Non-ordered lower and upper functions in second order impulsive periodic problems

Irena Rachůnková* and Milan Tvrdý†

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Summary. In this paper, using the lower/upper functions argument, we establish new existence results for the nonlinear impulsive periodic boundary value problem

$$u'' = f(t,u,u'), \tag{1.1}$$

$$u(t_i+) = \mathrm{J}_i(u(t_i)), \quad u'(t_i+) = \mathrm{M}_i(u'(t_i)), \quad i = 1,2,\dots,m, \tag{1.2}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{1.3}$$

where $f \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$ and J_i, $\mathrm{M}_i \in \mathbb{C}(\mathbb{R})$. The main goal of the paper is to obtain the results in the case that the lower/upper functions σ_1/σ_2 associated with the problem are not well-ordered, i.e. $\sigma_1 \not\leq \sigma_2$ on $[0,T]$.

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0. Introduction

In this paper we provide new conditions for f_i, J_i, M_i, $i = 1, 2, \dots, m$, which guarantee the existence of a solution of the nonlinear impulsive periodic boundary value problem (1.1)–(1.3). We have studied this problem in [11] using arguments based on the existence of a well-ordered pair $\sigma_1 \leq \sigma_2$ on $[0, T]$ of lower/upper functions σ_1/σ_2 associated with the problem. Such assumption corresponds to requirements imposed by Hu Shouchuan and Lakshmikantham [6] (see also Bainov and Simeonov [1]), Erbe and Liu Xinzhi [5], Liz and Nieto [7], [8], Dong Yujun [4] and Zhang Zhitao [12] who have investigated the problems of the type (1.1)–(1.3). Note that a similar problem with different impulse conditions was recently treated by Cabada, Nieto, Franco and Trofimchuk [2]. However, their principal assumption was that of the existence of well-ordered pair of lower/upper functions, as well.

Here, we consider problem (1.1)–(1.3) in a more complicated case. Particularly, we assume that there are only lower/upper functions to (1.1)–(1.3) which are not well-ordered, i.e.

$$\sigma_1(\tau) > \sigma_2(\tau) \quad \text{for some} \quad \tau \in [0, T]. \tag{0.1}$$

As far as we know, up to now there has been delivered no existence result for any kind of second order impulsive problems having only lower/upper functions in this setting. The first step in this direction we did in [10] where we worked for $m = 1$ with strict lower/upper functions and where we computed the Leray-Schauder degree of certain auxiliary operators related to the problem (1.1)–(1.3).

Throughout the paper we keep the following notation and conventions: For a real valued function u defined a.e. on $[0, T]$, we put

$$\|u\|_\infty = \sup_{t \in [0,T]} |u(t)| \quad \text{and} \quad \|u\|_1 = \int_0^T |u(s)| \, \mathrm{d}s.$$

For a given interval $J \subset \mathbb{R}$, by $\mathbb{C}(J)$ we denote the set of real valued functions which are continuous on J. Furthermore, $\mathbb{C}^1(J)$ is the set of functions having continuous first derivatives on J and $\mathbb{L}(J)$ is the set of functions which are Lebesgue integrable on J.

Let $m \in \mathbb{N}$ and let $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$ be a division of the interval $[0, T]$. We denote $\mathrm{D} = \{t_1, t_2, \ldots, t_m\}$ and define $\mathbb{C}^1_{\mathrm{D}}[0, T]$ as the set of functions $u : [0, T] \mapsto \mathbb{R}$,

$$u(t) = \begin{cases} u_{[0]}(t) & \text{if } \ t \in [0, t_1], \\ u_{[1]}(t) & \text{if } \ t \in (t_1, t_2], \\ \ldots & \qquad \ldots \\ u_{[m]}(t) & \text{if } \ t \in (t_m, T], \end{cases}$$

where $u_{[i]} \in \mathbb{C}^1[t_i, t_{i+1}]$ for $i = 0, 1, \ldots, m$. Moreover, $\mathbb{AC}^1_{\mathrm{D}}[0, T]$ stands for the set of functions $u \in \mathbb{C}^1_{\mathrm{D}}[0, T]$ having first derivatives absolutely continuous on each subinterval (t_i, t_{i+1}), $i = 0, 1, \ldots, m$. For $u \in \mathbb{C}^1_{\mathrm{D}}[0, T]$ and $i = 1, 2, \ldots, m+1$ we write

$$u'(t_i) = u'(t_i-) = \lim_{t \to t_i-} u'(t), \quad u'(0) = u'(0+) = \lim_{t \to 0+} u'(t) \tag{0.2}$$

and $\|u\|_{\mathrm{D}} = \|u\|_\infty + \|u'\|_\infty$. Note that the set $\mathbb{C}^1_{\mathrm{D}}[0, T]$ becomes a Banach space when equipped with the norm $\|.\|_{\mathrm{D}}$ and with the usual algebraic operations.

We say that $f : [0, T] \times \mathbb{R}^2 \mapsto \mathbb{R}$ satisfies the *Carathéodory conditions* on $[0, T] \times \mathbb{R}^2$ if (i) for each $x \in \mathbb{R}$ and $y \in \mathbb{R}$ the function $f(., x, y)$ is measurable on $[0, T]$; (ii) for almost every $t \in [0, T]$ the function $f(t, ., .)$ is continuous on $\mathbb{R}^2$; (iii) for each compact set $K \subset \mathbb{R}^2$ there is a function $m_K(t) \in \mathbb{L}[0, T]$ such that $|f(t, x, y)| \le m_K(t)$ holds for a.e. $t \in [0, T]$ and all $(x, y) \in K$. The set of functions satisfying the Carathéodory conditions on $[0, T] \times \mathbb{R}^2$ will be denoted by $\mathrm{Car}([0, T] \times \mathbb{R}^2)$.

Given a Banach space $\mathbb{X}$ and its subset M, let $\mathrm{cl}(M)$ and ∂M denote the closure and the boundary of M, respectively.

Let Ω be an open bounded subset of $\mathbb{X}$. Assume that the operator $\mathrm{F}:\ \mathrm{cl}(\Omega)\mapsto\mathbb{X}$ is completely continuous and $\mathrm{F}\,u\neq u$ for all $u\in\partial\,\Omega$. Then $\deg(\mathrm{I}-\mathrm{F},\Omega)$ denotes the *Leray-Schauder topological degree* of $\mathrm{I}-\mathrm{F}$ with respect to Ω, where I is the identity operator on $\mathbb{X}$. For the definition and properties of the degree see e.g. [3] or [9].

1. Formulation of the problem and main assumptions

Here we study the existence of solutions to the problem

$$u'' = f(t,u,u'), \tag{1.1}$$

$$u(t_i+) = \mathrm{J}_i(u(t_i)),\quad u'(t_i+) = \mathrm{M}_i(u'(t_i)),\quad i=1,2,\dots,m, \tag{1.2}$$

$$u(0)=u(T),\quad u'(0)=u'(T), \tag{1.3}$$

where $u'(t_i)$ are understood in the sense of (0.2), $f\in\mathrm{Car}([0,T]\times\mathbb{R}^2)$, $\mathrm{J}_i\in\mathbb{C}(\mathbb{R})$ and $\mathrm{M}_i\in\mathbb{C}(\mathbb{R})$.

1.1. Definition. A *solution of the problem* (1.1)–(1.3) is a function $u\in\mathbb{AC}^1_{\mathrm{D}}[0,T]$ which satisfies the impulsive conditions (1.2), the periodic conditions (1.3) and for a.e. $t\in[0,T]$ fulfils the equation (1.1).

