Strictly increasing solutions of non-autonomous difference equations arising in hydrodynamics

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Abstract. The paper provides conditions sufficient for the existence of strictly increasing solutions of the second-order non-autonomous difference equation

$$x(n+1) = x(n) + \left(\frac{n}{n+1}\right)^2 \Big(x(n) - x(n-1) + h^2 f(x(n))\Big), \quad n \in \mathbb{N},$$

where $h > 0$ is a parameter and f is Lipschitz continuous and has three real zeros $L_0 < 0 < L$.

In particular we prove that for each sufficiently small $h > 0$ there exists a solution $\{x(n)\}_{n=0}^{\infty}$ such that $\{x(n)\}_{n=1}^{\infty}$ is increasing, $x(0) = x(1) \in (L_0, 0)$ and $\lim_{n\to\infty} x(n) > L$. The problem is motivated by some models arising in hydrodynamics.

Keywords. Non-autonomous second-order difference equation, strictly increasing solutions, discretization.

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1 Formulation of problem

We will investigate the following second-order non-autonomous difference equation

$$x(n+1) = x(n) + \left(\frac{n}{n+1}\right)^2 \Big(x(n) - x(n-1) + h^2 f(x(n))\Big), \quad n \in \mathbb{N}, \tag{1.1}$$

where f is supposed to fulfil

$$L_0 < 0 < L, \quad f \in \mathrm{Lip}_{\mathrm{loc}}\,[L_0, \infty), \quad f(L_0) = f(0) = f(L) = 0, \tag{1.2}$$

$$xf(x) < 0 \text{ for } x \in (L_0, L) \setminus \{0\}, \quad f(x) \geq 0 \text{ for } x \in (L, \infty), \tag{1.3}$$

$$\exists \bar{B} \in (L_0, 0) \text{ such that } \int_{\bar{B}}^{L} f(z)\,\mathrm{d}z = 0. \tag{1.4}$$

Let us note that $f \in \mathrm{Lip_{loc}}\,[L_0, \infty)$ means that for each $[L_0, A] \subset [L_0, \infty)$ there exists $K_A > 0$ such that $|f(x) - f(y)| \le K_A|x - y|$ for all $x, y \in [L_0, A]$. A simple example of a function f satisfying (1.2)–(1.4) is $f(x) = c(x - L_0)x(x - L)$, where c is a positive constant.

A sequence $\{x(n)\}_{n=0}^{\infty}$ which satisfies (1.1) is called a solution of equation (1.1). For each values $B, B_1 \in [L_0, \infty)$ there exists a unique solution $\{x(n)\}_{n=0}^{\infty}$ of equation (1.1) satisfying the initial conditions

$$x(0) = B, \quad x(1) = B_1. \tag{1.5}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called a solution of problem (1.1), (1.5).

In [17] we have shown that equation (1.1) is a discretization of differential equations which generalize some models arising in hydrodynamics or in the nonlinear field theory, see [5], [6], [8], [12]. Increasing solutions of (1.1), (1.5) with $B = B_1 \in (L_0, 0)$ have a fundamental role in these models. Therefore, in [17], we have described the set of all solutions of problem (1.1), (1.6), where

$$x(0) = B, \quad x(1) = B, \quad B \in (L_0, 0). \tag{1.6}$$

In this paper, using [17], we will prove that for each sufficiently small $h > 0$ there exists at least one $B \in (L_0, 0)$ such that the corresponding solution of problem (1.1), (1.6) fulfils

$$x(0) = x(1), \quad \lim_{n\to\infty} x(n) > L, \quad \{x(n)\}_{n=1}^{\infty} \text{ is increasing.} \tag{1.7}$$

Note that an autonomous case of (1.1) was studied in [16]. We would like to point out that recently there has been a huge interest in studying the existence of monotonous and nontrivial solutions of nonlinear difference equations. For papers during last three years see for example [1], [2], [4], [9]–[11], [13]–[15], [19], [20]–[24]. A lot of other interesting references can be found therein.

2 Four types of solutions

Here we present some results of [17] which we need in next sections. In particular, we will use the following definitions and lemmas.

