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CONSTRUCTIVE METHOD FOR INVESTIGATION OF SOLUTIONS TO STATE-DEPENDENT IMPULSIVE BOUNDARY VALUE PROBLEMS

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Abstract. In this paper we investigate the nonlinear system of differential equations

$$u'(t) = f(t, u(t)),\ \text{a.e. } t \in [a, b] \subset \mathbb{R},$$

subject to the state-dependent impulse condition

$$u(t+) - u(t-) = \gamma(u(t-)),\ \text{where}\quad g(t, u(t-)) = 0,$$

and the linear boundary condition

$$Au(a) + Cu(b) = d.$$

Here f and γ are given continuous vector-functions, g is a continuous scalar function, A, C are constant matrices, and d is a constant vector. The impulse instants $t \in (a, b)$ are unknown and they depend on a solution u, because they are determined by the equation $g(t, u(t-)) = 0$. We discuss not only the existence of solutions of the problem but also present an approximate construction of solutions. Note that we have found no previous numerical results for state-dependent impulsive boundary value problems in the literature.

1. Introduction

We consider the nonlinear system of differential equations

$$u'(t) = f(t, u(t)),\ \text{a.e. } t \in [a, b] \subset \mathbb{R}, \tag{1.1}$$

with continuous $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$. Equation (1.1) is subject to the *state-dependent* impulse condition

$$u(t+) - u(t-) = \gamma(u(t-)), \quad \text{where} \quad g(t, u(t-)) = 0. \tag{1.2}$$

Here $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ and $g : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ are continuous, and the impulse instants $t \in (a, b)$ in (1.2) are unknown. These instants are called state-dependent because they depend on a solution u through the equation $g(t, u(t-)) = 0$. Impulsive problem (1.1), (1.2) is investigated together with the linear boundary condition

$$Au(a) + Cu(b) = d, \tag{1.3}$$

where d is a constant vector, and A, C are constant matrices which can be singular and which satisfy

$$\operatorname{rank}[A, C] = n.$$

For classical monographs about impulsive problems see [4, 17, 37]. Studies of real life problems with state-dependent impulsive effects can be found in [15, 19, 20, 38, 40]. Many papers are devoted to state-dependent impulsive initial value problems, where the existence, stability and other asymptotic properties of solutions have been studied, e.g. [1–3, 9–11, 14, 16]. We can also refer to state-dependent impulsive periodic problems, e.g. [5–7, 12, 18, 21]. In contrast to that there are only few papers dealing with other types of state-dependent impulsive boundary value problems, see [8,13,22–26]. Namely, most of the results in the literature devoted to boundary value problems concern fixed-times impulses. A reason for the lack of results for state-dependent impulsive boundary value problems lies in the fact that state-dependent impulses significantly change properties of boundary value problems, which is explained in more details

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in [25]. In addition, we have found no numerical results for state-dependent impulsive boundary value problems. This is our motivation for the investigation of problem (1.1)–(1.3).

Definition 1. A left-continuous vector-function $u : [a, b] \to \mathbb{R}^n$ is called a *solution* of problem (1.1)–(1.3) if there exist $p \in \mathbb{N}$ and $t_i \in (a, b)$, $i = 1, \dots, p$, such that:

- $a < t_1 < t_2 < \ldots < t_p < b$,
- the restrictions $u|_{[a,t_1]}$, $u|_{(t_1,t_2]}$, ..., $u|_{(t_p,b]}$ have continuous derivatives,
- u satisfies (1.1) for $t \in [a, b], t \neq t_i$, $i = 1, \dots, p$,
- u satisfies (1.2) for $t = t_i$, i.e. $u(t_i+) - u(t_i) = \gamma(u(t_i))$, $\quad g(t_i, u(t_i)) = 0$, $i = 1, \dots, p$,
- u fufils the boundary conditions (1.3).

The set

$$G = \{(t, x) \in [a, b] \times \mathbb{R}^n : g(t, x) = 0\} \tag{1.4}$$

is called a *barrier*.

Wee see that if u satisfies condition (1.2) for $t = t_i \in (a, b)$, then u has an intersection point $(t_i, u(t_i))$ with the barrier G, and in addition, u has a jump of the size $\gamma(u(t_i))$ at the point t_i.

We focus our attention to the case where $p = 1$, that is u has a unique intersection point with the barrier G, and then we use the technique suggested in [27], which makes possible to discuss the solvability of problem (1.1)–(1.3) as well as to find approximate solutions. This approach is based on a construction of two simple parametrized model problems (3.3), (3.4) and (3.5), (3.6). We give conditions which guarantee that if the parameters t_1, z, λ, η belong to some bounded sets (cf. Section 3), then solutions of these parametrized model problems can be obtained as limits of uniformly convergent sequences of successive approximations (3.8) and (3.17). Equations in the parametrized model problems contain functional perturbation terms which essentially depend on the parameters and which together with the original boundary conditions (1.3) and the barrier (1.4) generate a system of algebraic *determining* equations (4.2). Numerical values of the parameters should be found from (4.2) in the bounded sets mentioned above. A solution of problem (1.1)–(1.3) is then constructed (see (4.1)) by means of such solutions of problems (3.3), (3.4) and (3.5), (3.6) which have the values of parameters satisfying (4.2). Consequently, the infinite-dimensional problem (1.1)–(1.3) is reduced to the finite-dimensional algebraic system (4.2).

In practice, we investigate system (4.2), where explicitly determined successive approximations are written instead of their limits (cf. (5.1)). Then the solvability of (4.2) can be checked more easily and we get approximate solutions of problem (1.1)–(1.3) and error estimates using for example Maple 14. By our knowledge this is the first numerical-analytic method for this type of impulsive problems. This method can be applied on problems with linear as well as with nonlinear boundary conditions which has been demonstrated on problems without impulses in [27–30, 32–34]. In addition, we can work with barriers in the form $g(t, x) = 0$. Note, that the papers [22–26] are applicable only to problems with barriers in the form $t = g(x)$. Example in Section 6 shows that the method provides also multiplicity results.

2. Notation and symbols

In the sequel, for any vector $x = \mathrm{col}(x_1, ..., x_n) \in \mathbb{R}^n$ the obvious notation $|x| = \mathrm{col}(|x_1|, ..., |x_n|)$ is used and inequalities between vectors are understood component-wise. The same convention is adopted implicitly for operations 'max' and 'min'. The symbols 1_n and 0_n stand respectively for the unit and zero matrix of dimension n, and $r(K)$ denotes the maximal, in modulus, eigenvalue of a square matrix K.

Definition 2. For any non-negative vector $\varrho \in \mathbb{R}^n$ under a *component-wise ϱ-neighbourhood* of a point $z \in \mathbb{R}^n$ we understand

$$B(z, \varrho) := \{\xi \in \mathbb{R}^n : |\xi - z| \le \varrho\}\,.$$

Similarly, for a compact connected set $\Omega \subset \mathbb{R}^n$, we define its component-wise ϱ-neighbourhood by putting

$$B(\Omega, \varrho) := \underset{\xi \in \Omega}{\cup} B(\xi, \varrho).$$

Definition 3. For two compact connected sets $D_a \subset \mathbb{R}^n$ and $D_b \subset \mathbb{R}^n$, introduce the set

$$D_{a,b} := (1-\theta)z + \theta\eta,\ z \in D_a, \eta \in D_b, \theta \in [0,1], \tag{2.1}$$

and its component-wise ϱ-neighbourhood

$$D := B(D_{a,b}, \varrho). \tag{2.2}$$

For a compact set $D \subset \mathbb{R}^n$, a closed interval $[a,b] \subset \mathbb{R}$, a continuous function $f : [a,b] \times D \to \mathbb{R}^n$, and an $n \times n$ matrix K with non-negative entires, we write

$$f \in \mathrm{Lip}(K, D), \tag{2.3}$$

if the inequality

$$|f(t,u) - f(t,v)| \le K\,|u - v|$$

holds for all $u, v \in D$ and $t \in [a,b]$. In addition, we introduce the vector

$$\delta_{[a,b],D}(f) := \frac{1}{2}\left[\max_{(t,x)\in[a,b]\times D} f(t,x) - \min_{(t,x)\in[a,b]\times D} f(t,x)\right]. \tag{2.4}$$

We recall some subsidary statements which are needed below. Let us put $\alpha_0(t; a, b) = 1$ for $t \in [a,b]$, and for $m \in \mathbb{N}$, define

$$\alpha_m(t;a,b) = \left(1 - \frac{t-a}{b-a}\right)\int_a^t \alpha_{m-1}(s;a,b)\,\mathrm{d}s + \frac{t-a}{b-a}\int_t^b \alpha_{m-1}(s;a,b)\,\mathrm{d}s, \quad t \in [a,b]. \tag{2.5}$$

Clearly

$$\alpha_1(t;a,b) = 2\,(t-a)\left(1 - \frac{t-a}{b-a}\right),\ |\alpha_1(t;a,b)| \le \frac{b-a}{2}, \quad t \in [a,b]. \tag{2.6}$$

Lemma 4. ([29], Lemma 3.16). *Functions α_m from (2.5) are positive, continuous and fulfil the estimate*

$$\alpha_m(t;a,b) \le \frac{10}{9}\left(\frac{3(b-a)}{10}\right)^{m-1}\alpha_1(t;a,b), \quad t \in [a,b],\ m \in \mathbb{N}. \tag{2.7}$$

Lemma 5. ([29], Lemma 3.13). *Let $\tilde{f} : [a,b] \to \mathbb{R}^n$ be a continuous function. Then*

$$\left|\int_a^t \left(\tilde{f}(\tau) - \frac{1}{b-a}\int_a^b \tilde{f}(s)\,\mathrm{d}s\right)\mathrm{d}\tau\right| \le \frac{1}{2}\alpha_1(t;a,b)\left(\max_{s\in[a,b]}\tilde{f}(s) - \min_{s\in[a,b]}\tilde{f}(s)\right), \quad t \in [a,b]. \tag{2.8}$$