1.2. Definition. A function $\sigma_1\in\mathbb{AC}^1_{\mathrm{D}}[0,T]$ is called a *lower function of the problem* (1.1)–(1.3) if

$$\sigma_1''(t)\geq f(t,\sigma_1(t),\sigma_1'(t))\quad\text{for a.e.}\quad t\in[0,T], \tag{1.4}$$

$$\sigma_1(t_i+) = \mathrm{J}_i(\sigma_1(t_i)),\quad \sigma_1'(t_i+)\geq \mathrm{M}_i(\sigma_1'(t_i)),\quad i=1,2,\dots,m, \tag{1.5}$$

$$\sigma_1(0)=\sigma_1(T),\quad \sigma_1'(0)\geq\sigma_1'(T). \tag{1.6}$$

Similarly, a function $\sigma_2 \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ is an *upper function of the problem* (1.1)–(1.3) if

$$\sigma_2''(t) \le f(t, \sigma_2(t), \sigma_2'(t)) \quad \text{for a.e.} \quad t \in [0,T], \tag{1.7}$$

$$\sigma_2(t_i+) = \mathrm{J}_i(\sigma_2(t_i)), \quad \sigma_2'(t_i+) \le \mathrm{M}_i(\sigma_2'(t_i)), \quad i = 1, 2, \ldots, m, \tag{1.8}$$

$$\sigma_2(0) = \sigma_2(T), \quad \sigma_2'(0) \le \sigma_2'(T). \tag{1.9}$$

A straightforward illustration of Definition 1.2 is the following proposition providing a simple example of conditions ensuring the existence of lower and upper functions for (1.1)–(1.3).

1.3. Proposition. *Let* $\alpha_0 \in R$. *For* $i = 1, 2, \ldots, m$ *assume that* $\mathrm{M}_i(0) = 0$, $\alpha_i = \mathrm{J}_i(\alpha_{i-1})$ *where* $\alpha_m = \alpha_0$, $f(t, \alpha_0, 0) \le 0$ *for a.e.* $t \in (0, t_1)$, $f(t, \alpha_i, 0) \le 0$ *for a.e.* $t \in (t_i, t_{i+1})$, *and put* $\sigma_1(t) = \alpha_0$ *on* $[0, t_1]$, $\sigma_1(t) = \alpha_i$ *on* $(t_i, t_{i+1}]$. *Then* σ_1 *is a lower function of* (1.1)–(1.3).

Let $\beta_0 \in R$. *For* $i = 1, 2, \ldots, m$ *assume that* $\mathrm{M}_i(0) = 0$, $\beta_i = \mathrm{J}_i(\beta_{i-1})$ *where* $\beta_m = \beta_0$, $f(t, \beta_0, 0) \ge 0$ *for a.e.* $t \in (0, t_1)$, $f(t, \beta_i, 0) \ge 0$ *for a.e.* $t \in (t_i, t_{i+1})$, *and put* $\sigma_2(t) = \beta_0$ *on* $[0, t_1]$, $\sigma_2(t) = \beta_i$ *on* $(t_i, t_{i+1}]$. *Then* σ_2 *is an upper function of* (1.1)–(1.3).

1.4. Remark. In particular, if $\mathrm{M}_i(0) = 0$, $\mathrm{J}_i(\alpha_0) = \alpha_0$, $\mathrm{J}_i(\beta_0) = \beta_0$ for $i = 1, 2, \ldots, m$ and $f(t, \alpha_0, 0) \le 0$, $f(t, \beta_0, 0) \ge 0$ for a.e. $t \in [0,T]$, then $\sigma_1(t) = \alpha_0$ and $\sigma_2(t) = \beta_0$, $t \in [0,T]$, are respectively lower and upper functions of (1.1)–(1.3).

1.5. Assumptions. In the paper we work with the following assumptions:

$$\left.\begin{array}{l} 0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T < \infty,\ \mathrm{D} = \{t_1, t_2, \dots, t_m\}, \\ f \in \mathrm{Car}([0,T] \times \mathbb{R}^2),\ \mathrm{J}_i \in \mathbb{C}(\mathbb{R}),\ \mathrm{M}_i \in \mathbb{C}(\mathbb{R}),\ i = 1, 2, \dots, m; \end{array}\right\} \tag{1.10}$$

$$\sigma_1 \text{ and } \sigma_2 \text{ are respectively lower and upper functions of (1.1)–(1.3);} \tag{1.11}$$

$$\left.\begin{array}{l} x > \sigma_1(t_i) \implies \mathrm{J}_i(x) > \mathrm{J}_i(\sigma_1(t_i)), \\ x < \sigma_2(t_i) \implies \mathrm{J}_i(x) < \mathrm{J}_i(\sigma_2(t_i)), \quad i = 1, 2, \dots, m; \end{array}\right\} \tag{1.12}$$

$$\left.\begin{array}{l} y \le \sigma_1'(t_i) \implies \mathrm{M}_i(y) \le \mathrm{M}_i(\sigma_1'(t_i)), \\ y \ge \sigma_2'(t_i) \implies \mathrm{M}_i(y) \ge \mathrm{M}_i(\sigma_2'(t_i)), \quad i = 1, 2, \dots, m. \end{array}\right\} \tag{1.13}$$

1.6. Operator reformulation of (1.1)–(1.3). Let $G(t,s)$ be the Green function of the Dirichlet boundary value problem $u'' = 0$, $u(0) = u(T) = 0$, i.e.

$$G(t,s) = \begin{cases} \dfrac{t\,(s-T)}{T} & \text{if } 0 \le t \le s \le T, \\ \dfrac{s\,(t-T)}{T} & \text{if } 0 \le s < t \le T, \end{cases}$$

and let the operator $\mathrm{F} : \mathbb{C}^1_{\mathrm{D}}[0,T] \mapsto \mathbb{C}^1_{\mathrm{D}}[0,T]$ be defined by

$$\begin{aligned} (\mathrm{F}\,u)(t) = u(0) + u'(0) - u'(T) + \int_0^T G(t,s)\, f(s, u(s), u'(s))\, \mathrm{d}s \\ - \sum_{i=1}^m \frac{\partial G}{\partial s}(t,t_i)\,(\mathrm{J}_i(u(t_i)) - u(t_i)) + \sum_{i=1}^m G(t,t_i)\,(\mathrm{M}_i(u'(t_i)) - u'(t_i)). \end{aligned} \tag{1.14}$$

Then, as in [10, Lemma 3.1], we can prove that F is completely continuous. Moreover, we can check that each fixed point u of F is a solution of (1.1)–(1.3).

In the proof of our main result we will need the next proposition which concerns the case of well-ordered lower/upper functions and which follows from [11, Corollary 3.5].

1.7. Proposition. *Assume that* (1.10) *holds and let α and β be respectively lower* *and upper functions of* (1.1)–(1.3) *such that*

$$\alpha(t) < \beta(t) \quad \textit{for } t \in [0,T] \quad \textit{and} \quad \alpha(\tau+) < \beta(\tau+) \quad \textit{for } \tau \in \mathrm{D}, \tag{1.15}$$

$$\alpha(t_i) < x < \beta(t_i) \implies \mathrm{J}_i(\alpha(t_i)) < \mathrm{J}_i(x) < \mathrm{J}_i(\beta(t_i)), \quad i = 1, 2, \dots, m \tag{1.16}$$

and

$$\left.\begin{aligned} &y \le \alpha'(t_i) \implies \mathrm{M}_i(y) \le \mathrm{M}_i(\alpha'(t_i)), \\ &y \ge \beta'(t_i) \implies \mathrm{M}_i(y) \ge \mathrm{M}_i(\beta'(t_i)), \quad i = 1, 2, \dots, m. \end{aligned}\right\} \tag{1.17}$$

Further, let $h \in \mathbb{L}[0,T]$ *be such that*

$$|f(t,x,y)| \le h(t) \qquad \textit{for a.e. } t \in [0,T] \quad \textit{and all } (x,y) \in [\alpha(t), \beta(t)] \times \mathbb{R} \tag{1.18}$$

and let the operator F *be defined by* (1.14). *Finally, for* $r \in (0,\infty)$ *denote*

$$\Omega(\alpha,\beta,r) = \{u \in \mathbb{C}^1_{\mathrm{D}}[0,T] : \alpha(t) < u(t) < \beta(t) \quad \textit{for } t \in [0,T], \tag{1.19}$$
$$\alpha(\tau+) < u(\tau+) < \beta(\tau+) \quad \textit{for } \tau \in \mathrm{D}, \ \|u'\|_\infty < r\}.$$

Then $\deg(\mathrm{I} - \mathrm{F}, \Omega(\alpha,\beta,r)) = 1$ *whenever* $\mathrm{F}\,u \ne u$ *on* $\partial\Omega(\alpha,\beta,r)$ *and*

$$r > \|h\|_1 + \frac{\|\alpha\|_\infty + \|\beta\|_\infty}{\Delta}, \quad \textit{where } \Delta = \min_{i=1,2,\dots,m+1} (t_i - t_{i-1}). \tag{1.20}$$

Proof. Using the Mean Value Theorem, we can show that

$$\|u'\|_\infty \le \|h\|_1 + \frac{\|\alpha\|_\infty + \|\beta\|_\infty}{\Delta} \tag{1.21}$$

holds for each $u \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ fulfilling $\alpha(t) < u(t) < \beta(t)$ for $t \in [0,T]$ and $\alpha(\tau+) < u(\tau+) < \beta(\tau+)$ for $\tau \in \mathrm{D}$. Thus, if we denote by c the right-hand side of (1.21), we can follow the proof of [11, Corollary 3.5]. □

2. A priori estimates

The proof of our main existence result (Theorem 3.1) is based on the evaluation of the topological degree of a proper auxiliary operator by means of Proposition 1.7. To this aim we need a priori estimates for certain sets of functions which are provided in this section.

