Approximation of differential problems with singularities and time discontinuities

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Abstract. The paper deals with the singular mixed boundary value problem

$$(t^\mu u'(t))' + t^\mu f(t, u(t)) = 0, \quad \lim_{t\to 0+} t^\mu u'(t) = 0, \quad u(T) = A,$$

where $\mu \in \mathbb{N}$, $\mu \geq 2$, $[0,T] \subset \mathbb{R}$, $A \in [0,\infty)$. For $s_1, \ldots, s_r \in (0,T]$ and $J = (0,T] \setminus \{s_1, \ldots, s_r\}$ we assume that $f(t,x)$ is continuous on the set $J \times (0,\infty)$ and may have singularities at $t = 0$ and $x = 0$ and integrable discontinuities at $t = s_i$, $i = 1, \ldots, r$. We provide a new approach giving the existence of positive solutions of the above singular problem by means of a sequence of its discretizations. As an application we present new existence results for singular problems arising in the theory of shallow membrane caps.

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1 Formulation of problem

The paper deals with the differential equation

$$(t^\mu u'(t))' + t^\mu f(t, u(t)) = 0, \tag{1.1}$$

$\mu \in \mathbb{N}$, $\mu \geq 2$, subject to the mixed boundary conditions

$$\lim_{t\to 0+} t^\mu u'(t) = 0, \quad u(T) = A, \tag{1.2}$$

where $[0,T] \subset \mathbb{R}$, $A \in [0,\infty)$. We are interested in solutions of (1.1), (1.2) which are positive on $(0,T)$. In order to get such solutions, we will investigate possible discretizations of problem (1.1), (1.2). Due to the positivity of the solutions, we investigate the function f of (1.1) just on the set $[0,T] \times [0,\infty)$.

Consider a finite number of points $s_1, \ldots, s_r \in (0,T]$ and denote $J = (0,T] \setminus \{s_1, \ldots, s_r\}$. We write $f \in C(J \times (0,\infty))$ if f is continuous on the set $J \times (0,\infty)$.

Let $[a,b] \subset \mathbb{R}$. We will also work with the following sets:

- $C[a,b]$ ($C(a,b)$) — the set of continuous functions on $[a,b]$ (on (a,b));
- $AC[a,b]$ — the set of absolutely continuous functions on $[a,b]$;
- $AC(a,b)$ — the set of functions $f \in AC[c,d]$ for each $[c,d] \subset (a,b)$;
- $AC^1[a,b]$ — the set of functions having absolutely continuous first derivative on $[a,b]$;
- $AC^1(a,b)$ — the set of functions $f \in AC^1[c,d]$ for each $[c,d] \subset (a,b)$;
- $L[a,b]$ — the set of Lebesgue integrable functions on $[a,b]$.

We say that $f \in C(J \times (0,\infty))$ has integrable discontinuities at $t = s_i$, $i = 1, \ldots, r$, if for each $[a,T] \subset (0,T]$ and for each compact set $\mathcal{K} \subset (0,\infty)$ there is a function $m_{a,\mathcal{K}} \in L[a,T]$ such that

$$|f(t,x)| \le m_{a,\mathcal{K}}(t) \quad \text{for a.e. } t \in [a,T] \text{ and all } x \in \mathcal{K}.$$

In what follows we will assume:

$$\begin{cases} f \in C(J \times (0,\infty)) \text{ may have integrable discontinuities} \\ \text{at } t = s_i,\ i = 1, \ldots, r, \\ f(t,x) \text{ may have singularities at } t = 0 \text{ and } x = 0. \end{cases} \tag{1.3}$$

Definition 1.1 A function $f(t,x)$ has a *time singularity at* $t = 0$, if there exists $x \in (0,\infty)$ such that

$$\int_0^\varepsilon |f(t,x)| \mathrm{d}t = \infty \quad \text{for } \varepsilon \in (0,T).$$

A function $f(t,x)$ has a *space singularity at* $x = 0$, if

$$\limsup_{x \to 0+} |f(t,x)| = \infty \quad \text{for } t \in J.$$

Example 1.2 Let $a_0 \ge 0$, $b_0 > 0$, $\gamma > 1$. The function

$$f(t,x) = \frac{1}{8x^2} - \frac{a_0}{x} - b_0 t^{2\gamma - 4} \tag{1.4}$$

appears in an equation modelling shallow membrane caps, see [4] and [7]. We see that f fulfils (1.3) for $J = (0, T]$ and has a space singularity at $x = 0$. Moreover, for $\gamma \in (1, \frac{3}{2}]$, f has also a time singularity at $t = 0$, because $\int_0^\varepsilon t^{2\gamma-4}\mathrm{d}t = \infty$. Problem (1.1), (1.2) with f of the form (1.4) has been studied for $A > 0$ in [8] and [3] and for $A = 0$ in [10]. An equidistant discretization of this problem has been investigated in [12].

Example 1.3 Let $q \in L[0, T] \cap C(J)$, $q(t) > 0$ for $t \in J$. The function

$$f(t, x) = \frac{q(t)}{x^2} \tag{1.5}$$

has a space singularity at $x = 0$. Problem (1.1), (1.2) with f given by (1.5) describes a behaviour of symmetric circular membranes and, for $A > 0$ and $q \in C[0, 1]$, has been studied in [1].

Discretization of problem (1.1), (1.2).
Let $n \in \mathbb{N}$. For f satisfying condition (1.3), we can find points

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T, \tag{1.6}$$

such that

$$f(t_k, \cdot) \text{ is continuous on } (0, \infty) \text{ for } k = 1, \ldots, n-1. \tag{1.7}$$

The points (1.6) cannot be equidistant in general, and so we use variable steps and denote them by

$$h_k = t_k - t_{k-1}, \quad k = 1, \ldots, n, \tag{1.8}$$

and we get the following discretization of problem (1.1), (1.2):

$$\frac{1}{h_{k+1}}\Delta\left(\frac{t_k^\mu}{h_k}\Delta u_{k-1}\right) + t_k^\mu f(t_k, u_k) = 0, \quad k = 1, \ldots, n-1, \tag{1.9}$$

$$\Delta u_0 = 0, \quad u_n = A. \tag{1.10}$$

Here Δ denotes the forward difference operator, i.e. $\Delta u_{k-1} = u_k - u_{k-1}$.

The main goal of the paper is to present a new approach giving the existence of a positive solution of the singular problem (1.1), (1.2) by means of its proper non-equidistant discretizations. Questions about discrete problems associated with differential boundary value problems have been also discussed in [5], [9] and [14].

Definition 1.4 A function $y \in C[0, T] \cap AC^1(0, T)$ with $y > 0$ on $(0, T)$, which satisfies equation (1.1) for a.e. $t \in (0, T)$ and fulfils conditions (1.2), is called a *positive solution* of problem (1.1), (1.2).

Definition 1.5 A vector $(u_0, \dots, u_n) \in \mathbb{R}^{n+1}$ satisfying equation (1.9), conditions (1.10) and $u_k > 0$ for $k = 0, \dots, n-1$, is called a *positive solution* of problem (1.9), (1.10).

In order to get a positive solution of (1.1), (1.2) we construct the discrete problems (1.9), (1.10), $n \in \mathbb{N}$, and we prove that they have positive solutions. By means of a sequence of positive solutions of the discrete problems (1.9), (1.10), $n \in \mathbb{N}$, we get a sequence of approximate functions which converges for $n \to \infty$ to a positive solution of the singular differential problem (1.1), (1.2). Such approach was used for equidistant discretization and $f(t,x)$ continuous on $(0,1] \times (0,\infty)$ in [12]. Here we generalize the results of [12] for functions $f(t,x)$ which can be quickly growing for large x and need not be continuous on $(0,1] \times (0,\infty)$ and for non-equidistant discretizations. Moreover, in (1.2), we consider $A = 0$ as well as $A > 0$. As an application we present new existence results for singular problems which cover problem (1.5), (1.2). We emphasize that this problem has not been solved before for $A = 0$.

