Properties of the set of positive solutions to Dirichlet boundary value problems with time singularities

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Abstract

The paper investigates the structure and properties of the set $\mathcal{S}$ of all positive solutions to the singular Dirichlet boundary value problem $u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = f(t, u(t), u'(t))$, $u(0) = 0$, $u(T) = 0$. Here $a \in (-\infty, -1)$ and f satisfies the local Carathéodory conditions on $[0, T] \times \mathcal{D}$, where $\mathcal{D} = [0, \infty) \times \mathbb{R}$. It is shown that $\mathcal{S}_c = \{u \in \mathcal{S} : u'(T) = -c\}$ is nonempty and compact for each $c \geq 0$ and $\mathcal{S} = \cup_{c\geq 0}\mathcal{S}_c$. The uniqueness of the problem is discussed. Having a special case of the problem, we introduce an ordering in $\mathcal{S}$ showing that the difference of any two solutions in $\mathcal{S}_c$, $c \geq 0$, keeps its sign on $[0, T]$. The application on the equation $v''(t) + \frac{k}{t}v'(t) = \psi(t) + g(t, v(t))$, $k \in (1, \infty)$, is given here.

Mathematics Subject Classification 2010: 34B18, 34B16, 34A12

Key words: Singular ordinary differential equation of the second order, time singularities, set of all positive solutions, uniqueness, blow-up solutions.

1 Introduction

We consider the singular Dirichlet boundary value problem

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = f(t, u(t), u'(t)), \tag{1}$$

$$u(0) = 0, \quad u(T) = 0, \tag{2}$$

where $a \in (-\infty, -1)$. For $\mathcal{D} = [0, \infty) \times \mathbb{R}$ we assume that f satisfies the local Carathéodory conditions on $[0, T] \times \mathcal{D}$ ($f \in Car([0, T] \times \mathcal{D})$), that is

(i) $f(\cdot, x, y) : [0, T] \to \mathbb{R}$ is measurable for all $(x, y) \in \mathcal{D}$,

(ii) $f(t, \cdot, \cdot) : \mathcal{D} \to \mathbb{R}$ is continuous for a.e. $t \in [0, T]$,

(iii) for each compact set $\mathcal{U} \subset \mathcal{D}$ there exists a function $m_{\mathcal{U}} \in L^1[0, T]$ such that

$$|f(t, x, y)| \le m_{\mathcal{U}}(t) \text{ for a.e. } t \in [0, T] \text{ and all } (x, y) \in \mathcal{U}.$$

Equation (1) has a time singularity at $t = 0$ due to the differential operator on its left hand side. This operator has an equivalent form $(t^{-a}(t^a u)')'$ and, after the substitution $x(t) = t^a u(t)$ it takes the form $(t^{-a}x')'$. It is shown in [18], that such type of operators appears cf. in the study of phase transitions of Van der Waals fluids [4], [10], [16], [25], in population genetics, where it appears in models for the spatial distribution of the genetic composition of a population [8], [9], in the homogenenous nucleation theory [1], in relativistic cosmology for description of particles which can be treated as domains in the universe [20], in the nonlinear field theory [11], in particular, when describing bubbles generated by scalar fields of the Higgs type in the Minkowski spaces [7].

We say that $u : [0, T] \to \mathbb{R}$ is *a positive solution of problem* (1), (2) if $u \in AC^1[0, T]$, $u > 0$ on $(0, T)$, u satisfies the boundary conditions (2) and (1) holds for a.e. $t \in [0, T]$.

Clearly, for each positive solution u of problem (1), (2) there exists $c \ge 0$ such that

$$u'(T) = -c. \tag{3}$$

We denote the set of all positive solutions of problem (1), (2), (3) by $\mathcal{S}_c$ and prove that $\mathcal{S}_c$ is nonempty and compact for each $c \ge 0$.

In literature, there is a lot of results about the existence of solutions of various singular problems, for monographs see e.g. [2], [3], [14], [15], [21], [22], [23]. Here, we provide besides the solvability of problem (1), (2), the deeper study of the set of all its positive solutions. Our main goal is to prove the properties of the set $\mathcal{S} = \cup_{c\ge0}\mathcal{S}_c$. In particular, having a special case of (1), we introduce some ordering in $\mathcal{S}$ showing that the difference of any two solutions in $\mathcal{S}_c$, $c \ge 0$, keeps its sign on $[0, T]$. Then we prove that there exist minimal and maximal solutions $u_{c,min}, u_{c,max} \in \mathcal{S}_c$ for each $c \ge 0$. If the interior of the set $\{(t, x) \in \mathbb{R}^2 : 0 \le t \le T,\ u_{c,min}(t) \le x \le u_{c,max}(t)\}$ is nonempty, we prove the interesting result that this interior is covered by graphs of other solutions of $\mathcal{S}_c$ for each $c > 0$. The uniqueness of solutions of problem (1), (2), (3) is discussed and we prove two uniqueness results. The first one is generic and need not the Lipschitz continuity of f. At the end of the paper we provide the application of the results obtained for solutions of problem (1), (2) onto the equation $v'' + \frac{k}{t}v' = \psi(t) + g(t, v)$, satisfying $v(T) = 0$. In contrast to the literature, [12], [13], [17], [19], [24], our solutions are unbounded at the left end point $t = 0$ of $[0, T]$ (see condition (25)).

We work with the following conditions on f in (1).

(H_1) $f \in Car([0, T] \times \mathcal{D})$, where $\mathcal{D} = [0, \infty) \times \mathbb{R}$.

(H_2) There exists $\varphi \in L^1[0,T]$ such that

$$0 < \varphi(t) \le f(t,x,y) \quad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathcal{D}.$$

(H_3) For a.e. $t \in [0,T]$ and all $(x,y) \in \mathcal{D}$ the estimate

$$f(t,x,y) \le h(t,x,|y|),$$

is fulfilled, where $h \in Car([0,T] \times [0,\infty)^2)$, $h(t,x,z)$ is nondecreasing in the variables x, z, and

$$\lim_{x\to\infty} \frac{1}{x} \int_0^T h(t,x,x)\,\mathrm{d}t = 0.$$

Let us by $L^1[0,T]$ denote the set of functions which are (Lebesgue) integrable on $[0,T]$ equipped with the norm $\|x\|_1 = \int_0^T |x(t)|\,\mathrm{d}t$. Moreover, let us by $C[0,T]$ and $C^1[0,T]$ denote the set of functions being continuous on $[0,T]$, and having continuous first derivative on $[0,T]$, respectively. The norm on $C[0,T]$ and $C^1[0,T]$ is defined as $\|x\|_\infty = \max_{t\in[0,T]} |x(t)|$ and $\|x\|_\infty + \|x'\|_\infty$, respectively. Further, we denote by $AC^1[0,T]$ the set of functions which have absolutely continuous first derivatives on $[0,T]$, while $AC^1_{loc}(0,T]$ is the set of functions having absolutely continuous derivatives on each compact subinterval of $(0,T]$. Finally, for $J \subset \mathbb{R}$ we denote by $PC^1(J)$ the set of functions continuous on J and having piecewise continuous derivatives on J.

The paper is organized as follows. Section 2 is devoted to the study of three operators associated to problem (1), (2). In Section 3 we prove the existence and properties of positive solutions of (1), (2). Section 4 deals with the special case of problem (1), (2) and presents the structure and further properties of the set of all positive solutions. Section 5 contains some blow-up results. Throughout the paper $a \in (-\infty, -1)$.

2 Operators

In order to prove the properties of the sets $\mathcal{S}$ and $\mathcal{S}_c$, $c \ge 0$, we will introduce three operators $\mathcal{H}$, $\mathcal{K}_{t_0,A}$ and $\mathcal{L}_c$ acting on $C^1[0,T]$. To do this we will need an auxiliary function $\tilde{f} : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ given by the formula

$$\tilde{f}(t,x,y) = \begin{cases} f(t,x,y) & \text{if } x \ge 0 \\ f(t,0,y) & \text{if } x < 0. \end{cases}$$

Under conditions $(H_1) - (H_3)$, $\tilde{f}$ satisfies

$(\tilde{H}_1)$ $\tilde{f} \in Car([0,T] \times \mathbb{R}^2)$.

$(\tilde{H}_2)$ There exists $\varphi \in L^1[0,T]$ such that

$$0 < \varphi(t) \le \tilde{f}(t,x,y) \ \text{ for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathbb{R}^2.$$

$(\tilde{H}_3)$ For a.e. $t \in [0,T]$ and all $(x,y) \in \mathbb{R}^2$ the estimate

$$\tilde{f}(t,x,y) \le h(t,|x|,|y|),$$

is fulfilled, where h is given in (H_3).

Now, we put

$$(\mathcal{H}x)(t) = t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1}\tilde{f}(\xi,x(\xi),x'(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s. \tag{4}$$

Further, for each $t_0 \in (0,T)$ and $A \ge 0$ we define

$$(\mathcal{K}_{t_0,A}x)(t) = \frac{t}{t_0}\frac{T^{-a-1}-t^{-a-1}}{T^{-a-1}-t_0^{-a-1}}\max\{0, A-(\mathcal{H}x)(t_0)\} + (\mathcal{H}x)(t), \tag{5}$$

and for each $c \ge 0$ we define

$$(\mathcal{L}_c x)(t) = t\frac{cT^{a+1}}{|a+1|}(T^{-a-1}-t^{-a-1}) + (\mathcal{H}x)(t). \tag{6}$$

The following lemmas will be needed in our proofs.

Lemma 1 *Let $p \in L^1[0,T]$. Then the inequalities*

$$\left|t^{-a-1}\int_t^T s^{a+1}p(s)\,\mathrm{d}s\right| \le \int_t^T |p(s)|\,\mathrm{d}s, \tag{7}$$

$$\left|\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1}p(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s\right| \le \frac{1}{|a+1|}\int_t^T |p(s)|\,\mathrm{d}s \tag{8}$$

are fulfilled for $t \in [0,T]$.