2.1. Lemma. *Let* $\rho_1 \in (0,\infty)$, $\widetilde{h} \in \mathbb{L}[0,T]$, $\mathrm{M}_i \in \mathbb{C}(\mathbb{R})$, $i = 1,2,\dots,m$. *Then there exists* $d \in (\rho_1,\infty)$ *such that the estimate*

$$\|u'\|_\infty < d \tag{2.1}$$

is valid for each function $u \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ *satisfying* (1.3),

$$|u'(\xi_u)| < \rho_1 \quad \textit{for some} \ \ \xi_u \in [0,T], \tag{2.2}$$

$$u'(t_i+) = \mathrm{M}_i(u'(t_i)), \quad i = 1,2,\dots,m, \tag{2.3}$$

and

$$|u''(t)| < \widetilde{h}(t) \ \ \textit{for a.e.} \ \ t \in [0,T]. \tag{2.4}$$

Proof. Suppose that $u \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ satisfies (1.3) and (2.2)–(2.4). Since $\mathrm{M}_i \in \mathbb{C}(\mathbb{R})$ for $i = 1,2,\dots,m$, we have

$$b_i(a) := \sup_{|y|<a} |\,\mathrm{M}_i(y)| < \infty \quad \text{for} \ \ a \in (0,\infty), \ i = 1,2,\dots,m. \tag{2.5}$$

Furthermore, due to (1.3), we can assume that $\xi_u \in (0,T]$, i.e. there is $j \in \{1,2,\dots,m+1\}$ such that $\xi_u \in (t_{j-1},t_j]$. We will distinguish 3 cases: either $j = 1$ or $j = m+1$ or $1 < j < m+1$.

Let $j = 1$. Then, using (2.2) and (2.4), we obtain

$$|u'(t)| < a_1 \ \ \text{on} \ \ [0,t_1], \tag{2.6}$$

where $a_1 = \rho_1 + \|\widetilde{h}\|_1$. Hence, in view of (2.5), we have $|u'(t_1+)| < b_1(a_1)$, wherefrom, using (2.4), we deduce that $|u'(t)| < b_1(a_1) + \|\widetilde{h}\|_1$ for $t \in (t_1, t_2]$. Continuing by induction, we get $|u'(t)| < a_{i+1} = b_i(a_i) + \|\widetilde{h}\|_1$ on $(t_i, t_{i+1}]$ for $i = 2, \dots, m$, i.e.

$$\|u'\|_\infty < d := \max\{a_i : i = 1, 2, \dots, m+1\}. \tag{2.7}$$

Assume that $j = m+1$. Then, using (2.2) and (2.4), we obtain

$$|u'(t)| < a_{m+1} \quad \text{on} \quad (t_m, T], \tag{2.8}$$

where $a_{m+1} = \rho_1 + \|\widetilde{h}\|_1$. Furthermore, due to (1.3), we have $|u'(0)| < a_{m+1}$ which together with (2.4) yields that (2.6) is true with $a_1 = a_{m+1} + \|\widetilde{h}\|_1$. Now, proceeding as in the case $j = 1$, we show that (2.7) is true also in the case $j = m+1$.

Assume that $1 < j < m+1$. Then (2.2) and (2.4) yield $|u'(t)| < a_{j+1} = \rho_1 + \|\widetilde{h}\|_1$ on $(t_j, t_{j+1}]$. If $j < m$, then $|u'(t)| < a_{j+2} = b_{j+1}(a_{j+1}) + \|\widetilde{h}\|_1$ on $(t_{j+1}, t_{j+2}]$. Proceeding by induction we get (2.8) with $a_{m+1} = b_m(a_m) + \|\widetilde{h}\|_1$, wherefrom (2.7) again follows as in the previous case. □

2.2. Lemma. *Let $\rho_0, d \in (0, \infty)$ and $\mathrm{J}_i \in \mathbb{C}(\mathbb{R})$, $i = 1, 2, \dots, m$. Then there exists $c \in (\rho_0, \infty)$ such that the estimate*

$$\|u\|_\infty < c \tag{2.9}$$

is valid for each $u \in \mathbb{C}^1_{\mathrm{D}}[0, T]$ and each $\widetilde{\mathrm{J}}_i \in \mathbb{C}(\mathbb{R})$, $i = 1, 2, \dots, m$, satisfying (1.3), (2.1),

$$u(t_i+) = \widetilde{\mathrm{J}}_i(u(t_i)), \quad i = 1, 2, \dots, m, \tag{2.10}$$

$$|u(\tau_u)| < \rho_0 \quad \textit{for some} \quad \tau_u \in [0, T] \tag{2.11}$$

and

$$\sup\{|\,\mathrm{J}_i(x)|:\,|x|<a\}<b \implies \sup\{|\widetilde{\mathrm{J}}_i(x)|:\,|x|<a\}<b \tag{2.12}$$

$$\textit{for}\;\; i=1,2,\dots,m,\;\; a\in(0,\infty),\;\; b\in(a,\infty).$$

Proof. We will argue similarly as in the proof of Lemma 2.1. Suppose that $u \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ satisfies (1.3), (2.1), (2.10), (2.11) and that $\widetilde{\mathrm{J}}_i \in \mathbb{C}(\mathbb{R})$, $i=1,2,\dots,m$, satisfy (2.12). Due to (1.3) we can assume that $\tau_u \in (0,T]$, i.e. there is $j \in \{1,2,\dots,m+1\}$ such that $\tau_u \in (t_{j-1},t_j]$. We will consider three cases: $j=1$, $j=m+1$, $1<j<m+1$. If $j=1$, then (2.1) and (2.11) yield $|u(t)|<a_1=\rho_0+d\,T$ on $[0,t_1]$. In particular, $|u(t_1)|<a_1$. Since $\mathrm{J}_1\in\mathbb{C}(\mathbb{R})$, we can find $b_1(a_1)\in(a_1,\infty)$ such that $|\,\mathrm{J}_1(x)|<b_1(a_1)$ for all $x\in(-a_1,a_1)$ and consequently, by (2.12), also $|\widetilde{\mathrm{J}}_1(x)|<b_1(a_1)$ for all $x\in(-a_1,a_1)$. Therefore, by (2.1), $|u(t)|<|u(t_1+)|+d\,T=|\widetilde{\mathrm{J}}_1(u(t_1))|+d\,T<a_2=b_1(a_1)+d\,T$ on $(t_1,t_2]$. Proceeding by induction we get $b_i(a_i)\in(a_i,\infty)$ such that $|u(t)|<a_{i+1}=b_i(a_i)+d\,T$ for $t\in(t_i,t_{i+1}]$ and $i=2,\dots,m$. As a result, (2.9) is true with $c=\max\{a_i:\,i=1,2,\dots,m+1\}$. Analogously we would proceed in the remaining cases $j=m+1$ or $1<j<m+1$. □

Finally, we will need two estimates for functions u satisfying one of the following conditions:

$$u(s_u)<\sigma_1(s_u) \quad\text{and}\quad u(t_u)>\sigma_2(t_u) \quad\text{for some}\;\; s_u,t_u\in[0,T], \tag{2.13}$$

$$u\ge\sigma_1 \;\;\text{on}\;\; [0,T] \quad\text{and}\quad \inf_{t\in[0,T]}|u(t)-\sigma_1(t)|=0, \tag{2.14}$$

$$u\le\sigma_2 \;\;\text{on}\;\; [0,T] \quad\text{and}\quad \inf_{t\in[0,T]}|u(t)-\sigma_2(t)|=0. \tag{2.15}$$