2 Solvability of discrete problems

Linear discrete problems.
Assume that $n \in \mathbb{N}$ and $g \in L[0,T]$. Let us choose the points of (1.6) in such a way that

$$g(t_k) \in \mathbb{R}, \quad k = 1, \dots, n-1.$$

For h_k given by (1.8), consider the linear difference equation

$$\frac{1}{h_{k+1}} \Delta \left(\frac{t_k^\mu}{h_k} \Delta u_{k-1} \right) + g(t_k) = 0, \quad k = 1, \dots, n-1, \tag{2.1}$$

and the corresponding homogeneous equation

$$\frac{1}{h_{k+1}} \Delta \left(\frac{t_k^\mu}{h_k} \Delta u_{k-1} \right) = 0, \quad k = 1, \dots, n-1, \tag{2.2}$$

subject to the boundary conditions

$$\Delta u_0 = 0, \quad u(T) = 0. \tag{2.3}$$

Since problem (2.2), (2.3) has just the trivial solution, there exists its Green function. If we put

$$P(t_k) = \sum_{i=1}^{k} \frac{h_i}{t_i^\mu}, \quad k = 1, \dots, n,$$

then the Green function G can be written in the form

$$G(t_k, s_i) = h_{i+1} \begin{cases} P(t_k) - P(T) & \text{for } 0 < s_i \le t_k \le T, \\ P(s_i) - P(T) & \text{for } 0 \le t_k < s_i \le T, \end{cases} \tag{2.4}$$

where $h_{n+1} = h_n$, $s_i = \sum_{j=1}^{i} h_j$, $t_k = \sum_{j=1}^{k} h_j$, $i, k = 1, \dots, n$. We can check that

$$G(T, s_i) = 0, \quad \Delta G(t_0, s_i) = 0, \quad i = 1, \dots, n, \tag{2.5}$$

$$\frac{1}{h_{i+1}} \Delta \left(\frac{t_k^{\mu}}{h_k} \Delta G(t_{k-1}, s_i) \right) = \delta_{ik}, \quad i, k = 1, \dots, n-1, \tag{2.6}$$

hold. Moreover, if we denote $M_0 = T/t_1^{\mu}$, we have

$$-M_0 h_{i+1} < G(t_k, s_i) < 0, \quad i = 1, \dots, n-1, \quad k = 0, \dots, n-1. \tag{2.7}$$

Lemma 2.1 *Problem* (2.1), (2.3) *has a unique solution* $(u_0, \dots, u_n) \in \mathbb{R}^{n+1}$. *The solution* $(u_0, \dots, u_n)$ *has the form*

$$u_k = -\sum_{i=1}^{n-1} G(t_k, s_i) g(s_i), \quad k = 0, \dots, n. \tag{2.8}$$

Proof. Since the homogeneous problem (2.2), (2.3) has just the trivial solution, the nonhomogeneous problem (2.1), (2.3) has a unique solution. Let us show that this solution is given by (2.8). By virtue of (2.5) we get

$$u_n = -\sum_{i=1}^{n-1} G(T, s_i) g(s_i) = 0,$$

$$\Delta u_0 = u_1 - u_0 = -\sum_{i=1}^{n-1} G(t_1, s_i) g(s_i) + \sum_{i=1}^{n-1} G(t_0, s_i) g(s_i)$$

$$= -\sum_{i=1}^{n-1} \Delta G(t_0, s_i) g(s_i) = 0.$$

Hence $(u_0, \dots, u_n)$ satisfies condition (2.3). Further, using equality (2.6), we obtain

$$\frac{1}{h_{k+1}} \Delta \left(\frac{t_k^{\mu}}{h_k} \Delta u_{k-1} \right) = \frac{1}{h_{k+1}} \sum_{i=1}^{n-1} h_{i+1} \left(\frac{1}{h_{i+1}} \Delta \left(\frac{t_k^{\mu}}{h_k} \Delta G(t_{k-1}, s_i) \right) g(s_i) \right)$$

$$= -\frac{1}{h_{k+1}} \sum_{i=1}^{n-1} h_{i+1} \delta_{ik} g(s_i) = -g(t_k), \quad k = 1, \dots, n-1.$$

Therefore $(u_0, \dots, u_n)$ satisfies equation (2.1). □

Nonlinear discrete problems.
Now, we will study the solvability of the nonlinear singular discrete problem (1.9), (1.10). To this end we will use lower and upper functions.

Definition 2.2 The vector $(\alpha_0, \dots, \alpha_n) \in \mathbb{R}^{n+1}$ is called a *lower function* of problem (1.9), (1.10) if

$$\frac{1}{h_{k+1}} \Delta \left(\frac{t_k^\mu}{h_k} \Delta \alpha_{k-1} \right) + t_k^\mu f(t_k, \alpha_k) \geq 0, \quad k = 1, \dots, n-1, \tag{2.9}$$

$$\Delta \alpha_0 \geq 0, \quad \alpha_n \leq A. \tag{2.10}$$

Definition 2.3 The vector $(\beta_0, \dots, \beta_n) \in \mathbb{R}^{n+1}$ is called an *upper function* of problem (1.9), (1.10) if

$$\frac{1}{h_{k+1}} \Delta \left(\frac{t_k^\mu}{h_k} \Delta \beta_{k-1} \right) + t_k^\mu f(t_k, \beta_k) \leq 0, \quad k = 1, \dots, n-1, \tag{2.11}$$

$$\Delta \beta_0 \leq 0, \quad \beta_n \geq A. \tag{2.12}$$

The next theorem contains the lower and upper functions method which is based on the assumption that there exists a well ordered couple of lower and upper functions to a problem under consideration. This method for regular discrete problems can be found e.g. in [2], [6], [11] and for singular discrete problems with equidistant points $t_0, \dots, t_n$ in [12].

Theorem 2.4 *Assume that conditons* (1.3), (1.6) *and* (1.7) *hold. Let* $(\alpha_0, \dots, \alpha_n)$ *and* $(\beta_0, \dots, \beta_n)$ *be, respectively, a lower and an upper function of problem* (1.9), (1.10) *with*

$$0 < \alpha_k \leq \beta_k, \quad k = 1, \dots, n-1. \tag{2.13}$$

Then problem (1.9), (1.10) *has a positive solution* $(u_0, \dots, u_n)$ *satisfying*

$$\alpha_k \leq u_k \leq \beta_k, \quad k = 0, \dots, n. \tag{2.14}$$

Proof. We argue similarly as in the proof of Theorem 3.3 in [12]. For $k \in \{1, \dots, n-1\}$, $x \in \mathbb{R}$ define a function

$$\tilde{f}(t_k, x) = \begin{cases} f(t_k, \beta_k) - \frac{x - \beta_k}{x - \beta_k + 1} & \text{if } x > \beta_k, \\ f(t_k, x) & \text{if } \alpha_k \leq x \leq \beta_k, \\ f(t_k, \alpha_k) + \frac{\alpha_k - x}{\alpha_k - x + 1} & \text{if } x < \alpha_k. \end{cases}$$

We see that $\tilde{f}(t_k, \cdot) \colon \mathbb{R} \to \mathbb{R}$ is continuous for $k = 1, \dots, n-1$ and there exists $M > 0$ such that

$$|\tilde{f}(t_k, x)| \leq M \quad \text{for } k = 1, \dots, n-1, \ x \in \mathbb{R}.$$