Proof. Inequality (7) follows from the relation

$$\left|t^{-a-1}\int_t^T s^{a+1}p(s)\,\mathrm{d}s\right| \le t^{-a-1}\int_t^T s^{a+1}|p(s)|\,\mathrm{d}s \le \int_t^T |p(s)|\,\mathrm{d}s.$$

Since

$$\left|\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1}p(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s\right| \le \int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1}|p(\xi)|\,\mathrm{d}\xi\right)\mathrm{d}s$$

and integration by parts gives (note that $a+1<0$)

$$\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1}|p(\xi)|\,\mathrm{d}\xi\right)\mathrm{d}s = \frac{t^{-a-1}}{a+1}\int_t^T s^{a+1}|p(s)|\,\mathrm{d}s - \frac{1}{a+1}\int_t^T |p(s)|\,\mathrm{d}s$$
$$\leq -\frac{1}{a+1}\int_t^T |p(s)|\,\mathrm{d}s = \frac{1}{|a+1|}\int_t^T |p(s)|\,\mathrm{d}s,$$

we see that (8) holds for $t\in[0,T]$. □

Lemma 2 *Let (H_1), (H_2) hold. Then*
(a) $\mathcal{H}: C^1[0,T]\to C^1[0,T]$,
(b) $\mathcal{H}$ is completely continuous.

Proof.
(a) Let $x\in C^1[0,T]$. We see that $(\mathcal{H}x)\in C^1(0,T]$. Since $\tilde f$ fulfils conditions $(\tilde H_1)$ and $(\tilde H_2)$, the functions

$$\varphi_1(t) := \int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1}\tilde f(\xi,x(\xi),x'(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s,$$

$$\varphi_2(t) := \int_t^T \xi^{a+1}\tilde f(\xi,x(\xi),x'(\xi))\,\mathrm{d}\xi$$

are continuous, positive and decreasing on $(0,T]$. Hence there exist $\lim_{t\to 0+}\varphi_1(t)$, $\lim_{t\to 0+}\varphi_2(t)$ and, by (7), (8),

$$0\leq \lim_{t\to 0+}\varphi_1(t)\leq \frac{1}{|a+1|}\int_0^T \tilde f(\xi,x(\xi),x'(\xi))\,\mathrm{d}\xi,$$

$$0\leq \lim_{t\to 0+} t^{-a-1}\varphi_2(t)\leq \int_0^T \tilde f(\xi,x(\xi),x'(\xi))\,\mathrm{d}\xi.$$

Since $(\mathcal{H}x)(t)=t\varphi_1(t)$ and $(\mathcal{H}x)'(t)=\varphi_1(t)-t^{-a-1}\varphi_2(t)$ for $t\in(0,T]$, we conclude that $(\mathcal{H}x)\in C^1[0,T]$.

(b) We start to prove that $\mathcal{H}$ is continuous. To this end let $\{x_n\}\subset C^1[0,T]$ be convergent to x in $C^1[0,T]$. Denote

$$r_n(t)=\tilde f(t,x_n(t),x_n'(t))-\tilde f(t,x(t),x'(t)) \ \text{ for a.e. } t\in[0,T] \text{ and all } n\in\mathbb{N}.$$

Then (7), (8) yield

$$|(\mathcal{H}x_n)(t)-(\mathcal{H}x)(t)|\leq t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1}|r_n(\xi)|\,\mathrm{d}\xi\right)\mathrm{d}s\leq \frac{T\|r_n\|_1}{|a+1|},$$

$$|(\mathcal{H}x_n)'(t)-(\mathcal{H}x)'(t)|\leq \int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1}|r_n(\xi)|\,\mathrm{d}\xi\right)\mathrm{d}s+t^{-a-1}\int_t^T s^{a+1}|r_n(s)|\,\mathrm{d}s$$

$$\leq \left(\frac{1}{|a+1|}+1\right)\|r_n\|_1$$

for $t \in [0,T]$ and $n \in \mathbb{N}$. In particular,

$$\begin{aligned}\|\mathcal{H}x_n - \mathcal{H}x\|_\infty &\leq \frac{T\|r_n\|_1}{|a+1|},\\ \|(\mathcal{H}x_n)' - (\mathcal{H}x)'\|_\infty &\leq \left(\frac{1}{|a+1|}+1\right)\|r_n\|_1\end{aligned}$$

for $n \in \mathbb{N}$. If we prove that $\lim_{n\to\infty}\|r_n\|_1 = 0$, then the above inequalities guarantee that $\mathcal{H}$ is a continuous operator. From the fact that $\tilde{f} \in Car([0,T]\times\mathbb{R}^2)$ and $\{x_n\}$ is bounded in $C^1[0,T]$, it follows that

$$|\tilde{f}(t, x_n(t), x_n'(t)| \leq \rho(t) \text{ for a.e. } t \in [0,T] \text{ and all } n \in \mathbb{N},$$

where $\rho \in L^1[0,T]$. Since

$$\lim_{n\to\infty} \tilde{f}(t, x_n(t), x_n'(t)) = \tilde{f}(t, x(t), x'(t)) \text{ for a.e. } t \in [0,T],$$

the Lebesgue dominated convergence theorem yields $\lim_{n\to\infty}\|r_n\|_1 = 0$.

Now, we choose a bounded set $\Omega \subset C^1[0,T]$ and prove that the set $\mathcal{H}(\Omega)$ is relatively compact in $C^1[0,T]$. The boundedness of Ω implies the existence of $\mu \in L^1[0,T]$ such that

$$|\tilde{f}(t, x(t), x'(t))| \leq \mu(t) \text{ for a.e. } t \in [0,T] \text{ and all } x \in \Omega.$$

Therefore, by (7), (8), we get

$$|(\mathcal{H}x)(t)| \leq t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1}\mu(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s \leq \frac{T\|\mu\|_1}{|a+1|},$$

$$\begin{aligned}|(\mathcal{H}x)'(t)| &\leq \int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1}\mu(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s + t^{-a-1}\int_t^T s^{a+1}\mu(s)\,\mathrm{d}s\\ &\leq \left(\frac{1}{|a+1|}+1\right)\|\mu\|_1\end{aligned}$$

for $t \in [0,T]$ and $x \in \Omega$. We have proved that the set $\mathcal{H}(\Omega)$ is bounded in $C^1[0,T]$. We now show that the set $\{x' : x \in \mathcal{H}(\Omega)\}$ is equicontinuous on $[0,T]$. For a.e. $t \in [0,T]$ and all $x \in \Omega$ we have that

$$|(\mathcal{H}x)''(t)| \leq t^{-a-2}\int_t^T s^{a+1}\mu(s)\,\mathrm{d}s + |a+1|t^{-a-2}\int_t^T s^{a+1}\mu(s)\,\mathrm{d}s + \mu(t).$$

Since, by (8),

$$t^{-a-2}\left(\int_t^T s^{a+1}\mu(s)\,\mathrm{d}s\right) \in L^1[0,T],$$

there exists a majorant function $\mu^* \in L^1[0,T]$ such that $|(\mathcal{H}x)''(t)| \le \mu^*(t)$ for a.e. $t \in [0,T]$ and all $x \in \Omega$. As a result the set $\{x' : x \in \mathcal{H}(\Omega)\}$ is equicontinuous on $[0,T]$ and consequently, the set $\mathcal{H}(\Omega)$ is relatively compact in $C^1[0,T]$ by the Arzelà-Ascoli theorem. □

Lemma 3 *Let* (H_1), (H_2) *hold. Then*

(a) the operator $\mathcal{K}_{t_0,A} : C^1[0,T] \to C^1[0,T]$ *is completely continuous for each* $t_0 \in (0,T)$ *and* $A \ge 0$*;*

(b) the operator $\mathcal{L}_c : C^1[0,T] \to C^1[0,T]$ *is completely continuous for each* $c \ge 0$.

Proof.

(a) Let us choose $t_0 \in (0,T)$ and $A \ge 0$. Since $\mathcal{H}$ is completely continuous by Lemma 2, it suffices to prove that an operator $\mathcal{Q} : C^1[0,T] \to C^1[0,T]$ given by

$$(\mathcal{Q}x)(t) = \frac{t}{t_0} \frac{T^{-a-1} - t^{-a-1}}{T^{-a-1} - t_0^{-a-1}} \max\{0, A - (\mathcal{H}x)(t_0)\}$$

is completely continuous. The continuity of $\mathcal{Q}$ follows from the inequality

$$|\max\{0, A - (\mathcal{H}x)(t_0)\} - \max\{0, A - (\mathcal{H}y)(t_0)\}| \le |(\mathcal{H}x)(t_0) - (\mathcal{H}y)(t_0)|$$

for $x, y \in C^1[0,T]$. Let $\Omega \subset C^1[0,T]$ be bounded. Then the set $\{(\mathcal{H}x)(t_0) : x \in \Omega\}$ is bounded in $\mathbb{R}$, and therefore there exists a positive constant S such that $0 \le \max\{0, A - (\mathcal{H}x)(t_0)\} \le S$ for $x \in \Omega$. Hence the relations

$$\begin{aligned}
0 \le (\mathcal{Q}x)(t) &\le \frac{T^{-a}S}{t_0(T^{-a-1} - t_0^{-a-1})}, \\
|(Qx)'(t)| &= \left| \frac{T^{-a-1} + at^{-a-1}}{t_0(T^{-a-1} - t_0^{-a-1})} \max\{0, A - (\mathcal{H}x)(t_0)\} \right| \\
&\le \frac{T^{-a-1}(|a|+1)S}{t_0(T^{-a-1} - t_0^{-a-1})}, \\
|(\mathcal{Q}x)'(t_2) - (\mathcal{Q}x)'(t_1)| &= \left| \frac{a(t_2^{-a-1} - t_1^{-a-1})}{t_0(T^{-a-1} - t_0^{-a-1})} \max\{0, A - (\mathcal{H}x)(t_0)\} \right| \\
&\le \frac{|a||t_2^{-a-1} - t_1^{-a-1}|S}{t_0(T^{-a-1} - t_0^{-a-1})}
\end{aligned}$$

hold for $t, t_1, t_2 \in [0,T]$ and $x \in \Omega$. As a result the set $\{\mathcal{Q}x : x \in \Omega\}$ is bounded in $C^1[0,T]$, and since the function t^{-a-1} is continuous on $[0,T]$ and therefore it is uniformly continuous on this interval, the set $\{(\mathcal{Q}x)' : x \in \Omega\}$ is equicontinuous on $[0,T]$. By the Arzelà-Ascoli theorem, the set $\{\mathcal{Q}x : x \in \Omega\}$ is relatively compact in $C^1[0,T]$.