2.3. Lemma. *Assume that* $\sigma_1,\,\sigma_2\in\mathbb{AC}^1_{\mathrm{D}}[0,T]$, J_i, M_i, $\widetilde{\mathrm{J}}_i\in\mathbb{C}(\mathbb{R})$, $i=1,2,\dots,m$, *satisfy* (1.12), (1.13) *and*

$$\left.\begin{aligned} x>\sigma_1(t_i) &\implies \widetilde{\mathrm{J}}_i(x)>\widetilde{\mathrm{J}}_i(\sigma_1(t_i))=\mathrm{J}_i(\sigma_1(t_i)),\\ x<\sigma_2(t_i) &\implies \widetilde{\mathrm{J}}_i(x)<\widetilde{\mathrm{J}}_i(\sigma_2(t_i))=\mathrm{J}_i(\sigma_2(t_i)), \quad i=1,2,\dots,m. \end{aligned}\right\} \tag{2.16}$$

Denote

$$B = \{u \in \mathbb{C}^1_{\mathrm{D}}[0,T] \,:\, u \textit{ satisfies } (1.3), (2.10), (2.3) \textit{ and one of the conditions } (2.13), (2.14), (2.15)\}. \tag{2.17}$$

Then each function $u \in B$ satisfies

$$\left.\begin{aligned} &|u'(\xi_u)| < \rho_1 \quad \textit{for some } \xi_u \in [0,T], \textit{ where} \\ &\rho_1 = \frac{2}{t_1}\,(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty) + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty + 1. \end{aligned}\right\} \tag{2.18}$$

Proof. • Part 1. Assume that $u \in B$ satisfies (2.13). There are 3 cases to consider:

Case A. If $\min\{\sigma_1(t), \sigma_2(t)\} \le u(t) \le \max\{\sigma_1(t), \sigma_2(t)\}$ for $t \in [0,T]$, then, by the Mean Value Theorem, there is $\xi_u \in (0, t_1)$ such that

$$|u'(\xi_u)| \le \frac{2}{t_1}\,(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty). \tag{2.19}$$

Case B. Assume that $u(s) > \sigma_1(s)$ for some $s \in [0,T]$ and denote $v = u - \sigma_1$. Due to (2.13) we have

$$v_* = \inf_{t \in [0,T]} v(t) < 0 \quad \text{and} \quad v^* = \sup_{t \in [0,T]} v(t) > 0. \tag{2.20}$$

We are going to prove that

$$v'(\alpha) = 0 \text{ for some } \alpha \in [0,T] \text{ or } v'(\tau+) = 0 \text{ for some } \tau \in \mathrm{D}\,. \tag{2.21}$$

Suppose, on the contrary, that (2.21) does not hold.

Let $v'(0) > 0$. Then, according to (1.3) and (1.6), $v'(T) > 0$, as well. Due to the assumption that (2.21) does not hold, this together with (1.5) yields that

$$0 < v'(t_m+) = u'(t_m+) - \sigma_1'(t_m+) \le \mathrm{M}_m(u'(t_m)) - \mathrm{M}_m(\sigma_1'(t_m)),$$

which is by (1.13) possible only if $u'(t_m) > \sigma_1'(t_m)$, i.e. $v'(t_m) > 0$. Continuing in this way on each $(t_i, t_{i+1}]$, $i = 0, 1, \ldots, m-1$, we get

$$v'(t) > 0 \text{ for } t \in [0,T] \quad \text{and} \quad v'(\tau+) > 0 \text{ for } \tau \in \mathrm{D}. \tag{2.22}$$

If $v(0) \geq 0$, then $v(t) > 0$ on $(0, t_1]$ due to (2.22). Further, it follows by (1.5), (2.10) and (2.16) that $u(t_1+) > \sigma_1(t_1+)$, i.e. $v(t_1+) > 0$. Continuing by induction we deduce that $v \geq 0$ on $[0,T]$, contrary to (2.20).

If $v(0) < 0$, then by (1.3) and (1.6) we have $v(T) < 0$. Further, by virtue of (2.22) we obtain $v < 0$ on $(t_m, T]$ and, in particular, $v(t_m+) < 0$. So, $\widetilde{\mathrm{J}}_m(u(t_m)) < \mathrm{J}_m(\sigma_1(t_m))$ wherefrom $u(t_m) \leq \sigma_1(t_m)$ follows, due to (2.16). Thus, we have $v < 0$ on (t_{m-1}, t_m). Continuing by induction we get $v \leq 0$ on $[0,T]$, contrary to (2.20).

Now, assume that $v'(0) < 0$. Then $v'(t_1) < 0$, i.e. $u'(t_1) < \sigma_1'(t_1)$ wherefrom, by (1.5), (1.13) and the assumption that (2.21) does not hold, the inequality $v'(t_1+) = u'(t_1+) - \sigma_1'(t_1+) < 0$ follows. Similarly as in the proof of (2.22) we show that

$$v'(t) < 0 \text{ for } t \in [0,T] \quad \text{and} \quad v'(\tau+) < 0 \text{ for } \tau \in \mathrm{D}. \tag{2.23}$$

Now, having (2.23), we consider as above two cases: $v(0) \geq 0$ and $v(0) < 0$, and construct a contradiction by means of analogous arguments.

So we have proved that (2.21) is true, which yields the existence of $\xi_u \in [0,T]$ having the property

$$|u'(\xi_u)| < \|\sigma_1'\|_\infty + 1. \tag{2.24}$$

Case C. If $u(s) < \sigma_2(s)$ for some $s \in [0,T]$, we put $v = u - \sigma_2$ and, using the properties of σ_2 instead of σ_1, we can argue as in Case B and show that there exists $\xi_u \in [0,T]$ such that

$$|u'(\xi_u)| < \|\sigma_2'\|_\infty + 1. \tag{2.25}$$

Taking into account (2.19), (2.24) and (2.25) we conclude that (2.18) is valid for any $u \in B$ fulfilling (2.13).

• PART 2. Let $u \in B$ satisfy (2.14). Then $u \geq \sigma_1$ on $[0,T]$ and either there is $\alpha_u \in [0,T]$ such that $u(\alpha_u) = \sigma_1(\alpha_u)$ or there is $t_j \in \mathrm{D}$ such that $u(t_j+) = \sigma_1(t_j+)$.

CASE A. Let the first possibility occur. If $\alpha_u \in (0,T) \setminus \mathrm{D}$, then necessarily $u'(\alpha_u) = \sigma_1'(\alpha_u)$. Consequently, the estimate (2.24) is valid. If $\alpha_u = 0$, then $\inf\{u(t) - \sigma_1(t) : t \in [0,T]\} = u(0) - \sigma_1(0) = u(T) - \sigma_1(T) = 0,$ which, by virtue of (1.3) and (1.6), implies $0 \leq u'(0) - \sigma_1'(0) \leq u'(T) - \sigma_1'(T) \leq 0$, i.e. $u'(0) = \sigma_1'(0)$ and the estimate (2.24) is valid with $\xi_u = 0$. If $\alpha_u = t_j$ for some $t_j \in \mathrm{D}$, then

$0 = u(t_j) - \sigma_1(t_j) = u(t_j+) - \sigma_1(t_j+)$. Having in mind that $u \geq \sigma_1$ on $[0,T]$, we get $u'(t_j+) \geq \sigma_1'(t_j+)$ and $u'(t_j) \leq \sigma_1'(t_j)$. On the other hand, with respect to (1.13), the last inequality gives also $\mathrm{M}_j(u'(t_j)) \leq \mathrm{M}_j(\sigma_1'(t_j))$, which leads to $\sigma_1'(t_j+) = u'(t_j+)$. Thus, (2.24) is fulfilled for some $\xi_u \in (t_j, t_{j+1})$ which is sufficiently close to t_j.