Consider the auxiliary regular difference equation

$$\frac{1}{h_{k+1}} \Delta \left(\frac{t_k^\mu}{h_k} \Delta v_{k-1} \right) + t_k^\mu \tilde{f}(t_k, v_k + A) = 0, \quad k = 1, \dots, n-1. \tag{2.15}$$

Denote $E = \{\mathbf{v} = (v_0, \dots, v_n) \in \mathbb{R}^{n+1} \colon \Delta v_0 = 0,\ v_n = 0\}$, and define $\|\mathbf{v}\| = \max\{|v_k| \colon k = 1, \dots, n-1\}$. Then E is a Banach space with $\dim E = n - 1$. Let the function G be given by formula (2.4). Define an operator $\mathcal{F} \colon E \to E$ by

$$(\mathcal{F}\mathbf{v})_k = -\sum_{i=1}^{n-1} G(t_k, s_i)\tilde{f}(s_i, v_i + A), \quad k = 0, \dots, n.$$

Estimate (2.7) implies

$$|(\mathcal{F}\mathbf{v})_k| < M_0 M \sum_{i=1}^{n-1} h_{i+1} < T M_0 M, \quad k = 0, \dots, n.$$

Therefore, by the Brouwer fixed point theorem, there is a fixed point $\mathbf{v}^*$ of the operator $\mathcal{F}$. By Lemma 2.1, the vector $\mathbf{v}^* = (v_0^*, \dots, v_n^*)$ is a solution of problem (2.15), (2.3). Put $u_k = v_k^* + A$, $k = 0, \dots, n$. Then we get (2.14) as in the proof of Theorem 3.3 in [12]. Consequently the vector $\mathbf{u} = (u_0, \dots, u_n)$ is a solution of problem (1.9), (1.10). □

3 Approximation principle

This section is devoted to the study of sequences of piece-wise linear functions which approximate solutions of the singular differential problem (1.1), (1.2). We describe a construction of such functions.

Remark 3.1 We want to point out that for an approximation of problem (1.1), (1.2) with f satisfying (1.3) we need, for $n \in \mathbb{N}$, the isolated time scale (1.6), where $t_k \notin \{s_1, \dots, s_r\}$, $k = 1, \dots, n-1$. Moreover, since we need to prove an approximation principle (Theorem 3.3) and, in particular, convergences (3.10) and (3.11), a choice of t_k depends on the fact whether $f(t, x)$ is unbounded for $t \to s_i$ or not. See Remarks 3.2 and 3.4. Consider, for example, the function

$$f(t, x) = \frac{x^3 + 1}{t\sqrt{(s_{i+1} - t)(t - s_i)}} \quad \text{for } t \in (s_i, s_{i+1}),\ x \in [0, \infty),\ i = 1, \dots, r-1.$$

Then $f \in C(J \times [0, \infty))$ and f is unbounded near each s_i, $i = 1, \dots, r$.

In Remark 3.2, we explain a choice of the time scale (1.6) for $n \in \mathbb{N}$.

Remark 3.2 Consider a non-negative functions $g_1 \in C(J \times (0, \infty))$ and $g_2 \in C(J \times [0, \infty))$ which are unbounded if $t \to s_i$, $i = 1, \dots, r$, $x \in (0, \infty)$. For $n \in \mathbb{N}$, choose points (1.6) such that $t_k \in J$, $k = 1, \dots, n-1$, and each interval (t_k, t_{k+1}) contains at most one $s_i \in \{s_1, \dots, s_r\}$. Moreover, if some $s_i \in (t_k, t_{k+1})$, then t_k and t_{k+1} are chosen such that

$$g_j(t_k, x) \le g_j(t, x) \quad \text{for } t \in (t_k, t_{k+1}),\ x \in (0, \infty),\ j = 1, 2. \tag{3.1}$$

For each sufficiently large $n \in \mathbb{N}$ we assume

$$\begin{cases} \text{conditions (1.6), (1.7) and (3.1) hold,} \\ \text{problem (1.9), (1.10) has a positive solution } (u_0, \dots, u_n). \end{cases} \tag{3.2}$$

Denote

$$v_k = \frac{t_k^\mu}{h_k} \Delta u_{k-1}, \quad k = 1, \dots, n, \tag{3.3}$$

and define

$$y^{[n]}(t) = u_k + \frac{\Delta u_k}{h_{k+1}}(t - t_k),\ t \in [t_k, t_{k+1}],\ k = 0, \dots, n-1, \tag{3.4}$$

$$\begin{cases} z^{[n]}(t) = 0,\ t \in [t_0, t_1], \\ z^{[n]}(t) = v_k + \dfrac{\Delta v_k}{h_{k+1}}(t - t_k),\ t \in [t_k, t_{k+1}],\ k = 1, \dots, n-1. \end{cases} \tag{3.5}$$

The main result of the paper is contained in the next theorem providing an approximation principle.

Theorem 3.3 *Let* $A = 0$ *and* (1.3) *hold. Assume that there exist functions* $\alpha, \beta \in C[0, T]$, $\alpha < \beta$ *on* $(0, T)$, $\beta(T) = 0$, *and non-negative functions* $g_1 \in C(J \times (0, \infty))$ *and* $g_2 \in C(J \times [0, \infty))$ *satisfying*

$$|f(t, x)| \le g_1(t, x) + g_2(t, x) \text{ for } t \in J, x \in [\alpha(t), \beta(t)], \tag{3.6}$$

where g_1 *is nonincreasing in its second variable,* g_1 *and* g_2 *have integrable discontinuities at* $t = s_i$, $i = 1, \dots, r$, *and at* $t = 0$, $t = s_i$, $i = 1, \dots, r$, *respectively, and*

$$\int_0^{\frac{T}{2}} t g_1(t, \alpha(t)) \mathrm{d}t < \infty. \tag{3.7}$$

Further assume that there exists $n^* \in \mathbb{N}$ *such that for each* $n \in \mathbb{N}$, $n \ge n^*$, *condition* (3.2) *is fulfilled and that*

$$\lim_{n \to \infty} \max\{h_k = t_k - t_{k-1} \colon k = 1, \dots, n\} = 0, \tag{3.8}$$

$$0 < \alpha(t_k) < u_k < \beta(t_k), \quad k = 1, \dots, n-1. \tag{3.9}$$

Then the following approximation principle is valid:

Let the sequences $\{y^{[n]}\}$ *and* $\{z^{[n]}\}$ *be given by* (3.4) *and* (3.5). *Then there exist their subsequences* $\{y^{[m]}\}$ *and* $\{z^{[m]}\}$ *such that*

$$\lim_{m \to \infty} y^{[m]}(t) = y(t) \quad \text{locally uniformly on } (0, T), \tag{3.10}$$

$$\lim_{m \to \infty} z^{[m]}(t) = z(t) \quad \text{locally uniformly on } [0, T), \tag{3.11}$$

$z(t) = t^\mu y'(t)$ *and* y *is a positive solution of the singular differential problem* (1.1), (1.2) *and*

$$\alpha(t) \le y(t) \le \beta(t) \quad \text{for } t \in [0, T]. \tag{3.12}$$

Proof.