Definition 2.1 Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (1.6) such that

$$\{x(n)\}_{n=1}^{\infty} \text{ is increasing}, \quad \lim_{n\to\infty} x(n) = 0. \tag{2.1}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *a damped solution.*

Definition 2.2 Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (1.6) which fulfils

$$\{x(n)\}_{n=1}^{\infty} \text{ is increasing}, \quad \lim_{n\to\infty} x(n) = L. \tag{2.2}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *a homoclinic solution.*

Definition 2.3 Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (1.6). Assume that there exists $b \in \mathbb{N}$, such that $\{x(n)\}_{n=1}^{b+1}$ is increasing and

$$x(b) \le L < x(b+1). \tag{2.3}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *an escape solution.*

Definition 2.4 Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (1.6). Assume that there exists $b \in \mathbb{N}$, $b > 1$, such that $\{x(n)\}_{n=1}^{b}$ is increasing and

$$0 < x(b) < L, \quad x(b+1) \le x(b). \tag{2.4}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *a non-monotonous solution.*

Lemma 2.5 [17] (On four types of solutions)
Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (1.6). *Then $\{x(n)\}_{n=0}^{\infty}$ is just one of the following four types:*

(I) $\{x(n)\}_{n=0}^{\infty}$ *is an escape solution;*

(II) $\{x(n)\}_{n=0}^{\infty}$ *is a homoclinic solution;*

(III) $\{x(n)\}_{n=0}^{\infty}$ *is a damped solution;*

(IV) $\{x(n)\}_{n=0}^{\infty}$ *is a non-monotonous solution.*

Lemma 2.6 [17] (Estimates of solutions)
Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (1.6). *Then there exists a maximal $b \in \mathbb{N} \cup \{\infty\}$ satisfying*

$$\begin{aligned} x(n) \in [B, L) \quad &\textit{for } n = 1, \dots, b, \quad \textit{if } b \in \mathbb{N}, \\ x(n) \in [B, L) \quad &\textit{for } n \in \mathbb{N}, \quad \textit{if } b = \infty. \end{aligned} \tag{2.5}$$

Further, if $b > 1$, then moreover

$$\{x(n)\}_{n=1}^{b} \quad \textit{is increasing}, \tag{2.6}$$

$$\Delta x(n) < h\sqrt{(L - 2L_0)M_0} + h^2 M_0 \tag{2.7}$$

for $n = 1, \dots, b-1$ if $b \in \mathbb{N}$, and for $n \in \mathbb{N}$ if $b = \infty$, where

$$M_0 = \max\{|f(x)|\colon x \in [L_0, L]\}. \tag{2.8}$$

In [17] we have proved that the set consisting of damped and non-monotonous solutions of problem (1.1), (1.6) is nonempty for each sufficiently small $h > 0$. This is contained in the next lemma.

Lemma 2.7 [17] (On the existence of non-monotonous or damped solutions) *Let $B \in (\bar{B}, 0)$, where $\bar{B}$ is defined by* (1.4). *There exists $h_B > 0$ such that if $h \in (0, h_B]$, then the corresponding solution $\{x(n)\}_{n=0}^{\infty}$ of problem* (1.1), (1.6) *is non-monotonous or damped.*

In Section 4 of this paper we prove that also the set of escape solutions of problem (1.1), (1.6) is nonempty for each sufficiently small $h > 0$. Note that in our next paper [18] we prove this assertion for the set of homoclinic solutions.

3 Properties of solutions

Now, we provide other properties of solutions important in the investigation of escape solutions.