CASE B. Let the second possibility occur, i.e. $u(t_j+) = \sigma_1(t_j+)$ for some $t_j \in \mathrm{D}$. According to (1.5) and (2.10), we have $\widetilde{\mathrm{J}}_j(u(t_j)) = \mathrm{J}_j(\sigma_1(t_j))$. Taking into account (2.16), we see that this can occur only if $u(t_j) \leq \sigma_1(t_j)$. On the other hand, by the assumption (2.14) we have $u \geq \sigma_1$ on $[0,T]$. Hence we conclude that $u(t_j) = \sigma_1(t_j)$ and so, arguing as before, we get (2.24) again.

To summarize: (2.18) holds for any $u \in B$ fulfilling (2.14).

• PART 3. Let $u \in B$ satisfy (2.15). Then using the properties of σ_2 instead of σ_1, we argue analogously to PART 2 and prove that (2.25) is valid for each $u \in B$ which satisfies (2.15). In particular, (2.18) holds for any $u \in B$ fulfilling (2.15). □

2.4. Lemma. *Assume that* $\sigma_1, \sigma_2 \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$, $\mathrm{J}_i, \widetilde{\mathrm{J}}_i \in \mathbb{C}(\mathbb{R})$, $i = 1, 2, \ldots, m$, *satisfy* (1.12) *and* (2.16). *Then*

$$\min\{\sigma_1(\tau_u+),\sigma_2(\tau_u+)\} \le u(\tau_u+) \le \max\{\sigma_1(\tau_u+),\sigma_2(\tau_u+)\} \qquad \text{for some} \;\; \tau_u \in [0,T) \tag{2.26}$$

is true for each $u \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ *fulfilling* (1.3), (2.10) *and one of the conditions* (2.13)–(2.15).

Proof. Assume, on the contrary, that there is $u \in B$ for which (2.26) does not hold. If $u(0) < \min\{\sigma_1(0),\sigma_2(0)\}$ then, taking into account the continuity of the functions u, σ_1 and σ_2 on $[0,t_1]$, we deduce that $u(t) < \min\{\sigma_1(t),\sigma_2(t)\}$ is true for each $t \in [0,t_1]$. Consequently, due to (2.16), we have $u(t_1+) < \min\{\sigma_1(t_1+),\sigma_2(t_1+)\}$. It is easy to see that proceeding by induction we get

$$u(t) < \min\{\sigma_1(t),\sigma_2(t)\} \quad \text{and} \quad u(\tau+) < \min\{\sigma_1(\tau+),\sigma_2(\tau+)\}$$

for each $t \in [0,T) \setminus \mathrm{D}$ and $\tau \in \mathrm{D}$, a contradiction to (2.13). Similarly, we can see that $u(0) > \max\{\sigma_1(0),\sigma_2(0)\}$ implies that

$$u(t) > \max\{\sigma_1(t),\sigma_2(t)\} \quad \text{and} \quad u(\tau+) > \max\{\sigma_1(\tau+),\sigma_2(\tau+)\}$$

hold for each $t \in [0,T) \setminus \mathrm{D}$ and $\tau \in \mathrm{D}$, again a contradiction to (2.13). The proof will be completed by an obvious observation that u can satisfy neither (2.14) nor (2.15) whenever it does not satisfy (2.26). □

3. Main results

Our main result is the following theorem which is the first known existence result for impulsive periodic problems with non-ordered lower and upper functions.

3.1. Theorem. *Assume that* (1.10)–(1.13) *and* (0.1) *hold and let* $h \in \mathbb{L}[0,T]$ *be such that*

$$|f(t,x,y)| \le h(t) \text{ for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathbb{R}^2. \tag{3.1}$$

Further, let

$$y\,\mathrm{M}_i(y) \ge 0 \text{ for } y \in \mathbb{R} \quad \text{and} \quad i = 1,2,\dots,m. \tag{3.2}$$

Then the problem (1.1)–(1.3) *has a solution* u *satisfying one of the conditions* (2.13)–(2.15).

Proof. • STEP 1. *We construct a proper auxiliary problem.*

Let σ_1 and σ_2 be respectively lower and upper functions of (1.1)–(1.3) and let ρ_1 be associated with them as in (2.18). Put

$$\widetilde{h}(t) = 2\,h(t) + 1 \text{ for a.e. } t \in [0,T]$$

and, by Lemma 2.1, find $d \in (\rho_1, \infty)$ satisfying (2.1). Furthermore, put $\rho_0 = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + 1$ and, by Lemma 2.2, find $c \in (\rho_0, \infty)$ fulfilling (2.9). In particular, we have

$$c > \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + 1. \tag{3.3}$$

Finally, for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$ define functions

$$\left.\begin{array}{ll} \widetilde{f}(t,x,y) = & \begin{array}{ll} f(t,x,y) - h(t) - 1 & \text{if } x \le -c-1, \\ f(t,x,y) + (x+c)\,(h(t)+1) & \text{if } -c-1 < x < -c, \\ f(t,x,y) & \text{if } -c \le x \le c, \\ f(t,x,y) + (x-c)\,(h(t)+1) & \text{if } c < x < c+1, \\ f(t,x,y) + h(t) + 1 & \text{if } x \ge c+1, \end{array} \end{array}\right\} \tag{3.4}$$

$$
\left.
\begin{array}{lll}
 & x & \text{if } \ x \le -c-1, \\
 & \mathrm{J}_i(-c)\,(c+1+x) - x\,(x+c) & \text{if } \ -c-1 < x < -c, \\
\widetilde{\mathrm{J}}_i(x) = & \mathrm{J}_i(x) & \text{if } \ -c \le x \le c, \\
 & \mathrm{J}_i(c)\,(c+1-x) + x\,(x-c) & \text{if } \ c < x < c+1, \\
 & x & \text{if } \ x \ge c+1, \quad i = 1,2,\dots,m,
\end{array}
\right\} \tag{3.5}
$$

and consider an auxiliary problem

$$
u'' = \widetilde{f}(t,u,u'), \quad (2.10), \quad (2.3), \quad (1.3). \tag{3.6}
$$

Due to (1.10), $\widetilde{f} \in \mathrm{Car}([0,T]\times\mathbb{R})$ and $\widetilde{\mathrm{J}}_i \in \mathbb{C}(\mathbb{R})$ for $i = 1,2,\dots,m$. According to (3.3)–(3.5) the functions σ_1 and σ_2 are respectively lower and upper functions of (3.6). By (3.1) we have

$$
|\widetilde{f}(t,x,y)| \le \widetilde{h}(t) \ \text{ for a.e. } \ t \in [0,T] \ \text{ and all } \ (x,y) \in \mathbb{R}^2 \tag{3.7}
$$

and

$$
\left.
\begin{array}{ll}
\widetilde{f}(t,x,y) < 0 & \text{for a.e. } \ t \in [0,T] \ \text{ and all } \ (x,y) \in (-\infty,-c-1] \times \mathbb{R}, \\
\widetilde{f}(t,x,y) > 0 & \text{for a.e. } \ t \in [0,T] \ \text{ and all } \ (x,y) \in [c+1,\infty) \times \mathbb{R}.
\end{array}
\right\} \tag{3.8}
$$

Furthermore, in view of (3.5), it is easy to check that the condition (2.12) is satisfied. Moreover, due to (1.12), we see that (2.16) holds if $|x| \le c$. We are going to show that (2.16) is valid also for $|x| > c$. First, assume that $x > c$. In this case it suffices to verify the first condition in (2.16). Let $i \in \{1,2,\dots,m\}$ be given. Notice that, due to (3.3) and (1.12), we have

$$
c > \max\{\sigma_1(t_i), \sigma_1(t_i+)\} \ge \mathrm{J}_i(\sigma_1(t_i)) \quad \text{and} \quad \mathrm{J}_i(c) > \mathrm{J}_i(\sigma_1(t_i)). \tag{3.9}
$$

In view of (1.5), (3.3) and (3.5), this yields that

$$
\widetilde{\mathrm{J}}_i(x) = x > \sigma_1(t_i+) = \mathrm{J}_i(\sigma_1(t_i)) = \widetilde{\mathrm{J}}_i(\sigma_1(t_i))
$$

holds for $x > c+1$, i.e. the first condition in (2.16) is satisfied also for $x > c+1$. If $x \in (c, c+1]$, then the values $\widetilde{\mathrm{J}}_i(x)$ are convex combinations of the values $\mathrm{J}_i(c)$ and

x, which both are according to (3.9) greater than $\mathrm{J}_i(\sigma_1(t_i))$, and so we can conclude that the first condition in (2.16) is satisfied for all $x \in (c, \infty)$. Similarly, we can prove that the second condition in (2.16) is satisfied for $x \in (-\infty, -c)$.

