, 1]$, every solution $u$ of $(4.7\lambda)$, (1.2) satisfies

$$\|u\|_0 < R, \quad \|u'\|_0 < \rho.$$

**Proof of proposition 4.** Let us choose an arbitrary fixed $s \in [s_0, s_1]$ and a number $R$ with

$$R > \max[r_1 + s_1 - \sigma, r_2 + M - s_0]. \quad (4.8)$$

Suppose that for some $\lambda \in [0, 1]$ and for a corresponding solution $u$ of $(4.7\lambda)$, (1.2) we can find $t_0 \in I$ such that $|u(t_0)| \geq R$. Let $u(t_0) \geq R$. Then, in view of (1.2),

$$u'(t_0) = 0, \quad u''(t_0) \leq 0$$

and by $(4.7\lambda)$, (1.2), (4.8),

$$u''(t_0) \geq (1 - \lambda)R + \lambda[s_0 + M - r_2] > 0,$$

a contradiction. Similarly, if $u(t_0) \leq -R$, then we get

$$0 \leq u''(t_0) \leq -(1 - \lambda)R + \lambda(s_1 - \sigma - R + r_1) < 0,$$

a contradiction. Therefore $\|u\|_0 < R$.

Further, since $f$ satisfies the Bernstein-Nagumo conditions, there exists a Nagumo function $\omega_R$ and $u'' \text{ sgn } u'' < \omega_R(|u'|) + R + S_2$ on $I$ and $u'' \text{ sgn } u'' > -\omega_R(|u'|) - R - S_2$ on $[a, c]$, where $S_2 = \max\{|s_0 - 1|, |s_1|\}$. We can easily check that $\tilde{\omega} = \omega_R + R + S_2$ is a Nagumo function, and so, using lemma 2 for $r = R$ and $\omega_r = \tilde{\omega}$ we can find a number $\rho = \rho(R, \tilde{\omega})$ such that $\|u\|_0 < \rho$.

Let us put

$$\text{dom } L = \{u \in C^2(I) : u(a) = u(c), \ u(b) = u(d)\},$$

$$L : \text{dom } L \rightarrow C^0(I), \quad u \mapsto u'',$$

$$N_s : C^1(I) \rightarrow C^0(I), \quad u \mapsto h(\cdot, u(\cdot), u'(\cdot)) - s,$$

$$G_s : C^1(I) \rightarrow C^0(I), \quad u \mapsto g(\cdot, u(\cdot), u'(\cdot)) - s,$$

$$I : C^1(I) \rightarrow C^1(I), \quad u \mapsto u.$$

Then problems (3.4), (1.2) or (4.6), (1.2) or (4.7\lambda), (1.2) can be written in the forms

$$\quad (L + N_s)u = 0, \quad (4.9)$$

or

$$\quad (L + G_s)u = 0, \quad (4.10)$$

or

$$\quad (L - (1 - \lambda)I + \lambda G_s)u = 0. \quad (4.11)$$

Similarly to the periodic case, it can be proved (see [6]), that $N_s$ and $G_s$ are $L$-compact on $C^1(I)$, so that the coincidence degree method (see [2]) can be applied to problems (4.9)-(4.11).
Let us consider two open bounded sets in $C^1(I)$:

$$\Omega = \{ u \in C^1(I) : -r_1 < u(t) < 0 \text{ for each } t \in I, \| u' \|_0 < \rho \}$$

and

$$\Omega_1 = \{ u \in C^1(I) : \| u \|_0 < R, \| u' \|_0 < \rho \},$$

where $R$ and $\rho$ are numbers in proposition 4.

**Proposition 5.** Let $s \in (\bar{s}, s_1]$. Then

$$d_L(L + N_s, \Omega) = \pm 1. \quad (4.12)$$

**Proof of proposition 5.** Suppose that $s \in (\bar{s}, s_1]$. Then, by proposition 4, for any $\lambda \in [0, 1]$, each solution $u$ of (4.11) belongs to $\Omega_1$ and so $u \notin \partial \Omega_1$. Further, for $\lambda = 0$, (4.11) has the form $(L - I)u = 0$ and since $\ker(C(L - I)) = \{0\}$, we get