(b) The assertion is a consequence of Lemma 2. □

Lemma 4 *Let* (H_1) *-* (H_3) *hold. Then for each* $t_0 \in (0,T)$ *and each* $A \geq 0$*, the set*

$$\mathcal{M} = \{x \in C^1[0,T] : x = \lambda \mathcal{K}_{t_0,A} x \text{ for some } \lambda \in [0,1]\}$$

is bounded.

Proof. Let us fix $t_0 \in (0,T)$ and $A \geq 0$ and let $x = \lambda \mathcal{K}_{t_0,A} x$ for some $\lambda \in [0,1]$. Then

$$\begin{aligned} x'(t) &= \lambda \frac{T^{-a-1} + a t^{-a-1}}{t_0 (T^{-a-1} - t_0^{-a-1})} \max\{0, A - (\mathcal{H}x)(t_0)\} \\ &\quad + \lambda \int_t^T s^{-a-2} \left(\int_s^T \xi^{a+1} \tilde{f}(\xi, x(\xi), x'(\xi)) \, \mathrm{d}\xi \right) \mathrm{d}s \\ &\quad - \lambda t^{-a-1} \int_t^T s^{a+1} \tilde{f}(s, x(s), x'(s)) \, \mathrm{d}s, \quad t \in [0,T]. \end{aligned}$$

Since $\tilde{f}$ fulfils $(\tilde{H}_3)$, we get

$$\begin{aligned} |x'(t)| &\leq \frac{T^{-a-1}(|a|+1)}{t_0 (T^{-a-1} - t_0^{-a-1})} \left(A + t_0 \int_{t_0}^T s^{-a-2} \left(\int_s^T \xi^{a+1} h(\xi, |x(\xi)|, |x'(\xi)|) \, \mathrm{d}\xi \right) \mathrm{d}s \right) \\ &\quad + \int_t^T s^{-a-2} \left(\int_s^T \xi^{a+1} h(\xi, |x(\xi)|, |x'(\xi)|) \, \mathrm{d}\xi \right) \mathrm{d}s \\ &\quad + t^{-a-1} \int_t^T s^{a+1} h(s, |x(s)|, |x'(s)|) \, \mathrm{d}s, \quad t \in [0,T]. \end{aligned}$$

Hence, by (7) and (8),

$$\begin{aligned} |x'(t)| &\leq \frac{T^{-a-1}(|a|+1)}{t_0 (T^{-a-1} - t_0^{-a-1})} \left(A + \frac{T}{|a+1|} \int_0^T h(\xi, \|x\|_\infty, \|x'\|_\infty) \, \mathrm{d}\xi \right) \\ &\quad + \frac{1}{|a+1|} \int_0^T h(\xi, \|x\|_\infty, \|x'\|_\infty) \, \mathrm{d}\xi \\ &\quad + \int_0^T h(\xi, \|x\|_\infty, \|x'\|_\infty) \, \mathrm{d}\xi, \quad t \in [0,T]. \end{aligned}$$

Therefore, since $x(t) = \int_0^t x'(s) \, \mathrm{d}s$ implies $\|x\|_\infty \leq T \|x'\|_\infty$, we have

$$\begin{aligned} 1 &\leq \frac{1}{\|x'\|_\infty} \left[\frac{T^{-a-1}(|a|+1)}{t_0 (T^{-a-1} - t_0^{-a-1})} \left(A + \frac{T}{|a+1|} \int_0^T h(\xi, T\|x'\|_\infty, \|x'\|_\infty) \, \mathrm{d}\xi \right) \right. \\ &\quad \left. + \left(\frac{1}{|a+1|} + 1 \right) \int_0^T h(\xi, T\|x'\|_\infty, \|x'\|_\infty) \, \mathrm{d}\xi \right]. \end{aligned}$$

Since

$$h(t,Tw,w) \le \begin{cases} h(t,w,w) & \text{if } T \le 1, \\ h(t,Tw,Tw) & \text{if } T > 1, \end{cases}$$

and since, by (H_3),

$$\lim_{w\to\infty} \frac{1}{\nu w} \int_0^T h(\xi,\nu w,\nu w)\,\mathrm{d}\xi = 0 \quad \text{for all } \nu > 0,$$

we have

$$\lim_{w\to\infty} \frac{1}{w} \int_0^T h(\xi,Tw,w)\,\mathrm{d}\xi = 0.$$

Consequently,

$$\lim_{w\to\infty} \frac{1}{w} \left[\frac{T^{-a-1}(|a|+1)}{t_0(T^{-a-1}-t_0^{-a-1})} \left(A + \frac{T}{|a+1|} \int_0^T h(\xi,Tw,w)\,\mathrm{d}\xi \right) \right.$$
$$\left. + \left(\frac{1}{|a+1|} + 1 \right) \int_0^T h(\xi,Tw,w)\,\mathrm{d}\xi \right] = 0,$$

which implies that there exists $S > 0$ such that

$$\frac{1}{w} \left[\frac{T^{-a-1}(|a|+1)}{t_0(T^{-a-1}-t_0^{-a-1})} \left(A + \frac{T}{|a+1|} \int_0^T h(\xi,Tw,w)\,\mathrm{d}\xi \right) \right.$$
$$\left. + \left(\frac{1}{|a+1|} + 1 \right) \int_0^T h(\xi,Tw,w)\,\mathrm{d}\xi \right] < 1 \quad \text{for each } w \ge S.$$

This gives that

$$\|x'\|_\infty < S, \quad \|x\|_\infty < ST \quad \text{for each } x \in \mathcal{M}.$$

□

Lemma 5 *Let (H_1) - (H_3) hold. Then for each $0 \le Q < \infty$, the set*

$$\mathcal{N} = \{x \in C^1[0,T] : x = \lambda \mathcal{L}_c x \text{ for some } \lambda \in [0,1] \text{ and some } c \in [0,Q]\}$$

is bounded in $C^1[0,T]$.

Proof. Let us fix $0 \le Q < \infty$ and let $x = \lambda \mathcal{L}_c x$ for some $\lambda \in [0,1]$ and some $c \in [0,Q]$. Then

$$\begin{aligned} x'(t) \;=\; & \lambda \frac{cT^{a-1}}{|a+1|}(T^{-a-1} + at^{-a-1}) \\ & + \lambda \int_t^T s^{-a-2} \left(\int_s^T \xi^{a+1} \tilde{f}(\xi,x(\xi),x'(\xi))\,\mathrm{d}\xi \right) \mathrm{d}s \\ & - \lambda t^{-a-1} \int_t^T s^{a+1} \tilde{f}(s,x(s),x'(s))\,\mathrm{d}s, \quad t \in [0,T]. \end{aligned}$$

By (7) and (8) and since $\tilde{f}$ fulfils $(\tilde{H}_3)$, we get

$$|x'(t)| \le \frac{Q(|a|+1)}{T^2|a+1|} + \left(\frac{1}{|a+1|}+1\right)\int_0^T h(\xi, \|x\|_\infty, \|x'\|_\infty)\,\mathrm{d}\xi, \quad t \in [0,T].$$

Since $\|x\|_\infty \le T\|x'\|_\infty$, we have

$$1 \le \frac{1}{\|x'\|_\infty}\left[\frac{Q(|a|+1)}{T^2|a+1|} + \left(\frac{1}{|a+1|}+1\right)\int_0^T h(\xi, T\|x'\|_\infty, \|x'\|_\infty)\,\mathrm{d}\xi\right].$$

Due to (H_3), we deduce as in the proof of Lemma 4, that there exists $W > 0$ such that

$$\|x'\|_\infty < W, \quad \|x\|_\infty < WT \quad \text{for each } x \in \mathcal{N}.$$

□

From Lemma 5 it follows immediately

Corollary 1 *Let (H_1) - (H_3) hold. Then for each $c \ge 0$, the set*

$$\mathcal{N}_c = \{x \in C^1[0,T] : x = \lambda\mathcal{L}_c x \text{ for some } \lambda \in [0,1]\}$$

is bounded in $C^1[0,T]$.

3 Structure of the set of positive solutions of problem (1), (2)

We are now in the position to prove the existence of a positive solution of problem (1), (2), (3). This result is proved by the following nonlinear alternative of Leray-Schauder type which follows for example from [6, Corollary 8.1].

Lemma 6 *Let X be a Banach space and let $\mathcal{F} : X \to X$ be a completely continuous operator. Then either the equation $\lambda\mathcal{F}x = x$ has a solution for each $\lambda \in [0,1]$ or the set*

$$\{x \in X : \lambda\mathcal{F}x = x \text{ for some } \lambda \in (0,1)\}$$

is unbounded.

Theorem 1 *Let $(H_1) - (H_3)$ hold. Then for each $c \geq 0$ there exists a positive solution of problem* (1), (2), (3).

