On a homoclinic point of an autonomous second-order difference equation

Lukáš Rachůnek and Irena Rachůnková

Department of Mathematics, Faculty of Science, Palacký University,
Tomkova 40, 77900 Olomouc, Czech Republic,
e-mail: rachunko@inf.upol.cz

Abstract. The paper deals with the second-order difference equation.

\[ x(n+1) = 2x(n) - x(n-1) + h^2 f(x(n)), \quad n \in \mathbb{N}, \]

where \( h > 0 \) is a parameter and \( f \) has continuous first derivative and three zeros on the real line. The main result is that for each sufficiently small \( h \) the above equation has a homoclinic point.

Keywords. Autonomous second-order difference equation, homoclinic point, strictly increasing solutions.

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1 Introduction

We consider the autonomous second-order difference equation

\[ x(n+1) = 2x(n) - x(n-1) + h^2 f(x(n)), \quad n \in \mathbb{N}, \quad (1.1) \]

where \( h \in (0, \infty) \) is a parameter. A sequence \( \{x(n)\}_{n=0}^{\infty} \) which satisfies (1.1) is called a solution of equation (1.1). We assume that

\[ L_0 < 0 < L, \quad f \in C^1[L_0, L], \quad f(L_0) = f(0) = f(L) = 0, \quad (1.2) \]
\[ xf(x) < 0 \text{ for } x \in (L_0, L) \setminus \{0\}, \quad f'(L_0) > 0, \quad f'(0) < 0, \quad f'(L) > 0, \quad (1.3) \]
\[ \exists \bar{B} \in (L_0, 0) \text{ such that } \int_{\bar{B}}^L f(z) \, dz = 0. \quad (1.4) \]

Equation (1.1) represents an autonomous discrete case of some models arising in hydrodynamics. See [7], [9], [13], [16]. For monographs dealing with difference equations we refer to [1], [2], [3], [8], [12], [14]. We mention also some recent papers...
investigating the solvability of second-order discrete boundary value problems, for example [4]–[6], [10], [11], [15], [17]–[24].

The main result of our paper is the existence of a homoclinic point of equation (1.1). The results presented here can be also useful when analyzing the discretization of corresponding boundary value problems for ordinary differential equations, in particular, by finite-difference methods. To elucidate the geometry of the dynamics of (1.1) it is convenient to convert it to an equivalent planar map. To this end we let $x_{1}^{n} = x(n-1)$, $x_{2}^{n} = x(n)$ and we obtain the equivalent first-order system of difference equations

$$
\begin{align*}
    x_{1}^{n+1} &= x_{2}^{n} \\
    x_{2}^{n+1} &= 2x_{2}^{n} - x_{1}^{n} + h^{2}f(x_{2}^{n}),
\end{align*}
$$

which can be written as the iteration of the map

$$
\left( \begin{array}{c}
    x_{1} \\
    x_{2}
\end{array} \right) \mapsto \left( \begin{array}{c}
    x_{2} \\
    2x_{2} - x_{1} + h^{2}f(x_{2})
\end{array} \right).
$$

Let us choose $B \in (L_0, 0)$ and denote

$$
\begin{align*}
x^{0} &= \left( \begin{array}{c}
    B \\
    B
\end{array} \right), \quad x = \left( \begin{array}{c}
    x_{1} \\
    x_{2}
\end{array} \right), \quad F \left( \begin{array}{c}
    x_{1} \\
    x_{2}
\end{array} \right) = \left( \begin{array}{c}
    x_{2} \\
    2x_{2} - x_{1} + h^{2}f(x_{2})
\end{array} \right).
\end{align*}
$$

Then (1.5) has the form $x \mapsto F(x)$, and the positive orbit $\gamma^{+}(x^{0})$ is the sequence

$$
\gamma^{+}(x^{0}) = \{x^{0}, F(x^{0}), \ldots, F^{n}(x^{0}), \ldots\}.
$$

The map $F$ is invertible and

$$
F^{-1} \left( \begin{array}{c}
    x_{1} \\
    x_{2}
\end{array} \right) = \left( \begin{array}{c}
    2x_{1} - x_{2} + h^{2}f(x_{1}) \\
    x_{1}
\end{array} \right).
$$

Hence the negative orbit $\gamma^{-}(x^{0})$ is the sequence

$$
\gamma^{-}(x^{0}) = \{x^{0}, F^{-1}(x^{0}), \ldots, F^{-n}(x^{0}), \ldots\},
$$

and the orbit $\gamma(x^{0}) = \gamma^{+}(x^{0}) \cup \gamma^{-}(x^{0})$ is uniquely determined for each $B \in (L_0, 0)$. Under the assumption that $h > 0$ is sufficiently small we prove that $(L, L)^{T}$ is a saddle point of $F$ and that there exists $B^{*} \in (L_0, L)$ such that $(B^{*}, B^{*})^{T}$ is a homoclinic point for $F$, that is the orbit $\gamma(x^{*})$, when $x^{*} = (B^{*}, B^{*})^{T}$, satisfies

$$
\lim_{n \to \infty} F^{n}(x^{*}) = \lim_{n \to \infty} F^{-n}(x^{*}) = \left( \begin{array}{c}
    L \\
    L
\end{array} \right).
$$