$$d_L(L - I, \Omega_1) = \pm 1.$$  

(See [2, proposition II.16].) Next, for $\lambda = 1$, (4.11) is equal to (4.10) and so, by the property of invariance under homotopy (see [2, p. 15]) we have

$$d_L(L + G_s, \Omega_1) = \pm 1.$$  

Now, using propositions 3 and 4, we get for each solution $u$ of (4.10) that $u \in \Omega$. Therefore, by the excess property [2, p. 15],

$$d_L(L + G_s, \Omega) = \pm 1.$$  

Since, $N_s = G_s$ on $\Omega$, we get

$$d_L(L + N_s, \Omega) = \pm 1.$$  

**Proposition 6.** Let $s \in (\bar{s}, s_1]$. Then

$$d_L(L + N_s, \Omega_1 \setminus \bar{\Omega}) = \pm 1. \quad (4.13)$$

**Proof of proposition 6.** Clearly $\Omega_1 \setminus \bar{\Omega}$ is a nonempty open bounded set in $C^1(I)$. Since problem (4.9) has no solution for $s < s_0$ (see proposition 2), it is an immediate consequence of the existence property (see [2, p. 16]) that, for $s < s_0$

$$d_L(L + N_s, \Omega_1) = 0. \quad (4.14)$$

On the other hand, by (4.3), for $s_0 - 1 < s \leq s_1$ any solution $u$ of (4.9) belongs to $\Omega_1$ and so $u \notin \partial \Omega_1$ (see proposition 4). Letting $s$ vary from $s_0 - 1$ to $s_1$ we can deduce by the property of invariance under homotopy that (4.14) holds for each $s \in (s_0 - 1, s_1]$. Now, for $s \in (\bar{s}, s_1]$, using (4.12) and (4.14), it follows from the additivity property of degree (see [2, p. 15]) that

$$d_L(L + N_s, \Omega_1 \setminus \bar{\Omega}) = \pm 1.$$  

Now, by means of the above propositions, we can complete the proof of theorem 3 as follows.

Proposition 2 and relation (3.3), together with the fact that any solution of (3.4), (1.2) belongs to $D(-r_1)$, imply assertion (a).
The relations (4.12) and (4.13) imply that, for \( s \in (s, s_1] \), equation (4.9) has at least one solution in \( \Omega \) and at least another one in \( \Omega_1 \setminus \Omega \). Since any solution of (4.9) belongs to \( D(-r_1) \) and \( s \) is arbitrary in \( (s_0, s_1] \), conclusion (c) is proved.

Finally, to prove (b), let \( (u_n)_1^n \) be a sequence in \( (s_0, s_1) \) which converges to \( s_0 \) and let \( u_n \) be a solution of (4.9) with \( s = t_n \). Using proposition 4, one gets \( (u_n)_1^n \) bounded in \( C^1(I) \) and hence in \( C^2(I) \) by the equation. By Ascoli’s theorem and the integrated form of the equation, one gets the existence of a converging subsequence of \( (u_n)_1^n \) whose limit is a solution \( u_0 \) of (4.9) with \( s = s_0 \). Clearly, \( u_0 \in D(-r_1) \) is a solution of (1.1), (1.2). Theorem 3 is proved.

Similarly we can prove theorem 4.

**Theorem 4.** Let \( f \in C^0(I \times \mathbb{R}^2) \) satisfy the Bernstein–Negumo conditions and let there exist \( r_1, r_2 \in (0, +\infty) \), \( s_1 \in \mathbb{R} \) such that for all \( t \in I \) the inequality (3.5) is fulfilled and for all \( s \geq s_1 \) any solution \( u_s \) of (1.1), (1.2) belonging to \( D(r_1) \) satisfies

\[-r_1 < u_s(t) \quad \text{for each } t \in I.\]

Then the number \( s_0 \) in theorem 2 is finite and

(a) for \( s > s_0 \), problem (1.1), (1.2) has no solution in \( D(r_1) \),

(b) for \( s = s_0 \), problem (1.1), (1.2) has at least one solution in \( D(r_1) \),

(c) for \( s \in [s_1, s_0) \), problem (1.1), (1.2) has at least two solutions in \( D(r_1) \).

**REFERENCES**


5. RACHŮNKOVÁ I., On a certain four-point problem, Preprint.