Now, put

$$A^* = 1 + \sum_{i=1}^{m} \max_{|x| \le c+1} |\widetilde{\mathrm{J}}_i(x)| \quad \text{and} \quad \sigma_3 = -A^*, \ \sigma_4 = A^* \ \text{ on } \ [0, T]. \tag{3.10}$$

By (3.5) and (3.10) we have $A^* \geq c + 2$ and the condition

$$\widetilde{\mathrm{J}}_i(x) = (-1)^k A^* \iff x = (-1)^k A^* \tag{3.11}$$

is true for $k = 1, 2$ and $i = 1, 2, \ldots, m$. According to Remark 1.4, (3.2) and (3.8), the functions σ_3 and σ_4 are respectively lower and upper functions of (3.6) which are well-ordered, i.e.

$$\sigma_3(t) < \sigma_4(t) \ \text{ for } \ t \in [0, T] \quad \text{and} \quad \sigma_3(\tau+) < \sigma_4(\tau+) \ \text{ for } \ \tau \in \mathrm{D}\,.$$

Similarly, since $A^* \geq c + 2$, we get by (3.3) the relations

$$\sigma_3(t) < \sigma_2(t) \ \text{ for } \ t \in [0, T], \quad \sigma_3(\tau+) < \sigma_2(\tau+) \ \text{ for } \ \tau \in \mathrm{D}$$

and

$$\sigma_1(t) < \sigma_4(t) \ \text{ for } \ t \in [0, T],, \quad \sigma_1(\tau+) < \sigma_4(\tau+) \ \text{ for } \ \tau \in \mathrm{D}\,.$$

To summarize, we have three pairs $\{\sigma_3, \sigma_4\}$, $\{\sigma_3, \sigma_2\}$ and $\{\sigma_1, \sigma_4\}$ of well-ordered lower and upper functions of the problem (3.6).

Having G from (1.14), define an operator $\widetilde{\mathrm{F}} : \mathbb{C}^1_{\mathrm{D}}[0, T] \mapsto \mathbb{C}^1_{\mathrm{D}}[0, T]$ by

$$\begin{aligned}(\widetilde{\mathrm{F}}u)(t) = u(0) + u'(0) - u'(T) + \int_0^T G(t, s)\, \widetilde{f}(s, u(s), u'(s))\, \mathrm{d}s \\ - \sum_{i=1}^{m} \frac{\partial G}{\partial s}(t, t_i)\, (\widetilde{\mathrm{J}}_i(u(t_i)) - u(t_i)) + \sum_{i=1}^{m} G(t, t_i)\, (\mathrm{M}_i(u'(t_i)) - u'(t_i)).\end{aligned} \tag{3.12}$$

Similarly as in [10, Lemma 3.1], we can show that $\widetilde{\mathrm{F}}$ is completely continuous. Moreover, we can check that u is a solution of (3.6) whenever $\widetilde{\mathrm{F}}u = u$.

- Step 2. *We prove the first a priori estimate for solutions of* (3.6).

Denote

$$\Omega_0 = \{u \in \mathbb{C}^1_{\mathrm{D}}[0,T] : \|u\|_\infty < A^*, \|u'\|_\infty < C^*\}, \tag{3.13}$$
$$\text{where } \; C^* = \frac{2A^*}{\Delta} + \|\widetilde{h}\|_1 + 1 \quad \text{and} \quad \Delta \text{ is defined in } \; (1.20).$$

By virtue of (1.19) and (3.10), we have $\Omega_0 = \Omega(\sigma_3, \sigma_4, C^*)$. We are going to prove that for each solution u of (3.6) the estimate

$$u \in \mathrm{cl}(\Omega_0) \implies u \in \Omega_0 \tag{3.14}$$

is true. To this aim, suppose that u is a solution of (3.6) and $u \in \mathrm{cl}(\Omega_0)$, i.e. $\|u\|_\infty \le A^*$ and $\|u'\|_\infty \le C^*$. By the Mean Value Theorem, there are $\xi_i \in (t_i, t_{i+1})$, $i = 1, 2, \ldots, m$, such that $|u'(\xi_i)| \le 2\,A^*/\Delta$. Hence, by (3.7), we get

$$\|u'\|_\infty < C^*, \tag{3.15}$$

where C^* is defined in (3.13). It remains to show that $\|u\|_\infty < A^*$. Assuming the contrary there are two cases to distinguish:

Case A. Let

$$\sup\{u(t) : \, t \in [0,T]\} = A^*. \tag{3.16}$$

Then, due to (1.3) and (3.11), there is $\tau \in [0,T)$ such that

$$u(\tau) = u(\tau+) = A^*. \tag{3.17}$$

Recall that $A^* \ge c + 2$. Consequently, (3.17) implies that

$$u(t) > c + 1 \;\text{ for }\; t \in [\tau, \tau + \delta] \tag{3.18}$$

is true for some $\delta > 0$. Furthermore, we have

$$u'(\tau+) = 0. \tag{3.19}$$

Indeed, if $\tau = 0$, then (1.3) and (3.16) give $u(0) = u(T) = A^*$ and $0 \geq u'(\tau+) = u'(0) = u'(T) \geq 0$. If $\tau \in \mathrm{D}$, then (3.16) and (3.17) imply $u'(\tau+) \leq 0$ and $u'(\tau) \geq 0$. As, by (3.2), the latter inequality yields also $u'(\tau+) \geq 0$, (3.19) is true. Finally, if $\tau \in (0,T) \setminus \mathrm{D}$, then the validity of (3.19) is evident.

Now, by (3.8) and (3.18), we obtain that $u''(t) > 0$ holds a.e. on $[\tau, \tau+\delta]$. Consequently, in view of (3.19), we have $u'(t) > u'(\tau+) = 0$ on $(\tau, \tau+\delta)$, a contradiction to (3.16) and (3.17).

Case B. If $\inf\{u(t) : t \in [0,T]\} = -A^*$, we construct a contradiction similarly as in Case A.

Therefore, $\|u\|_\infty < A^*$ holds for each solution u of (3.6). This together with (3.15) shows that the estimate (3.14) is valid for each solution u of (3.6).

- Step 3. *We prove the second a priori estimate for solutions of* (3.6).

Define sets

$$\Omega_1 = \{u \in \Omega_0 : u(t) > \sigma_1(t) \text{ for } t \in [0,T],\, u(\tau+) > \sigma_1(\tau+) \text{ for } \tau \in \mathrm{D}\},$$

$$\Omega_2 = \{u \in \Omega_0 : u(t) < \sigma_2(t) \text{ for } t \in [0,T],\, u(\tau+) < \sigma_2(\tau+) \text{ for } \tau \in \mathrm{D}\}$$

and $\widetilde{\Omega} = \Omega_0 \setminus \mathrm{cl}(\Omega_1 \cup \Omega_2)$. Then, by (0.1), $\Omega_1 \cap \Omega_2 = \emptyset$ and

$$\widetilde{\Omega} = \{u \in \Omega_0 : u \text{ satisfies (2.13)}\}. \tag{3.20}$$

Furthermore, with respect to (1.19) and (3.10) we have $\Omega_1 = \Omega(\sigma_1, \sigma_4, C^*)$ and $\Omega_2 = \Omega(\sigma_3, \sigma_2, C^*)$.

Consider c from Step 1. We are going to prove that the estimate

$$u \in \mathrm{cl}(\widetilde{\Omega}) \implies \|u\|_\infty < c \tag{3.21}$$

is valid for each solution u of (3.6). So, assume that u is a solution of (3.6) and $u \in \mathrm{cl}(\widetilde{\Omega})$. Then, due to (3.14), u fulfils one of the conditions (2.13), (2.14), (2.15) and so, by (2.17), $u \in B$. Since we have already proved that (2.16) holds, we can use Lemma 2.3 and get $\xi_u \in [0,T]$ such that (2.18) is true. Further, due to (1.3), (2.3) and (3.7), we can apply Lemma 2.1 to show that u satisfies the estimate (2.1). Finally, by Lemma 2.4, u satisfies (2.26) and hence also (2.11) with ρ_0 defined in Step 1. Moreover, let us recall that $\widetilde{\mathrm{J}}_i$, $i = 1, 2, \ldots, m$, verify the condition (2.12). Hence, by Lemma 2.2, we have (2.9), i.e. each solution u of (3.6) satisfies (3.21).