3. Parametrized model problems

Consider a parameter $t_1 \in (a,b)$, choose compact convex sets D_a, $D_{t_{1-}}$, $D_b \subset \mathbb{R}^n$, and define the set

$$D_{t_{1+}} := \left\{x + \gamma(x) :\ x \in D_{t_{1-}}\right\}.$$

Note, that the set $D_{t_{1+}}$ is obtained from $D_{t_{1-}}$ by a "shift" using the given vector of "jump" γ from (1.2). According to (2.1) and (2.2) we introduce the set

$$D_{a,t_{1-}} := (1-\theta)\,z + \theta\lambda, \quad z \in D_a, \lambda \in D_{t_{1-}},\ \theta \in [0,1],$$

and its component-wise ϱ^x-neighbourhood

$$D^x := B(D_{a,t_{1-}}, \varrho^x). \tag{3.1}$$

Similarly we introduce the set

$$D_{t_{1+},b} := (1-\theta)\,(\lambda + \gamma(\lambda)) + \theta\eta, \quad (\lambda + \gamma(\lambda)) \in D_{t_{1+}}, \eta \in D_b,\ \theta \in [0,1],$$

and its component-wise ϱ^y-neighbourhood

$$D^y := B(D_{t_1+,b}, \varrho^y). \tag{3.2}$$

Now, we consider the scalar parameter $t_1 \in (a, b)$ together with vector parameters $z \in D_a$, $\lambda \in D_{t_1-}$, $\eta \in D_b$, and instead of the impulsive boundary value problem (1.1)–(1.3) we will study the following two auxiliary parametrized boundary value problems on the intervals $[a, t_1]$ and $[t_1, b]$, respectively:

$$x'(t) = f(t, x(t)) + \frac{1}{t_1 - a}\left(\lambda - z - \int_a^{t_1} f(s, x(s))\,\mathrm{d}s\right), \tag{3.3}$$

$$x(a) = z, \quad x(t_1) = \lambda, \tag{3.4}$$

and

$$y'(t) = f(t, y(t)) + \frac{1}{b - t_1}\left(\eta - (\lambda + \gamma(\lambda)) - \int_{t_1}^{b} f(s, y(s))\,\mathrm{d}s\right), \tag{3.5}$$

$$y(t_1) = \lambda + \gamma(\lambda), \quad y(b) = \eta. \tag{3.6}$$

Definition 6. A vector function $x \in C^1[a, t_1]$ is called a *solution* of problem (3.3), (3.4), if x is a solution of the initial value problem

$$\text{Eq. (3.3) for } t \in [a, t_1], \quad x(a) = z,$$

and in addition x satisfies $x(t_1) = \lambda$. A vector function $y \in C^1[t_1, b]$ is called a *solution* of problem (3.5), (3.6), if y is a solution of the initial value problem

$$\text{Eq. (3.5) for } t \in [t_1, b], \quad y(t_1) = \lambda + \gamma(\lambda),$$

and in addition y satisfies $y(b) = \eta$.

I. Let us connect problem (3.3), (3.4) with the parametrized sequence of functions

$$x_0(t; t_1, z, \lambda) = \left(1 - \frac{t - a}{t_1 - a}\right) z + \frac{t - a}{t_1 - a}\lambda, \quad t \in [a, t_1], \tag{3.7}$$

$$x_m(t; t_1, z, \lambda) = z + \int_a^t f(s, x_{m-1}(s; t_1, z, \lambda))\,\mathrm{d}s \tag{3.8}$$

$$-\frac{t - a}{t_1 - a}\int_a^{t_1} f(s, x_{m-1}(s; t_1, z, \lambda))\,\mathrm{d}s + \frac{t - a}{t_1 - a}(\lambda - z), \quad t \in [a, t_1], \ m \in \mathbb{N}.$$

The following statement establishes the uniform convergence of sequence (3.8) to some parametrized limit function $x_\infty(t; t_1, z, \lambda)$.

Theorem 7. *Assume that:*
(i) *K_x and Q_x are matrices with non-negative elements such that*

$$r(Q_x) < 1, \quad Q_x = \frac{3(b - a)}{10} K_x. \tag{3.9}$$

(ii) *$t_1 \in (a, b)$, $z \in D_a$ and $\lambda \in D_{t_1-}$ are arbitrary fixed parameters.*
(iii) *There exists a non-negative vector ϱ^x such that $f \in \mathrm{Lip}(K_x, D^x)$, where D^x is from (3.1).*
(iv) *ϱ^x satisfies the inequality*

$$\varrho^x \geq \frac{b - a}{2}\delta_{[a,b],D^x}(f), \tag{3.10}$$

where $\delta_{[a,b],D^x}(f)$ is from (2.4) with D^x in place of D.

Then the following assertions are valid:

1. The functions x_m *in* (3.8) *are continuously differentiable on* $[a,t_1]$*, they have values in the domain* D^x *and satisfy the two-point separated boundary conditions* (3.4).

2. The sequence of functions x_m *in* (3.8) *converges uniformly on* $[a,t_1]$ *to the limit function* x_∞*:*

$$x_\infty(t;t_1,z,\lambda)=\lim_{m\to\infty}x_m(t;t_1,z,\lambda). \tag{3.11}$$

3. The limit function x_∞ *is a unique solution of problem* (3.3), (3.4).

4. The following error estimate holds:

$$\begin{aligned}&|x_\infty(t;t_1,z,\lambda)-x_m(t;t_1,z,\lambda)|\\ &\leqslant\frac{10}{9}\alpha_1(t;a,t_1)Q^m(1_n-Q)^{-1}\delta_{[a,b],D^x}(f),\quad t\in[a,t_1],\ m\in\mathbb{N},\end{aligned} \tag{3.12}$$

where $\alpha_1(t;a,t_1)$ *is from* (2.6) *with* t_1 *in place of* b.

Proof. We can argue similarly as in [27]. Assume that $t_1\in(a,b)$, $z\in D_a$ and $\lambda\in D_{t_1-}$ are fixed. It is easy to see from (3.7) that $x_0(t;t_1,z,\eta)$ belongs to D_{a,t_1-} for $t\in[a,t_1]$ as a convex combination of z and η. We use estimate (2.8) of Lemma 5 with t_1 in place of b. Then (3.8) for $m=0$ implies that

$$\begin{aligned}&|x_1(t;t_1,z,\lambda)-x_0(t;t_1,z,\lambda)|\\ &\le\frac{1}{2}\alpha_1(t;a,t_1)\left[\max_{s\in[a,t_1]}f(s,x_0(s;t_1,z,\lambda))-\min_{s\in[a,t_1]}f(s,x_0(s;t_1,z,\lambda))\right]\\ &\le\alpha_1(t;a,t_1)\delta_{[a,b],D^x}(f)\le\frac{b-a}{2}\delta_{[a,b],D^x}(f),\quad t\in[a,t_1],\end{aligned} \tag{3.13}$$

which means, according to (3.1) and (3.10), that $x_1(t;t_1,z,\lambda)$ belongs to D^x for $t\in[a,t_1]$. Using this and arguing by induction we can establish that

$$|x_m(t;t_1,z,\lambda)-x_0(t;t_1,z,\lambda)|\le\frac{b-a}{2}\delta_{[a,b],D^x}(f),\quad t\in[a,t_1],\ m\in\mathbb{N},$$

which yields that all functions in (3.8) are contained in the domain D^x.

Introduce the notation

$$r_{m+1}(t;t_1,z,\lambda)=|x_{m+1}(t;t_1,z,\lambda)-x_m(t;t_1,z,\lambda)|,\quad t\in[a,t_1],\ m\in\mathbb{N}. \tag{3.14}$$

Then, by (3.8),

$$\begin{aligned}r_2(t;t_1,z,\lambda)=\Bigg|&\left(1-\frac{t-a}{t_1-a}\right)\int_a^t(f(s,x_1(s;t_1,z,\lambda))-f(s,x_0(s;t_1,z,\lambda)))\,\mathrm{d}s\\ &-\frac{t-a}{t_1-a}\int_t^{t_1}(f(s,x_1(s;t_1,z,\lambda))-f(s,x_0(s;t_1,z,\lambda)))\,\mathrm{d}s\Bigg|,\quad t\in[a,t_1].\end{aligned}$$

Using the recurrence relation (2.5) with t_1 in place of b, the assumption $f\in\mathrm{Lip}(K_x,D^x)$ and inequalities (2.7), (3.13), we get

$$\begin{aligned}r_2(t;t_1,z,\lambda)&\le K_x\left[\left(1-\frac{t-a}{t_1-a}\right)\int_a^t\alpha_1(s;a,t_1)\,\mathrm{d}s+\frac{t-a}{t_1-a}\int_t^{t_1}\alpha_1(s;a,t_1)\,\mathrm{d}s\right]\delta_{[a,b],D^x}(f)\\ &\le K_x\alpha_2(t;a,t_1)\delta_{[a,b],D^x}(f)\le\frac{10}{9}Q_x\alpha_1(t;a,t_1)\delta_{[a,b],D^x}(f),\quad t\in[a,t_1].\end{aligned}$$

By induction we can establish that

$$\begin{aligned}r_{m+1}(t;t_1,z,\lambda)&\le K_x^m\alpha_{m+1}(t;a,t_1)\delta_{[a,b],D^x}(f)\\ &\le\frac{10}{9}Q_x^m\alpha_1(t;a,t_1)\delta_{[a,b],D^x}(f),\quad t\in[a,t_1],\ m\in\mathbb{N}.\end{aligned}$$