Proof. Fix $c \geq 0$ and put $X = C^1[0,T]$, $\mathcal{F} = \mathcal{L}_c$. By Lemmas 3(b), 6 and by Corollary 1, the operator $\mathcal{L}_c$ has a fixed point $u \in C^1[0,T]$. That is

$$\begin{aligned} u(t) &= t\frac{cT^{a+1}}{|a+1|}(T^{-a-1} - t^{-a-1}) \\ &\quad + t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1}\tilde{f}(\xi, u(\xi), u'(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s, \quad t \in [0,T]. \end{aligned}$$

Hence $u(0) = 0$, $u(T) = 0$ and, due to $(\tilde{H}_2)$, $u(t) > 0$ for $t \in (0,T)$. Therefore

$$\tilde{f}(t, u(t), u'(t)) = f(t, u(t), u'(t)) \quad \text{for a.e. } t \in [0,T].$$

Consequently,

$$\begin{aligned} u'(t) &= \frac{cT^{a+1}}{|a+1|}(T^{-a-1} + at^{-a-1}) \\ &\quad + \int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1} f(\xi, u(\xi), u'(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s \\ &\quad - t^{-a-1}\int_t^T s^{a+1} f(s, u(s), u'(s))\,\mathrm{d}s, \quad t \in [0,T], \end{aligned}$$

which yields (3). Since

$$u''(t) = caT^{a+1}t^{-a-2} + at^{-a-2}\int_t^T \xi^{a+1} f(\xi, u(\xi), u'(\xi))\,\mathrm{d}\xi + f(t, u(t), u'(t)) \quad (9)$$

for a.e. $t \in [0,T]$, inequality (8) gives $u'' \in L^1[0,T]$, and the direct computation shows that u satisfies equation (1) for a.e. $t \in [0,T]$. Thus u is a positive solution of problem $(1), (2), (3)$. □

Recall that $\mathcal{S}_c$ is the set of all positive solutions of problem $(1), (2), (3)$. By Theorem 1, for each $c \geq 0$, the set $\mathcal{S}_c$ is nonempty. Due to $\mathcal{S} = \cup_{c\geq 0}\mathcal{S}_c$, the cardinality of the set $\mathcal{S}$ is continuum. The following result gives the important property of solutions of problem $(1), (2), (3)$ that is used in further investigation of the set $\mathcal{S}$.

Lemma 7 *Let (H_1) - (H_3) hold. Then for each $0 \leq K \leq Q < \infty$, the set $\bigcup_{K\leq c\leq Q}\mathcal{S}_c$ is compact in $C^1[0,T]$.*

Proof. Let us choose $0 \le K \le Q < \infty$. We start to show that $u \in \mathcal{S}_c$ (i.e., u is a positive solution of problem (1), (2), (3)) if and only if u is a fixed point of operator $\mathcal{L}_c$.

($\Rightarrow$) Let u be a fixed point of $\mathcal{L}_c$. Then, due to the proof of Theorem 1, u is a positive solution of problem (1), (2), (3).

($\Leftarrow$) Let u be a positive solution of problem (1), (2), (3). Since $u > 0$ on $(0, T)$, we have

$$\tilde{f}(t, u(t), u'(t)) = f(t, u(t), u'(t)) \quad \text{for a.e. } t \in [0, T].$$

We can check that

$$\left(t^{a+2}\left(\frac{u(t)}{t}\right)'\right)' = t^{a+1}\left(u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t)\right) \quad \text{for a.e. } t \in [0, T],$$

and therefore the following equality

$$\left(t^{a+2}\left(\frac{u(t)}{t}\right)'\right)' = t^{a+1}\tilde{f}(t, u(t), u'(t)) \quad \text{for a.e. } t \in [0, T]$$

holds. We get by integration and by (2), (3) that

$$-cT^{a+1}t^{-a-2} - \left(\frac{u(t)}{t}\right)' = t^{-a-2}\int_t^T \xi^{a+1}\tilde{f}(\xi, u(\xi), u'(\xi))d\xi, \ t \in [0, T],$$

since

$$T^{a+2}\left(\frac{u(t)}{t}\right)'_{t=T} = -cT^{a+1}.$$

The next integration over $[t, T]$ yields that

$$\begin{aligned} u(t) \ &= \ t\frac{cT^{a+1}}{|a+1|}(T^{-a-1} - t^{-a-1}) \\ &\quad + t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1}\tilde{f}(\xi, u(\xi), u'(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s, \quad t \in [0, T]. \end{aligned}$$

Therefore u is a fixed point of operator $\mathcal{L}_c$.

Now, we are in the position to prove that the set $\bigcup_{K \le c \le Q} \mathcal{S}_c$ is compact in $C^1[0, T]$. Since $\mathcal{S}_c$ is the set of all fixed points of the operator $\mathcal{L}_c$, the boundedness of $\bigcup_{K \le c \le Q} \mathcal{S}_c$ in $C^1[0, T]$ follows from Lemma 5 with $\lambda = 1$ in $\mathcal{N}$. Therefore (H_1) gives that there exists $\mu^* \in L^1[0, T]$ such that

$$|f(t, u(t), u'(t))| \le \mu^*(t) \quad \text{for a.e. } t \in [0, T] \text{ and all } u \in \bigcup_{K \le c \le Q} \mathcal{S}_c. \tag{10}$$

Since (9) holds for $u \in \mathcal{S}_c$, we have by (10),

$$\left.\begin{aligned} |u''(t)| \le Q|a|T^{a+1}t^{-a-2} + |a|t^{-a-2}\int_t^T \xi^{a+1}\mu^*(\xi)\,\mathrm{d}\xi + \mu^*(t) \\ \text{for a.e. } t \in [0, T] \text{ and all } u \in \bigcup_{K \le c \le Q} \mathcal{S}_c. \end{aligned}\right\}$$

By inequality (8),

$$t^{-a-2}\int_t^T \xi^{a+1}\mu^*(\xi)\,\mathrm{d}\xi \in L^1[0,T],$$

and hence there exists a majorant function $p^* \in L^1[0,T]$ such that

$$|u''(t)| \le p^*(t) \quad \text{for a.e. } t\in[0,T] \text{ and all } u \in \bigcup_{K\le c\le Q} \mathcal{S}_c.$$

As a result, the set $\left\{u' : u \in \bigcup_{K\le c\le Q}\mathcal{S}_c\right\}$ is equicontinuous on $[0,T]$. We have proved that $\bigcup_{K\le c\le Q}\mathcal{S}_c$ is relatively compact in $C^1[0,T]$.

It remains to prove that $\bigcup_{K\le c\le Q}\mathcal{S}_c$ is closed in $C^1[0,T]$. To this end consider a sequence $\{u_n\} \subset \bigcup_{K\le c\le Q}\mathcal{S}_c$ converging in $C^1[0,T]$ to a function $u \in C^1[0,T]$. Note that $u_n \in \bigcup_{K\le c\le Q}\mathcal{S}_c$ implies $u_n \in \mathcal{S}_{c_n}$ for some $c_n \in [K,Q]$ and so, by the definition of the set $\mathcal{S}_c$, $c_n = -u_n'(T)$. Thus

$$\begin{aligned} u_n(t) &= -t\frac{u_n'(T)T^{a+1}}{|a+1|}(T^{-a-1}-t^{-a-1}) \\ &\quad + t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1} f(\xi,u_n(\xi),u_n'(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s,\ t\in[0,T],\ n\in\mathbb{N}. \end{aligned}$$

Letting $n\to\infty$ and using (10) and the Lebesgue dominated convergence theorem, we get

$$\begin{aligned} u(t) &= -t\frac{u'(T)T^{a+1}}{|a+1|}(T^{-a-1}-t^{-a-1}) \\ &\quad + t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1} f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s, \quad t\in[0,T]. \end{aligned}$$

Since $-u_n'(T)\in[K,Q]$, we have $-u'(T)\in[K,Q]$, and therefore it follows from the last equality that $u \in \mathcal{S}_{-u'(T)} \subset \bigcup_{K\le c\le Q}\mathcal{S}_c$. Consequently $\bigcup_{K\le c\le Q}\mathcal{S}_c$ is closed in $C^1[0,T]$. □

If $K=Q=c$ in Lemma 7, then the following result holds.

Corollary 2 *Let (H_1) - (H_3) hold. Then for each $c\ge 0$, the set $\mathcal{S}_c$ is compact in $C^1[0,T]$.*

In view of Corollary 2, we can define a bounded function

$$\beta(t) = \max\{u(t) : u\in\mathcal{S}_0\} \quad \text{for } t\in[0,T]. \tag{11}$$

We prove that for each $t_0\in(0,T)$

$$\{u(t_0) : u\in\mathcal{S}\setminus\mathcal{S}_0\} \supset (\beta(t_0),\infty). \tag{12}$$

This result is done in the next theorem.

Theorem 2 *Let* $(H_1)-(H_3)$ *hold. Then for each* $t_0 \in (0,T)$ *and each* $A > \beta(t_0)$ *there exists a positive solution* u *of problem* (1), (2) *satisfying* $u(t_0) = A$.

Proof. Fix $t_0 \in (0,T)$ and choose $A > \beta(t_0)$. Put $X = C^1[0,T]$, $\mathcal{F} = \mathcal{K}_{t_0,A}$. By Lemmas 3(a), 4 and 6, the operator $\mathcal{K}_{t_0,A}$ has a fixed point $u \in C^1[0,T]$. That is

$$u(t) = \frac{t}{t_0}\frac{T^{-a-1} - t^{-a-1}}{T^{-a-1} - t_0^{-a-1}} \max\{0, A - (\mathcal{H}u)(t_0)\} + (\mathcal{H}u)(t) \quad t \in [0,T],$$

where $\mathcal{H}$ is given in (4). We will consider two cases.

Case 1. Let $\max\{0, A - (\mathcal{H}u)(t_0)\} = 0$. That is $A \le (\mathcal{H}u)(t_0)$. Then $u(t) = (\mathcal{H}u)(t)$, which yields $u \in \mathcal{S}_0$, according to the proof Theorem 1. So, by (11), $u(t_0) = (\mathcal{H}u)(t_0) \le \beta(t_0) < A$, a contradiction.

Case 2. Let $\max\{0, A - (\mathcal{H}u)(t_0)\} > 0$. That is $A > (\mathcal{H}u)(t_0)$. Then

$$u(t) = \frac{t}{t_0}\frac{T^{-a-1} - t^{-a-1}}{T^{-a-1} - t_0^{-a-1}}(A - (\mathcal{H}u)(t_0)) + (\mathcal{H}u)(t), \quad t \in [0,T].$$

Hence $u(t) > 0$ for $t \in (0,T)$, $u(0) = 0$, $u(T) = 0$ and $u(t_0) = A - (\mathcal{H}u)(t_0)) + (\mathcal{H}u)(t_0) = A$. Further,

$$u'(t) = \frac{T^{-a-1} + at^{-a-1}}{t_0(T^{-a-1} - t_0^{-a-1})}(A - (\mathcal{H}u)(t_0)) + (\mathcal{H}u)'(t), \quad t \in [0,T],$$

$$u''(t) = \frac{a(-a-1)t^{-a-2}}{t_0(T^{-a-1} - t_0^{-a-1})}(A - (\mathcal{H}u)(t_0)) + (\mathcal{H}u)''(t) \quad \text{for a.e. } t \in [0,T].$$

Since the direct computation gives

$$\begin{aligned} u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) &= (\mathcal{H}u)''(t) + \frac{a}{t}(\mathcal{H}u)'(t) - \frac{a}{t^2}(\mathcal{H}u)(t) \\ &= \tilde{f}(t, u(t), u'(t)) = f(t, u(t), u'(t)) \quad \text{for a.e. } t \in [0,T], \end{aligned}$$

u is a positive solution of problem (1), (2) satisfying $u(t_0) = A$. □

Remark 1 Note that, due to Corollary 2, for each $t_0 \in (0,T)$ and each $c \ge 0$ the set $\{u(t_0) : u \in \mathcal{S}_c\}$ is a compact set in $\mathbb{R}$.