(1.7)
2 Fixed points

Due to (1.2) the map \( F \) given by (1.6) has three fixed points \((L_0, L_0)^T\), \((0, 0)^T\) and \((L, L)^T\) in the set \([L_0, L] \times [L_0, L]\). The Jacobian matrix of \( F \) has the form

\[
DF(x) = \begin{pmatrix}
0 & 1 \\
-1 & 2 + h^2 f'(x_2)
\end{pmatrix}.
\]

The assumption (1.3) gives \( \frac{1}{2} h^2 f'(L) =: \varepsilon > 0 \), and hence

\[
DF \begin{pmatrix} L \\ L \end{pmatrix} = \begin{pmatrix} 0 & 1 \\
-1 & 2 + 2\varepsilon
\end{pmatrix}
\]

has the eigenvalues \( \lambda_{1,2} = 1 + \varepsilon \pm \sqrt{\varepsilon^2 + 2\varepsilon} \). So, for a sufficiently small \( h > 0 \), one eigenvalue has modulus greater than 1 and the other less than 1. Therefore \((L, L)^T\) is an unstable hyperbolic fixed point—a saddle point. The same is true for \((L_0, L_0)^T\). On the other hand, (1.3) yields \( \frac{1}{2} h^2 f'(0) =: -\delta < 0 \), and hence

\[
DF \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\
-1 & 2 - 2\delta
\end{pmatrix}
\]

has the eigenvalues \( \lambda_{1,2} = 1 - \delta \pm i \sqrt{1 - (1 - \delta)^2} \) with moduli equal to 1. Therefore \((0, 0)^T\) is an elliptic fixed point which is a centre in the phase portrait of the approximate linear map

\[
x \mapsto DF \begin{pmatrix} 0 \\ 0 \end{pmatrix} x.
\]

The stability and type of the fixed point \((0, 0)^T\) of the nonlinear map \( x \rightarrow F(x) \) cannot be determined solely from linearization and the effects of the nonlinear terms in local dynamics must be accounted for.

3 Increasing solutions

For each values \( A_0, A_1 \in [L_0, L] \) there exists a unique solution \( \{x(n)\}_{n=0}^\infty \) of equation (1.1) satisfying the initial conditions

\[
x(0) = A_0, \quad x(1) = A_1.
\]

(3.1)

Such sequence \( \{x(n)\}_{n=0}^\infty \) is called a solution of problem (1.1), (3.1). In order to find a point \( x^* = (B^*, B^*)^T \) satisfying (1.7) we choose \( B \in (L_0, 0) \) and study solutions of problem (1.1), (3.2), where

\[
x(0) = B, \quad x(1) = B.
\]

(3.2)
Lemma 3.1 Let $B \in (L_0, 0)$ and let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (3.2). Then there exists $r \in \mathbb{N}$, $r > 1$, such that
\[
x(1) < x(2) < \cdots < x(r - 1) < 0 \leq x(r) ~ \text{if} ~ r > 2,
\]
\[
x(1) < 0 \leq x(2) ~ \text{if} ~ r = 2.
\]

Proof. Choose $B \in (L_0, 0)$ and consider the solution $\{x(n)\}_{n=0}^{\infty}$ of problem (1.1), (3.2). Then $\{x(n)\}_{n=0}^{\infty}$ fulfills
\[
\Delta x(n) = \Delta x(n - 1) + h^2 f(x(n)), \quad n \in \mathbb{N},
\]
\[
x(0) = B, \quad \Delta x(0) = 0,
\]
where $\Delta x(n - 1) = x(n) - x(n - 1)$ is the forward difference operator. By (1.3) and (3.2), we have $f(x(0)) = f(x(1)) = f(B) > 0$, and (3.5) yields $\Delta x(1) > 0$. Hence $x(1) < x(2)$. If $x(2) \geq 0$ we get (3.4). Otherwise $x(1) < x(2) < 0$ and we repeat the above arguments to get $\Delta x(3) > \Delta x(2)$ and $x(2) < x(3)$. If $x(3) \geq 0$, we put $r = 3$ and get (3.3). Otherwise we continue as before and prove that after a finite number $r$ of steps we get (3.3) and
\[
\Delta x(1) < \Delta x(2) < \cdots < \Delta x(r - 1).
\]