Therefore, in view of (3.14), choosing $j\in\mathbb{N}$, we have

$$|x_{m+j}(t;t_1,z,\lambda)-x_m(t;t_1,z,\lambda)| \tag{3.15}$$

$$\leq \sum_{i=1}^{j} r_{m+i}(t;t_1,z,\lambda) \leq \frac{10}{9}\alpha_1(t;a,t_1)\sum_{i=1}^{j} Q_x^{m+i-1}\delta_{[a,b],D^x}(f)$$
$$= \frac{10}{9}\alpha_1(t;a,t_1)Q_x^m\sum_{i=0}^{j-1} Q_x^i\delta_{[a,b],D^x}(f), \quad t\in[a,t_1],\ m\in\mathbb{N}.$$

Since, due to (3.9), the maximum eigenvalue of the matrix Q_x does not exceed the unity, we have

$$\sum_{i=0}^{j-1} Q_x^i \leq (1_n - Q_x)^{-1}, \ \lim_{m\to\infty} Q_x^m = 0_n.$$

Therefore, we conclude from (3.15) that the sequence $\{x_m(t;t_1,z,\lambda)\}_{m=0}^{\infty}$ from (3.8) uniformly converges in the domain $(t,t_1,z,\lambda)\in[a,t_1]\times[a,b]\times D_a\times D_{t_1-}$ to the limit function $x_\infty(t;t_1,z,\lambda)$. Since all functions of sequence (3.8) satisfy the boundary conditions (3.4) for all values of the introduced parameters, the limit function $x_\infty(t;t_1,z,\lambda)$ also satisfies these conditions. Passing to the limit as $m\to\infty$ in equality (3.8) we see that the limit function x_∞ satisfies on $[a,t_1]$ the integral equation

$$x(t) = z + \int_a^t f(s,x(s))\,\mathrm{d}s - \frac{t-a}{t_1-a}\int_a^{t_1} f(s,x(s))\,\mathrm{d}s + \frac{t-a}{t_1-a}(\lambda - z).$$

Consequently, x_∞ is a solution of problem (3.3), (3.4). Since x_∞ is a solution on $[a,t_1]$ of the initial value problem (3.3), $x(a)=z$, the uniqueness follows from $f\in\mathrm{Lip}(K_x,D^x)$. Passing to the limit as $j\to\infty$ in (3.15) we get estimation (3.12). □

II. Let us connect problem (3.5), (3.6) with the parametrized sequence of functions

$$y_0(t;t_1,\lambda,\eta) = \left(1-\frac{t-t_1}{b-t_1}\right)(\lambda+\gamma(\lambda)) + \frac{t-t_1}{b-t_1}\eta, \quad t\in[t_1,b], \tag{3.16}$$

$$y_m(t;t_1,\lambda,\eta) = (\lambda+\gamma(\lambda)) + \int_{t_1}^t f(s,y_{m-1}(s;t_1,\lambda,\eta))\,\mathrm{d}s \tag{3.17}$$
$$-\frac{t-t_1}{b-t_1}\int_{t_1}^b f(s,y_{m-1}(s;t_1,\lambda,\eta))\,\mathrm{d}s + \frac{t-t_1}{b-t_1}(\eta-(\lambda+\gamma(\lambda))), \quad t\in[t_1,b],\ m\in\mathbb{N}.$$

By analogy with Theorem 7 we establish the uniform convergence of sequence (3.17) to some parametrized limit function $y_\infty(t;t_1,\lambda,\eta)$.

Theorem 8. *Assume that:*
(i) K_y *and* Q_y *are matrices with non-negative elements such that*

$$r(Q_y) < 1, \quad Q_y = \frac{3(b-a)}{10}K_y.$$

(ii) $t_1\in(a,b)$, $\lambda\in D_{t_1-}$ *and* $\eta\in D_b$ *are arbitrary fixed parameters.*
(iii) *There exists a non-negative vector* ϱ^y *such that* $f\in\mathrm{Lip}(K_y,D^y)$, *where* D^y *is from* (3.2).
(iv) ϱ^y *satisfies the inequality*

$$\varrho^y \geq \frac{b-a}{2}\delta_{[a,b],D^y}(f),$$

where $\delta_{[a,b],D^y}(f)$ *is from* (2.4) *with* D^y *in place of* D.

Then the following assertions are valid:

1. The functions y_m *in* (3.17) *are continuously differentiable on* $[t_1,b]$, *they have values in the domain* D^y *and satisfy the two-point separated boundary conditions* (3.6).

2. The sequence of functions y_m in (3.17) *converges uniformly on* $[t_1, b]$ *to the limit function* y_∞

$$y_\infty(t; t_1, \lambda, \eta) = \lim_{m\to\infty} y_m(t; t_1, \lambda, \eta). \tag{3.18}$$

3. The limit function y_∞ is a unique solution of problem (3.5), (3.6).

4. The following error estimate holds:

$$|y_\infty(t; t_1, \lambda, \eta) - y_m(t; t_1, \lambda, \eta)|$$
$$\leqslant \frac{10}{9}\alpha_1(t; t_1, b) Q_y^m (1_n - Q_y)^{-1} \delta_{[a,b],D^y}(f), \quad t \in [t_1, b], \; m \in \mathbb{N}, \tag{3.19}$$

where $\alpha_1(t; t_1, b)$ is from (2.6) *with t_1 in place of a.*

Proof. We argue similarly as in the proof of Theorem 7. In particular, we show that the limit function y_∞ satisfies on $[t_1, b]$ the integral equation

$$y(t) = (\lambda + \gamma(\lambda)) + \int_{t_1}^{t} f(s, y(s))ds - \frac{t - t_1}{b - t_1}\int_{t_1}^{b} f(s, y(s))ds + \frac{t - t_1}{b - t_1}(\eta - (\lambda + \gamma(\lambda))).$$

□

4. System of algebraic determining equations

Let us find a relationship between the limit functions $x_\infty(t; t_1, z, \lambda)$, $y_\infty(t; t_1, z, \lambda)$ from Theorem 7 and Theorem 8 and a solution of the original impulsive boundary value problem (1.1)–(1.3).

Theorem 9. *Let the assumptions of Theorem 7 and Theorem 8 be fulfilled and let $x_\infty(t; t_1, z, \lambda)$ and $y_\infty(t; t_1, z, \lambda)$ be the limit functions given by* (3.11) *and* (3.18)*, respectively. Then the function*

$$u(t) = \begin{cases} x_\infty(t; t_1, z, \lambda) & \text{if } t \in [a, t_1], \\ y_\infty(t; t_1, \lambda, \eta) & \text{if } t \in (t_1, b], \end{cases} \tag{4.1}$$

is a solution of problem (1.1)–(1.3) *with exactly one impulse point t_1, if the parameters t_1, z, λ, η satisfy the system of algebraic "determining" equations*

$$\begin{cases} \lambda - z - \int_a^{t_1} f(s, x_\infty(s; t_1, z, \lambda))\,\mathrm{d}s = 0, \\ (\eta - (\lambda + \gamma(\lambda))) - \int_{t_1}^{b} f(s, y_\infty(s; t_1, \lambda, \eta))\,\mathrm{d}s = 0, \\ Az + C\eta = d, \\ g(t_1, \lambda) = 0, \end{cases} \tag{4.2}$$

and in addition

$$g(t, y_\infty(t; t_1, \lambda, \eta)) \neq 0, \quad t \in (t_1, b]. \tag{4.3}$$

Proof. Let u be given by (4.1). Assume that the parameters $t_1 \in (a, b)$, $z \in D_a$, $\lambda \in D_{t_1-}$ and $\eta \in D_b$ satisfy (4.2). Then, according to Theorems 7 and 8, equations (3.3), (3.5) and by the first two equations in (4.2), we get that the restrictions $u|_{[a,t_1]}$, $u|_{(t_1,b]}$ have continuous derivatives and

$$u'(t) = f(t, u(t)), \quad t \in [a, b], \; t \neq t_1.$$

Further, (3.4) and (3.6) yield

$$u(a) = z, \quad u(t_1-) = u(t_1) = \lambda, \quad u(t_1+) = \lambda + \gamma(\lambda), \quad u(b) = \eta.$$

We see that $u(t_1+) - u(t_1-) = \gamma(u(t_1-))$, where $g(t_1, u(t_1-)) = 0$ due to the fourth equation in (4.2). Hence, u satisfies the impulse conditions (1.2) for $t = t_1$. Further, by the third equation in (4.2), u fulfils the boundary conditions (1.3). Finally, (4.3) implies that u has exactly one intersection point $(t_1, u(t_1))$ with barrier (1.4). We have proved that u is a solution of (1.1)–(1.3) with $p = 1$ in Definition 1. □

Note that system (4.2) consists of $3n+1$ scalar equations for $3n+1$ scalar unknown parameters t_1, $z_1, \dots, z_n$, $\lambda_1, \dots, \lambda_n$, $\eta_1, \dots, \eta_n$.

Now, let the assumptions of Theorem 7 and Theorem 8 be fulfilled and let a vector-function u be a solution of problem (1.1)–(1.3) with $p = 1$ in Definition 1. Then there exists a unique point $t_1 \in (a, b)$ such that the restrictions $u|_{[a,t_1]}$, $u|_{(t_1,b]}$ have continuous derivatives and u has a unique jump at t_1

$$u(t_1+) - u(t_1) = \gamma(u(t_1)).$$

Therefore u can be written as

$$u(t) = \begin{cases} x(t) & \text{if } t \in [a, t_1], \\ y(t) & \text{if } t \in (t_1, b], \end{cases}$$

where x is a solution of the initial value problem

$$x'(t) = f(t, x(t)) \text{ for } t \in [a, t_1], \quad x(a) = u(a),$$

and y is a solution of the initial value problem

$$y'(t) = f(t, y(t)) \text{ for } t \in [t_1, b], \quad y(t_1) = u(t_1) + \gamma(u(t_1)).$$

If u satisfies

$$u(a) \in D_a, \quad u(t_1) \in D_{t_{1-}}, \quad u(b) \in D_b, \tag{4.4}$$

we get from Theorems 7, 8 and 9 that

$$x(t) = x_\infty(t; t_1, z, \lambda),\ t \in [a, t_1], \qquad y(t) = y_\infty(t; t_1, \lambda, \eta),\ t \in [t_1, b],$$

provided the parameters $t_1 \in (a, b)$, $z \in D_a$, $\lambda \in D_{t_{1-}}$ and $\eta \in D_b$ satisfy (4.2). Therefore the following assertion is valid.