Example 1 Let us choose $\alpha, \eta \in [0,1)$ and for a.e. $t \in [0,T]$ and all $x \in [0,\infty)$, $y \in \mathbb{R}$, define a function f by

$$f(t,x,y) = h_1(t) + h_2(t,x,y)x^\alpha + h_3(t,x,y)|y|^\eta.$$

Here $h_1 \in L^1[0,T]$, $h_1(t) > 0$ a.e. on $[0,T]$, h_2, h_3 are nonnegative, bounded and continuous on $[0,T] \times [0,\infty) \times \mathbb{R}$. Then f satisfies conditions $(H_1)-(H_3)$. To check it we take $M_i = \sup\{h_i(t,x,y) : t \in [0,T], x \in [0,\infty), y \in \mathbb{R}\}$, $i = 2,3$, $\varphi(t) = h_1(t)$ a.e. on $[0,T]$, $h(t,x,y) = \varphi(t) + M_1x^\alpha + M_2y^\eta$ for $t \in [0,T]$, $x \in [0,\infty)$, $y \in [0,\infty)$.

In order to prove that the set $\mathcal{S}_c$ contains only one solution of problem $(1), (2), (3)$ we will use the assumption

(H_4) $f(t,\cdot,\cdot) \in Lip_{loc}(\mathcal{D})$, for a.e. $t \in [0,T]$,

which means that for each compact set $\mathcal{U} \subset \mathcal{D}$ there exists a function $\ell_{\mathcal{U}} \in L^1[0,T]$ such that

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le \ell_{\mathcal{U}}(t) \sum_{i=1}^{2} |x_i - y_i|$$

for a.e. $t \in [0,T]$ and all $(x_1,x_2),(y_1,y_2) \in \mathcal{U}$.

Theorem 3 *Let $(H_1) - (H_4)$ hold. Then the set $\mathcal{S}_c$ is one-point for each $c \ge 0$.*

Proof. Choose an arbitrary $c \ge 0$. Theorem 1 guarantees that the set $\mathcal{S}_c$ is nonempty. Condition (H_4) implies that a solution u of equation (1) on $[0,T]$ satisfying conditions $u(T) = 0$, $u'(T) = -c$ is unique. $\square$

Example 2 Let $h_i \in L^1[0,T]$, $h_i(t) > 0$ a.e. on $[0,T]$, $i \in \{1,2,3\}$. For a.e. $t \in [0,T]$ and all $x \in [0,\infty)$, $y \in \mathbb{R}$, define a function f by

$$f(t,x,y) = h_1(t) + h_2(t)g_1(x) + h_3(t)g_2(y),$$

where g_1 and g_2 satisfy

$$g_1 \in PC^1[0,\infty), \quad g_2 \in PC^1(\mathbb{R}), \quad \lim_{x\to\infty} \frac{g_1(x)}{x} = 0, \quad \lim_{y\to\pm\infty} \frac{g_2(y)}{y} = 0.$$

Then f satisfies conditions $(H_1) - (H_4)$.

4 Special case of problem (1), (2)

In this section we consider the special case of equation (1), where the function f does not depend on u', that is $f(t,x,y) = f(t,x)$ and

$$u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) = f(t,u(t)). \tag{13}$$

Now we will work with the following assumptions on f:

(H_1^*) $f \in Car([0,T] \times [0,\infty))$.

(H_2^*) $0 < f(t,x)$ for a.e. $t \in [0,T]$ and all $x \in [0,\infty)$.

(H_3^*) $f(t,x)$ is increasing in x for a.e. $t \in [0,T]$ and

$$\lim_{x\to\infty} \frac{1}{x}\int_0^T f(t,x)\,\mathrm{d}t = 0.$$

Conditions $(H_1^*)-(H_3^*)$ guarantee that assumptions $(H_1)-(H_3)$ are fulfilled with $\varphi(t) = f(t,0)$ for a.e. $t \in [0,T]$. Therefore all results of Section 3 are applicable on problem (13), (2). For simplicity we denote again by $\mathcal{S}$ the set of all positive solutions of problem (13), (2) and by $\mathcal{S}_c$ the set $\{u \in \mathcal{S} : u'(T) = -c\}$, where $c \geq 0$. Note that if $u \in \mathcal{S}_c$, then

$$\begin{aligned} u(t) \;=\; & t\frac{cT^{a+1}}{|a+1|}(T^{-a-1} - t^{-a-1}) \\ & + t\int_t^T s^{-a-2}\left(\int_s^T \xi^{a+1} f(\xi, u(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s, \quad t \in [0,T]. \end{aligned}$$

Lemma 8 *Let $(H_1^*)-(H_3^*)$ hold. Assume that $c_1 > c_2 \geq 0$, $u_i \in \mathcal{S}_{c_i}$, $i = 1,2$. Then*

$$u_1(t) > u_2(t) \quad \textit{for } t \in (0,T).$$

Proof. Since $c_1 > c_2$, $u_1'(T) = -c_1$, $u_2'(T) = -c_2$ and $u_1(T) = u_2(T) = 0$, there exists $\delta > 0$ such that $u_1(t) > u_2(t)$ for $t \in (T-\delta, T)$. Assume that there exists $t_1 \in (0, T-\delta]$ such that $u_1(t_1) = u_2(t_1)$ and $u_1(t) > u_2(t)$ or $t \in (t_1, T)$. Then

$$\begin{aligned} 0 \;=\; (u_1-u_2)(t_1) = & \frac{t_1 T^{a+1}}{|a+1|}(c_1-c_2)(T^{-a-1} - t_1^{-a-1}) \\ & + t_1\int_{t_1}^T s^{-a-2}\left(\int_s^T \xi^{a+1}(f(\xi,u_1(\xi)) - f(\xi,u_2(\xi)))\,\mathrm{d}\xi\right)\mathrm{d}s. \end{aligned}$$

Since

$$\frac{t_1 T^{a+1}}{|a+1|}(c_1-c_2)(T^{-a-1} - t_1^{-a-1}) > 0$$

and, by (H_3^*),

$$t_1\int_{t_1}^T s^{-a-2}\left(\int_s^T \xi^{a+1}(f(\xi,u_1(\xi)) - f(\xi,u_2(\xi)))\,\mathrm{d}\xi\right)\mathrm{d}s > 0,$$

we get a contradiction. □

Lemma 9 *Let* $(H_1^*)-(H_3^*)$ *hold. Assume that* $c \geq 0$, $u_i \in \mathcal{S}_c$, $i = 1,2$. *Let* $u_1(t_0) > u_2(t_0)$ *for some* $t_0 \in (0,T)$. *Then either* $u_1(t) > u_2(t)$ *for* $t \in [0,T]$ *or there exists* $t^* \in (t_0, T]$ *such that* $u_1(t) > u_2(t)$ *for* $t \in (0,t^*)$ *and* $u_1(t) = u_2(t)$ *for* $t \in [t^*, T]$.

Proof. There exist $\ell_1, \ell_2 \in [0,T]$, $\ell_1 < t_0 < \ell_2$ such that $u_1(\ell_1) = u_2(\ell_1)$, $u_1(\ell_2) = u_2(\ell_2)$ and

$$u_1(t) > u_2(t) \quad \text{for } t \in (\ell_1, \ell_2). \tag{14}$$

Case 1. Let $\ell_1 > 0$ and $\ell_2 < T$. Then

$$\begin{aligned} 0 &= (u_1-u_2)(\ell_1) \\ &= \ell_1 \int_{\ell_1}^T s^{-a-2} \left(\int_s^T \xi^{a+1} (f(\xi, u_1(\xi)) - f(\xi, u_2(\xi)))\, \mathrm{d}\xi \right) \mathrm{d}s, \end{aligned} \tag{15}$$

$$\begin{aligned} 0 &= (u_1-u_2)(\ell_2) \\ &= \ell_2 \int_{\ell_2}^T s^{-a-2} \left(\int_s^T \xi^{a+1} (f(\xi, u_1(\xi)) - f(\xi, u_2(\xi)))\, \mathrm{d}\xi \right) \mathrm{d}s. \end{aligned} \tag{16}$$

Further,

$$\begin{aligned} 0 &\leq (u_1-u_2)'(\ell_1) \\ &= \int_{\ell_1}^T s^{-a-2} \left(\int_s^T \xi^{a+1} (f(\xi, u_1(\xi)) - f(\xi, u_2(\xi)))\, \mathrm{d}\xi \right) \mathrm{d}s \\ &\quad - \ell_1^{-a-1} \int_{\ell_1}^T s^{a+1} (f(s,u_1(s)) - f(s,u_2(s)))\, \mathrm{d}s, \end{aligned} \tag{17}$$

$$\begin{aligned} 0 &\geq (u_1-u_2)'(\ell_2) \\ &= \int_{\ell_2}^T s^{-a-2} \left(\int_s^T \xi^{a+1} (f(\xi, u_1(\xi)) - f(\xi, u_2(\xi)))\, \mathrm{d}\xi \right) \mathrm{d}s \\ &\quad - \ell_2^{-a-1} \int_{\ell_2}^T s^{a+1} (f(s,u_1(s)) - f(s,u_2(s)))\, \mathrm{d}s. \end{aligned} \tag{18}$$

Using (15) and (16) we deduce from (17) and (18)

$$\begin{aligned} 0 \; \leq \; & \frac{(u_1-u_2)'(\ell_1)}{\ell_1^{-a-1}} - \frac{(u_1-u_2)'(\ell_2)}{\ell_2^{-a-1}} = - \int_{\ell_1}^T s^{a+1} (f(s,u_1(s)) - f(s,u_2(s)))\, \mathrm{d}s \\ & + \int_{\ell_2}^T s^{a+1} (f(s,u_1(s)) - f(s,u_2(s)))\, \mathrm{d}s \\ & = - \int_{\ell_1}^{\ell_2} s^{a+1} (f(s,u_1(s)) - f(s,u_2(s)))\, \mathrm{d}s < 0, \end{aligned}$$

a contradiction.

