Investigation of solutions of state-dependent multi-impulsive boundary value problems

András Rontó, Irena Rachůnková, Miklós Rontó, Lukáš Rachůnek

Abstract

We discuss a reduction technique allowing one to combine an analysis of the existence of solutions with an efficient construction of approximate solutions for a state-dependent multi-impulsive boundary value problem which consists of the nonlinear system of differential equations

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = f\left(t, u(t)\right) \text{ for a.e. } t \in [a, b]\,,$$

subject to the state-dependent impulse condition

$$u(t+) - u(t-) = \gamma_t\left(u\left(t-\right)\right) \text{ for } t \in (a, b) \text{ such that } g\left(t, u\left(t-\right)\right) = 0,$$

and the nonlinear two-point boundary condition

$$V(u(a), u(b)) = 0.$$

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1 Problem setting

We consider the nonlinear system of differential equations

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = f\left(t, u(t)\right) \quad \text{for a.e. } t \in [a, b]\,, \tag{1.1}$$

with $-\infty < a < b < \infty$ and a continuous vector-function $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$. Eq. (1.1) is subject to the *state-dependent* impulse condition

$$u(t+) - u(t-) = \gamma_t\left(u\left(t-\right)\right) \text{ for } t \in (a, b) \text{ such that } g\left(t, u\left(t-\right)\right) = 0. \tag{1.2}$$

Here the impulse vector-functions $\gamma_t : \mathbb{R}^n \to \mathbb{R}^n$ and the barrier function $g : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ are continuous, and the impulse instants t in (1.2) are unknown since they depend on a solution u through the equation $g(t, u(t-)) = 0$. The impulsive problem (1.1), (1.2) is investigated together with the nonlinear two-point boundary condition

$$V(u(a), u(b)) = 0, \tag{1.3}$$

where the vector-function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. The set

$$G = \{(t, x) \in [a, b] \times \mathbb{R}^n : g(t, x) = 0\} \tag{1.4}$$

determined by the function g in (1.2) is called a *barrier.*

Studies of real life problems with state-dependent impulses can be found in [1]- [4], [13], [14]. But the majority of existence results on impulsive boundary value problems concern fixed-time impulses. This is due to the fact that state-dependent impulses significantly change properties of boundary value problems. This is explained in detail in [6] or [7], where new existence results for boundary value problems with state-dependent impulses as well as with fixed-time impulses are proven. The results about state-dependent impulses concern linear boundary conditions and barriers in the form $t = g(x)$ which are special cases of (1.3) and (1.4), respectively. Only solutions having exactly one intersection point with a barrier are discussed in the citied papers. Moreover, at present, according to the authors' knowledge, *no numerical results* for boundary value problems with state-dependent impulses are available in the literature, except of [5]. In particular, [5] contains the existence result for state-dependent impulsive boundary value problems with linear boundary conditions where a solution can have just one intersection point with a barrier.

Here we study solutions of the *fully nonlinear* problem (1.1)-(1.3) which are allowed to meet a barrier having the *general form* (1.4) *finitely many times.* Consequently, we specify solutions of problem (1.1)-(1.3) by the next definition.

Definition 1 Let $p \in \mathbb{N}$. A left continuous vector-function $u : [a,b] \to \mathbb{R}^n$ is called a *solution* of problem (1.1)-(1.3) with p jumps if (1.3) holds and there exist points $a < \tau_1 < \tau_2 < ... < \tau_p < b$, such that the restrictions

$$u\left|_{[a,\tau_1]}\right., u\left|_{(\tau_1,\tau_2]}\right., ..., u\left|_{(\tau_p,b]}\right.$$

have continuous derivatives and u satisfies (1.1) for $t \in [a,b] \setminus \{\tau_1, \tau_2, ..., \tau_p\}$, and in addition the conditions

$$g(\tau_i, u(\tau_i)) = 0,\ i = 1, ..., p, \quad g(t, u(t)) \neq 0,\ t \in [a,b] \setminus \{\tau_1, \ldots, \tau_p\}., \tag{1.5}$$

$$u(\tau_i+) - u(\tau_i) = \gamma_i\left(u\left(\tau_i\right)\right), \quad i = 1, ..., p, \tag{1.6}$$

are fulfilled.

2 Notation and subsidiary statements

1. For a vector $x = \mathrm{col}(x_1, ..., x_n) \in \mathbb{R}^n$ the notation $|x| = \mathrm{col}(|x_1|, ..., |x_n|)$ is used and the inequalities between vectors are understood *componentwise*. The same convention is adopted implicitly for the operations "max" and "min".
2. 1_n is the unit matrix of dimension n.
3. 0_n is the zero matrix of dimension n.
4. $r(K)$ is the maximal, in modulus, eigenvalue of a matrix K.
5. For $D \in \mathbb{R}^n$ and $f : [a,b] \times D \to \mathbb{R}^n$ the notation $f \in \mathrm{Lip}(K, D)$ means that there exists a square matrix K with non-negative entries satisfying the *componentwise Lipschitz condition*

$$|f(t, u_1) - f(t, u_2)| \leq K\,|u_1 - u_2|\,,\ t \in [a,b],\ u_1, u_2 \in D.$$

6. For a non-negative vector $\rho \in \mathbb{R}^n$, a *componentwise* $\rho-$neighbourhood of a point $z \in \mathbb{R}^n$ is defined as

$$B(z, \rho) := \{\nu \in \mathbb{R}^n : |\nu - z| \leq \rho\}\,. \tag{2.1}$$

7. For given two sets $D_1 \subset \mathbb{R}^n$ and $D_2 \subset \mathbb{R}^n$ we introduce the set

$$\mathcal{H}(D_1, D_2) := (1-\theta)z_1 + \theta z_2,\ z_1 \in D_1,\ z_2 \in D_2,\ \theta \in [0,1]\,. \tag{2.2}$$

8. On the base of a continuous vector-function $f : [a,b] \times D \to \mathbb{R}^n$, we introduce the vector

$$\delta(D) := \frac{1}{2}\left(\max_{(t,x)\in[a,b]\times D} f(t,x) - \min_{(t,x)\in[a,b]\times D} f(t,x)\right), \tag{2.3}$$

where $D \subset \mathbb{R}^n$ is a compact set.

Lemma 1 *([11], Lemma 3.13). Let $-\infty < t_0 < t_1 < \infty$ and*

$$\alpha_1(t; t_0, t_1) = 2\,(t - t_0)\left(1 - \frac{t - t_0}{t_1 - t_0}\right), \quad t \in [t_0, t_1]. \tag{2.4}$$

Then

$$|\alpha_1(t; t_0, t_1)| \leq \frac{t_1 - t_0}{2}, \quad t \in [t_0, t_1]\,, \tag{2.5}$$

and for an arbitrary continuous vector-function $z : [t_0, t_1] \to \mathbb{R}^n$, the estimate

$$\left|\int_{t_0}^{t} z(s)\,\mathrm{d}s - \frac{t - t_0}{t_1 - t_0}\int_{t_0}^{t_1} z(s)\,\mathrm{d}s\right| \leq \alpha_1(t; t_0, t_1)\ \frac{\max\limits_{s\in[t_0,t_1]} z(s) - \min\limits_{s\in[t_0,t_1]} z(s)}{2}, \quad t \in [t_0, t_1]\,, \tag{2.6}$$

is true.

Lemma 2 *([11], Lemma 3.16). Let for $t \in [t_0, t_1]$ the sequence of continuous functions $\{\alpha_m(t; t_0, t_1)\}_{m=1}^{\infty}$ be defined by the recurrence relation*

$$\alpha_{m+1}(t; t_0, t_1) = \left(1 - \frac{t - t_0}{t_1 - t_0}\right) \int_{t_0}^{t} \alpha_m(s; t_0, t_1)\, \mathrm{d}s + \frac{t - t_0}{t_1 - t_0} \int_{t}^{t_1} \alpha_m(s; t_0, t_1)\, \mathrm{d}s, \tag{2.7}$$

where $\alpha_1(t; t_0, t_1)$ is given in (2.4). Then for $m \in \mathbb{N}$, the estimate

$$\alpha_{m+1}(t; t_0, t_1) \leq \frac{10}{9} \left(\frac{3(t_1 - t_0)}{10}\right)^m \alpha_1(t; t_0, t_1), \quad t \in [t_0, t_1], \tag{2.8}$$

holds.

3 Sets of parameters and reduction to model types

Let $p \in \mathbb{N}$. The idea that we going to follow is to approximate a solution u of problem (1.1)-(1.3) with p jumps (see Definition 1) by suitable sequences of functions *separately* on the interval $[a, \tau_1]$ *preceding to unknown moments* $\tau_1, \tau_2, ..., \tau_p$ of jumps and then on the intervals

$$[\tau_1, \tau_2], [\tau_2, \tau_3], ..., [\tau_{p-1}, \tau_p], [\tau_p, \tau_{p+1}] := [\tau_p, b],$$

corresponding to the *after-jump evolution.* The jump moments are treated as parameters to be determined later. The key role in our analysis will be played by the values $\tau_1, \tau_2, ..., \tau_p$ and $\lambda^{[1]}, \lambda^{[2]}, ..., \lambda^{[k]}$, representing, respectively, the unknown jump moments of the solution and its pre-jump values, and $\xi, \lambda^{[p+1]}$ representing the values of the solution at the points a and b.