Theorem 10. *Under the assumptions of Theorems 7 and 8 system* (4.2) *determines all possible solutions u of problem* (1.1)–(1.3) *having exactly one impulse point and satisfying* (4.4).

Remark 11. The simpliest way for choosing parameter sets is to take a compact convex set $D_a \subset \mathbb{R}^n$ and then put

$$D_b = \{x + \gamma(x) : x \in D_a\}, \quad D_{t_{1-}} = D_a, \quad D_{t_{1+}} = D_b. \tag{4.5}$$

Then the convex linear combination $D_{a,t_{1-}}$ of vectors $z \in D_a$ and $\lambda \in D_{t_{1-}}$ (see (2.1)) is equal to D_a. Similarly $D_{t_{1+},b} = D_b$.

Suppose that (4.5) holds and the assumptions of Theorems 7 and 8 are satisfied. Further assume that system (4.2) has two different solutions in the set $(a, b) \times D_a \times D_a \times D_b$. The first solution consists of t_1^* and the triplet of vectors z^*, λ^*, η^* and and the second solution consists of $\tilde{t}_1$ and the triplet of vectors $\tilde{z}, \tilde{\lambda}, \tilde{\eta}$. Then we get from Theorems 7 and 8 the functions

$$x_\infty(t; t_1^*, z^*, \lambda^*), \quad y_\infty(t; t_1^*, \lambda^*, \eta^*), \quad x_\infty(t; \tilde{t}_1, \tilde{z}, \tilde{\lambda}), \quad y_\infty(t; \tilde{t}_1, \tilde{\lambda}, \tilde{\eta}).$$

Finally assume that

$$g(t, y_\infty(t; t_1^*, \lambda^*, \eta^*)) \neq 0,\ t \in (t_1^*, b], \quad g(t, y_\infty(t; \tilde{t}_1, \tilde{\lambda}, \tilde{\eta})) \neq 0,\ t \in (\tilde{t}_1, b].$$

Then problem (1.1)–(1.3) has two different solutions u^* and $\tilde{u}$

$$u^*(t) = \begin{cases} x_\infty(t; t_1^*, z^*, \lambda^*) & \text{if } t \in [a, t_1^*], \\ y_\infty(t; t_1^*, \lambda^*, \eta^*) & \text{if } t \in (t_1^*, b], \end{cases}$$

$$\tilde{u}(t) = \begin{cases} x_\infty(t; \tilde{t}_1, \tilde{z}, \tilde{\lambda}) & \text{if } t \in [a, \tilde{t}_1], \\ y_\infty(t; \tilde{t}_1, \tilde{\lambda}, \tilde{\eta}) & \text{if } t \in (\tilde{t}_1, b]. \end{cases}$$

The solution u^* has the unique impulse point t^* and the solution $\tilde{u}$ has the unique impulse point $\tilde{t}$. In addition $u^*(a), \tilde{u}(a) \in D_a$.

5. Approximation of solutions

The solvability of the determining system (4.2) can be established similarly to [32] by studying its approximate version

$$
\left\{
\begin{array}{l}
\lambda - z - \int\limits_a^{t_1} f(s, x_m(s; t_1, z, \lambda))\,\mathrm{d}s = 0, \\
(\eta - (\lambda + \gamma(\lambda))) - \int\limits_{t_1}^{b} f(s, y_m(s; t_1, \lambda, \eta))\,\mathrm{d}s = 0, \\
Az + C\eta = d, \\
g(t_1, \lambda) = 0,
\end{array}
\right.
\tag{5.1}
$$

with

$$
g(t, y_m(t; t_1, \lambda, \eta)) \neq 0, \quad t \in (t_1, b], \tag{5.2}
$$

which can be constructed explicitly for a fixed $m \in \mathbb{N}$.

Let the quartet $\left(\widehat{t}, \widehat{z}, \widehat{\lambda}, \widehat{\eta}\right) \in (a,b) \times D_a \times D_{\widehat{t}} \times D_b$ be a root of system (5.1) for a fixed $m \in \mathbb{N}$. Then the function

$$
\widehat{u}(t) = \left\{
\begin{array}{ll}
x_m\left(t; \widehat{t}, \widehat{z}, \widehat{\lambda}\right) & \text{if } t \in \left[a, \widehat{t}\right], \\
y_m\left(t; \widehat{t}, \widehat{\lambda}, \widehat{\eta}\right) & \text{if } t \in \left(\widehat{t}, b\right],
\end{array}
\right.
\tag{5.3}
$$

which satisfies (5.2) can be regarded as the m-th approximation to a solution of problem (1.1)–(1.3). The function $\widehat{u}$ has a unique impulse point $\widehat{t}$, where $\widehat{u}$ has the jump $\gamma(\widehat{\lambda})$. This is justified by the next estimates which follow directly from (3.12) and (3.19)

$$
\left| x_\infty(t; \widehat{t}, \widehat{z}, \widehat{\lambda}) - x_m(t; \widehat{t}, \widehat{z}, \widehat{\lambda}) \right| \leqslant
$$

$$
\leqslant \frac{10}{9} \alpha_1(t; a, \widehat{t}) Q_x^m \left(1_n - Q_x\right)^{-1} \delta_{[a,b], D^x}(f), \quad t \in \left[a, \widehat{t}\right], \ m \in \mathbb{N},
$$

$$
\left| y_\infty(t; \widehat{t}, \widehat{\lambda}, \widehat{\eta}) - y_m(t; \widehat{t}, \widehat{\lambda}, \widehat{\eta} \right| \leqslant
$$

$$
\leqslant \frac{10}{9} \alpha_1(t; \widehat{t}, b) Q_y^m \left(1_n - Q_y\right)^{-1} \delta_{[a,b], D^y}(f), \quad t \in \left[\widehat{t}, b\right], \ m \in \mathbb{N},
$$

where Q_x ,$Q_y, \delta_{[a,b],D^x}(f)$ and $\delta_{[a,b],D^y}(f)$ are given according to Theorems 7, 8.

It is worth to emphasise the role of unknown parameters whose values appearing in (5.3) are determined from (5.1):

- the vector $\widehat{z} \in D_a$ is an approximation of the initial value $u(a)$ of the solution u of (1.1)–(1.3),
- the value $\widehat{t} \in (a,b)$ is an approximation of the impulse point t_1 of u,
- the vector $\widehat{\lambda} \in D_{t_1-}$ is an approximation of $u(t_1)$,
- the vector $\widehat{\lambda} + \gamma(\widehat{\lambda}) \in D_{t_1+}$ is an approximation of $u(t_1+)$,
- the vector $\widehat{\eta} \in D_b$ is an approximation of $u(b)$.

The solvability analysis based on properties of equations (5.1) can be carried out by analogy to [31,35] on the base of topological degree methods, but it is not treated here.

Remark 12. The technique described above can be also applied to problem (1.1)–(1.3) with a piece-wise right-hand side f.

Note, that the most difficult part of our approach is the construction of the functions $x_m(t; t_1, z, \lambda)$ and $y_m(t; t_1, \lambda; , \eta)$ in (3.8) and (3.17). If the explicit integration in (3.8) and (3.17) is impossssible or difficult, one can use suitable modifications of (3.8) and (3.17), which at the expense of a certain loss in accuracy, lead one to iterations better suited for practical computations. In [27] it was mentioned two natural modifications of this kind which make the scheme more constructive, namely the version of "Frozen" parameters and the version of Polynomial interpolation used in [36].

If the version of "Frozen" parameters is used, then problem (1.1)–(1.3) can be solved as follows:

- Choose a compact convex set $D_a \subset \mathbb{R}^n$ and put $D_b = \{x + \gamma(x) : x \in D_a\}$, see Remark 11, (4.5). Check if the assumptions of Theorem 7 and Theorem 8 are fulfilled.
- For arbitrary parameters $(t_1, z, \lambda, \eta) \in (a, b) \times D_a \times D_a \times D_b$ find the first iterations $x_1(t; t_1, z, \lambda)$ and $y_1(t; t_1, \lambda, \eta)$ from (3.8) and (3.17), respectively.
- Put $m = 1$ in system (5.1) and find its solution $(\widehat{t}, \widehat{z}, \widehat{\lambda}, \widehat{\eta}) \in (a, b) \times D_a \times D_a \times D_b$.
- For arbitrary parameters $(t_1, z, \lambda, \eta) \in (a, b) \times D_a \times D_a \times D_b$ derive the second "frozen" iterations $\widehat{x}_2(t; t_1, z, \lambda)$ and $\widehat{y}_2(t; t_1, \lambda, \eta)$ using the functions $X_1(t) = x_1(t; \widehat{t}, \widehat{z}, \widehat{\lambda})$ and $Y_1(t) = y_1(t; \widehat{t}, \widehat{\lambda}, \widehat{\eta})$ in (3.8) and (3.17) with $m = 2$:

$$\widehat{x}_2(t; t_1, z, \lambda) = z + \int_a^t f(s, X_1(s))\,\mathrm{d}s - \frac{t-a}{t_1-a}\int_a^{t_1} f(s, X_1(s))\,\mathrm{d}s + \frac{t-a}{t_1-a}(\lambda - z), \quad t \in [a, t_1]\,, \tag{5.4}$$

and

$$\widehat{y}_2(t; t_1, \lambda, \eta) = \lambda + \gamma + \int_{t_1}^t f(s, Y_1(s))\,\mathrm{d}s - \frac{t-t_1}{b-t_1}\int_{t_1}^{b} f(s, Y_1(s)(s))\,\mathrm{d}s + \frac{t-t_1}{b-t_1}(\eta - \lambda - \gamma), \quad t \in [t_1, b]\,. \tag{5.5}$$