Assume on the contrary that $r$ is not finite, that is $x(n) < 0$ for each $n \in \mathbb{N}$. By (1.3), the inequality $f(x(n)) > 0$ holds for each $n \in \mathbb{N}$, and the sequence $\{\Delta x(n)\}_{n=1}^{\infty}$ is positive and increasing. Therefore
\[
\lim_{n \to \infty} \Delta x(n) > 0.
\]
The positivity of $\{\Delta x(n)\}_{n=1}^{\infty}$ implies that $\{x(n)\}_{n=1}^{\infty}$ is increasing. Since $\{x(n)\}_{n=1}^{\infty}$ is bounded above by 0, there exists a finite $\lim_{n \to \infty} x(n)$, contrary to (3.8). So, we have proved that (3.3) holds for some $r \in \mathbb{N}$. \hfill $\square$

Lemma 3.2 Let $B \in (L_0, 0)$ and let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (3.2). If $\{x(n)\}_{n=1}^{\infty}$ is increasing and $x(n) < L$ for $n \in \mathbb{N}$, then
\[
\lim_{n \to \infty} x(n) = L, \quad \lim_{n \to \infty} \Delta x(n) = 0.
\]

Proof. Since $\{x(n)\}_{n=1}^{\infty}$ is increasing and bounded above by $L$, there exists $\lim_{n \to \infty} x(n) = L_1 \leq L$. Consequently $\lim_{n \to \infty} \Delta x(n) = 0$. By Lemma 3.1, we have $0 < x(r) < x(r + 1)$ and $L_1 > 0$. If $L_1 < L$, then by virtue of (3.5), $\lim_{n \to \infty} \Delta x(n) = \lim_{n \to \infty} \Delta x(n - 1) + h^2 \lim_{n \to \infty} f(x(n))$, and hence $0 = 0 + h^2 f(L_1) < 0$, a contradiction. Therefore $L_1 = L$ and (3.9) is proved. \hfill $\square$

Definition 3.3 A solution satisfying the conditions of Lemma 3.2 is called a homoclinic solution.
Remark 3.4 Our main task is to prove the existence of a homoclinic solution \( \{x^*(n)\}_{n=0}^\infty \) of (1.1), (3.2) for some \( B = B^* \in (L_0, 0) \). Since \( L_0 < B \leq x^*(n) < L \) for \( n \in \mathbb{N} \cup \{0\} \), we may assume without loss of generality that

\[
f(x) = 0 \quad \text{for } x \in (-\infty, L_0) \cup (L, \infty).
\]

(3.10)

Note that if we have the above homoclinic solution and put \( x^* = (B^*, B^*)^T \), then the map \( F \) given by (1.6) satisfies (1.7), and hence the point \( (B^*, B^*)^T \) is a homoclinic point for \( F \).

In what follows (Sec. 3–6) we assume that, in addition to (1.2)–(1.4), \( f \) fulfills moreover (3.10).

Lemma 3.5 Let \( B \in (L_0, 0) \) and let \( \{x(n)\}_{n=0}^\infty \) be a solution of problem (1.1), (3.2). Assume that there exists \( b \in \mathbb{N}, b > 1 \), such that \( \{x(n)\}_{n=1}^b \) is increasing and

\[
x(b) < L < x(b+1) \quad \text{or} \quad x(b) = L.
\]

Then \( \{x(n)\}_{n=1}^\infty \) is increasing and

\[
\lim_{n \to \infty} x(n) = \infty, \quad \lim_{n \to \infty} \Delta x(n) = \Delta x(b) > 0.
\]

(3.12)

Proof. Choose \( B \in (L_0, 0) \) and consider the solution \( \{x(n)\}_{n=0}^\infty \) of problem (1.1), (3.2) which is increasing for \( 1 \leq n \leq b \). If the first condition in (3.11) holds, then \( \Delta x(b) > 0 \). Let \( x(b) = L \). Then (3.5) yields \( \Delta x(b) = \Delta x(b-1) + h^2 f(L) = \Delta x(b-1) > 0 \). Therefore (3.11) gives \( \Delta x(b) > 0 \) in both cases. By (3.10) and (3.11), \( f(x(b+1)) = 0 \). Consequently, by (3.5), \( \Delta x(b+1) = \Delta x(b) + h^2 f(x(b+1)) = \Delta x(b) \), and similarly \( \Delta x(n) = \Delta x(b) \) for \( n > b+1 \). This gives \( \lim_{n \to \infty} \Delta x(n) = \Delta x(b) > 0 \). Therefore \( \{x(n)\}_{n=1}^\infty \) is increasing and \( \lim_{n \to \infty} x(n) = \infty \).

Definition 3.6 A solution satisfying the conditions of Lemma 3.5 is called an escape solution.