Consider $(p+2)$ compact sets

$$\Omega_a,\ \Omega_1,\ \ldots,\ \Omega_p,\ \Omega_{p+1} \subset \mathbb{R}^n, \tag{3.1}$$

and by shifting with the jump γ_k from (1.6) define the sets

$$\Omega_k^+ := \{x + \gamma_k(x) : x \in \Omega_k\}, \quad k = 1, \ldots, p.$$

Let us focus on a solution u of problem (1.1)-(1.3) with p jumps by Definition 1 such that

$$u(a) \in \Omega_a,\ u(\tau_{p+1}) \in \Omega_{p+1},$$
$$u(\tau_k) \in \Omega_k,\ u(\tau_k+) \in \Omega_k^+,\ k = 1, ..., p.$$

To this end we choose vectors $\rho^{[a]}, \rho^{[1]}, \rho^{[2]}, \ldots, \rho^{[p]} \in \mathbb{R}^n$, and according to (2.1), (2.2), we construct the sets

$$\mathcal{U}_a := \bigcup_{v \in \mathcal{H}(\Omega_a, \Omega_1)} B(v, \rho^{[a]}), \tag{3.2}$$

$$\mathcal{U}_k := \bigcup_{v \in \mathcal{H}(\Omega_k^+, \Omega_{k+1})} B(v, \rho^{[k]}), \quad k = 1, \ldots, p. \tag{3.3}$$

Now, in Section 4 we study one auxiliary parametrized two-point BVP

$$\left.\begin{aligned} &\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x(t)), \quad t \in [a, \tau_1], \\ &x(a) = \xi,\ x(\tau_1) = \lambda^{[1]}, \end{aligned}\right\} \tag{3.4}$$

and then in Section 5 we discuss p auxiliary parametrized two-point BVPs

$$\left.\begin{aligned} &\frac{\mathrm{d}y^{[k]}(t)}{\mathrm{d}t} = f\left(t, y^{[k]}(t)\right), \quad t \in [\tau_k, \tau_{k+1}], \\ &y^{[k]}(\tau_k+) = \lambda^{[k]} + \gamma_k(\lambda^{[k]}),\ y^{[k]}(\tau_{k+1}) = \lambda^{[k+1]},\ k = 1, 2, ..., p, \end{aligned}\right\} \tag{3.5}$$

where

$$\left.\begin{aligned} &a < \tau_1 < .... < \tau_p < \tau_{p+1} = b, \quad \xi = \mathrm{col}(\xi_1, \xi_2, ..., \xi_n) \in \Omega_a, \\ &\lambda^{[k]} = \mathrm{col}(\lambda_1^{[k]}, \lambda_2^{[k]}, ..., \lambda_n^{[k]}) \in \Omega_k,\ k = 1, 2, ..., p+1, \end{aligned}\right\} \tag{3.6}$$

are considered as unknown parameters. We assume that $\mathcal{U}_a$ and $\mathcal{U}_k$ defined in (3.2) and (3.3) are domains for the values $x(t)$ and $y^{[k]}(t)$ of solutions to problems (3.4) and (3.5).

4 Iterations for the first pre-jump evolution

To study problem (3.4) on $[a,\tau_1]\times\mathcal{U}_a$, we introduce the parameterized sequences of vector-functions

$$x_m(t) := x_m(t;\tau_1,\xi,\lambda^{[1]}), \quad t\in[a,\tau_1]\,,$$

with the parameters $\tau_1,\xi,\lambda^{[1]}$ from (3.6) by the relations

$$x_0(t) = \xi + \frac{t-a}{\tau_1-a}\left(\lambda^{[1]}-\xi\right) = \left(1-\frac{t-a}{\tau_1-a}\right)\xi + \frac{t-a}{\tau_1-a}\lambda^{[1]}, \tag{4.1}$$

$$x_m(t) = x_0(t) + \int_a^t f(s,x_{m-1}(s))\,\mathrm{d}s - \frac{t-a}{\tau_1-a}\int_a^{\tau_1} f(s,x_{m-1}(s))\,\mathrm{d}s,\ m\in\mathbb{N}. \tag{4.2}$$

The following statement establishes the uniform convergence of sequence (4.2).

Theorem 1 *(Uniform convergence) Assume that there exist a non-negative vector $\rho^{[a]}$ and a matrix K_a such that*

$$\rho^{[a]} \geq \frac{b-a}{2}\delta(\mathcal{U}_a), \quad r(K_a) < \frac{10}{3(b-a)}, \tag{4.3}$$

where $\mathcal{U}_a$ and $\delta(\mathcal{U}_a)$ are defined according to (3.2) and (2.3), and let f satisfy in addition

$$f\in \mathrm{Lip}(K_a,\mathcal{U}_a). \tag{4.4}$$

Then, for any $m\geq 0$ and the parameters from (3.6)

1. *Vector-functions (4.2) are continuously differentiable on $[a,\tau_1]$ and*

$$x_m(a)=\xi,\ x_m(\tau_1)=\lambda^{[1]}, \quad \{x_m(t): t\in[a,\tau_1]\}\subset\mathcal{U}_a.$$

2. *There exists a vector-function x_∞ satisfying*

$$x_\infty(t) = \lim_{m\to\infty} x_m(t) \quad \text{uniformly on } [a,\tau_1]\,. \tag{4.5}$$

3. *The limit x_∞ is a unique continuously differentiable solution with values in $\mathcal{U}_a$ of the perturbed boundary value problem on $[a,\tau_1]$*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t,x(t)) \ + \frac{1}{\tau_1-a}\Psi_a, \tag{4.6}$$

$$x(a)=\xi,\ x(\tau_1)=\lambda^{[1]},$$

where $\Psi_a\in\mathbb{R}^n$ depending on the parameters $\tau_1,\xi,\lambda^{[1]}$ is given by the formula

$$\Psi_a := \lambda^{[1]} - \xi - \int_a^{\tau_1} f(s,x_\infty(s))\,\mathrm{d}s.$$

4. *This error estimate holds on $[a,\tau_1]$:*

$$|x_\infty(t)-x_m\,(t)| \leqslant \frac{10}{9}\alpha_1(t;a,\tau_1)Q_a^m\,(1_n-Q_a)\,\delta(\mathcal{U}_a),$$

where

$$Q_a = \frac{3\,(b-a)}{10}K_a \tag{4.7}$$

and $\alpha_1(t;a,\tau_1)$ is from (2.4).

Proof. Note, that (4.6) is an ordinary differential equation and differs from the original equation (1.1) by the additive constant forcing term $\Psi_a/(\tau_1 - a)$. We will argue similarly as in [11] using Lemmas 1 and 2. Since $\tau_1 \in (a,b)$, $\xi \in \Omega_a$, $\lambda^{[1]} \in \Omega_1$, we get by (2.2), (3.2) and (4.1),

$$\{x_0(t) : t \in [a,\tau_1]\} \subset \mathcal{H}(\Omega_a, \Omega_1) \subset \mathcal{U}_a.$$

Consider $m \in \mathbb{N}$ and assume that

$$\{x_m(t) : t \in [a,\tau_1]\} \subset \mathcal{U}_a.$$

Then, by (2.3), (2.5), (2.6), (4.2) and (4.3),

$$|x_{m+1}(t) - x_0(t)| \le \alpha_1(t; a, \tau_1)\, \delta(\mathcal{U}_a) \le \rho^{[a]}, \quad t \in [a,\tau_1]. \tag{4.8}$$

Hence

$$\{x_{m+1}(t) : t \in [a,\tau_1]\} \subset \mathcal{U}_a.$$

Further, using (2.7), (4.2), (4.4) and (4.8), we get

$$\begin{aligned}|x_2(t) - x_1(t)| &\le K_a \left(\left(1 - \frac{t-a}{\tau_1 - a}\right) \int_a^t |x_1(s) - x_0(s)|\, \mathrm{d}s + \frac{t-a}{\tau_1 - a} \int_t^{\tau_1} |x_1(s) - x_0(s)|\, \mathrm{d}s \right) \\ &\le K_a\, \delta(\mathcal{U}_a)\, \alpha_2(t; a, \tau_1), \quad t \in [a,\tau_1],\end{aligned}$$

and, by induction,

$$|x_{m+1}(t) - x_m(t)| \le K_a^m\, \delta(\mathcal{U}_a)\, \alpha_{m+1}(t; a, \tau_1), \quad t \in [a,\tau_1], \quad m \in \mathbb{N}.$$

Consequently, according to (2.8), (4.3) and (4.7), we obtain $r(Q_a) < 1$ and

$$|x_{m+1}(t) - x_m(t)| \le K_a^m\, \delta(\mathcal{U}_a) \frac{10}{9} \left(\frac{3(b-a)}{10}\right)^m \alpha_1(t; a, \tau_1) \le \frac{10}{9}\, Q_a^m\, \delta(\mathcal{U}_a), \quad t \in [a,\tau_1], \quad m \in \mathbb{N}. \tag{4.9}$$

Consider $m, j \in \mathbb{N}$. Then, (4.9) yields

$$|x_{m+j}(t) - x_m(t)| \le \frac{10}{9} \alpha_1(t; a, \tau_1)\, Q_a^m \sum_{i=0}^{j-1} Q_a^i\, \delta(\mathcal{U}_a) \le \frac{10}{9} \alpha_1(t; a, \tau_1)\, Q_a^m \left(1_n - Q_a\right)^{-1} \delta(\mathcal{U}_a), \quad t \in [a,\tau_1],$$

and the remaining assertions of Theorem 1 follow. □

5 Iterations for after-jumps evolutions

Let $k \in \{1, \dots, p\}$. To study problem (3.5) on $[\tau_k, \tau_{k+1}] \times \mathcal{U}_k$ we introduce the parameterized sequence of vector-functions

$$y_m^{[k]}(t) := y_m^{[k]}(t; \tau_k, \tau_{k+1}, \lambda^{[k]}, \lambda^{[k+1]}), \quad t \in [\tau_k, \tau_{k+1}],$$

with the parameters $\tau_k, \tau_{k+1}, \lambda^{[k]}, \lambda^{[k+1]}$ from (3.6) by the relations

$$\begin{aligned}y_0^{[k]}(t) &= (\lambda^{[k]} + \gamma_k(\lambda^{[k]})) + \frac{t - \tau_k}{\tau_{k+1} - \tau_k} \left(\lambda^{[k+1]} - \lambda^{[k]} - \gamma_k(\lambda^{[k]})\right) \\ &= \left(1 - \frac{t - \tau_k}{\tau_{k+1} - \tau_k}\right) (\lambda^{[k]} + \gamma_k(\lambda^{[k]})) + \frac{t - \tau_k}{\tau_{k+1} - \tau_k} \lambda^{[k+1]},\end{aligned} \tag{5.1}$$

$$y_m^{[k]}(t) = y_0^{[k]}(t) + \int_{\tau_k}^{t} f(s, y_{m-1}^{[k]}(s))\, \mathrm{d}s - \frac{t - \tau_k}{\tau_{k+1} - \tau_k} \int_{\tau_k}^{\tau_{k+1}} f(s, y_{m-1}^{[k]}(s))\, \mathrm{d}s, \ m \in \mathbb{N}. \tag{5.2}$$

Theorem 2 *(Uniform convergence) Assume that for $k \in \{1,2,...,p\}$ there exist a non-negative vector $\rho^{[k]}$ and a matrix K_k such that*

$$\rho^{[k]} \geq \frac{b-a}{2}\delta(\mathcal{U}_k), \quad r(K_k) < \frac{10}{3(b-a)}, \tag{5.3}$$

where $\mathcal{U}_k$ and $\delta(\mathcal{U}_k)$ are defined according to (3.3) and (2.3), and let f satisfy in addition

$$f \in \mathrm{Lip}(K_k, \mathcal{U}_k). \tag{5.4}$$

Then, for any $m \geq 0$, $k \in \{1,\dots,p\}$ and the parameters from (3.6)