Step 1. Boundedness of sequences $\{y^{[n]}\}$ and $\{z^{[n]}\}$.
Note that without loss of generality we can assume that for $x \in (0,\infty)$ the functions $g_1(t,x)$ and $g_2(t,x)$ are unbounded if $t \to s_i$, $i = 1,\dots,r$. Hence, for $n \in \mathbb{N}$, $n \geq n^*$, we can choose points (1.6) such that (1.7) and (3.1) are valid. By (3.2) there exists a positive solution $(u_0,\dots,u_n)$ of problem (1.9) and (1.10). Inserting (3.3) into equation (1.9) we get

$$\frac{1}{h_{k+1}}\Delta v_k = -t_k^{\mu} f(t_k,u_k), \quad k = 1,\dots,n-1. \tag{3.13}$$

Since $\Delta u_0 = v_1 = 0$, equations (3.3) and (3.13) can be written in the form

$$u_k = u_0 + \sum_{i=1}^{k} h_i \frac{v_i}{t_i^{\mu}}, \quad k = 1,\dots,n, \tag{3.14}$$

$$v_{k+1} = -\sum_{i=1}^{k} h_{i+1} t_i^{\mu} f(t_i,u_i), \quad k = 1,\dots,n-1. \tag{3.15}$$

By (1.10) and (3.9) we have

$$\max\{|u_k| \colon k = 0,\dots,n\} \leq \max\{\beta(t) \colon t \in [0,T]\} =: B.$$

Since $y^{[n]}(t)$ of (3.4) is a continuous piece-wise linear function and $y^{[n]}(t_k) = u_k$, $k = 0,\dots,n$, we get

$$\max\{|y^{[n]}(t)| \colon t \in [0,T]\} \leq B, \quad n \in \mathbb{N},\ n \geq n^*. \tag{3.16}$$

Choose an arbitrary $b \in \left(\frac{T}{2}, T\right)$. By (3.8) there is $n_0 \in \mathbb{N}$, $n_0 \geq n^*$, such that for each $n \in \mathbb{N}$, $n \geq n_0$, there is $b_n \in \{1,\dots,n\}$ such that

$$t_{b_n} \in (b,T), \quad \lim_{n\to\infty} t_{b_n} = b. \tag{3.17}$$

There is a function $\tilde{m}(t) \in L[0,T] \cap C(J)$ such that

$$|g_2(t,x)| \leq \tilde{m}(t) \quad \text{for } t \in J, x \in [\alpha(t),\beta(t)], \tag{3.18}$$

and due to (3.9), (3.7), (3.1), we can find $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$

$$\sum_{i=1}^{b_n} h_{i+1} t_i g_1(t_i,u_i) \leq 1 + \int_0^b t g_1(t,\alpha(t))\mathrm{d}t =: M_1,$$

$$\sum_{i=1}^{b_n} h_{i+1} g_2(t_i,u_i) \leq 1 + \int_0^T \tilde{m}(t)\mathrm{d}t =: M_2.$$

Clearly $M_2 \in (0,\infty)$. Let us show that $M_1 \in (0,\infty)$, as well. Since $b \in (\frac{T}{2}, T)$, we can write

$$\int_0^b tg_1(t,\alpha(t))dt = \int_0^{\frac{T}{2}} tg_1(t,\alpha(t))dt + \int_{\frac{T}{2}}^b tg_1(t,\alpha(t))dt.$$

By (3.7) we have

$$\int_0^{\frac{T}{2}} tg_1(t,\alpha(t))dt < \infty.$$

The assumption that g_1 is continuous on $J \times (0,\infty)$ and has integrable discontinuities at $t = s_i$, $s_i \in (0,T]$, $i = 1,\dots,r$ yields (see p. 2) that for each compact set $\mathcal{K} \subset (0,\infty)$ there exists a function $m_{\mathcal{K}} \in L[\frac{T}{2}, T]$ such that

$$|g_1(t,x)| \le m_{\mathcal{K}}(t) \quad \text{for a.e. } t \in [\frac{T}{2}, T] \text{ and all } x \in \mathcal{K}.$$

Let us put $\mathcal{K} = \{\alpha(t) : t \in [\frac{T}{2}, b]\}$. Then, by (3.9), $\mathcal{K} \subset (0,\infty)$ and moreover $\mathcal{K}$ is compact. Hence

$$|g_1(t,\alpha(t))| \le m_{\mathcal{K}}(t) \quad \text{for a.e. } t \in [\frac{T}{2}, b].$$

So,

$$\int_{\frac{T}{2}}^b tg_1(t,\alpha(t))dt \le b\int_{\frac{T}{2}}^b m_{\mathcal{K}}(t)dt < \infty.$$

Therefore

$$\int_0^b tg_1(t,\alpha(t))dt < \infty,$$

and consequently $M_1 \in (0,\infty)$. Further, by (3.9), (3.6), (3.15) and (3.17), we have for $k = 1,\dots,b_n$,

$$|v_k| \le \sum_{i=1}^k h_{i+1}t_i^\mu |f(t_i,u_i)| \le T^{\mu-1}\sum_{i=1}^k h_{i+1}t_i g_1(t_i,u_i) + T^\mu \sum_{i=1}^k h_{i+1}g_2(t_i,u_i)$$

$$\le T^{\mu-1}\sum_{i=1}^{b_n} h_{i+1}t_i g_1(t_i,u_i) + T^\mu \sum_{i=1}^{b_n} h_{i+1}g_2(t_i,u_i)$$

$$\le T^{\mu-1}M_1 + T^\mu M_2 =: M_3.$$

Since $z^{[n]}(t)$ of (3.5) is a continuous piece-wise linear function and $z^{[n]}(t_k) = v_k$, $k = 1,\dots,n$, $z^{[n]}(t) = 0$ on $[t_0,t_1]$, we get

$$\max\{|z^{[n]}(t)| \colon t \in [0,b]\} \le M_3, \quad n \in \mathbb{N},\ n \ge n_0. \tag{3.19}$$

Moreover, by (3.9), there exists $M_4 \in (0,\infty)$ such that

$$\max\{|z^{[n]}(t)| \colon t \in [0,b]\} \le M_4, \quad n \in \mathbb{N},\ n^* \le n \le n_0. \tag{3.20}$$

We have proved that the sequence $\{y^{[n]}\}$ is bounded on $[0,T]$ and the sequence $\{z^{[n]}\}$ is bounded on $[0,b]$ for each $b \in \left(\frac{T}{2}, T\right)$.

Step 2. Equicontinuity of sequences $\{y^{[n]}\}$ and $\{z^{[n]}\}$.
Consider $n \in \mathbb{N}$, $n \geq n^*$, $b \in \left(\frac{T}{2}, T\right)$ and b_n satisfying (3.17). Choose arbitrary $\tau_1, \tau_2 \in [0,b]$, $\tau_1 < \tau_2$. Then we can find $k, \ell \in \{1, \dots, b_n\}$, $k \leq \ell$, such that $\tau_1 \in [t_{k-1}, t_k)$, $\tau_2 \in (t_{\ell-1}, t_\ell]$ and, due to (3.5), (3.6) and (3.13),

$$
\begin{aligned}
&|z^{[n]}(\tau_2) - z^{[n]}(\tau_1)| \\
&\leq \sum_{i=k+1}^{\ell-1} \left|\frac{\Delta v_{i-1}}{h_i}\right| (t_i - t_{i-1}) + \left|\frac{\Delta v_{k-1}}{h_k}\right| (t_k - \tau_1) + \left|\frac{\Delta v_{\ell-1}}{h_\ell}\right| (\tau_2 - t_{\ell-1}) \\
&= \sum_{i=k+1}^{\ell-1} t_{i-1}^{\mu} |f(t_{i-1}, u_{i-1})| (t_i - t_{i-1}) + t_{k-1}^{\mu} |f(t_{k-1}, u_{k-1})| (t_k - \tau_1) \\
&\qquad + t_{\ell-1}^{\mu} |f(t_{\ell-1}, u_{\ell-1})| (\tau_2 - t_{\ell-1}) \\
&\leq \sum_{i=k+1}^{\ell-1} t_{i-1}^{\mu} \Big(g_1(t_{i-1}, u_{i-1}) + g_2(t_{i-1}, u_{i-1})\Big) h_i \\
&\quad + t_{k-1}^{\mu} \Big(g_1(t_{k-1}, u_{k-1}) + g_2(t_{k-1}, u_{k-1})\Big) (t_k - \tau_1) \\
&\quad + t_{\ell-1}^{\mu} \Big(g_1(t_{\ell-1}, u_{\ell-1}) + g_2(t_{\ell-1}, u_{\ell-1})\Big) (\tau_2 - t_{\ell-1}).
\end{aligned}
$$