Theorem 3.7 (On three types of solutions)

Let \( B \in (L_0, 0) \) and let \( \{x(n)\}_{n=0}^\infty \) be a solution of problem (1.1), (3.2). Then \( \{x(n)\}_{n=0}^\infty \) is just one of the following three types:

(I) \( \{x(n)\}_{n=0}^\infty \) is a homoclinic solution;

(II) \( \{x(n)\}_{n=0}^\infty \) is an escape solution;

(III) there exists \( b \in \mathbb{N}, b > 1 \), such that \( \{x(n)\}_{n=1}^b \) is increasing and

\[
0 < x(b) < L, \quad x(b+1) \leq x(b).
\]

(3.13)
**Proof.** Choose \( B \in (L_0, 0) \) and consider the solution \( \{x(n)\}_{n=0}^\infty \) of problem (1.1), (3.2). By Lemma 3.1, there exists \( r \in \mathbb{N}, r > 1 \), such that \( \{x(n)\}_{n=1}^r \) is increasing and \( x(r) \geq 0 \). Let \( x(r) \geq L \). Then, due to (3.3), (3.4) and Lemma 3.5, \( \{x(n)\}_{n=0}^\infty \) is an escape solution. Now, assume that \( x(r) < L \) and that \( \{x(n)\}_{n=0}^\infty \) is neither a homoclinic solution nor an escape solution. Then, by Lemma 3.2 and Lemma 3.5, the sequence \( \{x(n)\}_{n=1}^\infty \) cannot be increasing and cannot fulfill (3.9) or (3.11). Therefore there exists \( b \geq r \) such that \( \{x(n)\}_{n=1}^b \) is increasing and \( x(b+1) \leq x(b) \). Clearly \( x(b) < L \). Otherwise (3.5) gets \( x(b+1) > x(b) \), a contradiction. We have proved that \( \{x(n)\}_{n=0}^\infty \) is a solution of the type (III). \( \square \)

## 4 Estimates of solutions

**Lemma 4.1** Let \( B \in (L_0, 0) \) and let \( \{x(n)\}_{n=0}^\infty \) be a solution of problem (1.1), (3.2). If \( h > 0 \) is sufficiently small, then there exist constants \( r > 2, m \geq r \) and \( L_1 \in (0, L) \) such that

\[
egin{align*}
    x(1) &< x(2) < \cdots < x(r-1) < 0 \leq x(r) < \cdots < x(m) = L_1 \quad \text{if } m > r, \\
    x(1) &< x(2) < \cdots < x(r-1) < 0 < x(r) = L_1 \quad \text{if } m = r. 
\end{align*}
\]

Moreover

\[
\Delta x(j) < h\sqrt{2|B|M_0 + h^2M_0}, \quad j = 1, \ldots, m - 1, \tag{4.2}
\]

where \( M_0 = \max\{|f(x)|: x \in [L_0, L]\} \).

**Proof.** By Lemma 3.1 there exists \( r \in \mathbb{N}, r > 1 \) such that either (3.3) or (3.4) holds. In particular, we have \( x(1) < x(2) \). By (3.2) and (3.5), \( x(2) = B + h^2f(B) \leq B + h^2M_0 \). So, if we choose \( h \) such small that \( h^2M_0 < |B| \), we have \( x(1) < x(2) < 0 \). Consequently \( r > 2 \) holds, and inequalities in (3.3) are fulfilled. Multiplying (3.5) by \( \Delta x(n) + \Delta x(n-1) \), we obtain

\[
(\Delta x(n))^2 - (\Delta x(n-1))^2 = h^2 f(x(n))(x(n+1) - x(n-1)), \quad n \in \mathbb{N}. \tag{4.3}
\]

Summing (4.3) from 1 to \( r - 2 \), we have

\[
(\Delta x(r-2))^2 = h^2 \sum_{j=1}^{r-2} f(x(j))(x(j+1) - x(j-1)) < 2h^2|B|M_0,
\]

and

\[
\Delta x(r-2) < h\sqrt{2|B|M_0}. \tag{4.4}
\]

(i) Let \( x(r) = 0 \). By (3.5) we get \( \Delta x(r) = \Delta x(r-1) \leq \Delta x(r-2) + h^2M_0 \). Hence, (4.4) implies

\[
\Delta x(r-1) < h\sqrt{2|B|M_0 + h^2M_0}, \quad x(r+1) < h\sqrt{2|B|M_0 + h^2M_0}. \tag{4.5}
\]
(ii) Let \( x(r) > 0 \). By (3.5) we get \( \Delta x(r - 1) \leq \Delta x(r - 2) + h^2 M_0 \). Using (3.3) we get \( x(r) \leq x(r - 1) + \Delta x(r - 2) + h^2 M_0 < \Delta x(r - 2) + h^2 M_0 \). So, by (4.4), we can choose \( h > 0 \) such small that \( x(r) < L \) and \( \Delta x(r) = \Delta x(r - 1) + h^2 f(x(r)) < \Delta x(r - 1) \). Further, using (4.4), we obtain

\[
\Delta x(r - 1) < h \sqrt{2|B|M_0} + h^2 M_0, \quad x(r + 1) < 2 h \sqrt{2|B|M_0} + 2 h^2 M_0. \tag{4.6}
\]

Estimates (4.5) and (4.6) imply that we can find \( h > 0 \) such small that \( x(r + 1) < L \), as well. If \( x(r) \geq x(r + 1) \), we put \( m = r \).