1. *Vector-functions (5.2) are continuously differentiable on $[\tau_k, \tau_{k+1}]$ and*

$$y_m^{[k]}(\tau_k) = \lambda^{[k]} + \gamma_k(\lambda^{[k]}), \quad y_m^{[k]}(\tau_{k+1}) = \lambda^{[k+1]}, \quad \left\{y_m^{[k]}(t) : t \in [\tau_k, \tau_{k+1}]\right\} \subset \mathcal{U}_k.$$

2. *There exists a vector-function $y_\infty^{[k]}$ satisfying*

$$y_\infty^{[k]}(t) = \lim_{m\to\infty} y_m^{[k]}(t) \quad \textit{uniformly on } [\tau_k, \tau_{k+1}]. \tag{5.5}$$

3. *The limit $y_\infty^{[k]}$ is a unique continuously differentiable solution with values in $\mathcal{U}_k$ of the perturbed boundary value problem on $[\tau_k, \tau_{k+1}]$*

$$\frac{\mathrm{d}y^{[k]}(t)}{\mathrm{d}t} = f\left(t, y^{[k]}(t)\right) + \frac{1}{\tau_{k+1}-\tau_k}\Psi_k, \tag{5.6}$$

$$y^{[k]}(\tau_k) = \lambda^{[k]} + \gamma_k(\lambda^{[k]}), \quad y^{[k]}(\tau_{k+1}) = \lambda^{[k+1]},$$

where $\Psi_k \in \mathbb{R}^n$ depending on the parameters $\tau_k, \tau_{k+1}, \lambda^{[k]}, \lambda^{[k+1]}$ is given by the formula

$$\Psi_k := \lambda^{[k+1]} - \lambda^{[k]} - \gamma_k(\lambda^{[k]}) - \int_{\tau_k}^{\tau_{k+1}} f(s, y_\infty^{[k]}(s))\,\mathrm{d}s.$$

4. *This error estimate holds on $[\tau_k, \tau_{k+1}]$:*

$$\left|y_\infty^{[k]}(t) - y_m^{[k]}(t)\right| \leqslant \frac{10}{9}\alpha_1(t;\tau_k,\tau_{k+1}) Q_k^m \left(1_n - Q_k\right)^{-1} \delta(\mathcal{U}_k),$$

where

$$Q_k = \frac{3\,(b-a)}{10} K_k$$

and $\alpha_1(t;\tau_k,\tau_{k+1})$ is from (2.4).

Proof. Note, that (5.6) is an ordinary differential equation and differs from the original equation (1.1) by the additive constant forcing term $\Psi_k/(\tau_{k+1}-\tau_k)$. So, we can argue as in the proof of Theorem 1. □

Under the conditions of the above theorems, we can construct for $m \geq 0$ the vector-functions

$$u_m(t) := \begin{cases} x_m(t) & \text{if} \quad t \in [a, \tau_1]\,, \\ y_m^{[k]}(t) & \text{if} \quad t \in (\tau_k, \tau_{k+1}]\,,\ k = 1,2,...,p, \end{cases}$$

and their limit

$$u_\infty(t) := \begin{cases} x_\infty(t) & \text{if} \quad t \in [a, \tau_1]\,, \\ y_\infty^{[k]}(t) & \text{if} \quad t \in (\tau_k, \tau_{k+1}]\,,\ k = 1,2,...,p, \end{cases} \tag{5.7}$$

where

$$u_\infty : [a,b] \to \mathcal{U}_a \cup \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_p \subset \mathbb{R}^n$$

depends on the parameters $\tau_1, \ldots, \tau_p \in (a,b)$, $\xi \in \Omega_a$, $\lambda^{[k]} \in \Omega_k$, $k = 1, \ldots, p+1$. The vector-function (5.7) with suitable values of these parameters is a solution of the original impulsive boundary value problem (1.1)-(1.3) with p jumps. The appropriate values of the parameters are specified in the next section.

If some problem under investigation is such that conditions (4.3), (4.4) or (5.3), (5.4) are not fulfilled we suggest to modify an interval halving procedure to this problem. The interval halving procedure is described in [8], [9] or [10] for problems without impulses.

6 Determining equations

We note again, that equations (4.6) and (5.6) are ordinary differential equations that differ from the original equation (1.1) by constant forcing terms. This simple observation allows us to argue as in [11] and get that the limits x_∞ and $y_\infty^{[k]}$, $k = 1, 2, ..., p$, in Theorems 1 and 2 are related to the original impulsive boundary value (1.1)-(1.3) with p jumps in the following way.

Theorem 3 *Let the conditions of Theorems 1 and 2 be fulfilled and let x_∞ and $y_\infty^{[k]}$, $k = 1, \ldots, p$, be from (4.5) and (5.5). Then the following assertions hold.*

1. Assume that the system of algebraic determining equations for unknown parameters $\tau_1, \ldots, \tau_p$, ξ, $\lambda^{[k]}$, $k = 1, \ldots, p+1$,

$$\left.\begin{aligned} &\Psi_a := \lambda^{[1]} - \xi - \int_a^{\tau_1} f(s, x_\infty(s))\, ds = 0, \\ &\Psi_k := \lambda^{[k+1]} - \lambda^{[k]} - \gamma_k(\lambda^{[k]}) - \int_{\tau_k}^{\tau_{k+1}} f(s, y_\infty^{[k]}(s))\, ds = 0, \;\; k = 1, 2, ..., p, \\ &g(\tau_k, \lambda^{[k]}) = 0, \;\; k = 1, \ldots, p, \\ &V(\xi, \lambda^{[p+1]}) = 0, \end{aligned}\right\} \tag{6.1}$$

has a solution

$$\tau_1^*, \tau_2^*, ..., \tau_p^*, \;\; \xi^*, \;\; \lambda^{[k]*}, \;\; k = 1, \ldots, p+1, \tag{6.2}$$

where $a < \tau_1^* < \tau_1^* < \cdots < \tau_p^* < b$, $\xi^* \in \Omega_a$, $\lambda^{[k]*} \in \Omega_k$, $k = 1, \ldots, p+1$.

Finally, consider the limits x_∞^* and $y_\infty^{[k]*}$ determined in (4.5) and (5.5) by means of values (6.2) and assume that

$$\left.\begin{aligned} &g(t, x_\infty^*(t)) \neq 0, && t \in [a, \tau_1^*), \\ &g(t, y_\infty^{[k]*}(t)) \neq 0, && t \in [\tau_k^*, \tau_{k+1}^*), \;\; k = 1, 2, ..., p-1, \\ &g(t, y_\infty^{[p]*}(t) \neq 0, && t \in [\tau_p^*, b]. \end{aligned}\right\} \tag{6.3}$$

Then the vector-function

$$u_\infty^*(t) := \begin{cases} x_\infty^*(t) & \text{if} \quad t \in [a, \tau_1^*], \\ y_\infty^{[k]*}(t) & \text{if} \quad t \in (\tau_k^*, \tau_{k+1}^*], \;\; k = 1, 2, ..., p-1, \\ y_\infty^{[p]*}(t) & \text{if} \quad t \in (\tau_p^*, b] \end{cases} \tag{6.4}$$

is a solution of the impulsive boundary value (1.1)-(1.3) with p jumps at the moments $\tau_1^*, \tau_2^*, ..., \tau_p^*$.

2. If u is a solution of problem (1.1)-(1.3) with p jumps at the moments $\tau_1^*, \tau_2^*, ..., \tau_p^*$ such that

$$\left.\begin{gathered} \{u(t) : t \in [a, \tau_1^*]\} \subset \mathcal{U}_a, \quad \{u(t) : t \in (\tau_p^*, b]\} \subset \mathcal{U}_p, \\ \{u(t) : t \in (\tau_k^*, \tau_{k+1}^*]\} \subset \mathcal{U}_k, \;\; k = 1, 2, ..., p-1, \\ u(a) \in \Omega_a, \;\; u(b) \in \Omega_{p+1}, \;\; u(\tau_k^*) \in \Omega_k, \;\; k = 1, \ldots, p, \end{gathered}\right\} \tag{6.5}$$

then the values

$$\xi^* = u(a), \quad \tau_k^*, \quad \lambda^{[k]*} = u(\tau_k^*),\ k = 1, \ldots, p, \quad \lambda^{[p+1]*} = u(b)$$

necessarily satisfy the system of determining equations (6.1).

Remark 1 The system of algebraic determinig equations (6.1) consists of $(p+2)n+p$ scalar equations for $(p+2)n+p$ scalar unknown parameters (3.6). So, the number of equations coincides with the number of unknown parameters involved. Under the conditions of Theorem 3, system (6.1) with condition (6.3) allows one to determine all possible solutions u of problem (1.1)-(1.3) with values satisfying (6.5) and having p jumps. Consequently the argument based on Theorem 3 allows one to deal with multiple solutions of the problem.

Remark 2 In general, problem (1.1)-(1.3) can have a solution u with p jumps and another solution v with q jumps, where $p \neq q$. Therefore, if we want to find solutions of problem (1.1)-(1.3) having various number of jumps at intersection points with barrier (1.4), we follow these steps:

Step 1. Choose $p = 1$ and use our scheme with only one possible jump at the point τ_1. Then system (6.1) of $3n+1$ scalar algebraic equations has the form

$$\Psi_a = 0, \quad \Psi_1 = 0, \quad g(\tau_1, \lambda^{[1]}) = 0, \quad V(\xi, b) = 0. \tag{6.6}$$

1. If (6.6) has not a solution, then a solution u of problem (1.1)-(1.3) satisfying $u(a) \in \Omega_a$ and having one jump does not exist.

2. Assume that system (6.6) has a solution

$$\tau_1^* \in (a, b), \quad \xi^* \in \Omega_a, \quad \lambda^{[k]*} \in \Omega_k, \quad k = 1, 2, \tag{6.7}$$

   and let

$$\left.\begin{aligned} &g(t, x_\infty^*(t)) \neq 0, \quad t \in [a, \tau_1^*), \\ &g(t, y_\infty^{[1]*}(t) \neq 0, \quad t \in [\tau_1^*, b], \end{aligned}\right\} \tag{6.8}$$

   where x_∞^* and $y_\infty^{[1]*}$ are the limits determined in (4.5) and (5.5) by means of values (6.7). Then, according to (6.4), we can conclude that the vector-function

$$u_\infty^*(t) := \begin{cases} x_\infty^*(t) & \text{if} \quad t \in [a, \tau_1^*]\,, \\ y_\infty^{[1]*}(t) & \text{if} \quad t \in (\tau_1^*, b]. \end{cases}$$

   is a solution of the impulsive boundary value problem (1.1)-(1.3) with one jump at the moment τ_1^*.