Lemma 3.1 *Let $\{x(n)\}_{n=0}^{\infty}$ be an escape solution of problem* (1.1), (1.6). *Then $\{x(n)\}_{n=1}^{\infty}$ is increasing.*

Proof. Due to (1.1), $\{x(n)\}_{n=0}^{\infty}$ fulfils

$$\Delta x(n) = \left(\frac{n}{n+1}\right)^2 \Big(\Delta x(n-1) + h^2 f(x(n))\Big), \quad n \in \mathbb{N}. \tag{3.1}$$

According to Definition 2.3 there exists $b \in \mathbb{N}$, such that $\{x(n)\}_{n=1}^{b+1}$ is increasing and (2.3) holds. By (1.3) we get $f(x(b+1)) \geq 0$. Consequently, by (3.1) and (2.3), $\Delta x(b+1) \geq \left(\frac{b+1}{b+2}\right)^2 \Delta x(b) > 0$ and $f(x(b+2)) \geq 0$. Similarly $\Delta x(b+j) \geq \left(\frac{b+j}{b+1+j}\right)^2 \Delta x(b+j-1)$ and

$$\Delta x(b+j) \geq \left(\frac{b+1}{b+1+j}\right)^2 \Delta x(b), \quad j \in \mathbb{N}. \tag{3.2}$$

This yields that $\{x(n)\}_{n=1}^{\infty}$ is increasing. □

Lemma 3.2 *Assume that $f(x) = 0$ for $x > L$. Choose an arbitrary $\varrho > 0$. Let $B_1, B_2 \in (L_0, 0)$ and let $\{x(n)\}_{n=0}^{\infty}$ and $\{y(n)\}_{n=0}^{\infty}$ be a solution of problem* (1.1), (1.6) *with $B = B_1$ and $B = B_2$, respectively. Let K_L be the Lipschitz constant for f on $[L_0, L]$. Then*

$$|x(n) - y(n)| \leq |B_1 - B_2| \mathrm{e}^{\varrho^2 K_L}, \tag{3.3}$$

$$\left|\frac{\Delta x(n) - \Delta y(n)}{h}\right| \leq |B_1 - B_2| \varrho K_L \, \mathrm{e}^{\varrho^2 K_L}, \tag{3.4}$$

where $n \in \mathbb{N}$, $n \leq \frac{\varrho}{h}$.

Proof. By (3.1) we have

$$(j+1)^2\Delta x(j) - j^2\Delta x(j-1) = h^2 j^2 f(x(j)), \quad j \in \mathbb{N}. \tag{3.5}$$

Summing it for $j = 1, \dots, k$, we get by (1.6),

$$\Delta x(k) = h^2 \frac{1}{(k+1)^2} \sum_{j=1}^{k} j^2 f(x(j)), \quad k \in \mathbb{N}. \tag{3.6}$$

Summing it again for $k = 1, \dots, n-1$, we get

$$x(n) = B_1 + h^2 \sum_{k=1}^{n-1} \frac{1}{(k+1)^2} \sum_{j=1}^{k} j^2 f(x(j)), \quad n \in \mathbb{N},$$

and similarly

$$y(n) = B_2 + h^2 \sum_{k=1}^{n-1} \frac{1}{(k+1)^2} \sum_{j=1}^{k} j^2 f(y(j)), \quad n \in \mathbb{N}.$$

From this and by using summation by parts we easily obtain

$$|x(n) - y(n)| \le |B_1 - B_2| + h^2 \sum_{k=1}^{n-1} \frac{1}{(k+1)^2} \sum_{j=1}^{k} j^2 |f(x(j)) - f(y(j))|$$

$$\le |B_1 - B_2| + (n-1)h^2 K_L \sum_{j=1}^{n-1} |x(j) - y(j)|, \quad n \in \mathbb{N}.$$

By the discrete analogue of the Gronwall-Bellman inequality (see e.g. [7], Lemma 4.34), we get

$$|x(n) - y(n)| \le |B_1 - B_2| e^{(n-1)^2 h^2 K_L} \quad \text{for } n \in \mathbb{N},$$

which yields (3.3).