- Step 4. *We prove the existence of a solution to* (1.1)–(1.3).

Consider the operator $\widetilde{\mathrm{F}}$ defined by (3.12). We distinguish two cases: either $\widetilde{\mathrm{F}}$ has a fixed point in $\partial\widetilde{\Omega}$ or it has no fixed point in $\partial\widetilde{\Omega}$.

Assume that $\widetilde{\mathrm{F}}\,u = u$ for some $u \in \partial\widetilde{\Omega}$. Then u is a solution of (3.6) and, with respect to (3.21), we have $\|u\|_\infty < c$, which means, by (3.4) and (3.5), that u is a solution of (1.1)–(1.3). Furthermore, due to (3.14), u satisfies (2.14) or (2.15).

Now, assume that $\widetilde{\mathrm{F}}\,u \neq u$ for all $u \in \partial\widetilde{\Omega}$. Then $\widetilde{\mathrm{F}}u \neq u$ for all $u \in \partial\Omega_0 \cup \partial\Omega_1 \cup \partial\Omega_2$. If we replace f, h, J_i, $i = 1, 2, \ldots, m$, α, β and r respectively by $\widetilde{f}$, $\widetilde{h}$, $\widetilde{\mathrm{J}}_i$, $i = 1, 2, \ldots, m$, σ_3, σ_4 and C^* in Proposition 1.7, we see that the assumptions (1.15)–(1.18) and (1.20) are satisfied. Thus, by Proposition 1.7, we obtain that

$$\deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega(\sigma_3, \sigma_4, C^*)) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_0) = 1. \tag{3.22}$$

Similarly, we can apply Proposition 1.7 to show that

$$\deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega(\sigma_1, \sigma_4, C^*)) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_1) = 1 \tag{3.23}$$

and

$$\deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega(\sigma_3, \sigma_2, C^*)) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_2) = 1. \tag{3.24}$$

Using the additivity property of the Leray-Schauder topological degree we derive from (3.22)–(3.24) that

$$\deg(\mathrm{I} - \widetilde{\mathrm{F}}, \widetilde{\Omega}) = -1.$$

Therefore, $\widetilde{\mathrm{F}}$ has a fixed point $u \in \widetilde{\Omega}$. By (3.20) and (3.21) we have (2.13) and $\|u\|_\infty < c$. This together with (3.4) and (3.5) yields that u is a solution to (1.1)–(1.3) fulfilling (2.13). □

3.2. Remark. Let the assumptions of Theorem 3.1 be fulfilled and let, moreover, $\sigma_1 \geq \sigma_2$ on $[0, T]$. Put $\widetilde{\mathrm{J}}_i = \mathrm{J}_i$ for $i = 1, 2, \ldots, m$. Then, by Lemma 2.4, a solution u of (1.1)–(1.3) fulfils one of the conditions (2.13)–(2.15) if and only if it satisfies

$$\sigma_2(\tau_u+) \leq u(\tau_u+) \leq \sigma_1(\tau_u+) \quad \text{for some} \ \ \tau_u \in [0, T].$$

Now, let us compare the existence result provided by Theorem 3.1 which is applicable for non-ordered lower and upper function with the following one which has been proved by the authors in [11, Theorem 3.1] and which concerns the well-ordered case.

3.3. Theorem. *Assume that* (1.10), (1.11), (1.13) *and* $\sigma_1 \leq \sigma_2$ *on* $[0, T]$ *hold. Furthermore, let the conditions*

$$\sigma_1(t_i) \leq x \leq \sigma_2(t_i) \implies \mathrm{J}_i(\sigma_1(t_i)) \leq \mathrm{J}(x) \leq \mathrm{J}_i(\sigma_2(t_i)) \quad \textit{for} \ \ i = 1, 2, \ldots, m$$

and

$$|f(t,x,y)| \le \omega(|y|)\,(|y|+h(t)) \tag{3.25}$$
$$\text{for a.e. } t \in [0,T] \text{ and all } x \in [\sigma_1(t),\sigma_2(t)],\ |y| \ge 1,$$

be satisfied, where $h \in \mathbb{L}[0,T]$ *is nonnegative function,* $\omega \in \mathbb{C}([1,\infty))$ *is positive and*

$$\int_1^\infty \frac{ds}{\omega(s)} = \infty.$$

Then the problem (1.1)–(1.3) *has a solution* u *satisfying*

$$\sigma_1 \le u \le \sigma_2 \quad on \ [0,T].$$

□

Imposing assumptions ensuring the existence of constant or piecewise constant lower/upper functions for (1.1)–(1.3) in Theorems 3.1 and 3.3, we obtain simple effective existence criteria. The first couple of them deals with piecewise constant lower/upper functions.

3.4. Corollary. *Let* (1.10), (3.1) *and* (3.2) *hold. Assume that* α_i, $\beta_i \in \mathbb{R}$, $i = 0,1,\dots,m$, *fulfil the assumptions of Proposition* 1.3, $\alpha_0 > \beta_0$ *and that the implications*

$$x > \alpha_{i-1} \implies \mathrm{J}_i(x) > \alpha_i \quad and \quad x < \beta_{i-1} \implies \mathrm{J}_i(x) < \beta_i \tag{3.26}$$

are true for $i = 1,2,\dots,m$. *Then the problem* (1.1)–(1.3) *has a solution* u *for which there exist* $j \in \{0,1,\dots,m\}$ *and* $\tau_u \in [t_j,t_{j+1})$ *such that*

$$\beta_j \le u(\tau_u+) \le \alpha_j. \tag{3.27}$$

Proof. First, recall that $\alpha_0 = \alpha_m$, $\beta_0 = \beta_m$. Hence $\alpha_m > \beta_m$ and, in view of (3.26), $\alpha_i > \beta_i$ for all $i = 0,1,\dots,m$. Let the functions σ_1 and σ_2 be defined as in

Proposition 1.3. By this proposition they are respectively lower and upper functions of (1.1)–(1.3). Now, the existence of a solution u to (1.1)–(1.3) having the property (3.27) follows by Theorem 3.1 and Remark 3.2. □

3.5. Corollary. *Let* (1.10) *and* (3.2) *hold. Assume that* α_i, $\beta_i \in \mathbb{R}$, $i = 0, 1, \dots, m$, *fulfil the assumptions of Proposition* 1.3, $\alpha_0 \le \beta_0$ *and that the implications*

$$\alpha_{i-1} \le x \le \beta_{i-1} \implies \alpha_i \le \mathrm{J}_i(x) \le \beta_i \tag{3.28}$$

are true for $i = 1, 2, \dots, m$. *Let* σ_1 *and* σ_2 *be defined as in Proposition* 1.3 *and let* (3.25) *be fulfilled with* h *and* ω *from Theorem* 3.3. *Then the problem* (1.1)–(1.3) *has a solution* u *satisfying*

$$\alpha_i \le u(t) \le \beta_i \;\; \textit{for } t \in (t_i, t_{i+1}] \quad \textit{and} \quad i = 0, 1, \dots, m. \tag{3.29}$$

Proof. By Proposition 1.3, σ_1 and σ_2 are respectively lower and upper functions of (1.1)–(1.3) and, by (3.28), we have $\sigma_1 \le \sigma_2$ on $[0, T]$. Thus, by Theorem 3.3 there is a solution u of (1.1)–(1.3) fulfilling (3.29). □

The special case of constant lower/upper functions is considered in the second couple of corollaries.