Case 2. Let $\ell_1 > 0$ and $\ell_2 = T$. Then we get a contradiction immediately from (15) since (due to (14) and (H_3^*))

$$\ell_1 \int_{\ell_1}^{T} s^{-a-2} \left(\int_{s}^{T} \xi^{a+1} (f(\xi, u_1(\xi)) - f(\xi, u_2(\xi)))\,\mathrm{d}\xi \right) \mathrm{d}s > 0.$$

Case 3. Let $\ell_1 = 0$ and $\ell_2 < T$. Assume that there exists $\gamma \in (\ell_2, T)$ such that $u_1(\gamma) \neq u_2(\gamma)$. Consequently we can find t_1, t_2 such that $\ell_2 \le t_1 < \gamma < t_2 \le T$, $u_1(t_1) = u_2(t_1)$, $u_1(t_2) = u_2(t_2)$ and

$$(u_1 - u_2)(t) \cdot \operatorname{sgn}(u_1 - u_2)(\gamma) > 0 \quad \text{for } t \in (t_1, t_2).$$

Now, we can derive a contradiction as in Case 1. Therefore $u_1(t) = u_2(t)$ for $t \in [\ell_2, T]$ and the assertion is valid with $t^* = \ell_2$.

Case 4. Let $\ell_1 = 0$ and $\ell_2 = T$. Then the assertion is valid. $\square$

Let $c \ge 0$ and let $\mathcal{I} : \mathcal{S}_c \to \mathbb{R}$ be a functional defined by

$$\mathcal{I}(x) = \int_0^T x(t)\,\mathrm{d}t.$$

Then $\mathcal{I}$ is continuous and since $\mathcal{S}_c$ is compact by Corollary 2, there exist $u_{c,min}, u_{c,max} \in \mathcal{S}_c$ such that

$$\mathcal{I}(u_{c,min}) = \min\{\mathcal{I}(x) : x \in \mathcal{S}_c\}, \quad \mathcal{I}(u_{c,max}) = \max\{\mathcal{I}(x) : x \in \mathcal{S}_c\}.$$

It follows from Lemma 9, that if $u_1, u_2 \in \mathcal{S}_c$ and $u_1 \neq u_2$, then $\mathcal{I}(u_1) \neq \mathcal{I}(u_2)$. This together with the fact that $\mathcal{I}$ is increasing imply

$$u_{c,min}(t) \le u(t) \le u_{c,max}(t) \quad \text{for } t \in [0,T],\ u \in \mathcal{S}_c. \tag{19}$$

Besides, by Lemma 8, $u_{c_2,\max}(t) < u_{c_1,\min}(t)$ for $t \in (0,T)$ and $c_1 > c_2 \ge 0$. In particular, $c_i, c_j \in [0,\infty)$, $c_i \neq c_j$ and $\mathcal{S}_{c_1}$, $\mathcal{S}_{c_2}$ are not one-point sets imply

$$(u_{c_i,min}(t), u_{c_i,max}(t)) \cap (u_{c_j,min}(t), u_{c_j,max}(t)) = \emptyset \quad \text{for } t \in (0,T). \tag{20}$$

If $u_{c,min} = u_{c,max}$ on $[0,T]$ for some $c \ge 0$, then problem (13), (2), (3) has a unique solution. If it is not that case, the structure of the set $\mathcal{S}_c$ is described in the next theorem. Note that, by Lemma 9, if $u_{c,min} \neq u_{c,max}$, two possibilities can occur. Either $u_{c,min}(t) < u_{c,max}(t)$ for $t \in (0,T)$ or there exists $t^* \in (0,T)$ such that

$$u_{c,min}(t) < u_{c,max}(t),\ t \in (0,t^*), \quad u_{c,min}(t) = u_{c,max}(t),\ t \in [t^*,T].$$

In particular, if for some $c > 0$ the interior of the set $\{(t,x) \in \mathbb{R}^2 : 0 \le t \le T,\ u_{c,min}(t) \le x \le u_{c,max}(t)\}$ is nonempty, then it is covered by graphs of other functions of $\mathcal{S}_c$. This is contained in Theorem 4.

Theorem 4 *Let $(H_1^*)-(H_3^*)$ hold. Assume that there exists $t_0 \in (0,T)$ such that $u_{c,min}(t_0) < u_{c,max}(t_0)$ for some $c > 0$. Then for each $A \in (u_{c,min}(t_0), u_{c,max}(t_0))$ there exists $u \in \mathcal{S}_c$ satisfying $u(t_0) = A$.*

Proof. Since the function β given by (11) is equal to $u_{0,max}$, we have that $A > \beta(t_0)$. Therefore, by Theorem 2, there exists a positive solution u of problem (13), (2) satisfying $u(t_0) = A$. We prove that $u \in \mathcal{S}_c$ by contradiction.

Let $c_1 > c$ and $u \in \mathcal{S}_{c_1}$. Then, by Lemma 8, $u(t) > u_{c,max}(t)$ for $t \in (0,T)$, which contradicts that $u(t_0) = A < u_{c,max}(t_0)$.

Let $0 \le c_2 < c$ and $u \in \mathcal{S}_{c_2}$. Then, by Lemma 8, $u(t) < u_{c,min}(t)$ for $t \in (0,T)$, which contradicts that $u(t_0) = A > u_{c,min}(t_0)$. $\square$

Remark 2 Let us note that if the interior of the set $\{(t,x) \in \mathbb{R}^2 : 0 \le t \le T,\ u_{0,min}(t) \le x \le u_{0,max}(t)\}$ is nonempty, then we cannot apply Theorem 4 and the description of this set is an open problem.

The next two theorems give results on a unique solution of problem (13), (2), (3). The first theorem uses only the basic assumptions $(H_1^*)-(H_3^*)$ while the second one needs the additional assumption

(H_4^*) $f(t,\cdot) \in Lip_{loc}[0,\infty)$ for a.e. $t \in [0,T]$.

Theorem 5 *Let $(H_1^*)-(H_3^*)$ hold. Then problem (13), (2), (3) has a unique solution for each $c \in [0,\infty) \setminus \Gamma$, where $\Gamma \subset [0,\infty)$ is at most countable.*

Proof. Since problem (13), (2), (3) has a unique solution for some $c \in [0,\infty)$ if and only if $u_{c,min} = u_{c,max}$, we need to prove that the set $\Gamma := \{c \in [0,\infty) : u_{c,min} \ne u_{c,max}\}$ is at most countable.

For $t \in (0,T)$ we define

$$\Psi(t) = \{c \in (0,\infty) : u_{c,min}(t) < u_{c,max}(t)\}.$$

By Lemma 9, $\Psi(t_1) \supset \Psi(t_2)$ for $0 < t_1 < t_2 < T$. It follows from Theorem 2 and Lemma 8 that $\{u(t) \in \mathbb{R} : u \in \mathcal{S} \setminus \mathcal{S}_0\} = (\beta(t), \infty)$ for $t \in (0,T)$. Therefore Lemmas 8 and 9 and Theorem 4 yield $\beta = u_{0,max}$ and

$$\left\{ u(t) \in \mathbb{R} : u \in \bigcup_{0<c\le N} \mathcal{S}_c \right\} = (u_{0,max}(t), u_{N,max}(t)] \quad \text{for } t \in (0,T),\ N \in \mathbb{N}. \tag{21}$$

For $t \in (0,T)$, $N \in \mathbb{N}$ and $\varepsilon > 0$ let us put

$$\begin{aligned} \Psi_N(t) &= \{c \in (0,N] : u_{c,min}(t) < u_{c,max}(t)\}, \\ \Psi_{N,\varepsilon}(t) &= \{c \in (0,N] : u_{c,max}(t) - u_{c,min}(t) \ge \varepsilon\}. \end{aligned}$$

We claim that $\Psi_{N,\varepsilon}(t)$ is finite for $t \in (0,T)$, $N \in \mathbb{N}$ and $\varepsilon > 0$. Suppose, contrary to our claim, that there exist $t_0 \in (0,T)$, $N_0 \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that $\Psi_{N_0,\varepsilon_0}(t_0)$ is infinite. Then there exists a sequence $\{c_n\} \subset \Psi_{N_0,\varepsilon_0}(t_0)$, $c_i \neq c_j$ for $i \neq j$. Since $u_{c_n,\max}(t_0) - u_{c_n,min}(t_0) \geq \varepsilon_0$ for $n \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty}(u_{c_n,max}(t_0) - u_{c_n,min}(t_0)) = \infty,$$

which contradicts (cf. (20))

$$\sum_{n=1}^{\infty}(u_{c_n,max}(t_0) - u_{c_n,min}(t_0)) \leq u_{N_0,\max}(t_0) - u_{0,max}(t_0) < \infty$$

by (21) (for $t = t_0$ and $N = N_0$). Hence the set $\Psi_N(t)$ is at most countable for $t \in (0,T)$ and $N \in \mathbb{N}$ which follows from the equality $\Psi_N(t) = \bigcup_{n=1}^{\infty} \Psi_{N,\frac{1}{n}}(t)$. Since $\Psi(t) = \bigcup_{N=1}^{\infty} \Psi_N(t)$ we see that $\Psi(t)$ is at most countable for $t \in (0,T)$. Let $\{t_n\} \subset (0,T)$ be decreasing and let $\lim_{n\to\infty} t_n = 0$. We now show that

$$\Gamma \setminus \{0\} = \bigcup_{n=1}^{\infty} \Psi(t_n). \tag{22}$$

Let us choose $c \in \Gamma \setminus \{0\}$. Then $u_{c,min} \neq u_{c,max}$ and therefore there exists $\nu \in \mathbb{N}$ such that $u_{c,min}(t) < u_{c,max}(t)$ for $t \in (0,t_\nu)$ by Lemma 9. Hence $c \in \Psi(t_\nu)$ and since $\bigcup_{n=1}^{\infty} \Psi(t_n) \subset \Gamma \setminus \{0\}$, equality (22) holds. Using the fact that $\Psi(t_n)$ is at most countable for all $n \in \mathbb{N}$ it follows from (22) that the set Γ is at most countable. □

Theorem 6 *Let* $(H_1^*) - (H_4^*)$ *hold. Then problem* (13), (2), (3) *has a unique solution for each* $c \in [0,\infty)$.