3. If (6.8) is not fulfilled, that is $g(\cdot, x_\infty^*(\cdot))$ has one or more roots in (a, τ_1^*) or $g(\cdot, y_\infty^{[1]*}(\cdot))$ has one or more roots in (τ_1^*, b), then according to Definition 1 a solution u of problem (1.1)-(1.3) satisfying $u(a) \in \Omega_a$ and having one jump does not exist.

Step 2. If we failed in searching a solution with one jump, there is a possibility that there exists a solution of problem (1.1)-(1.3) with more intersection points with barrier (1.4) and with corresponding jumps at these points. Note, that a solution with more jumps can exist even in cases where we found a one-jump solution in Step 1. See Example 1. Therefore we repeat our scheme for higher p.

1. For $p = 2$ system (6.1) of $4n+2$ scalar algebraic equations has the form

$$\Psi_a = 0, \quad \Psi_k = 0, \quad g(\tau_k, \lambda^{[k]}) = 0,\ k = 1, 2, \quad V(\xi, b) = 0. \tag{6.9}$$

2. If (6.9) has not a solution, then a solution u of problem (1.1)-(1.3) satisfying $u(a) \in \Omega_a$ and having two jumps does not exist.

3. Assume that system (6.9) has a solution

$$\tau_1^*, \tau_2^* \in (a, b), \quad \xi^* \in \Omega_a, \quad \lambda^{[k]*} \in \Omega_k, \quad k = 1, 2, 3, \tag{6.10}$$

and let

$$\left.\begin{aligned} &g(t, x^*_\infty(t)) \neq 0, \quad t \in [a, \tau^*_1), \\ &g(t, y^{[1]*}_\infty(t) \neq 0, \quad t \in [\tau^*_1, \tau^*_2), \\ &g(t, y^{[2]*}_\infty(t) \neq 0, \quad t \in [\tau^*_2, b], \end{aligned}\right\} \tag{6.11}$$

where x^*_∞ and $y^{[k]*}_\infty$, $k = 1, 2$, are determined in (4.5) and (5.5) by means of values (6.10). Then, due to (6.4), we can conclude that the vector-function

$$u^*_\infty(t) := \begin{cases} x^*_\infty(t) & \text{if} \quad t \in [a, \tau^*_1]\,, \\ y^{[1]*}_\infty(t) & \text{if} \quad t \in (\tau^*_1, \tau^*_2], \\ y^{[2]*}_\infty(t) & \text{if} \quad t \in (\tau^*_2, b], \end{cases}$$

is a solution of the impulsive boundary value problem (1.1)-(1.3) with two jumps at the moments τ^*_1, τ^*_2.

4. If (6.11) is not fulfilled, then a solution u of problem (1.1)-(1.3) satisfying $u(a) \in \Omega_a$ and having two jumps does not exist.

5. We continue our calculations for $p \geq 3$.

Practical realization of this computation is discussed in Sections 7 and 8 and illustrated in Section 9.

7 Approximation of solutions

Let us fix $m \in \mathbb{N}$. The solvability of the determining algebraic system (6.1) can be established by studying its approximate version

$$\left.\begin{aligned} &\Psi_{a,m} := \lambda^{[1]} - \xi - \int_a^{\tau_1} f(s, x_m(s))\,\mathrm{d}s = 0, \\ &\Psi_{k,m} := \lambda^{[k+1]} - \lambda^{[k]} - \gamma_k(\lambda^{[k]}) - \int_{\tau_k}^{\tau_{k+1}} f(s, y^{[k]}_m(s))\,\mathrm{d}s = 0, \; k = 1, 2, ..., p, \\ &g(\tau_k, \lambda^{[k]}) = 0, \quad k = 1, \ldots, p, \\ &V(\xi, \lambda^{[p+1]}) = 0, \end{aligned}\right\} \tag{7.1}$$

with the additional conditions

$$\left.\begin{aligned} &g(t, x_m(t)) \neq 0, \quad t \in [a, \tau_1), \\ &g(t, y^{[k]}_m(t)) \neq 0, \quad t \in [\tau_k, \tau_{k+1}), \; k = 1, 2, ..., p-1, \\ &g(t, y^{[p]}_m(t)) \neq 0, \quad t \in [\tau_p, b], \end{aligned}\right\} \tag{7.2}$$

where

$$a < \tau_1 < \tau_2 < \cdots < \tau_{p-1} < \tau_p < \tau_{p+1} = b.$$

Clearly, the approximate determining system (7.1) is obtained from the exact system (6.1) by replacing the limits x_∞ and $y^{[k]}_\infty$ from (4.5) and (5.5) by the iterations x_m and $y^{[k]}_m$ from (4.2) and (5.2), respectively. It is important, that all the terms involved in (7.1) and (7.2) can be constructed explicitly.

Assume that the values

$$\widehat{\tau}_1, \widehat{\tau}_2, ..., \widehat{\tau}_p \in (a, b)\,, \quad \widehat{\xi} \in \Omega_a, \quad \widehat{\lambda}^{[k]} \in \Omega_k, \quad k = 1, \ldots, p+1, \tag{7.3}$$

are a solution of system (7.1). Consider the vector-functions $\widehat{x}_m$ and $\widehat{y}^{[k]}_m$ determined in (4.2) and (5.2) by means of values (7.3). If $\widehat{x}_m$ and $\widehat{y}^{[k]}_m$ satisfy (7.2) with $\tau_k = \widehat{\tau}_k$, $k = 1, \ldots, p$, then the vector-function

$$\widehat{u}(t) := \begin{cases} \widehat{x}_m(t), & t \in [a, \widehat{\tau}_1]\,, \\ \widehat{y}^{[k]}_m(t), & t \in (\widehat{\tau}_k, \widehat{\tau}_{k+1}], \; k = 1, 2, ..., p-1, \\ \widehat{y}^{[p]}_m(t), & t \in (\widehat{\tau}_p, b]. \end{cases} \tag{7.4}$$

undergoes the jump of the value $\gamma_k(\widehat{\lambda}^{[k]})$ at the moment $\widehat{\tau}_k$, $k = 1,\dots,p$. Recalling Theorems 1 and 2, we have the estimates for the limits $\widehat{x}_\infty$ and $\widehat{y}_\infty^{[k]}$

$$|\widehat{x}_\infty - \widehat{x}_m(t)| \leqslant \frac{10}{9}\alpha_1(t;a,\widehat{\tau}_1 - a)Q_a^m\,(1_n - Q_a)\,\delta(\mathcal{U}_a), \tag{7.5}$$

$$\left|\widehat{y}_\infty^{[k]}(t) - \widehat{y}_m^{[k]}(t)\right| \leqslant \frac{10}{9}\alpha_1(t;\widehat{\tau}_k,\widehat{\tau}_{k+1} - \widehat{\tau}_k)Q_k^m\,(1_n - Q_k)^{-1}\,\delta(\mathcal{U}_k),\ k = 1,\dots,p. \tag{7.6}$$

Estimates (7.5), (7.6) allow one to regard (7.4) as the $m-$th approximation to a solution of problem (1.1)-(1.3) with p jumps. The solvability analysis based on the properties of the approximate determining system (7.1) can be carried out using the topological degree methods as it is done in [11] or [12] for problems without impulses. This topic is not treated here.

8 Frozen parameter scheme

The simplest way how to choose the parameter sets (3.1) is to take a compact convex set

$$\Omega_a \subset \mathbb{R}^n$$

and put

$$\Omega_a = \Omega_1,\ \Omega_1^+ = \Omega_2,\ \dots,\ \Omega_{p-1}^+ = \Omega_p,\ \Omega_p^+ = \Omega_{p+1}, \tag{8.1}$$

where

$$\Omega_k^+ = \{x + \gamma_k(x) : x \in \Omega_k\},\ k = 1,\dots,p. \tag{8.2}$$

Then the sets, which are included in (3.2) have the form

$$\mathcal{U}_a = \underset{v\in\Omega_a}{\cup} B(v,\rho^{[a]}),\quad \mathcal{U}_k = \underset{v\in\Omega_k^+}{\cup} B(v,\rho^{[k]}),\quad k = 1,\dots,p. \tag{8.3}$$

If the assumptions of Theorems 1 and 2 are fulfilled on the sets (8.1)-(8.3) we suggest the following algorithm for an approximate solution of problem (1.1)-(1.3) with p jumps using *frozen parameters.*

1. Due to (4.1) introduce a vector-function x_0 depending on the parameters $\tau_1, \xi, \lambda^{[1]}$ from (3.6). Then calculate the first iteration x_1 by means of (4.2). Similarly, for $k = 1,\dots,p$, use formula (5.1) and introduce a vector-function $y_0^{[k]}$ which depends on the parameters $\tau_k, \tau_{k+1}, \lambda^{[k]}, \lambda^{[k+1]}$ from (3.6). Then calculate the first iteration $y_1^{[k]}$ by means of (5.2).

2. Put $m = 1$ in the approximate determining system (7.1), find its solution called *first frozen parameters* and write this solution as in (7.3).

3. By means of x_1 and $y_1^{[k]}$, $k = 1,\dots,p$, constructed in Step 1 and by the first frozen parameters found in Step 2, introduce the vector-functions

$$\begin{aligned}
&X_1(t) := x_1(t;\widehat{\tau}_1,\widehat{\xi},\widehat{\lambda}^{[1]}),\ t\in[a,\tau_1]\,,\\
&Y_1^{[k]}(t) := y_1^{[k]}(t;\widehat{\tau}_k,\widehat{\tau}_{k+1},\widehat{\lambda}^{[k]},\widehat{\lambda}^{[k+1]}),\ t\in[\tau_k,\tau_{k+1}]\,,\ k = 1,\dots,p-1,\\
&Y_1^{[p]}(t) := y_1^{[p]}(t;\widehat{\tau}_p,b,\widehat{\lambda}^{[p]},\widehat{\lambda}^{[p+1]}),\ t\in[\tau_p,b].
\end{aligned}$$

4. Define the *second frozen iterations*

$$\begin{aligned}
&\widehat{x}_2(t) := \widehat{x}_2(t;\tau_1,\xi,\lambda^{[1]}),\quad t\in[a,\tau_1],\\
&\widehat{y}_2^{[k]}(t) := \widehat{y}_2^{[k]}\big(t;\tau_k,\tau_{k+1},\lambda^{[k]},\lambda^{[k+1]}\big),\ t\in[\tau_k,\tau_{k+1}],\ k = 1,\dots,p,
\end{aligned}$$

according to (4.1), (4.2) and (5.1), (5.2) as follows:

$$\widehat{x}_2(t) = x_0(t) + \int_a^t f(s, X_1(s))\,\mathrm{d}s - \tfrac{t-a}{\tau_1 - a}\int_a^{\tau_1} f(s, X_1(s))\,\mathrm{d}s,\ t \in [a, \tau_1],$$

$$\widehat{y}_2^{[k]}(t) = y_0^{[k]}(t) + \int_{\tau_k}^{t} f(s, Y_1^{[k]}(s))\,\mathrm{d}s - \frac{t - \tau_k}{\tau_{k+1} - \tau_k}\int_{\tau_k}^{\tau_{k+1}} f(s, Y_1^{[k]}(s)\,\mathrm{d}s,$$

$$t \in [\tau_k, \tau_{k+1}], \quad k = 1, \ldots, p.$$

So, the first iterations x_1 and $y_1^{[k]}$ in (4.2) and (5.2) are replaced by X_1 and $Y_1^{[k]}$ introduced in Step 3, respectively.