Let \( x(r) < x(r + 1) \). If \( x(r + 1) \geq x(r + 2) \) or \( x(r + 2) \geq L \), we put \( m = r + 1 \).

Let \( x(r) < x(r + 1) < x(r + 2) < L \). If \( x(r + 2) \geq x(r + 3) \) or \( x(r + 3) \geq L \), we put \( m = r + 2 \). Otherwise we continue as before. Due to Theorem 3.7, after a finite number of steps, we get \( m > r + 2 \) fulfilling (4.1).

According to (3.7), the finite sequence \( \{\Delta x(j)\}_{j=1}^{m-1} \) is increasing. Similarly, by (1.3), \( f(x(r)) \leq 0 \) and \( f(x(j)) < 0 \) for \( j = r + 1, \ldots, m \), provided \( m \geq r + 1 \). Therefore, by (3.5), \( \Delta x(r - 1) \geq \Delta x(r) \). If \( m > r + 1 \), the finite sequence \( \{\Delta x(j)\}_{j=r}^{m-1} \) is decreasing. Consequently (4.5) and (4.6) give (4.2). \( \square \)

**Lemma 4.2** Choose an arbitrary \( c > 0 \). Let \( B_1, B_2 \in (L_0, 0) \) and let \( \{x(n)\}_{n=0}^{\infty} \) and \( \{y(n)\}_{n=0}^{\infty} \) be a solution of problem (1.1), (3.2) with \( B = B_1 \) and \( B = B_2 \), respectively.

Then

\[
|x(n) - y(n)| \leq |B_1 - B_2| e^{cK_0} \quad \text{for } n \in \mathbb{N}, \quad n \leq \frac{c}{h} + 1, \tag{4.7}
\]

where \( K_0 \) is the Lipschitz constant for \( f \) on \( [L_0, L] \).

**Proof.** By (3.5) we have \( \Delta x(k) = \Delta x(k - 1) + h^2 f(x(k)), \quad k \in \mathbb{N} \). Summing it from 1 to \( k \), we get by (3.2), \( \Delta x(k) = h^2 \sum_{j=1}^{k} f(x(j)), \quad k \in \mathbb{N} \). Summing it again from 1 to \( n - 1 \), we get

\[
x(n) = B_1 + h^2 \sum_{k=1}^{n-1} \sum_{j=1}^{k} f(x(j)), \quad n \in \mathbb{N},
\]

and similarly

\[
y(n) = B_2 + h^2 \sum_{k=1}^{n-1} \sum_{j=1}^{k} f(y(j)), \quad n \in \mathbb{N}.
\]

Therefore

\[
|x(n) - y(n)| \leq |B_1 - B_2| + h^2 \sum_{k=1}^{n-1} \sum_{j=1}^{k} |f(x(j)) - f(y(j))| + \sum_{j=1}^{n} |x(j) - y(j)|, \quad n \in \mathbb{N}.
\]

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By the discrete analogue of the Gronwall-Bellman inequality (see e.g. [8], Lemma 4.34), we get

$$|x(n) - y(n)| \leq |B_1 - B_2| e^{(n-1)^2 h^2 K_0}$$
for $n \in \mathbb{N}$.

So, (4.7) is proved. \qed

5 Existence of non-monotonous solutions

Definition 5.1 A solution of problem (1.1), (3.2) satisfying conditions (III) of Theorem 3.7 is called a non-monotonous solution.

Lemma 5.2 Let $\{x(n)\}_{n=0}^{\infty}$ be a non-monotonous solution. Then there exists $c \in \mathbb{N}, c \geq b$, such that $\{x(n)\}_{n=b}^{c+1}$ is decreasing and