Proof. Since the assumptions $(H_1^*) - (H_4^*)$ guarantee that the assumptions $(H_1) - (H_4)$ of Theorem 3 are fulfilled, there exists a unique solution of problem (13), (2), (3) for each $c \in [0,\infty)$. □

The following result deals with the existence of a positive solution u of problem (13), (2) satisfying the extra condition $\max\{u(t) : t \in [0,T]\} = A$. Note that for positive solutions u of problem (13), (2) we have $\|u\|_\infty = \max\{u(t) : t \in [0,T]\}$.

Theorem 7 *Let* $(H_1^*) - (H_3^*)$ *hold. Then for each* $A > \|u_{0,max}\|_\infty$ *there exists a positive solution* u *of problem* (13), (2) *such that* $\|u\|_\infty = A$.

Proof. Suppose the assertion of the theorem is false. Then there exists $A > \|u_{0,max}\|_\infty$ such that

$$\|u\|_\infty \neq A \quad \text{for all } u \in \mathcal{S} \setminus \mathcal{S}_0. \tag{23}$$

Put

$$\mathcal{U}_- = \{v \in \mathcal{S} \setminus \mathcal{S}_0 : \|v\|_\infty < A\}, \ \ \mathcal{U}_+ = \{u \in \mathcal{S} \setminus \mathcal{S}_0 : \|u\|_\infty > A\}.$$

Then $\mathcal{S} \setminus \mathcal{S}_0 = \mathcal{U}_- \cup \mathcal{U}_+$ and $\mathcal{U}_- \cap \mathcal{U}_+ = \emptyset$. Let

$$A_- = \sup\{\|v\|_\infty : v \in \mathcal{U}_-\}, \quad A_+ = \inf\{\|u\|_\infty : u \in \mathcal{U}_+\}.$$

Then $A_- \le A \le A_+$ and there exist sequences $\{v_n\} \subset \mathcal{U}_-$ and $\{u_n\} \subset \mathcal{U}_+$ such that $\{\|v_n\|_\infty\}$ is increasing, $\{\|u_n\|_\infty\}$ is decreasing and $\lim_{n\to\infty} \|v_n\|_\infty = A_-$, $\lim_{n\to\infty} \|u_n\|_\infty = A_+$. Hence, by Lemmas 8 and 9, the inequality $v_n \le v_{n+1} \le u_{n+1} \le u_n$ is fulfilled on $[0,T]$ for each $n \in \mathbb{N}$. Then

$$0 > v_n'(T) \ge v_{n+1}'(T) \ge u_{n+1}'(T) \ge u_n'(T) \ \text{ for } n \in \mathbb{N},$$

which yields

$$v_n, u_n \in \bigcup_{K \le c \le Q} \mathcal{S}_c \text{ for } n \in \mathbb{N}, \ \text{ where } K := -v_1'(T) > 0,\, Q := -u_1'(T) \ge K.$$

Since $\bigcup_{K \le c \le Q} \mathcal{S}_c$ is compact in $C^1[0,T]$ by Lemma 7, there exist $v, u \in \bigcup_{K \le c \le Q} \mathcal{S}_c$ such that $\lim_{n\to\infty} v_n = v$, $\lim_{n\to\infty} u_n = u$ in $C^1[0,T]$. Hence v, u are solutions of problem (13), (2) and $\|v\|_\infty = A_-$, $\|u\|_\infty = A_+$. In view of relation (23) we have $A_- < A < A_+$. Since $u(0) = u(T) = 0$ and $\|u\|_\infty = A_+$, there exists $t_0 \in (0,T)$ such that $u(t_0) = A_+$ and $u \le A_+$ on $[0,t_0)$, $u < A_+$ on $(t_0,T]$. Let us choose $B \in (A_-, A_+)$. Then, by Theorem 2, there is a solution w of problem (13), (2) satisfying $w(t_0) = B$. Lemmas 8 and 9 guarantee that $v < w < u$ on $(0,t_0]$ and $v \le w \le u$ on $(t_0,T]$. In addition, $w(t) < u(t)$ on a right neighbourhood of $t = t_0$ because $w(t_0) < u(t_0)$. Consequently, $\|w\|_\infty \in (A_-, A_+)$, which contradicts the definition of A_- and A_+. □

Example 3 Let us choose $\alpha \in [0,1)$ and for a.e. $t \in [0,T]$ and all $x \in [0,\infty)$, define the function f by

$$f(t,x) = h_1(t) + h_2(t,x)x^\alpha,$$

or

$$f(t,x) = h_1(t) + h_2(t,x)\frac{x}{\ln(x+2)},$$

where $h_1 \in L^1[0,T]$, $h_1 > 0$ a.e. on $[0,T]$, h_2 is nonnegative, bounded and continuous on $[0,T] \times [0,\infty)$ and increasing in x. Then f satisfies conditions $(H_1^*) - (H_3^*)$. To check it we take $M = \max\{h_2(t,x) : t \in [0,T], x \in [0,\infty)\}$, and then we get

$$0 \le \lim_{x\to\infty} \frac{1}{x}\left(\int_0^T h_1(t)\mathrm{d}t + x^\alpha \int_0^T h_2(t,x)\mathrm{d}t\right) \le TM \lim_{x\to\infty} x^{\alpha-1} = 0,$$

or

$$0 \le \lim_{x\to\infty} \frac{1}{x}\left(\int_0^T h_1(t)\mathrm{d}t + \frac{x}{\ln(x+2)}\int_0^T h_2(t,x)\mathrm{d}t\right) \le TM \lim_{x\to\infty} \frac{1}{\ln(x+2)} = 0.$$

Example 4 Let $h_i \in L^1[0,T]$, $h_i > 0$ a.e. on $[0,T]$, $i \in \{1,2\}$. For a.e. $t \in [0,T]$ and all $x \in [0,\infty)$, define a function f by

$$f(t,x) = h_1(t) + h_2(t)g(x),$$

where $g \in PC^1[0,\infty)$ is increasing and $\lim_{x\to\infty} \frac{g(x)}{x} = 0$. Then f satisfies conditions $(H_1^*) - (H_4^*)$. We can choose for example $g(x) = x^\alpha$ for $x \in [0,1]$ and $g(x) = x^\eta$ for $x \in (1,\infty)$, where $\alpha \in [1,\infty)$ and $\eta \in (0,1)$.

5 Blow-up results

In this section we provide new blow-up results for positive solutions of the equation

$$v''(t) + \frac{k}{t}v'(t) = \psi(t) + g(t,v(t)), \tag{24}$$

where $k \in (1,\infty)$ and ψ, g satisfy the following assumptions.

(H_1°) $t^k\psi \in L^1[0,T]$ and $\psi > 0$ a.e. on $[0,T]$.

(H_2°) $g \in Car([0,T] \times [0,\infty))$.

(H_3°) $0 \le g(t,x) \le \phi(x)$, for a.e. $t \in [0,T]$ and all $x \in [0,\infty)$,

where $\phi \in C[0,\infty)$ is nondecreasing on $[0,\infty)$, and

$$\lim_{x\to\infty} \frac{\phi(x)}{x} = 0.$$

In particular, we consider the boundary conditions

$$\lim_{t\to 0+} v(t) = \infty, \quad v(T) = 0, \tag{25}$$

and define *a positive solution of problem* (24), (25) as a function $u \in AC^1_{loc}(0,T]$ such that $u > 0$ on $(0,T)$, u satisfies the boundary conditions (25) and (24) holds for a.e. $t \in [0,T]$.

Theorem 8 *Let* $(H_1^\circ)-(H_3^\circ)$ *hold. Then for each* $c\geq 0$ *there exists a positive solution* v *of problem* (24), (25) *satisfying*

$$v'(T)=-c. \tag{26}$$

Proof. Since equation (24) has an equivalent form $(t^k v')'=t^k(\psi(t)+g(t,v))$, we see that, after the substitution

$$k=-a,\quad v(t)=t^a u(t)\quad \text{for } t\in(0,T], \tag{27}$$

equation (24) transforms to the equation

$$(t^{-a}(t^a u(t))')'=t^{-a}(\psi(t)+g(t,t^a u(t))), \tag{28}$$

and consequently to equation (13) with

$$\begin{gathered} f(t,x)=t^{-a}(\psi(t)+g(t,t^a x))=t^k(\psi(t)+g(t,t^{-k}x)) \\ \text{for a.e. } t\in[0,T] \text{ and all } x\in[0,\infty), \end{gathered} \tag{29}$$

where $a\in(-\infty,-1)$.