If $k+1 > \ell - 1$, we put $\sum_{i=k+1}^{\ell-1} = 0$. By (3.7), (3.18) and $\mu \geq 2$, for each $\varepsilon > 0$, there exists $n_\varepsilon \geq n^*$ such that for each $n \geq n_\varepsilon$,

$$|z^{[n]}(\tau_2) - z^{[n]}(\tau_1)| \leq \int_{\tau_1}^{\tau_2} t^{\mu} \Big(g_1(t, \alpha(t)) + \tilde{m}(t)\Big) \mathrm{d}t + \varepsilon.$$

Moreover there exists $\delta > 0$ such that if $\tau_2 - \tau_1 < \delta$, then $|z^{[n]}(\tau_2) - z^{[n]}(\tau_1)| < \varepsilon$ for $n = n^*, \dots, n_\varepsilon$, and

$$\int_{\tau_1}^{\tau_2} t^{\mu} \Big(g_1(t, \alpha(t)) + \tilde{m}(t)\Big) \mathrm{d}t < \varepsilon.$$

We have proved that the sequence $\{z^{[n]}\}$ is equicontinuous on $[0,b]$.

Choose an arbitrary $a \in (0,b)$. By (3.8), there is $n_0 \in \mathbb{N}$, $n_0 \geq n^*$, such that for each $n \in \mathbb{N}$, $n \geq n_0$, there is $a_n \in \{1, \dots, n\}$ such that

$$t_{a_n} \in (0,a), \quad \lim_{n \to \infty} t_{a_n} = a. \tag{3.21}$$

Choose arbitrary $\tau_1, \tau_2 \in [a,b]$, $\tau_1 < \tau_2$. By (3.16) and (3.21), we find $k, \ell \in \{a_n, \dots, b_n\}$, $k \leq \ell$, such that $\tau_1 \in [t_{k-1}, t_k)$, $\tau_2 \in (t_{\ell-1}, t_\ell]$ and, due to (3.4), (3.3), (3.19) and (3.20),

$$|y^{[n]}(\tau_2) - y^{[n]}(\tau_1)|$$

$$\le \sum_{i=k+1}^{\ell-1} \left|\frac{v_i}{t_i^\mu}\right| (t_i - t_{i-1}) + \left|\frac{v_k}{t_k^\mu}\right| (t_k - \tau_1) + \left|\frac{v_\ell}{t_\ell^\mu}\right| (\tau_2 - t_{\ell-1})$$

$$< \frac{1}{a^\mu}(M_3 + M_4)(\tau_2 - \tau_1).$$

Having in mind that each function $y^{[n]}$ is continuous on $[a,b]$ for $n = n^*, \dots, n_0$, we have proved that the sequence $\{y^{[n]}\}$ is equicontinuous on $[a,b]$.

Step 3. Convergence of sequences $\{y^{[n]}\}$ and $\{z^{[n]}\}$.
Choose arbitrary $b \in \left(\frac{T}{2}, T\right)$ and $a \in (0,b)$. By Steps 1, 2 and the Arzelà-Ascoli theorem we can choose subsequences $\{y^{[m]}\} \subset \{y^{[n]}\}$ and $\{z^{[m]}\} \subset \{z^{[n]}\}$ such that

$$\lim_{m\to\infty} y^{[m]}(t) = y(t) \quad \text{uniformly on } [a,b],$$

$$\lim_{m\to\infty} z^{[m]}(t) = z(t) \quad \text{uniformly on } [0,b].$$

Since $a, b \in (0,T)$ are arbitrary, we use the diagonalization theorem (see e.g. [13]) and get that these subsequences can be chosen in such a way that they fulfil (3.10) and (3.11). Consequently,

$$y \in C(0,T), \quad z \in C[0,T], \quad z(0) = 0. \tag{3.22}$$

Now choose $c \in (0,T)$. By (3.8) there is a sequence $\{t_{c_m}\} \subset (0,T)$ which fulfils $\lim_{m\to\infty} t_{c_m} = c$. By (3.9) we have for $m \in \mathbb{N}$, $m \ge n^*$,

$$\alpha(t_{c_m}) \le y^{[m]}(t_{c_m}) = u_{c_m} \le \beta(t_{c_m}),$$

and letting $m \to \infty$ we obtain $\alpha(c) \le y(c) \le \beta(c)$. Having in mind that $c \in (0,T)$ is arbitrary, we get

$$\alpha(t) \le y(t) \le \beta(t), \quad t \in (0,T). \tag{3.23}$$

Step 4. Properties of limits y and z.
By (3.14) and (3.15), we get

$$y^{[m]}(t_k) = y^{[m]}(0) + \sum_{i=1}^{k} h_i \frac{z^{[m]}(t_i)}{t_i^\mu}, \quad k = 1, \dots, m, \tag{3.24}$$

$$z^{[m]}(t_{k+1}) = -\sum_{i=1}^{k} h_{i+1} t_i^\mu f(t_i, y^{[m]}(t_i)), \quad k = 1, \dots, m-1. \tag{3.25}$$

Assume that $0 < a^* < a < b < b^* < T$. By (3.17) and (3.21), $a_m \in (a^*, a)$, $b_m \in (b, b^*)$ for each sufficiently large m. According to Step 3, conditions (3.10) and (3.11) are satisfied, and we have

$$\lim_{m\to\infty} y^{[m]}(t_{a_m-1}) = y(a), \quad \lim_{m\to\infty} y^{[m]}(t_{b_m}) = y(b),$$

$$\lim_{m\to\infty} z^{[m]}(t_{a_m}) = z(a), \quad \lim_{m\to\infty} z^{[m]}(t_{b_m+1}) = z(b).$$

Denote

$$\varrho_m = \max\{|z^{[m]}(t_i) - z(t_i)| \colon i = a_m, \dots, b_m\}.$$

Then, by (3.11), the equality $\lim_{m\to\infty} \varrho_m = 0$ holds. Using (3.24), we get

$$y^{[m]}(t_{b_m}) = y^{[m]}(t_{a_m-1}) + \sum_{i=a_m}^{b_m} h_i \frac{z(t_i)}{t_i^\mu} + \sum_{i=a_m}^{b_m} \frac{h_i}{t_i^\mu}(z^{[m]}(t_i) - z(t_i)),$$

and letting $m \to \infty$, we obtain

$$y(b) = y(a) + \int_a^b \frac{z(\tau)}{\tau^\mu} \mathrm{d}\tau.$$

For $i = 1, \dots, m-1$, let us put

$$\tilde{f}_m(t) = t_i^\mu f(t_i, y^{[m]}(t_i)), \quad t \in [t_i, t_{i+1}).$$

By (3.6) and (3.18), we have for each sufficiently large $m \in \mathbb{N}$,

$$|\tilde{f}_m(t)| \le t^\mu \Big(g_1(t, \alpha(t)) + \tilde{m}(t) + 1\Big) \quad \text{for } t \in J.$$

Further, using (3.10), we obtain

$$\lim_{m\to\infty} \tilde{f}_m(t) = t^\mu f(t, y(t)) \quad \text{for } t \in J.$$