$$x(c) > 0 > x(c+1) \quad \text{or} \quad x(c) = 0. \quad (5.1)$$

Proof. Consider $b$ of Theorem 3.7 (III). If $x(b+1) < 0$, we put $b = c$ and (3.13) yields $x(c) > 0 > x(c+1)$. Clearly $\{x(n)\}_{n=b}^{c+1}$ is decreasing. If $x(b+1) = 0$, then for $b+1 = c$ we have $x(b) > x(c) = 0$. Further, by (3.5) and (3.13), $\Delta x(c) = \Delta x(c-1) + h^2 f(x(c)) = \Delta x(c-1) < 0$. So, $x(c+1) < 0$ and $\{x(n)\}_{n=b}^{c+1}$ is decreasing. Let $x(b+1) > 0$. Then (3.5) and (3.13) yield $\Delta x(b+1) = \Delta x(b) + h^2 f(x(b+1)) < \Delta x(b) \leq 0$, and hence $x(b+2) < x(b+1)$. We see that $\{x(n)\}_{n=b}^{c+1}$ and $\{\Delta x(n)\}_{n=b}^{c}$ are decreasing as long as $x(c) \geq 0$. If $x(n) > 0$ for all $n > b$, then $\lim_{n \to \infty} \Delta x(n) < 0$ which gives $\lim_{n \to \infty} x(n) = -\infty$, a contradiction. Therefore a finite $c$ fulfilling (5.1) must exist. \qed

Theorem 5.3 Let $B \in (\bar{B}, 0)$. There exists $h_B > 0$ such that if $h \in (0, h_B]$, then the corresponding solution $\{x(n)\}_{n=0}^{\infty}$ of problem (1.1), (3.2) is non-monotonous.

Proof. Choose $B \in (\bar{B}, 0)$. Then, by (1.3) and (1.4), we can find $\varepsilon > 0$ such that

$$\int_{B}^{L} f(z) \, dz + \varepsilon < 0. \quad (5.2)$$

Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (3.2) with this $B$.

(i) Assume that $\{x(n)\}_{n=0}^{\infty}$ is an escape solution. Then there exists $b \in \mathbb{N}, b > 1$, such that $\{x(n)\}_{n=1}^{b}$ is increasing and (3.11) holds. Therefore, summing (4.3) from 1 to $b-1$ and multiplying by $\frac{1}{2}$, we get

$$0 < \frac{1}{2} \left( \frac{\Delta x(b-1)}{h} \right)^2 \leq \sum_{j=1}^{b-1} f(x(j)) \frac{x(j+1) - x(j-1)}{2}. \quad (5.3)$$
By (1.2), $f$ integrable on $[L_0, L]$ and hence there exists $\delta > 0$ such that if $(x(j+1) - x(j-1))/2 < \delta$, then
\[
\left| \sum_{j=1}^{b-1} f(x(j)) \frac{x(j+1) - x(j-1)}{2} - \int_B^L f(z) \, dz \right| < \varepsilon. \tag{5.4}
\]

Let $h_B \in (0, 1)$ be such that
\[
h_B < \frac{\delta}{\sqrt{2|B|M_0 + M_0}}, \tag{5.5}
\]
where $M_0 = \max\{|f(x)|: x \in [L_0, L]\}$. Now, choose $h \in (0, h_B]$. Then (4.2) implies $(x(j+1) - x(j-1))/2 < \delta$ and we obtain (5.4). Consequently, by (5.2), (5.3) and (5.4), we get
\[
0 < \int_B^L f(z) \, dz + \varepsilon < 0,
\]
a contradiction. So, $\{x(n)\}_{n=0}^{\infty}$ is not an escape solution provided $B \in (\bar{B}, 0)$ and $h \in (0, h_B]$.

(ii) Assume that $\{x(n)\}_{n=0}^{\infty}$ is a homoclinic solution, that is $\{x(n)\}_{n=1}^{\infty}$ is increasing, $x(n) < L$ for $n \in \mathbb{N}$, and (3.9) holds. Summing (4.3) from 1 to $n$ and multiplying by $1/2$, we get
\[
0 < \frac{1}{2} \left( \Delta x(n) \right)^2 = \sum_{j=1}^{n} f(x(j)) \frac{x(j+1) - x(j-1)}{2}, \quad n \in \mathbb{N}. \tag{5.6}
\]

Let $\delta$, $h_B$ and $h$ be as in part (i). By (3.9), we can choose $n_0 \in \mathbb{N}$ such that $|x(n+1) - L| < \delta$ for $n \geq n_0$. Then, as in part (i), we conclude that
\[
\left| \sum_{j=1}^{n} f(x(j)) \frac{x(j+1) - x(j-1)}{2} - \int_B^L f(z) \, dz \right| < \varepsilon, \quad n \geq n_0. \tag{5.7}
\]

By (5.2), (5.6) and (5.7) we get a contradiction as in (i). We have proved that $\{x(n)\}_{n=0}^{\infty}$ is not a homoclinic solution provided $B \in (\bar{B}, 0)$ and $h \in (0, h_B]$.