3.6. Corollary. *Let* (1.10), (3.1) *and* (3.2) *hold. Assume that there are* α, $\beta \in \mathbb{R}$ *such that* $\alpha > \beta$,

$$f(t, \alpha, 0) \le 0 \le f(t, \beta, 0) \;\; \textit{for a.e. } t \in [0, T], \tag{3.30}$$

$$\mathrm{J}_i(\alpha) = \alpha, \quad \mathrm{J}_i(\beta) = \beta \;\; \textit{for } i = 1, 2, \dots, m \tag{3.31}$$

and

$$x > \alpha \implies \mathrm{J}_i(x) > \alpha, \quad x < \beta \implies \mathrm{J}_i(x) < \beta$$

are true for $i = 1, 2, \ldots, m$. *Then the problem* (1.1)–(1.3) *has a solution* u *such that* $\beta \le u(t_u+) \le \alpha$ *for some* $t_u \in [0, T)$.

Proof follows from Theorem 3.1 if we take into account Remarks 1.4 and 3.2. □

Similarly, using Theorem 3.3 and Proposition 1.3, we get for the well-ordered case (cf. also [11, Corollary 3.3]):

3.7. Corollary. *Let* (1.10) *and* (3.2) *hold. Assume that there are* α, $\beta \in \mathbb{R}$ *such that* $\alpha \le \beta$, (3.30), (3.31) *and*

$$\alpha \le x \le \beta \implies \alpha \le \mathrm{J}_i(x) \le \beta \quad \text{for } i = 1, 2, \ldots, m \tag{3.32}$$

are true. Let (3.25) *be satisfied with* $\sigma_1 = \alpha$, $\sigma_2 = \beta$ *on* $[0, T]$ *and* h, ω *from Theorem* 3.3. *Then the problem* (1.1)–(1.3) *has a solution* u *such that* $\alpha \le u \le \beta$ *on* $[0, T]$. □

Corollary 3.7 extends the scalar case of [4, Corollary 2]. In particular, it applies to the closing example of [4], i.e. to (1.1)–(1.3) with an arbitrary division $\mathrm{D} = \{t_1, t_2, \ldots, t_m\}$ of $[0, T]$ and

$$f(t, x, y) = \mathrm{tg}(x) + \frac{1}{2}\, t^2 + y^2, \quad \mathrm{J}_i(x) = x, \quad \mathrm{M}_i(y) = y + \sin(2\, y)$$

for $t \in [0, T]$, $x, y \in \mathbb{R}$ and $i = 1, 2, \ldots, m$.

3.8. Example. Consider the problem (1.1)–(1.3) with an arbitrary division $\mathrm{D} = \{t_1, t_2, \ldots, t_m\}$ of $[0, T]$. For a.e. $t \in [0, T]$ and for all $x, y \in \mathbb{R}$, $i = 1, 2, \ldots, m$, define

$$f(t, x, y) = \sum_{k=1}^{n} \frac{p(t) - |x - 2|\, \sin(\frac{5}{2}\, \pi x) + y}{\sqrt{|t - c_k|}} + q(t)\, y^2,$$
$$\mathrm{J}_i(x) = x + a_i\, (x - 1)\, (x - 2)\, (x - 3), \quad \mathrm{M}_i(y) = y + y\, \sin(b_i\, y).$$

Here we assume that p, $q \in \mathbb{L}_\infty[0, T]$, $\|p\|_\infty \le 1$, $a_i \in [-\frac{1}{2}, 4]$, $b_i \in \mathbb{R}$ for $i = 1, 2, \ldots, m$ and $c_k \in (0, T) \setminus \mathrm{D}$, $k = 1, 2, \ldots, n$, $n \in \mathbb{N}$. Let us put $\alpha = 1$ and $\beta = 3$.

Then we can check that (3.30)–(3.32) hold. Moreover, for a.e. $t \in [0,T]$ and for all $x \in [1,3]$, $y \in \mathbb{R}$, we have

$$|f(t,x,y)| \le \sum_{k=1}^{n} \frac{2+|y|}{\sqrt{|t-c_k|}} + \|q\|_\infty \, y^2 \le \omega(|y|)\,(|y|+h(t)),$$

where

$$\omega(s) = 2 + s\,(1+\|q\|_\infty) \ \text{ for } \ s \in \mathbb{R} \quad \text{and} \quad h(t) = \sum_{k=1}^{n} \frac{1}{\sqrt{|t-c_k|}} \ \text{ for a.e. } \ t \in [0,T],$$

i.e. ω and h fulfil the assumptions of Theorem 3.3. We summarize that the assumptions of Corollary 3.7 are satisfied and hence the given problem has a solution u such that $1 \le u \le 3$ on $[0,T]$.

Notice that our function $f \in \mathrm{Car}([0,T]\times\mathbb{R}^2)$ is not continuous in t on $(0,T)\setminus \mathrm{D}$ as needed in [1], [4], [6], [8], [12] and does not satisfy the classical Nagumo growth conditions imposed in [4]–[7] and [12]. Moreover, f is not monotonous and the impulse functions J_i and M_i do not satisfy neither the monotonicity conditions from [1], [2], [5]–[8] nor the implications (3.2) and (3.32) with strict inequalities as needed in [4, Corollary 2]. Therefore, none of these previous papers can give an existence result which apply to this example.

Our main result is illustrated by the next example.

3.9. Example. Consider the problem (1.1)–(1.3) with an arbitrary division $\mathrm{D} = \{t_1, t_2, \dots, t_m\}$ of $[0,T]$. For a.e. $t \in [0,T]$ and for all $x, y \in \mathbb{R}$, $i = 1,2,\dots,m$ define

$$f(t,x,y) = \sum_{k=1}^{n} \frac{1}{\sqrt{|t-c_k|}} \left(p(t) + 1 + \frac{1-5\,x}{x^2+1}\right) + q(t)\ \mathrm{arctg}(y) \tag{3.33}$$

and

$$\mathrm{J}_i(x) = x + a_i\,(x^3 - x), \ \ \mathrm{M}_i(y) = y + y\,\sin(b_i\,y),$$

where $p \in \mathbb{L}_\infty[0,T]$, $\|p\|_\infty \le 1$, $q \in \mathbb{L}[0,T]$, $a_i \in [0,\infty)$, $b_i \in \mathbb{R}$ and $c_k \in (0,T)\setminus \mathrm{D}$, $k = 1,2,\dots,n$, $n \in \mathbb{N}$. We see that for $\alpha = 1$ and $\beta = 0$ the assumptions of Corollary

3.6 are fulfilled. Hence the given problem has a solution u for which $u(\tau_u+) \in [0,1]$ for some $\tau_u \in [0,T]$.

3.10. Remark. According to Definition 1.2 and Corollary 3.6, the problem (1.1)–(1.3) in Example 3.9 has just two constant upper functions $\sigma_2 = 0$ and $\widetilde{\sigma}_2 = -1$ on $[0,T]$ and the unique constant lower function $\sigma_1 = 1$ on $[0,T]$. It means that a well-ordered couple of them does not exist. As all the previous papers rely on the existence of a well-ordered couple of lower/upper functions and provide only existence criteria based on constant lower/upper functions (see [4, Corollary 2 and Theorem 4], [12, Theorem 4.1] or Corollary 3.7 in this paper), it is apparent that they cannot decide about the solvability in Example 3.9. Moreover, an existence decision for Example 3.9 cannot be obtained neither by means of our Corollary 3.5, where well-ordered piecewise constant lower/upper functions are needed. Indeed, if for some $j \in \{1,2,\dots,m\}$ the equalities $p(t) = 1$ and $p(t) = -1$ hold on some subsets of $[t_{j-1}, t_j)$ of positive measure, then, by (3.33), the inequalities

$$f(t,\alpha_j,0) \le 0 \le f(t,\beta_j,0)$$

can be satisfied a.e. on $[t_{j-1}, t_j)$ only if $\beta_j \le \frac{1}{5}$ and $\alpha_j \in [1, \frac{3}{2}]$, i.e. only if $\beta_j < \alpha_j$. It means that our problem in Example 3.9 does not even have a well-ordered pair of piecewise constant lower/upper functions.

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Irena Rachůnková, Department of Mathematics, Palacký University, 779 00 OLOMOUC, Tomkova 40, Czech Republic (`e-mail: rachunko@risc.upol.cz`)

Milan Tvrdý, Mathematical Institute, Academy of Sciences of the Czech Republic, 115 67 PRAHA 1, Žitná 25, Czech Republic (`e-mail: tvrdy@math.cas.cz`)