- For $m = 2$ modify system (5.1) by means of the second "frozen" iterations $\widehat{x}_2(t; t_1, z, \lambda)$ and $\widehat{y}_2(t; t_1, \lambda, \eta)$. Find a solution of the modified system

$$\begin{cases} \lambda - z - \int\limits_a^{t_1} f(s, \widehat{x}_2\,(s; t_1, z, \lambda))\,\mathrm{d}s = 0, \\ (\eta - (\lambda + \gamma(\lambda))) - \int\limits_{t_1}^{b} f(s, \widehat{y}_2\,(s; t_1, \lambda, \eta))\,\mathrm{d}s = 0, \\ Az + C\eta = d, \\ g(t_1, \lambda) = 0. \end{cases} \tag{5.6}$$

  in the set $(a, b) \times D_a \times D_a \times D_b$ and denote it again $(\widehat{t}, \widehat{z}, \widehat{\lambda}, \widehat{\eta})$.
- For arbitrary parameters $(t_1, z, \lambda, \eta) \in (a, b) \times D_a \times D_a \times D_b$ derive the third "frozen" iterations $\widehat{x}_3(t; t_1, z, \lambda)$ and $\widehat{y}_3(t; t_1, \lambda, \eta)$ using the functions $X_2(t) = x_2(t; \widehat{t}, \widehat{z}, \widehat{\lambda})$ and $Y_2(t) = y_2(t; \widehat{t}, \widehat{\lambda}, \widehat{\eta})$ in (3.8) and (3.17) with $m = 3$.
- For $m = 3$ modify system (5.1) by means of the third "frozen" iterations $\widehat{x}_3(t; t_1, z, \lambda)$ and $\widehat{y}_3(t; t_1, \lambda, \eta)$. Find a solution of the modified system in the set $(a, b) \times D_a \times D_a \times D_b$ and denote it again $(\widehat{t}, \widehat{z}, \widehat{\lambda}, \widehat{\eta})$.
- Similarly derive further "frozen" iterations.
- If for some $m \in \mathbb{N}$ the m-th and $(m-1)$-st "frozen" iterations are close enough, check the inequality

$$g(t, Y_m(t)) \neq 0, \quad t \in (\widehat{t}, b]. \tag{5.7}$$

  If (5.7) is fulfilled, then the function

$$\widehat{u}(t) = \begin{cases} X_m(t) & \text{if } t \in \left[a, \widehat{t}\right], \\ Y_m(t) & \text{if } t \in \left(\widehat{t}, b\right], \end{cases}$$

  can be regarded as the m-th approximation of a solution u of problem (1.1)–(1.3) having a unique imupulse point and the initial value $u(a) \in D_a$. If (5.7) is not fulfilled, then another set D_a should be chosen.

	1. iteration	2. iteration	3. iteration	4. iteration
t_1	0.377367167	0.377366182	0.377366354	0.377366355
z_1	−8.437535639	−8.437471330	−8.437478608	−8.437478618
z_2	−3.968767820	−3.968735665	−3.968739304	−3.968739309
λ_1	−2.493949925	−2.493944384	−2.493945315	−2.493945318
λ_2	−3.935836303	−3.935836303	−3.935817921	−3.935817931
η_1	0.007600000	0.007600000	0.007600000	0.007600002
η_2	−4.024145297	−4.024123042	−4.024126787	−4.024126798

TABLE 1. Approximate values of parameters for the first solution of problem (6.1)–(6.3).

6. EXAMPLE

Let us apply the numerical-analytic approach described above to the system

$$\begin{cases} u_1'(t) = u_2^2(t) - \dfrac{t}{5}u_1(t) + \dfrac{t^3}{100} - \dfrac{t^2}{25}, \\ u_2'(t) = \dfrac{t^2}{10}u_2(t) + \dfrac{t}{8}u_1(t) - \dfrac{21t^3}{800} + \dfrac{t}{16} + \dfrac{1}{5}, \quad \text{a.e. } t \in \left[0, \dfrac{1}{2}\right]. \end{cases} \tag{6.1}$$

Equation (6.1) is subject to the state-dependent impulse conditions

$$\begin{cases} u_1(t+) - u_1(t-) = 0.5, \\ u_2(t+) - u_2(t-) = -0.1, \quad \text{where} \quad \left(u_1(t) + \dfrac{1}{2}\right)^2 + u_2(t) - \dfrac{1}{25} = 0. \end{cases} \tag{6.2}$$

The impulsive problem (6.1)–(6.2) is investigated with the boundary condition

$$\begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} u_1\left(\frac{1}{2}\right) \\ u_2\left(\frac{1}{2}\right) \end{bmatrix} = \begin{bmatrix} -0.1212 \\ 0.0019 \end{bmatrix}. \tag{6.3}$$

We are interested in solutions of problem (6.1)–(6.3) according to Definition 1 with $p = 1$. Therefore our solution is a left-continuous vector function $u : [0, \frac{1}{2}] \to \mathbb{R}^2$, $u = \mathrm{col}(u_1, u_2)$, which has a unique intersection point with a barrier G, where

$$G = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \left(x_1 + \frac{1}{2}\right)^2 + x_2 - \frac{1}{25} = 0 \right\}. \tag{6.4}$$

Accordingly there exists a unique $t_1 \in (0, \frac{1}{2})$ satisfying

$$\left(u_1(t_1) + \frac{1}{2}\right)^2 + u_2(t_1) - \frac{1}{25} = 0. \tag{6.5}$$

Further, the restrictions $u|_{[0,t_1]}$, $u|_{(t_1,b]}$ have continuous derivatives, u satisfies (6.1) for $t \in [0, \frac{1}{2}]$, $t \neq t_1$, and has a jump at t_1. The size of the jump is given by the constant vector

$$\gamma = \mathrm{col}(0.5, -0.1).$$

Finally, u fulfils (6.3).

We describe in detail the individual steps of our method.

Step 1. Application of Theorem 7.
For $a = 0$ and $b = \frac{1}{2}$ assume that $t_1 \in \left(0, \frac{1}{2}\right)$ is a parameter and put (see Remark 11)

$$D_0 = D_{t_1-} = D_{0,t_1-} = \{(x_1, x_2) : -8.44 \le x_1 \le 0.15,\ -4.0 \le x_2 \le 0.15\}. \tag{6.6}$$

	1. iteration	2. iteration	3. iteration	4. iteration
t_1	0.181450919	0.181450845	0.181450845	0.181450846
z_1	-0.492769263	-0.492769235	-0.492769235	-0.492769235
z_2	0.003615368	0.003615383	0.003615383	0.003615383
λ_1	-0.491120618	-0.491120590	-0.491120590	-0.491120590
λ_2	0.039921157	0.039921156	0.039921156	0.039921156
η_1	0.007600000	0.007600000	0.007600000	0.007600000
η_2	0.010065508	0.010065542	0.010065542	0.010065542

TABLE 2. Approximate values of parameters for the second solution of problem (6.1)–(6.3).

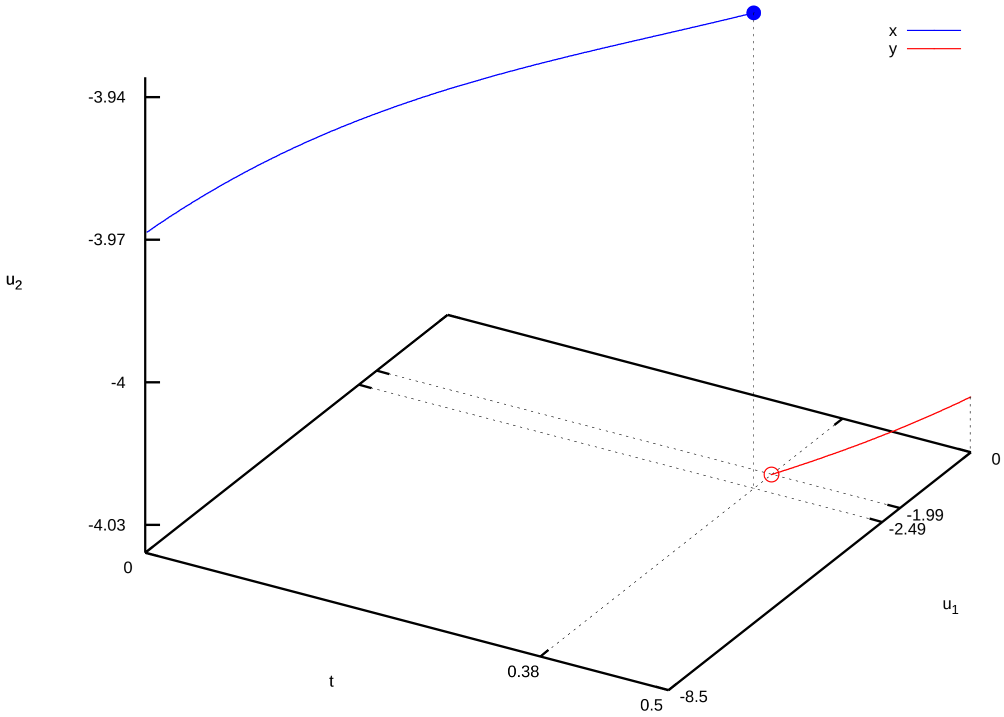

FIGURE 1. The first solution (u_1, u_2) of problem (6.1)–(6.3).