5. Put $m = 2$, modify the approximate system of determining equations (7.1) by substituting there the second frozen iterations $\widehat{x}_2$ and $\widehat{y}_2^{[k]}$ from Step 4. The resulting modified system of $(p+2)n + p$ scalar algebraic equations has the form

$$\left.\begin{aligned}
&\widehat{\Psi}_{a,2} := \lambda^{[1]} - \xi - \int_a^{\tau_1} f(s, \widehat{x}_2(s))\,\mathrm{d}s = 0,\\
&\widehat{\Psi}_{k,2} := \lambda^{[k+1]} - \lambda^{[k]} - \gamma_k(\lambda^{[k]}) - \int_{\tau_k}^{\tau_{k+1}} f(s, \widehat{y}_2^{[k]}(s))\,\mathrm{d}s = 0,\ k = 1, 2, ..., p,\\
&g(\tau_k, \lambda^{[k]}) = 0, \quad k = 1, \ldots, p,\\
&V(\xi, \lambda^{[p+1]}) = 0.
\end{aligned}\right\} \tag{8.4}$$

Find a solution of (8.4) called the *second frozen parameters* and write this solution as in (7.3).

6. By means of $\widehat{x}_2$ and $\widehat{y}_2^{[k]}$, $k = 1, \ldots, p$, constructed in Step 4 and by the second frozen parameters found in Step 5, introduce the vector-functions

$$X_2(t) := \widehat{x}_2(t; \widehat{\tau}_1, \widehat{\xi}, \widehat{\lambda}^{[1]}),\ t \in [a, \tau_1],$$

$$Y_2^{[k]}(t) := \widehat{y}_2^{[k]}(t; \widehat{\tau}_k, \widehat{\tau}_{k+1}, \widehat{\lambda}^{[k]}, \widehat{\lambda}^{[k+1]}),\ t \in [\tau_k, \tau_{k+1}],\ k = 1, \ldots, p-1,$$

$$Y_2^{[p]}(t) := \widehat{y}_2^{[p]}(t; \widehat{\tau}_p, b, \widehat{\lambda}^{[p]}, \widehat{\lambda}^{[p+1]}),\ t \in [\tau_p, b].$$

7. Define the *third frozen iterations*

$$\widehat{x}_3(t) := \widehat{x}_3(t; \tau_1, \xi, \lambda^{[1]}), \quad t \in [a, \tau_1],$$

$$\widehat{y}_3^{[k]}(t) := \widehat{y}_3^{[k]}\big(t; \tau_k, \tau_{k+1}, \lambda^{[k]}, \lambda^{[k+1]}\big),\ t \in [\tau_k, \tau_{k+1}],\ k = 1, \ldots, p,$$

by

$$\widehat{x}_3(t) = x_0(t) + \int_a^t f(s, X_2(s))\,\mathrm{d}s - \tfrac{t-a}{\tau_1 - a}\int_a^{\tau_1} f(s, X_2(s))\,\mathrm{d}s,\ t \in [a, \tau_1],$$

$$\widehat{y}_3^{[k]}(t) = y_0^{[k]}(t) + \int_{\tau_k}^{t} f(s, Y_2^{[k]}(s))\,\mathrm{d}s - \tfrac{t-\tau_k}{\tau_{k+1}-\tau_k}\int_{\tau_k}^{t_{k+1}} f(s, Y_2^{[k]}(s)\,\mathrm{d}s,$$

$$t \in [\tau_k+, \tau_{k+1}], \quad k = 1, \ldots, p.$$

Hence, by analogy, the second iterations x_2 and $y_2^{[k]}$ in (4.2) and (5.2) are replaced by X_2 and $Y_2^{[k]}$, respectively.

8. Put $m = 3$, modify system (7.1) by substituting there the third frozen iterations $\widehat{x}_3$ and $\widehat{y}_3^{[k]}$ from Step 7. The resulting modified system of $(p+2)n + p$ scalar algebraic equations has the form

$$\left.\begin{aligned}
&\widehat{\Psi}_{a,3} := \lambda^{[1]} - \xi - \int_a^{\tau_1} f(s, \widehat{x}_3(s))\,\mathrm{d}s = 0,\\
&\widehat{\Psi}_{k,3} := \lambda^{[k+1]} - \lambda^{[k]} - \gamma_k(\lambda^{[k]}) - \int_{\tau_k}^{\tau_{k+1}} f(s, \widehat{y}_3^{[k]}(s))\,\mathrm{d}s = 0,\ k = 1, 2, ..., p,\\
&g(\tau_k, \lambda^{[k]}) = 0, \quad k = 1, \ldots, p,\\
&V(\xi, \lambda^{[p+1]}) = 0.
\end{aligned}\right\} \tag{8.5}$$

Find a solution of (8.5) called the *third frozen parameters* and write this solution as in (7.3).

9. Continue in a similar manner and derive higher frozen parameters and higher frozen iterations. If, for some $m > 2$, the $m-$th and $(m-1)$-th frozen parameters are close enough, we put

$$X_m(t) := \widehat{x}_m(t; \widehat{\tau}_1, \widehat{\xi}, \widehat{\lambda}^{[1]}),\ t \in [a, \widehat{\tau}_1]\,,$$
$$Y_m^{[k]}(t) := \widehat{y}_m^{[k]}(t; \widehat{\tau}_k, \widehat{\tau}_{k+1}, \widehat{\lambda}^{[k]}, \widehat{\lambda}^{[k+1]}),\ t \in [\widehat{\tau}_k, \widehat{\tau}_{k+1}]\,,\ k = 1, \ldots, p-1,$$
$$Y_m^{[p]}(t) := \widehat{y}_m^{[p]}(t; \widehat{\tau}_p, b, \widehat{\lambda}^{[p]}, \widehat{\lambda}^{[p+1]}),\ t \in [\widehat{\tau}_p, b],$$

and according to (1.5) verify the condition

$$\left.\begin{aligned} &g(t, X_m(t)) \neq 0,\ t \in [a, \widehat{\tau}_1), \\ &g(t, Y_m^{[k]}(t)) \neq 0,\ t \in [\widehat{\tau}_k, \widehat{\tau}_{k+1}),\ k = 1, 2, ..., p-1, \\ &g(t, Y_m^{[p]}(t)) \neq 0,\ t \in [\widehat{\tau}_p, b]. \end{aligned}\right\} \tag{8.6}$$

If (8.6) is fulfilled, then the function

$$\widehat{u}(t) := \begin{cases} X_m(t), & t \in [a, \widehat{\tau}_1]\,, \\ Y_m^{[k]}(t), & t \in (\widehat{\tau}_k, \widehat{\tau}_{k+1}],\ k = 1, 2, ..., p-1, \\ Y_m^{[p]}(t), & t \in (\widehat{\tau}_p, b], \end{cases} \tag{8.7}$$

is regarded as the m-th approximation of a solution u of problem (1.1)-(1.3) with $u(a) \in \Omega_a$ and p jumps.

If (8.6) is not satisfied, then we discuss the frozen parameter scheme with other numbers of jumps as in Remark 2.

Remark 3 We see that all sets (8.1)-(8.3) are determined by the set Ω_a containing possible starting points of solutions to problem (1.1)-(1.3) with p jumps and by the vectors $\rho^{[a]}$, $\rho^{[k]}$, $k = 1, \ldots, p$. A choise of Ω_a can follow from a given practical problem which is modelled by (1.1)-(1.3). Alternatively, assumptions imposed on the set Ω_a are just those in Theorems 1, 2 and 3 and we can try more possibilities for its choise. In the both cases it is useful to start our computation directly at $m = 0$, where no iterations are needed and one works with x_0 and $y_0^{[k]}$ from (4.1) and (5.1) only. Being piecewise linear functions, these zero-th approximations are very rough but, nevertheless, they are usually helpful as a preliminary shot. In particular, the roots $\widehat{\xi}$, $\widehat{\tau}_1, \ldots, \widehat{\tau}_p$, $\widehat{\lambda}^{[1]}, \ldots, \widehat{\lambda}^{[p+1]}$ of the zero-th approximate determining system which consists of $(p+2)n + p$ scalar algebraic equations and has the form

$$\left.\begin{aligned} &\Psi_{a,0} := \lambda^{[1]} - \xi - \int_a^{\tau_1} f(s, x_0(s))\,\mathrm{d}s = 0, \\ &\Psi_{k,0} := \lambda^{[k+1]} - \lambda^{[k]} - \gamma_k(\lambda^{[k]}) - \int_{\tau_k}^{\tau_{k+1}} f(s, y_0^{[k]}(s))\,\mathrm{d}s = 0,\ k = 1, 2, ..., p, \\ &g(\tau_k, \lambda^{[k]}) = 0, \quad k = 1, \ldots, p, \\ &V(\xi, \lambda^{[p+1]}) = 0, \end{aligned}\right\} \tag{8.8}$$

can provide a hint helping one to choose the set Ω_a as a neighbourhood of $\widehat{\xi}$ in a suitable way and avoid unnecessary computations on sets that might possibly be excessively large.

Another possible algorithm which could be adopted for practical computations of approximate solutions for problem (1.1)-(1.3) is the scheme with a polynomial interpolation presented in [12] for a non-impulsive Dirichlet problem.