By (3.6) and (3.3) we have for $n \in \mathbb{N}$, $n \le \frac{\varrho}{h}$,

$$\left| \frac{\Delta x(n) - \Delta y(n)}{h} \right| \le h \frac{1}{(n+1)^2} \sum_{j=1}^{n} j^2 |f(x(j)) - f(y(j))|$$

$$\le h K_L \sum_{j=1}^{n} |x(j) - y(j)| \le |B_1 - B_2| \varrho K_L \, e^{\varrho^2 K_L}.$$

□

4 Existence of escape solutions

Lemma 4.1 *Assume that* $C \in (L_0, \bar{B})$ *and* $\{B_k\}_{k=1}^{\infty} \subset (L_0, C)$. *Let* $\{x_k(n)\}_{n=0}^{\infty}$ *be a solution of problem* (1.1), (1.6) *with* $B = B_k$, $k \in \mathbb{N}$. *For* $k \in \mathbb{N}$ *choose a maximal* $b_k \in \mathbb{N} \cup \{\infty\}$ *such that* $x_k(n) \in [B_k, L)$ *for* $n = 1, \ldots, b_k$ *if* b_k *is finite, and for* $n \in \mathbb{N}$ *if* $b_k = \infty$, *and* $\{x_k(n)\}_{n=1}^{b_k}$ *is increasing if* $b_k > 1$. *Then there exists* $h^* > 0$ *such that for any* $h \in (0, h^*]$, *there exists a unique* $\gamma_k \in \mathbb{N}$, $\gamma_k < b_k$, *such that*

$$x_k(\gamma_k) \geq C, \quad x_k(\gamma_k - 1) < C. \tag{4.1}$$

Moreover, if the sequence $\{\gamma_k\}_{k=1}^{\infty}$ *is unbounded, then there exists* $\ell \in \mathbb{N}$ *such that the solution* $\{x_\ell(n)\}_{n=0}^{\infty}$ *of problem* (1.1), (1.6) *with* $B = B_\ell \in (L_0, \bar{B})$ *is an escape solution.*

Proof. Choose $h_0 > 0$ such that

$$h_0\sqrt{(L - 2L_0)M_0} + h_0^2 M_0 < |C|. \tag{4.2}$$

For $k \in \mathbb{N}$ denote by $\{x_k(n)\}_{n=0}^{\infty}$ a solution of problem (1.1), (1.6) with $B = B_k$. The existence of b_k is guaranteed by Lemma 2.6. By Lemma 2.5, $\{x_k(n)\}_{n=0}^{\infty}$ is just one of the types (I)–(IV), and if $h \in (0, h_0]$, then the monotonicity of $\{x_k(n)\}_{n=0}^{b_k}$ yields a unique $\gamma_k \in \mathbb{N}$, $\gamma_k < b_k$, satisfying (4.1).

For $h \in (0, h_0)$, consider the sequence $\{\gamma_k\}_{k=1}^{\infty}$ and assume that it is unbounded. Then we have

$$\lim_{k\to\infty} \gamma_k = \infty. \tag{4.3}$$

(Otherwise we take a subsequence.) Assume on the contrary that for any $k \in \mathbb{N}$, $\{x_k(n)\}_{n=0}^{\infty}$ is not an escape solution. Choose $k \in \mathbb{N}$. If $\{x_k(n)\}_{n=0}^{\infty}$ is damped, then by Definition 2.1, we have $b_k = \infty$ and

$$x_k(b_k) := \lim_{k\to\infty} x_k(n) = 0, \quad \Delta x_k(b_k) := \lim_{k\to\infty} \Delta x_k(n) = 0. \tag{4.4}$$

If $\{x_k(n)\}_{n=0}^{\infty}$ is homoclinic, then by Definition 2.2, we have $b_k = \infty$ and

$$x_k(b_k) := \lim_{k\to\infty} x_k(n) = L, \quad \Delta x_k(b_k) := \lim_{k\to\infty} \Delta x_k(n) = 0. \tag{4.5}$$

If $\{x_k(n)\}_{n=0}^{\infty}$ is non-monotonous, then by Definition 2.4, we have $b_k < \infty$ and

$$x_k(b_k) \in (0, L), \quad \Delta x_k(b_k) \leq 0. \tag{4.6}$$

To summarize if $\{x_k(n)\}_{n=0}^{\infty}$ is not an escape solution, then by (4.4), (4.5) and (4.6), we have

$$x_k(b_k) \in [0, L], \quad \Delta x_k(b_k) \leq 0. \tag{4.7}$$

Since $\Delta x_k(0) = 0$, there exists $\bar{\gamma}_k \in \mathbb{N}$ satisfying

$$\gamma_k \leq \bar{\gamma}_k < b_k, \quad \Delta x_k(\bar{\gamma}_k) = \max\{\Delta x_k(j) \colon \gamma_k \leq j \leq b_k - 1\}. \tag{4.8}$$