To introduce the set D^x from (3.1) choose the vector

$$\varrho^x = \operatorname{col}(2.46,\ 0.2)\,.$$

Consequently the ϱ^x-neighbourhood D^x of the set $D_{a,t_{1-}}$ is

$$D^x = \{(x_1, x_2) : -10.9 \le x_1 \le 2.61,\ -4.2 \le x_2 \le 0.35\}\,.$$

Let $f = \operatorname{col}(f_1, f_2)$, where

$$f_1(t, x_1, x_2) = x_2^2(t) - \frac{t}{5}x_1(t) + \frac{t^3}{100} - \frac{t^2}{25}, \quad f_2(t, x_1, x_2) = \frac{t^2}{10}x_2(t) + \frac{t}{8}x_1(t) - \frac{21t^3}{800} + \frac{t}{16} + \frac{1}{5},$$

and

$$K_x = \begin{bmatrix} \frac{1}{10} & \frac{42}{5} \\ \frac{1}{16} & \frac{1}{40} \end{bmatrix}.$$

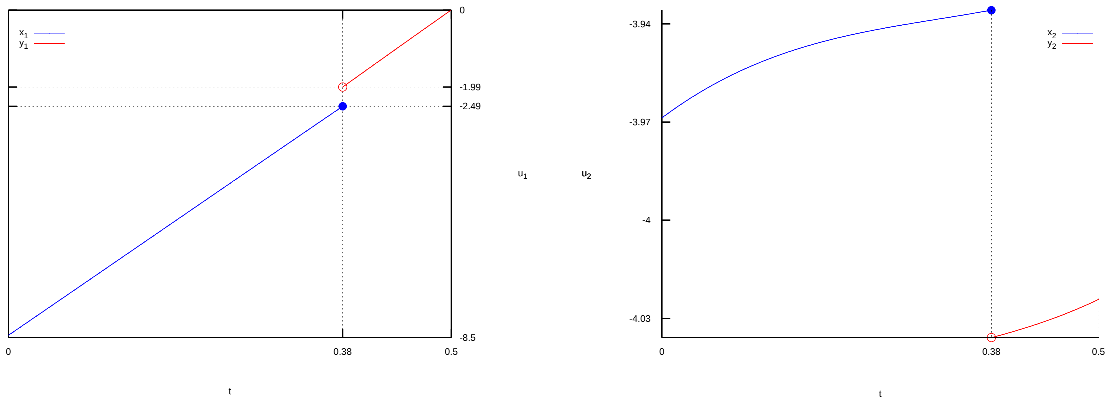

FIGURE 2. Component u_1 (left) and component u_2 (right) of the first solution of problem (6.1)–(6.3).

Direct computations show that $f \in \operatorname{Lip}(K_x, D^x)$ (see (2.3)). Since, according to (3.9),

$$Q_x = \frac{3}{20} K_x = \begin{bmatrix} \frac{3}{200} & \frac{63}{50} \\ \frac{3}{320} & \frac{3}{800} \end{bmatrix},$$

the maximal (in modulus) eigenvalue of Q_x satisfies the inequality in (3.9), in particular,

$$r(Q_x) = 0.05714152202 < 1.$$

Moreover,

$$\delta_{[0,\frac{1}{2}],D^x}(f) = \frac{1}{2}\left[\max_{(t,x)\in[0,\frac{1}{2}]\times D^x} f(t,x) - \min_{(t,x)\in[0,\frac{1}{2}]\times D^x} f(t,x)\right] = \begin{bmatrix} 9.4955000 \\ 0.4790625 \end{bmatrix},$$

$$\varrho^x = \begin{bmatrix} 2.46 \\ 0.2 \end{bmatrix} \geq \frac{1}{4}\delta_{[0,\frac{1}{2}],D^x}(f) = \begin{bmatrix} 2.373875000 \\ 0.119765625 \end{bmatrix},$$

which yields (3.10). So, all conditions of Theorem 7 are fulfilled, and the sequence of parametrized functions (3.8) for this example is convergent.

Step 2. Application of Theorem 8.
According to Remark 11 put

(6.7) $$D_{\frac{1}{2}} = D_{t_{1+}} = D_{t_{1+},\frac{1}{2}} = \{(y_1, y_2) : -7.94 \leq y_1 \leq 0.7,\ -4.15 \leq y_2 \leq 0.05\}.$$

To introduce the set D^y from (3.2) choose the vector

$$\varrho^y := \operatorname{col}(2.63;\ 0.15).$$

Consequently the ϱ^y-neighbourhood D^y of the set $D_{t_{1+},\frac{1}{2}}$ is

$$D^y = \{(y_1, y_2) : -10.57 \leq y_1 \leq 3.33,\ -4.3 \leq y_2 \leq 0.2\}.$$

By analogy, computations give that $f \in \operatorname{Lip}(K_y, D^y)$, where

$$K_y = \begin{bmatrix} \frac{1}{10} & \frac{43}{5} \\ \frac{1}{16} & \frac{1}{40} \end{bmatrix},$$

$$Q_y = \frac{3}{20} K_y = \begin{bmatrix} \frac{3}{200} & \frac{129}{100} \\ \frac{3}{320} & \frac{3}{800} \end{bmatrix},\ r(Q_y) = 0.06137034234 < 1,$$

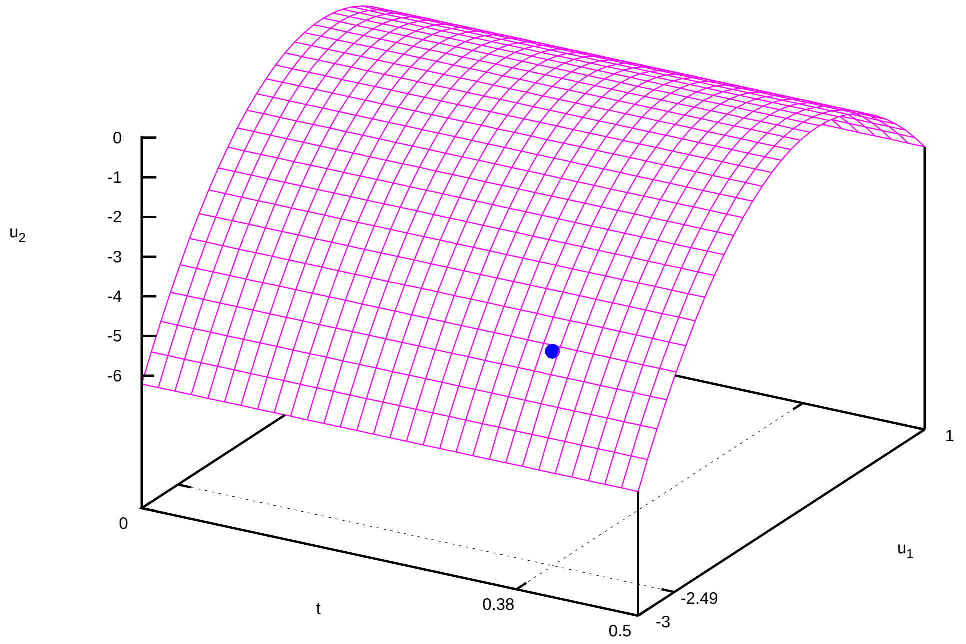

FIGURE 3. Barrier G: $(x_1+\frac{1}{2})^2+x_2-\frac{1}{25}=0$ and its intersection point $(t_1, u_1(t_1), u_2(t_1))$ with the first solution of problem (6.1)–(6.3).

and

$$\delta_{[0,\frac{1}{2}],D^y}(f) = \frac{1}{2}\left[\max_{(t,y)\in[0,\frac{1}{2}]\times D^y} f(t,y) - \min_{(t,y)\in[0,\frac{1}{2}]\times D^y} f(t,y)\right] = \begin{bmatrix} 9.940000 \\ 0.490625 \end{bmatrix},$$

$$\varrho^y = \begin{bmatrix} 2.63 \\ 0.15 \end{bmatrix} \geq \frac{1}{4}\delta_{[0,\frac{1}{2}],D^y}(f) = \begin{bmatrix} 2.48500000 \\ 0.12265625 \end{bmatrix}.$$

So, all conditions of Theorem 8 are fulfilled, and the sequence of functions (3.17) for this example is convergent.