9 Examples

Example 1 *Two jumps.* Put $n = 2$ and apply the numerical-analytic approach desribed above to the system of two differential equations on the interval $[0, 0.5]$

$$\frac{\mathrm{d}u_1(t)}{\mathrm{d}t} = u_2^2(t) - u_1^2(t) + t, \quad \frac{\mathrm{d}u_2(t)}{\mathrm{d}t} = u_1^2(t) - u_2^2(t) - t. \tag{9.1}$$

Put $p = 2$, consider the barrier

$$G = \{(t, x) \in [0, 0.5] \times \mathbb{R}^2 : x_1 - 7.233\bar{3}\, t^2 + 2.368\bar{3}\, t - 0.04 = 0\} \tag{9.2}$$

and the state-dependent impulse conditions at two unknown points τ_1 and τ_2

$$\left.\begin{array}{ll} u_1(\tau_1+) - u_1(\tau_1) = 0.01, & u_2(\tau_1+) - u_2(\tau_1) = -0.01, \\ u_1(\tau_2+) - u_1(\tau_2) = 0.015, & u_2(\tau_2+) - u_2(\tau_2) = -0.015, \end{array}\right\} \tag{9.3}$$

where, by (1.5), τ_1 and τ_2 have to satisfy

$$\left.\begin{array}{ll} u_1(\tau_k) - 7.233\bar{3}\, \tau_k^2 + 2.368\bar{3}\, \tau_k - 0.04 = 0, & k = 1, 2, \\ u_1(t) - 7.233\bar{3}\, t^2 + 2.368\bar{3}\, t - 0.04 \neq 0, & t \in [0, 0.5] \setminus \{\tau_1, \tau_2\}. \end{array}\right\} \tag{9.4}$$

Finally, consider the nonlinear boundary conditions

$$u_1^2(0) + u_2(0.5) + 0.125 = 0, \quad u_1^2(0.5) + u_2(0) - 0.015625 = 0. \tag{9.5}$$

We are interested in a solution of problem (9.1), (9.3), (9.5) as defined in Definition 1 with $n = p = 2$. Let us describe in detail individual steps of our method. Here $a = 0$, $b = 0.5$ and $f = \mathrm{col}(f_1, f_2)$, where

$$f_1(t, x_1, x_2) = x_2^2 - x_1^2 + t, \quad f_2(t, x_1, x_2) = x_1^2 - x_2^2 - t. \tag{9.6}$$

The impulse vector-functions γ_1 and γ_2 in (1.6) are constant here

$$\gamma_1 = \mathrm{col}(0.01, -0.01), \quad \gamma_2 = \mathrm{col}(0.015, -0.015), \tag{9.7}$$

the barrier function g has the form

$$g(t, x) = x_1 - 7.233\bar{3}\, t^2 + 2.368\bar{3}\, t - 0.04, \tag{9.8}$$

and the boundary vector-function $V = \mathrm{col}(V_1, V_2)$ can be written as

$$V_1(x_1, x_2, y_1, y_2) = x_1^2 + y_2 + 0.125, \quad V_2(x_1, x_2, y_1, y_2) = x_2^2 + y_1 - 0.015625. \tag{9.9}$$

Introduce the zero-th iterations x_0, $y_0^{[k]}$, $k = 1, 2$, solve system (8.8) of 10 algebraic equations and obtain the roots $\widehat{\tau}_1, \widehat{\tau}_2, \widehat{\xi}_1, \widehat{\xi}_2, \widehat{\lambda}_1^{[1]}, \widehat{\lambda}_2^{[1]} \lambda_1^{[2]}, \widehat{\lambda}_2^{[2]}, \widehat{\lambda}_1^{[3]}, \widehat{\lambda}_2^{[3]}$, presented in the first column of Table 1. Having

$$\widehat{\xi}_1 = -0.1059217222, \quad \widehat{\xi}_2 = 0.01369007648, \tag{9.10}$$

choose

$$\Omega_0 = [-0.14, 0.04] \times [-0.18, 0.03] = \Omega_1,$$

and then, by (8.1), (8.2), (9.7), get

$$\begin{aligned} \Omega_1^+ &= [-0.15, 0.05] \times [-0.19, 0.04] = \Omega_2, \\ \Omega_2^+ &= [-0.165, 0.065] \times [-0.205, 0.055] = \Omega_3. \end{aligned}$$

Now, choose the vectors

$$\rho^{[0]} = \mathrm{col}(0.1, 0.1), \quad \rho^{[1]} = \mathrm{col}(0.15, 0.15), \quad \rho^{[2]} = \mathrm{col}(0.1, 0.15),$$

and using (8.3) construct the sets

$$\begin{aligned} \mathcal{U}_0 &= [-0.24, 0.14] \times [-0.28, 0.13], \\ \mathcal{U}_1 &= [-0.30, 0.20] \times [-0.34, 0.19], \\ \mathcal{U}_2 &= [-0.265, 0.165] \times [-0.355, 0.205]. \end{aligned}$$

Maple computations give that conditions (4.3) and (4.4) are fulfilled with the matrix

$$K_0 =$$

and conditions (5.3) and (5.4), $k = 1, 2$, are fulfilled with the matrices

$$K_1 = \quad K_2 =$$

Therefore the frozen parameter scheme suggested in Section 8 can be applied using both symbolic and numerical Maple computations.

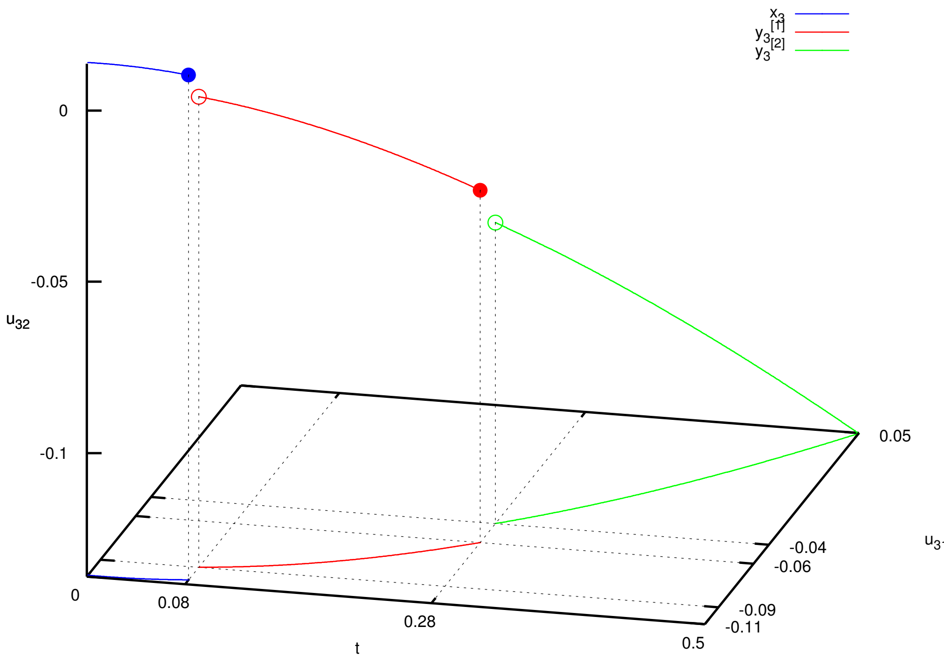

Figure 1: Third approximation $\widehat{u}_3$ of a solution to problem (9.1), (9.3), (9.5)

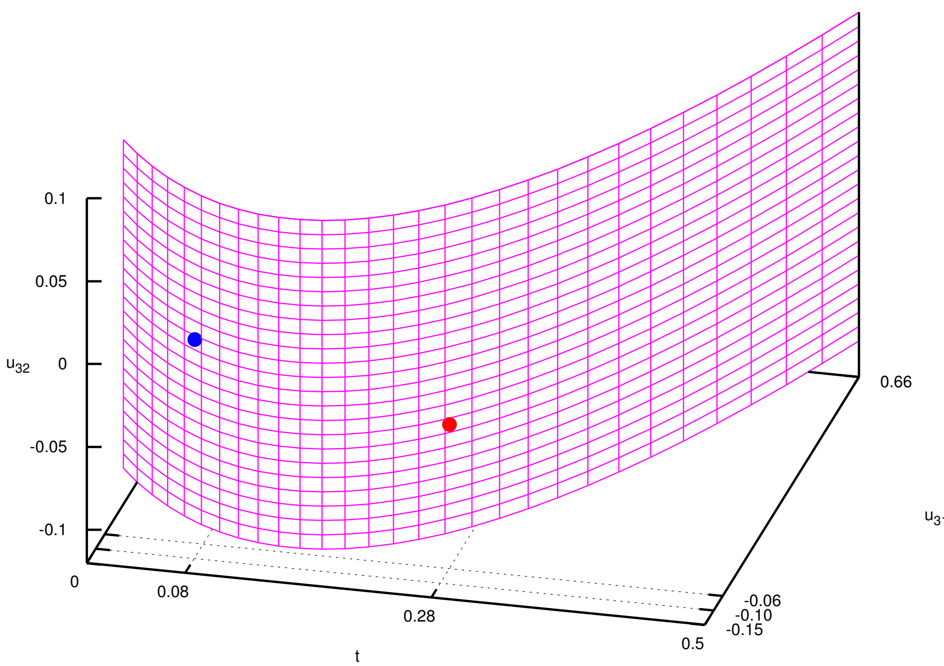

Figure 2: Barrier (9.2)

1. Calculate the first iterations x_1, $y_1^{[1]}$ and $y_1^{[2]}$.

2. Put $m = 1$ and solve system (7.1) which consists of 10 scalar algebraic equations with the unknowns $\tau_1, \tau_2, \xi_1, \xi_2, \lambda_1^{[1]}, \lambda_2^{[1]} \lambda_1^{[2]}, \lambda_2^{[2]}, \lambda_1^{[3]}, \lambda_2^{[3]}$. Under the restrictions $\tau_1 \in (0.00001, 0.25)$, $\tau_2 \in (0.25, 0.5)$ numerical computations by Maple give the roots (first frozen parameters) written in the second column of Table 1.