Since (3.25) yields

$$z^{[m]}(t_{b_m+1}) = z^{[m]}(t_{a_m}) - \sum_{i=a_m}^{b_m} h_{i+1} t_i^\mu f(t_i, y^{[m]}(t_i)),$$

we get for $m \to \infty$, due to (3.7) and the Lebesgue dominated convergence theorem,

$$z(b) = z(a) - \int_a^b \tau^\mu f(\tau, y(\tau)) \mathrm{d}\tau.$$

Since $a, b \in (0, T)$ are arbitrary, we can write

$$y(t) = y(a) + \int_a^t \frac{z(\tau)}{\tau^\mu} \mathrm{d}\tau, \quad t \in (0, T), \tag{3.26}$$

$$z(t) = z(a) - \int_a^t \tau^\mu f(\tau, y(\tau)) \mathrm{d}\tau, \quad t \in (0, T). \tag{3.27}$$

Equality (3.26) gives $y'(t) = z(t)/t^\mu$ for $t \in (0, T)$, and then equality (3.27) can be written in the form

$$t^\mu y'(t) = a^\mu y'(a) - \int_a^t \tau^\mu f(\tau, y(\tau)) \mathrm{d}\tau, \quad t \in (0, T).$$

Due to (3.22), we have $\lim_{t\to 0+} t^\mu y'(t) = 0$, and hence,

$$t^\mu y'(t) = -\int_0^t \tau^\mu f(\tau, y(\tau))\mathrm{d}\tau \quad \text{for } t \in [0,T). \tag{3.28}$$

We have proved that $y \in AC^1(0,T)$ fulfils equation (1.1) for a.e. $t \in (0,T)$ and satisfies the first condition in (1.2). If we integrate equation (3.28), we get for $t \in (0,a)$

$$y(t) = y(a) + \int_t^a \frac{1}{\tau^\mu} \int_0^\tau s^\mu f(s, y(s))\mathrm{d}s\mathrm{d}\tau.$$

Denote

$$\tilde{h}(\tau) = \frac{1}{\tau^\mu} \int_0^\tau s^\mu f(s, y(s))\mathrm{d}s.$$

Due to (3.6) and (3.7), we see that $\tilde{h} \in L[0,a]$ and so $y \in C[0,T)$. Since $\alpha, \beta \in C[0,T]$ and $\alpha(T) = \beta(T) = 0$, we get by (3.23), $\lim_{t\to T-} y(t) = 0$. Therefore, putting $y(T) = 0$ yields that $y \in C[0,T]$ satisfies the second condition in (1.2). Finally, by (3.9) and (3.23), $y(t) > 0$ for $t \in (0,T)$. We have proved that y is a positive solution of problem (1.1), (1.2). □

In the next theorem, we consider a simplier case, where $A > 0$ and

$$\begin{cases} f \in C(J \times (0,\infty)) \text{ may have integrable discontinuities} \\ \text{at } t = 0,\ t = s_i,\ i = 1, \dots, r, \\ f \text{ may have a singularity at } x = 0. \end{cases} \tag{3.29}$$

In Remark 3.4 we explain a choice of the time scale (1.6) for $n \in \mathbb{N}$.

Remark 3.4 If (3.29) holds, we find a function $g_1 \in C(J \times (0,\infty))$ with integrable discontinuities at $t = 0$, $t = s_i$, $i = 1, \dots, r$, which is unbounded if $t \to s_i$, $i = 1, \dots, r$, $x \in (0,\infty)$, and fulfils

$$|f(t,x)| \le g_1(t,x) \quad \text{for } t \in J,\ x \in (0,\infty).$$

Then, for $n \in \mathbb{N}$, we choose points (1.6) satisfying (1.7) and (3.1) for $j = 1$.

The next theorem states that under (3.29) the convergence interval for $\{y^{[m]}\}$ and $\{z^{[m]}\}$ can be extended to T.

Theorem 3.5 *Let $A > 0$ and* (3.29) *hold. Assume that conditions* (3.2) *and* (3.8) *are fulfilled for $j = 1$ and g_1 of Remark* 3.4. *Further assume that there exist functions $\alpha, \beta \in C[0,T]$ satisfying* (3.9) *and $\alpha(0) > 0$. Then the following approximation principle is valid:*

Let the sequences $\{y^{[n]}\}$ and $\{z^{[n]}\}$ be given by (3.4) *and* (3.5). *Then there exist their subsequences $\{y^{[m]}\}$ and $\{z^{[m]}\}$ such that*

$$\lim_{m\to\infty} y^{[m]}(t) = y(t) \quad \textit{locally uniformly on } (0,T], \tag{3.30}$$

$$\lim_{m\to\infty} z^{[m]}(t) = t^\mu y'(t) \quad \textit{locally uniformly on } [0,T], \tag{3.31}$$

and y is a positive solution of the singular differential problem (1.1), (1.2) *satisfying* (3.12). *Moreover $y \in AC^1(0,T]$.*

Proof. For $t \in J$, put $\tilde{m}(t) = \sup\{|f(t,x)| : x \in [\alpha(t), \beta(t)]\}$. Since $\alpha(t) > 0$ on $[0,T)$, we get by (3.29) that $\tilde{m} \in L[0,T]$. We argue similarly as in Steps 1 and 2 of the proof of Theorem 3.3 and get that the sequences $\{y^{[n]}\}$ and $\{z^{[n]}\}$ are bounded on $[0,T]$. Moreover, $\{y^{[n]}\}$ is equicontinuous on $(0,T]$ and $\{z^{[n]}\}$ on $[0,T]$. Therefore, we can find their subsequences $\{y^{[m]}\}$ and $\{z^{[m]}\}$ which fulfil (3.30) and (3.31). Consequently, $y \in C(0,T]$, $z \in C[0,T]$, $y(T) = A$, $z(0) = 0$. The arguments of Steps 3 and 4 of the proof of Theorem 3.3 yield $\alpha(t) \le y(t) \le \beta(t)$ for $t \in (0,T]$, and

$$t^\mu y'(t) = -\int_0^t \tau^\mu f(\tau, y(\tau))\mathrm{d}\tau, \quad t \in [0,T].$$

This implies that $AC^1(0,T]$ is a positive solution of problem (1.1), (1.2). □

4 Solvability of singular membrane problems

Choose $A, r_0 \in [0,\infty)$, assume that $q \in L[0,T] \cap C(J)$, and consider the problem

$$(t^3 u'(t))' + t^3 q(t)\left(\frac{1}{u^2(t)} - r_0 u(t)\right) = 0, \tag{4.1}$$

$$\lim_{t\to 0+} t^3 u'(t) = 0, \quad u(T) = A. \tag{4.2}$$

This problem is a special case of (1.1), (1.2), where $\mu = 3$ and

$$f(t,x) = q(t)\left(\frac{1}{x^2} - r_0 x\right) \quad \text{for } t \in J,\ x \in (0,\infty). \tag{4.3}$$

We see that f satisfies (3.29). Therefore, we can use Theorem 3.3 or Theorem 3.5 to get a solvability of problem (4.1), (4.2). This problem was studied in [1] for $r_0 = 0$, $A > 0$ and $q \in C[0,1]$. It describes a behavior of symmetric circular membranes. We prove that corresponding discretizations of this problem are solvable. We discuss three cases:

$$A > 0,\ r_0 > 0 \quad \text{or} \quad A > 0,\ r_0 = 0 \quad \text{or} \quad A = 0,\ r_0 \ge 0.$$