Therefore, by virtue of Theorem 3.7, $\{x(n)\}_{n=0}^{\infty}$ has to be a non-monotonous solution provided $B \in (\bar{B}, 0)$ and $h \in (0, h_B]$. \qed

6 Existence of escape solutions

**Theorem 6.1** Let $B_{es} \in (L_0, \bar{B})$. There exists $h_{B_{es}} > 0$ such that if $h \in (0, h_{B_{es}}]$, then the corresponding solution $\{x(n)\}_{n=0}^{\infty}$ of problem (1.1), (3.2) with $B = B_{es}$ is an escape solution.
Proof. Choose \( B_{es} \in (L_0, B) \). Then, by (1.3) and (1.4), we can find \( \varepsilon > 0 \) and \( c_0 > 0 \) such that
\[
\int_{B_{es}}^L f(z) \, dz - \varepsilon = c_0^2.  \tag{6.1}
\]
Let \( \{x(n)\}_{n=0}^\infty \) be a solution of problem (1.1), (3.2) with \( B = B_{es} \).

(i) Assume that \( \{x(n)\}_{n=0}^\infty \) is a non-monotonous solution. Then there exists \( b \in \mathbb{N} \) such that \( \{x(n)\}_{n=1}^b \) is increasing and (3.13) holds. As in the proof of Theorem 5.3 we can find \( \delta > 0 \) such that (5.4) holds with \( B = B_{es} \). Choose \( h_B \in (0, 1) \) such that (5.5) is valid. Assume that \( h \in (0, h_B] \). We derive (5.3) as in the proof of Theorem 5.3. Consequently we get
\[
\frac{\Delta x(b - 1)}{h\sqrt{2}} = \sqrt{\frac{b-1}{\sum_{j=1}^{b-1} f(x(j)) x(j+1) - x(j)} - \frac{2}{2}} > \sqrt{\int_{B_{es}}^L f(z) \, dz - \varepsilon} = c_0 > 0. \tag{6.2}
\]

Further, by (1.3) and (3.13) it holds \( f(x(b)) < 0 \), \( \Delta x(b-1) > 0 \) and \( \Delta x(b) \leq 0 \). Therefore (3.5) leads to \( |\Delta x(b)| + \Delta x(b-1) = h^2|f(x(b))| \) and
\[
\frac{\Delta x(b - 1)}{h} \leq hM_0, \quad M_0 = \max\{|f(x)|: x \in [L_0, L]\}.
\]
Choose \( h_{B_{es}} \in (0, h_B] \) such that \( h_{B_{es}}M_0 < c_0 \). Then for each \( h \in (0, h_{B_{es}}] \) we get \( \Delta x(b - 1)/h < c_0 \), contrary to (6.2). So, \( \{x(n)\}_{n=0}^\infty \) is not a non-monotonous solution provided \( B_{es} \in (L_0, B) \) and \( h \in (0, h_{B_{es}}] \).

(ii) Assume that \( \{x(n)\}_{n=0}^\infty \) is a homoclinic solution. We choose \( h_B \in (0, 1) \) and \( n_0 \in \mathbb{N} \) as in the proof of Theorem 5.3 part (ii) and arguing similarly we get (5.6) and (5.7) with \( B = B_{es} \). Using (6.1) we obtain
\[
\frac{\Delta x(n)}{h\sqrt{2}} = \sqrt{\frac{n}{\sum_{j=1}^{n} f(x(j)) x(j+1) - x(j)} - \frac{2}{2}} > \sqrt{\int_{B_{es}}^L f(z) \, dz - \varepsilon} = c_0 > 0 \quad \text{for } n \geq n_0. \tag{6.3}
\]

By (3.9), for any fixed \( h \in (0, h_B] \), we have
\[
\lim_{n \to \infty} \frac{\Delta x(n)}{h} = 0,
\]
contrary to (6.3). Put \( h_{B_{es}} = h_B \). Then \( \{x(n)\}_{n=0}^\infty \) cannot be a homoclinic solution provided \( B_{es} \in (L_0, B) \) and \( h \in (0, h_{B_{es}}] \).

Therefore, by virtue of Theorem 3.7, \( \{x(n)\}_{n=0}^\infty \) has to be an escape solution provided \( B_{es} \in (L_0, B) \) and \( h \in (0, h_{B_{es}}] \). \( \square \)
7 Existence of homoclinic solutions

**Theorem 7.1** Let \( f \) fulfil (1.2)–(1.4). There exists \( h_0 > 0 \) such that for each \( h^* \in (0, h_0] \) there exists \( B^* \in (L_0, 0) \) such that the corresponding solution \( \{ x(n) \}_{n=0}^{\infty} \) of problem (1.1), (3.2) with \( B = B^* \) is a homoclinic solution.

**Proof.** First, assume that, in addition, \( f \) fulfils (3.10). By Theorems 5.3 and 6.1 there exists \( h_0 \in (0, 1) \) such that if we choose an arbitrary \( h \in (0, h_0) \), then it holds:

(a) Let \( B_{\text{non}} \in (\bar{B}, 0) \). Then the corresponding solution \( \{ x_{\text{non}}(n) \}_{n=0}^{\infty} \) of problem (1.1), (3.2) with \( B = B_{\text{non}} \) is a non-monotonous solution.