Consider (3.5) with $x = x_k$. By dividing it by j^2, multiplying such obtained equality by $x_k(j+1) - x_k(j-1)$ and summing in j from 1 to n we get

$$\begin{aligned} &(\Delta x_k(n))^2 - h^2 \sum_{j=1}^{n} f(x_k(j))(x_k(j+1) - x_k(j-1)) \\ &= -\sum_{j=1}^{n} \frac{2j+1}{j^2} \Delta x_k(j)(x_k(j+1) - x_k(j-1)), \quad n \in \mathbb{N}. \end{aligned} \tag{4.9}$$

Denote

$$E_k(n+1) = (\Delta x_k(n))^2 - h^2 \sum_{j=1}^{n} f(x_k(j))(x_k(j+1) - x_k(j-1)). \tag{4.10}$$

Then we get

$$E_k(n+1) = -\sum_{j=1}^{n} \frac{2j+1}{j^2} \Delta x_k(j)(x_k(j+1) - x_k(j-1)), \quad n \in \mathbb{N}. \tag{4.11}$$

Let us put $n = \gamma_k - 1$ and $n = b_k - 1$ to (4.11) and subtract. By (4.7) and (4.8) we get

$$\begin{aligned} E_k(\gamma_k) - E_k(b_k) &= \sum_{j=\gamma_k}^{b_k-1} \frac{2j+1}{j^2} \Delta x_k(j)(x_k(j+1) - x_k(j-1)) \\ &\le 2\frac{2\gamma_k+1}{\gamma_k^2} \Delta x_k(\bar{\gamma}_k)(L - L_0). \end{aligned} \tag{4.12}$$

Let us put $n = \gamma_k - 1$ and $n = b_k - 1$ to (4.10) and subtract. We get

$$\begin{aligned} E_k(\gamma_k) - E_k(b_k) &= (\Delta x_k(\gamma_k - 1))^2 - (\Delta x_k(b_k - 1))^2 \\ &+2h^2 \sum_{j=\gamma_k}^{b_k-1} f(x_k(j)) \frac{x_k(j+1) - x_k(j-1)}{2}. \end{aligned} \tag{4.13}$$

Choose $\varepsilon > 0$ and $h_1 > 0$ such that

$$\varepsilon < \frac{1}{2} \int_C^L f(z)\,\mathrm{d}z, \quad h_1 M_0 < \sqrt{\varepsilon}. \tag{4.14}$$

Let $b_k < \infty$. Then (4.6) holds. Since $\Delta x_k(b_k - 1) > 0$, $f(x_k(b_k)) < 0$ and $\Delta x_k(b_k) \le 0$, (3.1) yields

$$\left(\frac{b_k+1}{b_k}\right)^2 |\Delta x_k(b_k)| + \Delta x_k(b_k - 1) = h^2 |f(x_k(b_k))|,$$

and hence

$$0 < \Delta x_k(b_k - 1) \le -h^2 f(x_k(b_k)) < h^2 M_0 < h\sqrt{\varepsilon} \quad \text{for } h \in (0, h_1]. \tag{4.15}$$

Clearly, if $b_k = \infty$, then by (4.4) and (4.5), inequality (4.15) holds, as well. Having in mind (1.2) and (1.3), we deduce similarly as in the proof of Theorem 2.7 that there exists $\delta > 0$ such that if

$$\frac{x_k(j+1) - x_k(j-1)}{2} < \delta, \quad j = \gamma_k, \ldots, b_k - 1, \tag{4.16}$$

then

$$\sum_{j=\gamma_k}^{b_k-1} f(x_k(j)) \frac{x_k(j+1) - x_k(j-1)}{2} > \int_C^L f(z)\,\mathrm{d}z - \varepsilon. \tag{4.17}$$

Let $h_2 > 0$ be such that

$$h_2 \left(\sqrt{(L - 2L_0)M_0} + h_2 M_0 \right) < \delta. \tag{4.18}$$

If $h \in (0, h_2]$, then (2.7) implies (4.16) and hence (4.17) holds.