3. Introduce vector-functions

$$X_1 = \mathrm{col}(X_{11}, X_{12}), \quad Y_1^{[1]} = \mathrm{col}(Y_{11}^{[1]}, Y_{12}^{[1]}), \quad Y_1^{[2]} = \mathrm{col}(Y_{11}^{[2]}, Y_{12}^{[2]})$$

as follows. Using the first frozen parameters put

$$\begin{aligned} X_1(t) &:= x_1(t; \widehat{\tau}_1, \widehat{\xi}, \widehat{\lambda}^{[1]}),\ t \in [0, \widehat{\tau}_1], \\ Y_1^{[1]}(t) &:= y_1^{[1]}(t; \widehat{\tau}_1, \widehat{\tau}_2, \widehat{\lambda}^{[1]}, \widehat{\lambda}^{[2]}),\ t \in [\widehat{\tau}_1, \widehat{\tau}_2], \\ Y_1^{[2]}(t) &:= y_1^{[2]}(t; \widehat{\tau}_2, 0.5, \widehat{\lambda}^{[2]}, \widehat{\lambda}^{[3]}),\ t \in [\widehat{\tau}_2, 0.5], \end{aligned}$$

and get their componentwise form:

$$\begin{aligned} X_{11}(t) &= -0.1049467336 - 1.356666667 \cdot 10^{-11} t^3 + 0.5026541335\, t^2 \\ &\quad -0.01092495168\, t,\ t \in [0, \widehat{\tau}_1], \\ X_{12}(t) &= 0.01362740290 + 1.356666667 \cdot 10^{-11}\, t^3 - 0.5026541335\, t^2 \\ &\quad +0.01092495181 t,\ t \in [0, \widehat{\tau}_1], \\ Y_{11}^{[1]}(t) &= -0.09496693961 + 0.5158065535\, t^2 - 0.01171732438\, t,\ t \in [\widehat{\tau}_1, \widehat{\tau}_2], \\ Y_{12}^{[1]}(t) &= 0.00364760892 - 0.5158065535 t^2 + 0.01171732445\, t,\ t \in [\widehat{\tau}_1, \widehat{\tau}_2], \\ Y_{11}^{[2]}(t) &= -0.07909248307 + 0.5362580815 t^2 - 0.02055510225\, t,\ t \in [\widehat{\tau}_2, 0.5], \\ Y_{12}^{[2]}(t) &= -0.0122268475 - 0.5362580815\, t^2 + 0.02055510169\, t,\ t \in [\widehat{\tau}_2, 0.5]. \end{aligned}$$

4. Define the second frozen iterations $\widehat{x}_2$, $\widehat{y}_2^{[1]}$, $\widehat{y}_2^{[2]}$.

5. Put $m = 2$, solve system (8.4) of 10 scalar algebraic equations and get the second frozen parameters written in the third column of Table 1.

6. Using the second frozen parameters put

$$\begin{aligned} X_2(t) &:= \widehat{x}_2(t; \widehat{\tau}_1, \widehat{\xi}, \widehat{\lambda}^{[1]}),\ t \in [0, \widehat{\tau}_1], \\ Y_2^{[1]}(t) &:= \widehat{y}_2^{[1]}(t; \widehat{\tau}_1, \widehat{\tau}_2, \widehat{\lambda}^{[1]}, \widehat{\lambda}^{[2]}),\ t \in [\widehat{\tau}_1, \widehat{\tau}_2], \\ Y_2^{[2]}(t) &:= \widehat{y}_2^{[2]}(t; \widehat{\tau}_2, 0.5, \widehat{\lambda}^{[2]}, \widehat{\lambda}^{[3]}),\ t \in [\widehat{\tau}_2, 0.5], \end{aligned}$$

7. Define the third frozen iterations $\widehat{x}_3$, $\widehat{y}_3^{[1]}$, $\widehat{y}_3^{[2]}$.

8. Put $m = 3$, solve system (8.5) of 10 scalar algebraic equations and get the third frozen parameters written in the last column of Table 1.

9. Using the third frozen parameters put

$$\begin{aligned} X_3(t) &:= \widehat{x}_3(t; \widehat{\tau}_1, \widehat{\xi}, \widehat{\lambda}^{[1]}),\ t \in [0, \widehat{\tau}_1], \\ Y_3^{[1]}(t) &:= \widehat{y}_3^{[1]}(t; \widehat{\tau}_1, \widehat{\tau}_2, \widehat{\lambda}^{[1]}, \widehat{\lambda}^{[2]}),\ t \in [\widehat{\tau}_1, \widehat{\tau}_2], \\ Y_3^{[2]}(t) &:= \widehat{y}_3^{[2]}(t; \widehat{\tau}_2, 0.5, \widehat{\lambda}^{[2]}, \widehat{\lambda}^{[3]}),\ t \in [\widehat{\tau}_2, 0.5], \end{aligned}$$

and show that condition (8.6) with $p = 2$ holds for $m = 3$. More precisely, for $\widehat{\tau}_1 = 0.07955623539$ and $\widehat{\tau}_2 = 0.2787337381$

$$X_{31}(t) - 7.233\bar{3}\,t^2 + 2.368\bar{3}\,t - 0.04 \neq 0,\ t \in [0, \widehat{\tau}_1),$$
$$Y^{[1]}_{31}(t) - 7.233\bar{3}\,t^2 + 2.368\bar{3}\,t - 0.04 \neq 0,\ t \in [\widehat{\tau}_1, \widehat{\tau}_2),$$
$$Y^{[2]}_{31}(t) - 7.233\bar{3}\,t^2 + 2.368\bar{3}\,t - 0.04 \neq 0,\ t \in [\widehat{\tau}_2, 0.5].$$

Consequently, the vector-function

$$\widehat{u}_3(t) = \begin{cases} X_3(t) & \text{if} \quad t \in [0, \widehat{\tau}_1]\,, \\ Y^{[1]}_3(t) & \text{if} \quad t \in [\widehat{\tau}_1, \widehat{\tau}_2]\,, \\ Y^{[2]}_3(t) & \text{if} \quad t \in [\widehat{\tau}_2, 0.5]\,, \end{cases} \tag{9.11}$$

is the third approximation to a solution of problem (9.1), (9.3), (9.5).

The graph and its orthogonal projection of the third approximation $\widehat{u}_3$ of a solution to problem (9.1), (9.3), (9.5) are on Fig.1 while Fig. 2 shows the graph of barrier (9.2) and the points where it is intersected by the graph of $\widehat{u}_3$.

Table 1. Frozen parameters to problem (9.1), (9.3), (9.5)

Variable	$m=0$	$m=1$	$m=2$	$m=3$
$\widehat{\tau}_1$	0.08032359386	0.07955621663	0.07955621664	0.07955623539
$\widehat{\tau}_2$	0.278089169	0.2787337541	0.2787337541	0.2787337381
$\widehat{\xi}_1$	−0.1059217222	−0.1049467336	−0.1049467336	−0.1049467573
$\widehat{\xi}_2$	0.01369007648	0.01362740290	0.01362740290	0.01362740444
$\widehat{\lambda}^{[1]}_1$	−0.1035644481	−0.1026344871	−0.1026344871	−0.1026345099
$\widehat{\lambda}^{[1]}_2$	0.01133280237	0.01131515641	0.01131515641	0.01131515702
$\widehat{\lambda}^{[2]}_1$	−0.05922824314	−0.05815864985	−0.05815864985	−0.05815867642
$\widehat{\lambda}^{[2]}_2$	−0.03300340255	−0.03316068084	−0.03316068084	−0.03316067649
$\widehat{\lambda}^{[3]}_1$	0.04398776554	0.04469448620	0.04469448620	0.04469446897
$\widehat{\lambda}^{[3]}_2$	−0.1362194112	−0.1360138169	−0.1360138169	−0.1360138219

Substituting approximation (9.11) into system (9.1), we obtain a residual estimated as follows:

$$\max_{t\in[a,\widehat{\tau}_1]} \left|X'_{31}(t) - X^2_{32}(t) + X^2_{31}(t) - t\right| = 6 \times 10^{-10},$$
$$\max_{t\in[0,\widehat{\tau}_1]} \left|X'_{32}(t) - X^2_{31}(t) + X^2_{32}(t) + t\right| = 3 \times 10^{-10},$$

$$\max_{t\in[\widehat{\tau}_1,\widehat{\tau}_2]} \left|\frac{dY^{[1]}_{31}(t)}{dt} - Y^{[1]^2}_{32}(t) + Y^{[1]^2}_{31}(t) - t\right| = 1 \times 10^{-8},$$
$$\max_{t\in[\widehat{\tau}_1,\widehat{\tau}_2]} \left|\frac{dY^{[1]}_{32}(t)}{dt} - Y^{[1]^2}_{31}(t) + Y^{[1]^2}_{32}(t) + t\right| = 1 \times 10^{-8},$$

$$\max_{t\in[\widehat{\tau}_2,b]} \left|\frac{dY^{[2]}_{31}(t)}{dt} - Y^{[2]^2}_{32}(t) + Y^{[2]^2}_{31}(t) - t\right| = 1 \times 10^{-8},$$
$$\max_{t\in[\widehat{\tau}_2,b]} \left|\frac{dY^{[2]}_{32}(t)}{dt} - Y^{[2]^2}_{31}(t) + Y^{[2]^2}_{32}(t) + t\right| = 1 \times 10^{-8}.$$

One jump. Now, consider system (9.1) on the interval $[0, 0.5]$ with the boundary conditions (9.5) and barrier (9.2) and search for a solution with just one jump. So, we have now $p = 1$ and and the state-dependent impulse condition at one unknown point τ_1

$$u_1(\tau_1+) - u_1(\tau_1) = 0.01, \qquad u_2(\tau_1+) - u_2(\tau_1) = -0.01, \tag{9.12}$$

where, by (1.5), τ_1 satisfies

$$\begin{aligned} &u_1(\tau_1) - 7.233\bar{3}\,\tau_1 + 2.368\bar{3}\,\tau_1 - 0.04 = 0, \\ &u_1(t) - 7.233\bar{3}\,t^2 + 2.368\bar{3}\,t - 0.04 \neq 0, \quad t \in [0, 0.5] \setminus \{\tau_1\}. \end{aligned}$$

Let us find if there exists also a solution of problem (9.1), (9.12), (9.5). Calculation of approximate roots of the corresponding determining system (7.1) with $p = 1$ yields for $m = 3$ the third frozen parameters to problem (9.1), (9.12), (9.5)

$$\begin{aligned} &\widehat{\tau}_1 = 0.3084130198, \\ &\widehat{\xi}_1 = -0.04961902764, && \widehat{\xi}_2 = 0.008218427251, \\ &\widehat{\lambda}_1^{[1]} = -0.002400361867, && \widehat{\lambda}_2^{[1]} = -0.03900023852, \\ &\widehat{\lambda}_1^{[2]} = 0.08606144752, && \widehat{\lambda}_2^{[2]} = -0.1274620479. \end{aligned}$$

Using the third frozen parameters put

$$\begin{aligned} &X_3(t) := \widehat{x}_3(t; \widehat{\tau}_1, \widehat{\xi}, \widehat{\lambda}^{[1]}),\ t \in [a, \widehat{\tau}_1]\,, \\ &Y_3^{[1]}(t) := \widehat{y}_3^{[1]}\big(t; \widehat{\tau}_1, 0.5, \widehat{\lambda}^{[1]}, \lambda^{[2]}\big),\ t \in [\widehat{\tau}_1, 0.5], \end{aligned}$$

and show that condition (8.6) with $p = 1$ holds for $m = 3$. Consequently, the vector-function

$$\widehat{u}_3(t) = \begin{cases} X_3(t) & \text{if} \quad t \in [0, \widehat{\tau}_1]\,, \\ Y_3^{[1]}(t) & \text{if} \quad t \in [\widehat{\tau}_1, 0.5]\,, \end{cases}$$

is the third approximation to a solution of problem (9.1), (9.12), (9.5).