We check that f satisfies conditions $(H_1)-(H_3)$, where we put $f(t,x)$ instead of $f(t,x,y)$. Clearly $f(\cdot,x):[0,T]\to\mathbb{R}$ is measurable for all $x\in[0,\infty)$ and $f(t,\cdot):[0,\infty)\to\mathbb{R}$ is continuous for a.e. $t\in[0,T]$. Assumption (H_3°) implies that there exists $A>0$ such that $\phi(x)\leq\phi(A)+x$ for all $x\geq 0$. Consider a compact set $\mathcal{U}\subset[0,\infty)$ and put $B_{\mathcal{U}}:=\max\{x: x\in\mathcal{U}\}$. Then for a.e. $t\in[0,T]$ and all $x\in\mathcal{U}$

$$f(t,x)\leq t^k(\psi(t)+\phi(t^{-k}x))\leq t^k(\psi(t)+\phi(A))+x\leq t^k(\psi(t)+\phi(A))+B_{\mathcal{U}},$$

where $t^k(\psi(t)+\phi(A))+B_{\mathcal{U}}=:m_{\mathcal{U}}\in L^1[0,T]$. Hence f fulfils (H_1). Assumptions (H_1°) and (H_3°) yield $0<t^k\psi(t)\leq f(t,x)$ for a.e. $t\in[0,T]$ and all $x\in[0,\infty)$. So, f satisfies (H_2). Finally, by (H_3°),

$$f(t,x)\leq h(t,x){:=}t^k(\psi(t)+\phi(t^{-k}x))\quad \text{for a.e. } t\in[0,T] \text{ and all } x\in[0,\infty),$$

and for any $\varepsilon>0$ there exists $S>0$ such that $\phi(x)/x<\varepsilon$ for all $x\geq S$. If we put $V=T^k S$, then

$$t^{-k}x\geq T^{-k}x\geq S\quad \text{for all } x\geq V,$$

$$\frac{\phi(t^{-k}x)}{t^{-k}x}<\varepsilon\quad\text{and}\quad\int_0^T\frac{\phi(t^{-k}x)}{t^{-k}x}\mathrm{d}t<\varepsilon T\quad\text{for all } x\geq V.$$

This yields

$$\lim_{x\to\infty}\int_0^T\frac{\phi(t^{-k}x)}{t^{-k}x}\mathrm{d}t=0,$$

and consequently,

$$\lim_{x\to\infty}\frac{1}{x}\int_0^T h(t,x)\mathrm{d}t=\lim_{x\to\infty}\frac{1}{x}\int_0^T t^k\psi(t)\mathrm{d}t+\lim_{x\to\infty}\int_0^T\frac{\phi(t^{-k}x)}{t^{-k}x}\mathrm{d}t=0.$$

We have proved that f satisfies conditions $(H_1)-(H_3)$.

Therefore results of Section 3 are valid for problem (28), (2) and we will modify them for problem (24), (25). Denote again by $\mathcal{S}$ the set of all positive solutions of problem (28), (2) and let

$$\mathcal{S}_c=\{u\in\mathcal{S}:u'(T)=-c\},\quad c\ge 0.$$

Put $c_0=cT^{-a}$ and choose $u\in\mathcal{S}_{c_0}$. Then v from (27) is positive on $(0,T)$ and satisfies equation (24) for a.e. $t\in[0,T]$. Further, $v(T)=T^a u(T)=0$, $v'(T)=aT^{a-1}u(T)+T^a u'(T)=-T^a c_0=-c$. Hence v satisfies (26) and the second condition in (25). It remains to prove the first condition in (25). According to the proof of Lemma 7, we have

$$\begin{aligned}u(t)\;&=\;t\frac{c_0T^{a+1}}{|a+1|}(T^{-a-1}-t^{-a-1})\\&\quad+t\int_t^T s^{-a-2}\left(\int_s^T\xi^{a+1}f(\xi,u(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s,\quad t\in[0,T],\end{aligned}$$

and hence

$$\lim_{t\to0+}\frac{u(t)}{t}=\frac{c_0}{|a+1|}+\int_0^T s^{-a-2}\left(\int_s^T\xi^{a+1}f(\xi,u(\xi))\,\mathrm{d}\xi\right)\mathrm{d}s=:a_0\in(0,\infty),$$

due to (8), (29), (H_1°)–(H_3°). Therefore

$$\lim_{t\to0+}v(t)=\lim_{t\to0+}\frac{u(t)}{t}\cdot t^{a+1}=a_0\lim_{t\to0+}t^{a+1}=\infty.$$

□

Denote the set of all positive solutions of problem (24), (25) by $\mathcal{R}$ and put

$$\mathcal{R}_c=\{v\in\mathcal{R}:v(T)=-c\},\quad c\ge 0.$$

Then the proof of Theorem 8 yields the following lemma.

Lemma 10 *Let $(H_1^\circ)-(H_3^\circ)$ hold. Assume that functions u and v fulfil (27). Then $v\in\mathcal{R}_c$ if and only if $u\in\mathcal{S}_{c_0}$ for $c_0=T^{-a}c$ and $c\ge 0$.*

Theorem 9 *Let $(H_1^\circ)-(H_3^\circ)$ hold. Then the set $\mathcal{R}_c$ is nonempty for each $c \ge 0$. If in addition $g(t,\cdot) \in Lip_{loc}[0,\infty)$, then the set $\mathcal{R}_c$ is one-point for each $c \ge 0$.*

Proof. The assertion follows from Theorem 8, Lemma 10 and Theorem 3. □

Due to Lemma 10 we can define a function γ

$$\gamma(t) = \max\{v(t) : v \in \mathcal{R}_0\} = \max\{t^a u(t) : u \in \mathcal{S}_0\} \quad \text{for } t \in (0,T]. \tag{30}$$

Theorem 10 *Let $(H_1^\circ)-(H_3^\circ)$ hold. Then for each $t_0 \in (0,T)$ and each $B > \gamma(t_0)$ there exists a positive solution v of problem (24), (25) satisfying $v(t_0) = B$.*

Proof. Choose $t_0 \in (0,T)$ and $B > \gamma(t_0)$. Put $A = t_0^{-a}B$. Then $A > t_0^{-a}\gamma(t_0) = t_0^{-a}\max\{v(t_0) : v \in \mathcal{R}_0\} = t_0^{-a}\max\{t_0^a u(t_0) : u \in \mathcal{S}_0\} = \beta(t_0)$ and, by Theorem 2, there exists a positive solution u of problem (13), (2) satisfying $u(t_0) = A$. Consider v satisfying (27). By Lemma 10, v is a positive solution of problem (24), (25). Clearly $v(t_0) = t_0^a u(t_0) = t_0^a A = B$. □

Example 5 Let us choose $k \in (1,\infty)$, $\alpha \in [0,k+1)$, $\eta \in [0,1)$ and define the functions ψ, g by

$$\psi(t) = h_1(t)t^{-\alpha}, \quad g(t,x) = h_2(t,x)x^\eta, \quad t \in [0,T],\ x \in [0,\infty),$$

where $h_1 \in C[0,T]$, $h_1(t) > 0$ for a.e. $t \in [0,T]$ and $h_2 \in C([0,T]\times[0,\infty))$ is nonnegative and bounded. Then ψ satisfies condition (H_1°) and g satisfies (H_2°) and (H_3°) with $\phi(x) = Mx^\eta$, where $M = \sup\{h_2(t,x) : t \in [0,T], x \in [0,\infty)\}$.

Now, assume moreover

(H_4°) $g(t,x)$ is increasing in x for a.e. $t \in [0,T]$.

Conditions $(H_1^\circ)-(H_4^\circ)$ guarantee that the function f of (29) satisfies conditions $(H_1^*)-(H_3^*)$ as well as conditions $(H_1)-(H_3)$ with $\varphi(t) = t^k\psi(t)$ for a.e. $t \in [0,T]$. Therefore now, all results of the both Sections 3 and 4 are valid (with the exception of Theorem 6) for problem (28), (2) and can be modified for problem (24), (25). For example, Lemma 10 and Theorem 4 yield next two assertions.

Lemma 11 *Let $(H_1^\circ)-(H_4^\circ)$ hold. Assume that $c \ge 0$. Then there exist $v_{c,min}, v_{c,max} \in \mathcal{R}_c$ such that*

$$v_{c,min}(t) \le v(t) \le v_{c,max}(t) \quad \textit{for } t \in (0,T],\ v \in \mathcal{R}_c. \tag{31}$$

Proof. Consider $u_{c_0,min}, u_{c_0,max} \in \mathcal{S}_{c_0}$, where $c_0 = T^{-a}c$. According to (19) we have

$$u_{c_0,min}(t) \le u(t) \le u_{c_0,max}(t) \quad \text{for } t \in [0,T],\ u \in \mathcal{S}_{c_0}. \tag{32}$$

If we put

$$v_{c,min} = t^a u_{c_0,min},\ v_{c,max} = t^a u_{c_0,max},\ v = t^a u, \tag{33}$$

we get by Lemma 10 that $v_{c,min}, v_{c,max}, v \in \mathcal{R}_c$ and (32) yields (31). □

Theorem 11 *Let (H_1°)–(H_4°) hold. Assume that there exists $t_0 \in (0,T)$ such that $v_{c,min}(t_0) < v_{c,max}(t_0)$ for some $c > 0$.*
Then for each $B \in (v_{c,min}(t_0), v_{c,max}(t_0))$ there exists $v \in \mathcal{R}_c$ satisfying $v(t_0) = B$.

Proof. Choose $B \in (v_{c,min}(t_0), v_{c,max}(t_0))$ and put $A = t_0^{-a}B$. Put

$$c_0 = T^{-a}c, \quad u_{c_0,min}(t) = t^{-a}v_{c,min}(t), u_{c_0,max}(t) = t^{-a}v_{c,max}(t),\ t \in (0,T].$$

By Lemma 10, $u_{c_0,min}, u_{c_0,max} \in \mathcal{S}_{c_0}$. Since

$$u_{c_0,min}(t_0) = t_0^{-a}v_{c,min}(t_0) < A < t_0^{-a}v_{c,max}(t_0) = u_{c_0,max}(t_0),$$

Theorem 4 gurantees that there exists $u \in \mathcal{S}_{c_0}$ satisfying $u(t_0) = A$. Put $v = t^a u$ for $t \in (0,T]$. Then $v(t_0) = t_0^a u(t_0) = t_0^a A = B$. Lemma 10 yields $v \in \mathcal{R}_c$. □

Remaining assertions of Section 4 can be modified for problem (24), (25) similarly.

Example 6 Consider the functions ψ, g of Example 5 and assume moreover that the function h_2 is increasing in x for a.e. $t \in [0,T]$. Then ψ, g satisfy conditions (H_1°)–(H_4°).

Acknowledgements
This research was supported by the Council of Czech Goverment MSM6198959214.

The authors are very grateful to the referees for the careful reading and useful comments and suggestions which greatly improved the paper.

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