Case 1.
Let $A > 0$, $r_0 > 0$. Assume

$$q(t) \ge 0 \quad \text{for } t \in J. \tag{4.4}$$

For $n \in \mathbb{N}$, choose points $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ such that

$$q(t_k) \in [0, \infty), \quad k = 1, \ldots, n-1, \tag{4.5}$$

and, for $h_k = t_k - t_{k-1}$, consider the following discretization of problem (4.1), (4.2):

$$\frac{1}{h_{k+1}} \Delta \left(\frac{t_k^3}{h_k} \Delta u_{k-1} \right) + t_k^3 q(t_k) \left(\frac{1}{u_k^2} - r_0 u_k \right) = 0, \quad k = 1, \ldots, n-1, \tag{4.6}$$

$$\Delta u_0 = 0, \quad u(T) = A. \tag{4.7}$$

Theorem 4.1 *Let $A > 0$, $r_0 > 0$ and (4.5) hold. Then, there are constants $0 < \nu < c$ such that, for each $n \in \mathbb{N}$, problem (4.6), (4.7) has a positive solution $(u_0, \ldots, u_n)$ satisfying (3.9), where*

$$\alpha(t) = \nu, \quad \beta(t) = c, \quad t \in [0, T]. \tag{4.8}$$

Proof. Let $n \in \mathbb{N}$ be arbitrary. Choose $\nu \in (0, A]$, $c \in [A, \infty)$, and consider α, β given by (4.8). Denote

$$\alpha_k = \alpha(t_k), \quad \beta_k = \beta(t_k), \quad k = 0, \ldots, n. \tag{4.9}$$

If $\nu^3 \leq 1/r_0$ and $c^3 \geq 1/r_0$, we can check that, for each $n \in \mathbb{N}$, the vectors $(\alpha_0, \ldots, \alpha_n)$ and $(\beta_0, \ldots, \beta_n)$ are lower and upper functions of problem (4.6), (4.7) and satisfy (2.13). According to (4.3), (3.29) and (4.5), we can use Theorem 2.4 and get a positive solution $(u_0, \ldots, u_n)$ of problem (4.6), (4.7) satisfying (3.9). □

Case 2.
Let $A > 0$, $r_0 = 0$. Assume

$$\exists K > 0\colon\ 0 \leq q(t) \leq K \quad \text{for } t \in J. \tag{4.10}$$

For $n \in \mathbb{N}$, choose points $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ such that

$$q(t_k) \in [0, K], \quad k = 1, \ldots, n-1, \tag{4.11}$$

and consider the corresponding discretization (4.6), (4.7).

Theorem 4.2 *Let $A > 0$, $r_0 = 0$ and (4.11) hold. Then, there is a constant $c > 0$ such that, for each $n \in \mathbb{N}$, problem (4.6), (4.7) has a positive solution $(u_0, \ldots, u_n)$ satisfying (3.9), where*

$$\alpha(t) = A, \quad \beta(t) = A + c(T^2 - t^2), \quad t \in [0, T]. \tag{4.12}$$

Proof. Let $n \in \mathbb{N}$ be arbitrary. Choose $c > 0$ and consider α, β given by (4.12) and use (4.9). We can show that for each sufficiently large $c > 0$

$$\frac{1}{h_{k+1}}\Delta\left(\frac{t_k^3}{h_k}\Delta\alpha_{k-1}\right) + t_k^3 q(t_k)\frac{1}{\alpha_k^2} = t_k^3 q(t_k)\frac{1}{A^2} \geq 0, \quad k = 1, \ldots, n-1,$$

$$\frac{1}{h_{k+1}}\Delta\left(\frac{t_k^3}{h_k}\Delta\beta_{k-1}\right) + t_k^3 q(t_k)\frac{1}{\beta_k^2}$$

$$= -\frac{c}{h_{k+1}}(t_{k+1}^4 + t_{k+1}^3 t_k - t_k^4 - t_k^3 t_{k-1}) + \frac{t_k^3 q(t_k)}{(A + c(T^2 - t_k^2))^2}$$

$$\leq -ct_k^3\left(7 - \frac{K}{cA^2}\right) \leq 0, \quad k = 1, \ldots, n-1.$$

Note that c does not depend on n. Moreover, $\Delta\alpha_0 = 0$, $\Delta\beta_0 = -ct_1^2 < 0$, $\alpha_n = \beta_n = A$. Hence, for each $n \in \mathbb{N}$, the vectors $(\alpha_0, \ldots, \alpha_n)$ and $(\beta_0, \ldots, \beta_n)$ are lower and upper functions of problem (4.6), (4.7) and satisfy (2.13). According to (4.3), (3.29), (4.11) and Theorem 2.4, there exists a positive solution $(u_0, \ldots, u_n)$ of problem (4.6), (4.7) satisfying (3.9). □

Case 3.
Let $A = 0$, $r_0 \geq 0$. Assume

$$\exists \varepsilon, K > 0\colon\ \varepsilon \leq q(t) \leq K \quad \text{for } t \in J. \tag{4.13}$$

For $n \in \mathbb{N}$, choose points $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ such that

$$q(t_k) \in [\varepsilon, K], \quad k = 1, \ldots, n-1, \tag{4.14}$$

and consider the corresponding discretization (4.6), (4.7). Denote

$$\omega_n = \min\{t_{k+1} - t_k\colon k = 0, \ldots, n\}, \quad \chi_n = \max\{t_{k+1} - t_k\colon k = 0, \ldots, n\}.$$

Theorem 4.3 *Let $A = 0$, $r_0 \geq 0$ and* (4.14) *hold. Assume*

$$\lim_{n\to\infty} \frac{\chi_n}{\omega_n} = c_0 \in (0, \infty). \tag{4.15}$$

Then, there are constants $0 < \nu < c$ such that, for each $n \in \mathbb{N}$, problem (4.6), (4.7) *has a positive solution $(u_0, \ldots, u_n)$ satisfying* (3.9), *where*

$$\alpha(t) = \nu(t + \nu)(T - t), \quad \beta(t) = c\sqrt{T^2 - t^2}, \quad t \in [0, T]. \tag{4.16}$$

Proof. Let $n \in \mathbb{N}$ be arbitrary. Choose $\nu > 0$, $c > \nu$ and consider α, β given by (4.16). We use (4.9) and show that if ν is sufficiently small and c sufficiently large, then for each $n \in \mathbb{N}$, the vectors $(\alpha_0, \ldots, \alpha_n)$ and $(\beta_0, \ldots, \beta_n)$ are lower and upper functions of problem (4.6), (4.7). We see that $\alpha_n = \beta_n = 0$,

$\Delta\beta_0 = c\sqrt{T^2 - t_1^2} - cT < 0$ and, $\Delta\alpha_0 = \nu t_1(T - (1+\nu)t_1) > 0$ if ν is sufficiently small. Further, we get for $k = 1, \dots, n-1$,

$$\frac{1}{h_{k+1}}\Delta\left(\frac{t_k^3}{h_k}\Delta\alpha_{k-1}\right) + t_k^3 q(t_k)\left(\frac{1}{\alpha_k^2} - r_0\alpha_k\right)$$

$$= \nu\left((T - \nu - t_k^2)(t_{k+1}^2 + t_{k+1}t_k + t_k^2) - (t_{k+1} + t_k)(t_{k+1}^2 + t_k^2) - t_k^3\frac{t_k - t_{k-1}}{t_{k+1} - t_k}\right)$$

$$+ t_k^3 q(t_k)\left(\frac{1}{\nu^2(t_k+\nu)^2(T-t_k)^2} - r_0\nu(t_k+\nu)(T-t_k)\right) =: \varphi(t_k, \nu).$$