Step 3. Starting functions and first iterations
Consider parameters $(t_1, z, \lambda, \eta) \in \left(0, \frac{1}{2}\right) \times D_0 \times D_0 \times D_{\frac{1}{2}}$, where

$$z = \operatorname{col}(z_1, z_2), \quad \lambda = \operatorname{col}(\lambda_1, \lambda_2), \quad \eta = \operatorname{col}(\eta_1, \eta_2).$$

By (3.7) and (3.16), the starting functions $x_0 = \operatorname{col}(x_{01}, x_{02})$ and $y_0 = \operatorname{col}(y_{01}, y_{02})$ have the form

$$x_{01}(t; t_1, z_1, \lambda_1) = \left(1 - \frac{t}{t_1}\right) z_1 + \frac{t}{t_1}\lambda_1, \quad t \in [0, t_1],$$

$$x_{02}(t; t_1, z_2, \lambda_2) = \left(1 - \frac{t}{t_1}\right) z_2 + \frac{t}{t_1}\lambda_2, \quad t \in [0, t_1],$$

$$y_{01}(t; t_1, \lambda_1, \eta_1) = \left(1 - \frac{t - t_1}{\frac{1}{2} - t_1}\right)(\lambda_1 + 0.5) + \frac{t - t_1}{\frac{1}{2} - t_1}\eta_1, \quad t \in [t_1, \frac{1}{2}],$$

$$y_{02}(t; t_1, \lambda_2, \eta_2) = \left(1 - \frac{t - t_1}{\frac{1}{2} - t_1}\right)(\lambda_2 - 0.1) + \frac{t - t_1}{\frac{1}{2} - t_1}\eta_2, \quad t \in [t_1, \frac{1}{2}].$$

The first iterations $x_1 = \mathrm{col}(x_{11}, x_{12})$ and $y_1 = \mathrm{col}(y_{11}, y_{12})$ can be found by symbolic computation on the base of Maple 14 from (3.8) and (3.17), where $m = 1$, $a = 0$, $b = \frac{1}{2}$. Due to relatively complicated expressions we present here only the function x_{11}

$$\begin{aligned} x_{11}(t;t_1,z_1,z_2,\lambda_1,\lambda_2) &= z_1 + \frac{1}{400}t^4 + \frac{1}{3}\left[\left(\frac{-z_2}{t_1}+\frac{\lambda_2}{t_1}\right)^2 + \frac{z_1}{5t_1} - \frac{\lambda_1}{t_1} - \frac{1}{25}\right]t^3 + \\ &+ \frac{1}{2}\left[2z_2\left(\frac{-z_2}{t_1}+\frac{\lambda_2}{t_1}\right) - \frac{z_1}{5}\right]t^2 + z_2^2 t - \\ &- \frac{t}{t_1}\left[\frac{1}{400}t_1^4 + \frac{1}{3}\left(\left(\frac{-z_2}{t_1}+\frac{\lambda_2}{t_1}\right)^2 + \frac{z_1}{t_1} - \frac{\lambda_1}{5t_1} - \frac{1}{25}\right)t_1^3 + \right. \\ &\left. + \frac{1}{2}\left(2z_2\left(\frac{-z_2}{t_1}+\frac{\lambda_2}{t_1}\right) - \frac{z_1}{5}\right)t_1^2 + z_2^2 t_1\right] + \frac{t(\lambda_1 - z_1)}{t_1}, \qquad t \in [0, t_1]. \end{aligned}$$

System (5.1) for $m = 1$ has the form

$$\left\{\begin{array}{l} \lambda_1 - z_1 - \displaystyle\int_0^{t_1} f_1(s; x_{11}(s;t_1,z_1,z_2,\lambda_1,\lambda_2), x_{12}(s;t_1,z_1,z_2,\lambda_1,\lambda_2))\,\mathrm{d}s = 0, \\ \lambda_2 - z_2 - \displaystyle\int_0^{t_1} f_2(s; x_{11}(s;t_1,z_1,z_2,\lambda_1,\lambda_2), x_{12}(s;t_1,z_1,z_2,\lambda_1,\lambda_2))\,\mathrm{d}s = 0, \\ (\eta_1 - (\lambda_1 + 0.5)) - \displaystyle\int_{t_1}^{\frac{1}{2}} f_1(s; y_{11}(s;t_1,\lambda_1,\lambda_2,\eta_1,\eta_2), y_{12}(s;t_1,\lambda_1,\lambda_2,\eta_1,\eta_2))\,\mathrm{d}s = 0, \\ (\eta_2 - (\lambda_2 - 0.1)) - \displaystyle\int_{t_1}^{\frac{1}{2}} f_2(s; y_{11}(s;t_1,\lambda_1,\lambda_2,\eta_1,\eta_2), y_{12}(s;t_1,\lambda_1,\lambda_2,\eta_1,\eta_2))\,\mathrm{d}s = 0, \\ \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & 0 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} -0.1212 \\ 0.0019 \end{bmatrix}, \\ \left(\lambda_1 + \frac{1}{2}\right)^2 + \lambda_2 - \frac{1}{25} = 0. \end{array}\right. \tag{6.8}$$

We see that system (6.8) is well defined and it consists of seven algebraic equations with unknown variables $t_1, z_1, z_2, \lambda_1, \lambda_2, \eta_1, \eta_2$, which are searched in the domain $(0, \frac{1}{2}) \times D_a \times D_a \times D_b$, cf. (6.6) and (6.7). For $z_1 \in [-8.44, -1]$, numerical computations give the roots which are written in the first column in Table 1. Substituting these roots into

$$x_{11}(t;t_1,z_1,z_2,\lambda_1,\lambda_2),\ x_{12}(t;t_1,z_1,z_2,\lambda_1,\lambda_2),\ y_{11}(t;t_1,\lambda_1,\lambda_2,\eta_1,\eta_2),\ y_{12}(t;t_1,\lambda_1,\lambda_2,\eta_1,\eta_2),$$

we get the functions

$$\begin{aligned} X_{11}(t) &= -8.437535639 + 15.71336319\,t + 0.4974130077\,t^2 - 1.060804186\,t^3 + 0.0025\,t^4, \\ X_{12}(t) &= -3.968767820 + 0.200096725\,t - 0.4960959775\,t^2 + 0.5239635733\,t^3 - 0.004380837\,t^4, \\ Y_{11}(t) &= -8.254109890 + 16.58414073\,t + 0.4271345180\,t^2 - 1.098402835\,t^3 + 0.0025\,t^4, \\ Y_{12}(t) &= -4.072409823 + 0.2001284496\,t - 0.4783214004\,t^2 + 0.5443348006\,t^3 - 0.004179165\,t^4, \end{aligned}$$

which are used in the next step for computation of second iterations.

Step 4. Second iterations

Assume now again that parameters $(t_1, z, \lambda, \eta) \in \left(0, \frac{1}{2}\right) \times D_0 \times D_0 \times D_{\frac{1}{2}}$ are arbitrary and using the functions $X_1 = \mathrm{col}(X_{11}, X_{12})$ and $Y_1 = \mathrm{col}(Y_{11}, Y_{12})$ derive the second iterations $\widehat{x}_2 = \mathrm{col}(\widehat{x}_{21}, \widehat{x}_{22})$ and $\widehat{y}_2 = \mathrm{col}(\widehat{y}_{21}, \widehat{y}_{22})$ from (5.4) and (5.5), where $a = 0$, $b = \frac{1}{2}$. Then, according to (5.6), solve the system

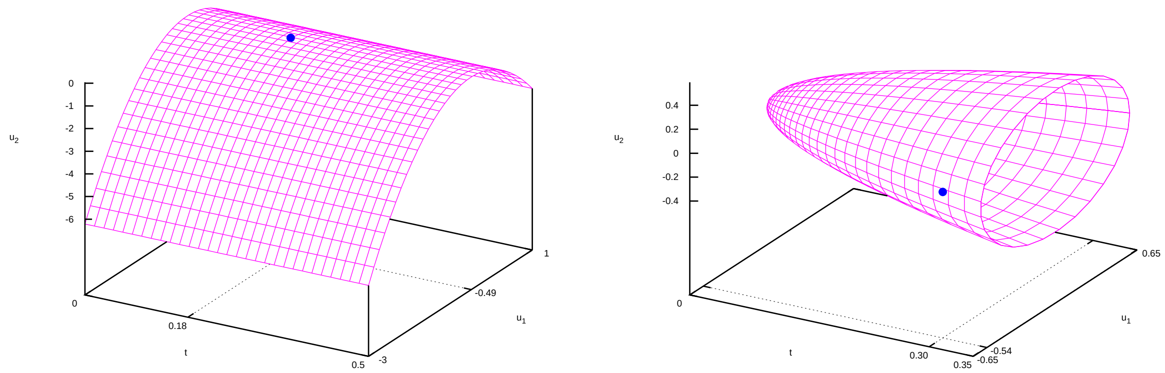

FIGURE 4. Left: Barrier G: $(x_1+\frac{1}{2})^2+x_2-\frac{1}{25}=0$ and its intersection point with the second solution of problem (6.1)–(6.3). Right: Barrier G_1: $x_1^2+x_2^2-t=0$ and its intersection point with a solution of problem (6.1)–(6.3) with G_1 and the jump $\operatorname{col}(0.55,-0.15)$.

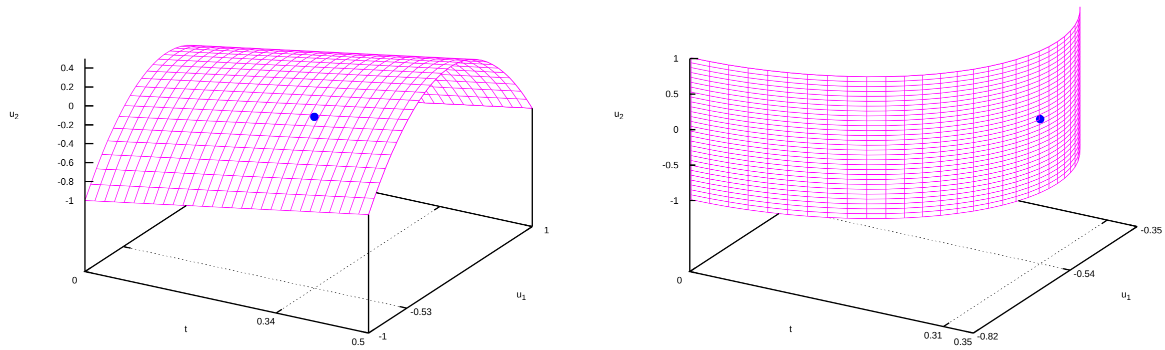

FIGURE 5. Left: Barrier G_2: $x_1^2+x_2-t=0$ and its intersection point with a solution of problem (6.1)–(6.3) with G_2 and the jump $\operatorname{col}(0.55,-0.15)$. Right: Barrier G_3: $\left(x_1+\frac{1}{2}\right)^2+t^2-\frac{1}{10}=0$ and its intersection point with a solution of problem (6.1)–(6.3) with G_3 and the jump $\operatorname{col}(0.55,-0.15)$.