Now, let us put $h^* = \min\{h_0, h_1, h_2\}$ and choose $h \in (0, h^*]$. Then, (4.2), (4.14), (4.18), (4.13)–(4.17) yield

$$\begin{aligned} E_k(\gamma_k) - E_k(b_k) &> -h^2\varepsilon + 2h^2 \left(\int_C^L f(z)\,\mathrm{d}z - \varepsilon \right) \\ &= 2h^2 \left(\int_C^L f(z)\,\mathrm{d}z - \frac{3}{2}\varepsilon \right) > h^2\varepsilon > 0. \end{aligned} \tag{4.19}$$

Finally, (4.12) and (4.19) imply

$$0 < h^2\varepsilon < E_k(\gamma_k) - E_k(b_k) \le 2\frac{2\gamma_k + 1}{\gamma_k^2} \Delta x_k(\bar{\gamma}_k)(L - L_0),$$

and

$$\frac{h^2\varepsilon}{2(L - L_0)} \cdot \frac{\gamma_k^2}{2\gamma_k + 1} < \Delta x_k(\bar{\gamma}_k).$$

Letting $k \to \infty$, we obtain by (4.3), that $\lim_{k\to\infty} \Delta x_k(\bar{\gamma}_k) = \infty$, contrary to (4.16). Therefore an escape solution $\{x_\ell(n)\}_{n=0}^\infty$ of problem (1.1), (1.6) with $B = B_\ell \in (L_0, \bar{B})$ must exist. □

Now, we are in a position to prove the next main result.

Theorem 4.2 (On the existence of escape solutions)
There exists $h^* > 0$ such that for any $h \in (0, h^*]$ there exists an escape solution $\{x_\ell(n)\}_{n=0}^\infty$ of problem (1.1), (1.6) *for some $B = B_\ell \in (L_0, \bar{B})$.*

Proof. Choose $h > 0$ $C \in (L_0, \bar{B})$ and let K_L be the Lipschitz constant for f on $[L_0, L]$. Consider a sequence $\{B_k\}_{k=1}^\infty \subset (L_0, C)$ such that $\lim_{k\to\infty} B_k = L_0$. Then, for each $m \in \mathbb{N}$ there exists $k_m \in \mathbb{N}$ such that

$$|B_{k_m} - L_0| < e^{-m^2 K_L}(C - L_0). \tag{4.20}$$

Let $x_0(0) = x_0(n) = L_0$ for $n \in \mathbb{N}$. Then the sequence $\{x_0(n)\}_{n=0}^{\infty}$ is the unique solution of problem (1.1), (1.6) with $B = L_0$. Let $\{x_k(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (1.6) with $B = B_k$, $k \in \mathbb{N}$, and let $\{\gamma_k\}_{k=1}^{\infty}$ be the sequence of Lemma 4.1. Then it suffices to prove that $\{\gamma_k\}_{k=1}^{\infty}$ is unbounded. According to Lemma 3.2, for each $m \in \mathbb{N}$,

$$|x_{k_m}(n) - x_0(n)| \le |B_{k_m} - L_0| e^{m^2 K_L}, \quad n \le \frac{m}{h}. \tag{4.21}$$

Consequently, (4.20) and (4.21) give

$$|x_{k_m}(n) - x_0(n)| \le C - L_0, \quad n \le \frac{m}{h},$$

and hence

$$x_{k_m}(n) \le C, \quad n \le \frac{m}{h}.$$

Therefore

$$\gamma_{k_m}(n) \ge \frac{m}{h}, \quad m \in \mathbb{N},$$

which yields that $\{\gamma_k\}_{k=1}^{\infty}$ is unbounded. Hence the assertion follows from Lemma 4.1. □

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