Let $t_{k+1} \leq \min\left\{\frac{T}{4}, 1\right\}$. Then, for each sufficiently small ν,

$$\varphi(t_k, \nu) > \frac{t_k^3}{\nu^2(t_k+\nu)^2(T-t_k)^2}(-\nu^3(c_0+1)T^4 + \varepsilon - \nu^3 K r_0 T^6) \geq 0.$$

Let $t_k > \min\left\{\frac{T}{4}, 1\right\}$. Then, for each sufficiently small ν,

$$\varphi(t_k, \nu) \geq \frac{T^3}{\nu^2(t_k+\nu)^2(T-t_k)^2}\left(-\nu^3(c_0+8) + \frac{\varepsilon}{4^3} - \nu^3 K r_0 T^6\right) > 0.$$

Hence, for each sufficiently small ν, the inequality $\varphi(t_k, \nu) \geq 0$, $k = 1, \dots, n-1$, is valid. This yields that $(\alpha_0, \dots, \alpha_n)$ is a lower function of problem (4.6), (4.7). Finally,

$$\frac{1}{h_{k+1}}\Delta\left(\frac{t_k^3}{h_k}\Delta\beta_{k-1}\right) + t_k^3 q(t_k)\left(\frac{1}{\beta_k^2} - r_0\beta_k\right)$$

$$= -\frac{c}{h_{k+1}}\left(t_{k+1}^3\frac{t_{k+1}+t_k}{\sqrt{T^2 - t_{k+1}^2} + \sqrt{T^2 - t_k^2}} - t_k^3\frac{t_k + t_{k-1}}{\sqrt{T^2 - t_k^2} + \sqrt{T^2 - t_{k-1}^2}}\right)$$

$$+ t_k^3 q(t_k)\left(\frac{1}{c^2(T^2 - t_k^2)} - r_0 c\sqrt{T^2 - t_k^2}\right) =: \psi(t_k, c), \quad k = 1, \dots, n.$$

Let $t_k \geq \frac{T}{2}$. Then, for sufficiently large c,

$$\psi(t_k, c) \leq -\frac{2ct_k^4}{h_{k+1}} \cdot \frac{\sqrt{T^2 - t_k^2} + \sqrt{T^2 - t_{k-1}^2} - \left(\sqrt{T^2 - t_{k+1}^2} + \sqrt{T^2 - t_k^2}\right)}{\left(\sqrt{T^2 - t_{k+1}^2} + \sqrt{T^2 - t_k^2}\right)\left(\sqrt{T^2 - t_k^2} + \sqrt{T^2 - t_{k-1}^2}\right)}$$

$$+ \frac{t_k^3 K}{c^2(T^2 - t_k^2)} \leq -\frac{ct_k^4}{h_{k+1}} \cdot \frac{T^2 - t_k^2 - (T^2 - t_{k+1}^2)}{4(T^2 - t_k^2)\sqrt{T^2 - t_{k-1}^2}} + \frac{t_k^3 K}{c^2(T^2 - t_k^2)}$$

$$\leq -\frac{ct_k^3}{(T^2 - t_k^2)\sqrt{T^2 - t_{k-1}^2}}\left(\frac{T}{4} - \frac{TK}{c^3}\right) < 0.$$

Let $t_k \le \frac{T}{2}$. Then, for sufficiently large c,

$$\psi(t_k, c) \le -\frac{2ct_k}{h_{k+1}} \cdot \frac{t_{k+1}^3 - t_k^3}{2\sqrt{T^2 - t_k^2}} + \frac{t_k^3 K}{c^2(T^2 - t_k^2)}$$

$$\le -\frac{ct_k(t_{k+1}^2 + t_{k+1}t_k + t_k^2)}{\sqrt{T^2 - t_k^2}} + \frac{t_k^3 K}{c^2(T^2 - t_k^2)} \le -ct_k^3\left(\frac{3}{T} - \frac{4K}{3c^3T^2}\right) \le 0.$$

Hence, $(\beta_0, \ldots, \beta_n)$ is an upper function of problem (4.6), (4.7) and satisfies (2.13). According to (4.3), (3.29), (4.14) and Theorem 2.4, there exists a positive solution $(u_0, \ldots, u_n)$ of problem (4.6), (4.7) satisfying (3.9). □

The main results about solvability of problem (4.1), (4.2) and about approximation of its solution are contained in the next two theorems.

Theorem 4.4 *Let $A > 0$, $r_0 \ge 0$. Assume that* (4.4) *holds if $r_0 > 0$ and that* (4.10) *holds if $r_0 = 0$. Then there exists a sequence $\{y^{[m]}\}$ of continuous piece-wise linear functions which converges locally uniformly on $(0, T]$ to a function $y \in AC^1(0, T]$, which is a positive solution of problem* (4.1), (4.2).

Proof. For $n \in \mathbb{N}$, we choose points (1.6) by Remark 3.4 and consider the discrete problem (4.6), (4.7), where $q(t_k) \in [0, \infty)$ if $r_0 > 0$, and $q(t_k) \in [0, K]$ if $r_0 = 0$, $k = 1, \ldots, n-1$. Moreover, the points $t_1, \ldots, t_n$ are chosen such that (3.8) is valid. By Theorem 4.1 or Theorem 4.2, for each $n \in \mathbb{N}$, problem (4.6), (4.7) has a positive solution $(u_0, \ldots, u_n)$ satisfying (3.9). Here α and β are of (4.8) for $r_0 > 0$ and of (4.12) for $r_0 = 0$. Hence $\alpha(0) > 0$. Define a sequence $\{y^{[n]}\}$ by (3.4). Then, by Theorem 3.5, there exists a subsequence $\{y^{[m]}\} \subset \{y^{[n]}\}$ satisfying (3.30), where the limit $y \in AC^1(0, T]$ is a positive solution of (4.1), (4.2). □

Theorem 4.5 *Let $A = 0$, $r_0 \ge 0$. Assume that* (4.13) *holds. Then there exists a sequence $\{y^{[m]}\}$ of continuous piece-wise linear functions which converges locally uniformly on $(0, T)$ to a function y, which is a positive solution of problem* (4.1), (4.2).

Proof. For $n \in \mathbb{N}$, we choose points (1.6) by Remark 3.2 and consider the discrete problem (4.6), (4.7), where $q(t_k) \in [\varepsilon, K]$, $k = 1, \ldots, n-1$. Moreover, the points $t_1, \ldots, t_n$ are chosen such that (3.8) and (4.15) are valid. By Theorem 4.3, for each $n \in \mathbb{N}$, problem (4.6), (4.7) has a positive solution $(u_0, \ldots, u_n)$ satisfying (3.9), where α and β are defined in (4.16). Hence $\beta(T) = 0$. Further we can put

$$g_1(t, x) = \frac{q(t)}{x^2}, \quad g_2(t, x) = r_0 q(t) x \quad \text{for } t \in J,\ x \in (0, \infty).$$

Then $g_1 \in C(J \times (0, \infty))$, $g_2 \in C(J \times [0, \infty))$, g_1 and g_2 have integrable discontinuities at $t = 0$, $t = s_i$, $i = 1, \dots, r$, g_1 is decreasing in x and

$$\int_0^{T/2} t g_1(t, \alpha(t)) \mathrm{d}t = \int_0^{T/2} \frac{t q(t)}{\nu^2 (t+\nu)^2 (T-t)^2} \mathrm{d}t \le \frac{2K}{T\nu^4} \int_0^{T/2} t \, \mathrm{d}t < \infty.$$

According to (4.3), inequality (3.6) holds. Define a sequence $\{y^{[n]}\}$ by (3.4). Then, by Theorem 3.3, there exists a subsequence $\{y^{[m]}\} \subset \{y^{[n]}\}$ satisfying (3.10), where the limit y is a positive solution of (4.1), (4.2). □

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