(b) Let \( B_{\text{es}} \in (L_0, \bar{B}) \). Then the corresponding solution \( \{ x_{\text{es}}(n) \}_{n=0}^{\infty} \) of problem (1.1), (3.2) with \( B = B_{\text{es}} \) is an escape solution.

Choose \( h \in (0, h_0] \), \( B_{\text{non}} \in (\bar{B}, 0) \) and the solution \( \{ x_{\text{non}}(n) \}_{n=0}^{\infty} \). By Lemma 5.2 there exists \( c \in \mathbb{N} \) satisfying (5.1) for \( x = x_{\text{non}} \). That is \( x_{\text{non}}(c+1) < 0 \). Let \( B \in (L_0, B_{\text{non}}) \) and \( \{ x(n) \}_{n=0}^{\infty} \) be the corresponding solution of problem (1.1), (3.2). By Lemma 4.2,

\[
|x_{\text{non}}(n) - x(n)| \leq |B_{\text{non}} - B|e^{cK_0}, \quad n \in \mathbb{N}, \quad n \leq c + 1.
\]  

(7.1)

Since \( h < 1 \), (7.1) yields for \( n = c + 1 \)

\[
|x_{\text{non}}(c+1) - x(c+1)| \leq |B_{\text{non}} - B|e^{cK_0}.
\]

Therefore we can find \( \delta > 0 \) such small that if \( B \in (B_{\text{non}} - \delta, B_{\text{non}}] \), then \( x(c+1) < 0 \). Consequently \( \{ x(n) \}_{n=0}^{\infty} \) is a non-monotonous solution. According to (b) there exists the minimal number \( B^* \in (L_0, B_{\text{non}}] \) such that for \( B \in (B^*, B_{\text{non}}] \) the corresponding solution \( \{ x(n) \}_{n=0}^{\infty} \) is a non-monotonous.

Let \( \{ x^*(n) \}_{n=0}^{\infty} \) be a solution of problem (1.1), (3.2) with \( B = B^* \). Assume that \( \{ x^*(n) \}_{n=0}^{\infty} \) is non-monotonous. Then using the same arguments as above we can find \( \delta > 0 \) such small that for \( B \in (B^* - \delta, B^*] \) the corresponding solution \( \{ x(n) \}_{n=0}^{\infty} \) is also non-monotonous. This contradicts the minimality of \( B^* \).

Assume that \( \{ x^*(n) \}_{n=0}^{\infty} \) is an escape solution. By Lemma 3.5 there exists \( b \in \mathbb{N} \) satisfying (3.11) for \( x = x^* \). That is \( x^*(b+1) > L \). Consider a solution \( \{ x(n) \}_{n=0}^{\infty} \) of problem (1.1), (3.2) for \( B \in (B^*, 0) \). We can use Lemma 4.2 again and get

\[
|x^*(n) - x(n)| \leq |B^* - B|e^{bK_0}, \quad n \in \mathbb{N}, \quad n \leq b + 1.
\]  

(7.2)

Since \( h < 1 \), (7.2) yields for \( n = b + 1 \)

\[
|x^*(b+1) - x(b+1)| \leq |B^* - B|e^{bK_0}.
\]

Therefore we can find \( \delta > 0 \) such small that if \( B \in [B^*, B^* + \delta) \) then \( x(b+1) > L \). This yields that \( \{ x(n) \}_{n=0}^{\infty} \) is an escape solution, contrary to the definition of \( B^* \).
We have proved that \( \{x^*(n)\}_{n=0}^{\infty} \) is a homoclinic solution of equation (1.1). Since \( L_0 < x^*(n) < L \) for \( n \in \mathbb{N} \), we can omit the assumption (3.10). □

**Remark 7.2** The proof of Theorem 7.1 implies that the homoclinic point \( x^* = (B^*, B^*)^T \) of equation (1.1) can be found in the following way. Choose \( h \in (0, h_0] \). We have two subsets \( M_{\text{non}} \) and \( M_{\text{es}} \) of the interval \((L_0, 0)\). \( M_{\text{non}} \) consists of all \( B \) such that the corresponding solutions \( \{x(n)\}_{n=0}^{\infty} \) of problem (1.1), (3.2) are non-monotonous. \( M_{\text{non}} \) is non-empty by Theorem 5.3 and open by Lemma 4.2. \( M_{\text{es}} \) consists of all \( B \) such that the corresponding solutions \( \{x(n)\}_{n=0}^{\infty} \) of problem (1.1), (3.2) are escape solutions. \( M_{\text{es}} \) is non-empty by Theorem 6.1 and open by Lemma 4.2. Each \( B^* \) lying on the common boundary of \( M_{\text{non}} \) and \( M_{\text{es}} \) forms the homoclinic point \( x^* = (B^*, B^*)^T \) satisfying (1.7).

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**References**