$$
\left\{
\begin{aligned}
&\lambda_1 - z_1 - \int_0^{t_1} f_1(s;\widehat{x}_{21}(s;t_1,z_1,z_2,\lambda_1,\lambda_2),\widehat{x}_{22}(s;t_1,z_1,z_2,\lambda_1,\lambda_2))\,\mathrm{d}s = 0,\\
&\lambda_2 - z_2 - \int_0^{t_1} f_2(s;\widehat{x}_{21}(s;t_1,z_1,z_2,\lambda_1,\lambda_2),\widehat{x}_{22}(s;t_1,z_1,z_2,\lambda_1,\lambda_2))\,\mathrm{d}s = 0,\\
&(\eta_1-(\lambda_1+0.5)) - \int_{t_1}^{\frac{1}{2}} f_1(s;\widehat{y}_{21}(s;t_1,\lambda_1,\lambda_2,\eta_1,\eta_2),\widehat{y}_{22}(s;t_1,\lambda_1,\lambda_2,\eta_1,\eta_2))\,\mathrm{d}s = 0,\\
&(\eta_2-(\lambda_2-0.1)) - \int_{t_1}^{\frac{1}{2}} f_2(s;\widehat{y}_{21}(s;t_1,\lambda_1,\lambda_2,\eta_1,\eta_2),\widehat{y}_{22}(s;t_1,\lambda_1,\lambda_2,\eta_1,\eta_2))\,\mathrm{d}s = 0,\\
&\begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & 0 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} -0.1212 \\ 0.0019 \end{bmatrix},\\
&\left(\lambda_1+\frac{1}{2}\right)^2+\lambda_2-\frac{1}{25}=0.
\end{aligned}
\right.
\tag{6.9}
$$

Note that system (6.9) for unknown values $t_1, z_1, z_2, \lambda_1, \lambda_2, \eta_1, \eta_2$ has to be solved numerically and it is considerably simpler than to solve (5.1) with $m = 2$. The roots of (6.9) are written in the second column of Table 1. By putting these values into $\widehat{x}_2(t, t_1, z, \lambda)$ and $\widehat{y}_2(t, t_1, \lambda, \eta)$, we get the functions

$$\begin{aligned}
X_{21}(t) = &- 8.43747133 + 15.75100318\, t + 0.04961612083\, t^2 + 0.2650485212\, t^3\\
&- 1.111749125\, t^4 + 0.1405463792\, t^5 - 0.08702093740\, t^6\\
&+ 0.03984063677\, t^7 - 0.5738497521 \cdot 10^{-3} t^8 + 0.213241475 \cdot 10^{-5} t^9,\\
X_{22}(t) = &- 3.968735665 + 0.1999734728\, t - 0.4960959774\, t^2 + 0.5224312056\, t^3\\
&+ 0.01398407462\, t^4 - 0.0364420242\, t^5 + 0.878480955 \cdot 10^{-2} t^6 - 0.6258338571 \cdot 10^{-4} t^7,\\
Y_{21}(t) = &- 8.241206558 + 16.58434465\, t + 0.01040592499\, t^2 + 0.1930215965\, t^3\\
&- 1.175096781\, t^4 + 0.1400768463\, t^5 - 0.08715111796\, t^6\\
&+ 0.04289976329\, t^7 - 0.568716202 \cdot 10^{-3} t^8 + 0.1940602001 \cdot 10^{-5} t^9,\\
Y_{22}(t) = &- 4.072094611 + 0.2000007263\, t - 0.4846318681\, t^2 + 0.5552588696\, t^3\\
&+ 0.01178866493\, t^4 - 0.03702649888\, t^5 + 0.912433001 \cdot 10^{-2} t^6 - 0.5970235357 \cdot 10^{-4} t^7,
\end{aligned}$$

which are used for the computations of third iterations.

Step 5. Higher iterations
The higher iterations can be obtained by analogy. For $m = 3$ and $m = 4$ the corresponding values of parameters are written in the third and fourth column in Table 1, respectively. If we derive the functions $X_4 = \mathrm{col}(X_{41}(t), X_{42}(t))$ and $Y_4 = \mathrm{col}(Y_{41}(t), Y_{42}(t))$, the Maple computations show that inequality (5.7) is fulfilled for $m = 4$. More precisely, for $\widehat{t} = 0.377366355$ and for each $t \in (\widehat{t}, 0.5]$, the value of $(Y_{41}(t) + 1/2)^2 + Y_{42}(t) - 1/25$ is strictly negative and belongs to the interval $[-4, -1]$. Consequently, the function

$$\widehat{u}(t) = \begin{cases} X_4(t) & \text{if } t \in \left[0, \widehat{t}\,\right], \\ Y_4(t) & \text{if } t \in \left(\widehat{t}, \frac{1}{2}\right], \end{cases}$$

is the fourth approximation of the first solution of problem (6.1)–(6.3). The graph of the solution is on Figure 1 and its components are on Figure 2. Figure 3 shows barrier (6.4) and its intersection point with the solution.

Note, that if we substitute the approximation $\widehat{u}$ of the first solution into system (6.1), we obtain the following residual:

$$\max_{t \in [0, \widehat{t}\,]} \left| X'_{41}(t) - X^2_{42}(t) + \frac{t}{5} X_{41}(t) - \frac{t^3}{100} + \frac{t^2}{25} \right| = 1.1 \cdot 10^{-7},$$

$$\max_{t \in [0, \widehat{t}\,]} \left| X'_{42}(t) - \frac{t^2}{10} X_{42}(t) - \frac{t}{8} X_{41}(t) + \frac{21}{800} t^3 - \frac{1}{16} t - \frac{1}{5} \right| = 3.1 \cdot 10^{-8},$$

$$\max_{t \in [\widehat{t}, \frac{1}{2}]} \left| Y'_{41}(t) - Y^2_{42}(t) + \frac{t}{5} Y_{41}(t) - \frac{t^3}{100} + \frac{t^2}{25} \right| = 4.0 \cdot 10^{-8},$$

$$\max_{t \in [\widehat{t}, \frac{1}{2}]} \left| Y'_{42}(t) - \frac{t^2}{10} Y_{42}(t) - \frac{t}{8} Y_{41}(t) + \frac{21}{800} t^3 - \frac{1}{16} t - \frac{1}{5} \right| = 6.6 \cdot 10^{-9}.$$

Step 6. Second solution
The Maple computations show that, for $z_1 \in [-1, 0]$, system (6.8) has other roots, which leads to the second solution of problem (6.1)–(6.3). The approximate values of parameters and functions X_m, Y_m, $m = 1, 2, 3, 4$, can be found similarly as for the first solution. The values of parameters are written in Table 2. We can check that inequality (5.7) is fulfilled for $m = 4$. More precisely, for $\widehat{t} = 0.1814508455$

and for each $t \in (\widehat{t}, 0.5]$, the value of $(Y_{41}(t)+1/2)^2 + Y_{42}(t) - 1/25$ is strictly positive and belongs to the interval $[0.16, 0.23]$. Consequently, the function

$$\widehat{u}(t) = \begin{cases} X_4(t) & \text{if } t \in \left[0, \widehat{t}\right], \\ Y_4(t) & \text{if } t \in \left(\widehat{t}, \frac{1}{2}\right], \end{cases}$$

is the fourth approximation of the second solution of problem (6.1)–(6.3). The left picture of Figure 4 shows barrier (6.4) and its intersection point with the second solution.

Note, that if we substitute the approximation $\widehat{u}$ of the second solution into system (6.1), we obtain the following residual:

$$\max_{t\in\left[0,\widehat{t}\right]} \left| X'_{41}(t) - X^2_{42}(t) + \frac{t}{5}X_{41}(t) - \frac{t^3}{100} + \frac{t^2}{25} \right| = 3 \cdot 10^{-11},$$

$$\max_{t\in\left[0,\widehat{t}\right]} \left| X'_{42}(t) - \frac{t^2}{10}X_{42}(t) - \frac{t}{8}X_{41}(t) + \frac{21}{800}t^3 - \frac{1}{16}t - \frac{1}{5} \right| = 4 \cdot 10^{-12},$$

$$\max_{t\in\left[\widehat{t},\frac{1}{2}\right]} \left| Y'_{41}(t) - Y^2_{42}(t) + \frac{t}{5}Y_{41}(t) - \frac{t^3}{100} + \frac{t^2}{25} \right| = 1 \cdot 10^{-10},$$

$$\max_{t\in\left[\widehat{t},\frac{1}{2}\right]} \left| Y'_{42}(t) - \frac{t^2}{10}Y_{42}(t) - \frac{t}{8}Y_{41}(t) + \frac{21}{800}t^3 - \frac{1}{16}t - \frac{1}{5} \right| = 1.15 \cdot 10^{-10}.$$

Step 7. Other barriers

Finally, we discuss problem (6.1)–(6.3) with the jump

$$\gamma = \operatorname{col}(0.55, -0.15),$$

and with other barriers, namely

$$G_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 - t = 0\}, \tag{6.10}$$

or

$$G_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2 - t = 0\}, \tag{6.11}$$

or

$$G_3 = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 + 1/2)^2 + t^2 - 1/10 = 0\}. \tag{6.12}$$

In all three cases, the third and fourth approximations of a solution are very close and inequality (5.7) is fulfilled for $m = 4$. Barrier (6.10) and its intersection point with a solution of problem (6.1)–(6.3), where G is replaced by G_1 is on the right part of Figure 4. Barrier (6.11) and (6.12) and its intersection point with a solution of problem (6.1)–(6.3), where G is replaced by G_2 and G_3 is on the left and right part of Figure 4, respectively.

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