Example 2 *Three jumps.* We apply our technique to the same system (9.1) on the interval $[0, 0.5]$ with the same boundary conditions (9.5) but with a different barrier and three jumps. So, put $p = 3$, consider the barrier

$$G = \big\{(t, x) \in [0, 0.5] \times \mathbb{R}^2 : x_2 + 474.9999931\ t^4 - 476.6666597\ t^3 + 147.2499979\ t^2 \\ -14.43333319\ t + 0.2 = 0\big\} \tag{9.13}$$

and the state-dependent impulse conditions at three unknown points τ_1, τ_2 and τ_3

$$\left.\begin{aligned} &u_1(\tau_1+) - u_1(\tau_1) = 0.01, && u_2(\tau_1+) - u_2(\tau_1) = -0.01, \\ &u_1(\tau_2+) - u_1(\tau_2) = 0.015, && u_2(\tau_2+) - u_2(\tau_2) = -0.015, \\ &u_1(\tau_3+) - u_1(\tau_3) = -0.0012, && u_2(\tau_3+) - u_2(\tau_3) = 0.0012, \end{aligned}\right\} \tag{9.14}$$

where, by (1.5), τ_1, τ_2 and τ_3 have to satisfy

$$\left.\begin{aligned} &u_2(\tau_k) + 474.9999931\,\tau_k^4 - 476.6666597\,\tau_k^3 + 147.2499979\,\tau_k^2 \\ &\qquad -14.43333319\,\tau_k + 0.2 = 0, \quad k = 1, 2, 3, \\ &u_2(t) + 474.9999931\ t^4 - 476.6666597\ t^3 + 147.2499979\ t^2 \\ &\qquad -14.43333319\ t + 0.2 \neq 0, \quad t \in [0, 0.5] \setminus \{\tau_1, \tau_2, \tau_3\}. \end{aligned}\right\} \tag{9.15}$$

We are interested in a solution of problem (9.1), (9.14), (9.5) as defined in Definition 1 with $n = 2$, $p = 3$. The impulse vector-functions γ_1, γ_2 and γ_2 in (1.6) are constant here

$$\gamma_1 = \mathrm{col}(0.01, -0.01), \quad \gamma_2 = \mathrm{col}(0.015, -0.015), \quad \gamma_3 = \mathrm{col}(-0.0012, 0.0012), \tag{9.16}$$

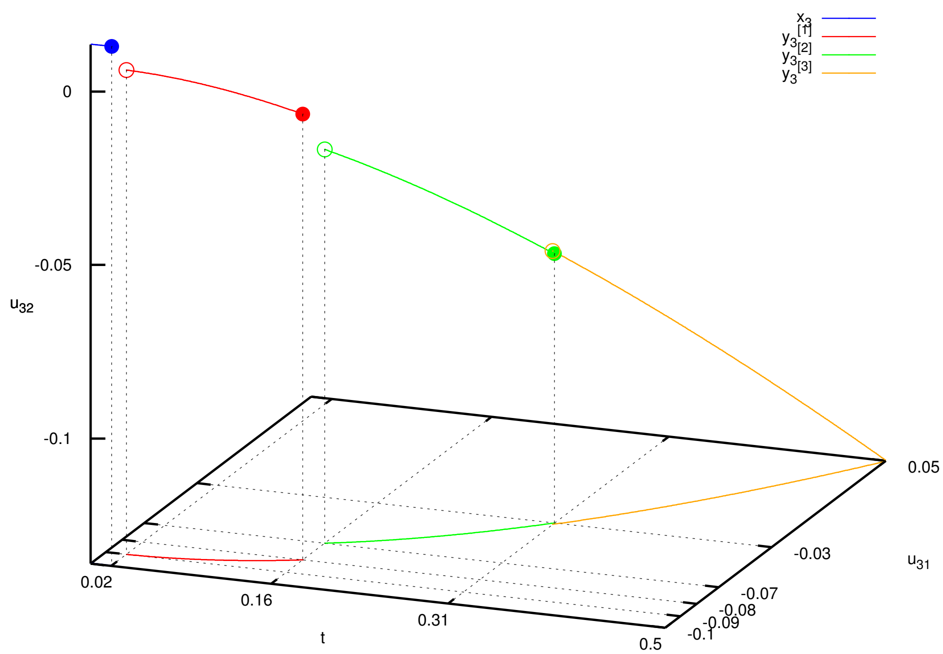

Figure 3: Third approximation $\widehat{u}_3$ of a solution to problem (9.1), (9.14), (9.5)

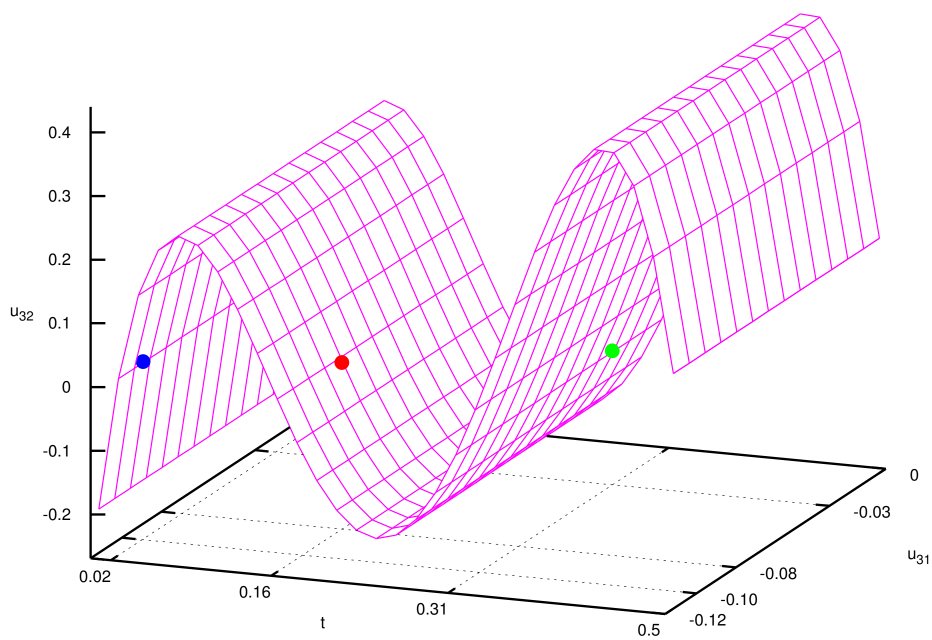

Figure 4: Barrier (9.13)

the barrier function g has the form

$$g(t,x) = x_2 + 474.9999931\ t^4 - 476.6666597\ t^3 + 147.2499979\ t^2 \\ -14.43333319\ t + 0.2. \tag{9.17}$$

Introduce the zero-th iterations x_0, $y_0^{[k]}$, $k = 1,2,3$, solve system (8.8) of 13 scalar algebraic equations and obtain the roots

$$\widehat{\tau}_1, \widehat{\tau}_2, \widehat{\tau}_3, \widehat{\xi}_1, \widehat{\xi}_2, \widehat{\lambda}_1^{[1]}, \widehat{\lambda}_2^{[1]} \lambda_1^{[2]}, \widehat{\lambda}_2^{[2]}, \widehat{\lambda}_1^{[3]}, \widehat{\lambda}_2^{[3]}, \widehat{\lambda}_1^{[4]}, \widehat{\lambda}_2^{[4]},$$

presented in the first column of Table 2. Having

$$\widehat{\xi}_1 = -0.1032363917, \quad \widehat{\xi}_2 = 0.01351486331, \tag{9.18}$$

choose

$$\Omega_0 = [-0.14, 0.04] \times [-0.18, 0.03] = \Omega_1,$$

and then, by (8.1), (8.2), (9.16), get

$$\begin{aligned} \Omega_1^+ &= [-0.15, 0.05] \times [-0.19, 0.04] = \Omega_2, \\ \Omega_2^+ &= [-0.165, 0.065] \times [-0.205, 0.055] = \Omega_3, \\ \Omega_3^+ &= [-0.265, 0.165] \times [-0.305, 0.155] = \Omega_4. \end{aligned}$$

Now, choose the vectors

$$\rho^{[0]} = \mathrm{col}(0.1, 0.1), \quad \rho^{[1]} = \mathrm{col}(0.15, 0.15), \quad \rho^{[2]} = \mathrm{col}(0.1, 0.15), \quad \rho^{[3]} = \mathrm{col}(0.1, 0.1),$$

and using (8.3) construct the sets

$$\begin{aligned} \mathcal{U}_0 &= [-0.24, 0.14] \times [-0.28, 0.13], \\ \mathcal{U}_1 &= [-0.30, 0.20] \times [-0.34, 0.19], \\ \mathcal{U}_2 &= [-0.265, 0.165] \times [-0.355, 0.205], \\ \mathcal{U}_3 &= [-0.365, 0.265] \times [-0.405, 0.255]. \end{aligned}$$

Maple computations give that the conditions of Theorems 1 and 2 are fulfilled and hence the frozen parameter scheme in Section 8 can be applied. Approximate roots of (7.1) are given in Table 2.

Table 2. Frozen parameters to problem (9.1), (9.14), (9.5)

Variable	$m=0$	$m=1$	$m=2$	$m=3$
$\widehat{\tau}_1$	0.01786682459	0.01786194368	0.01786194368	0.01786194376
$\widehat{\tau}_2$	0.1570192449	0.1570251944	0.1570251944	0.1570251942
$\widehat{\tau}_3$	0.3110673961	0.3110731609	0.3110731609	0.3110731609
$\widehat{\xi}_1$	−0.1032363917	−0.1025716094	−0.1025716097	−0.1025716097
$\widehat{\xi}_2$	0.01351486331	0.01347022529	0.01347022531	0.01347022531
$\widehat{\lambda}_1^{[1]}$	−0.1032639810	−0.1025968930	−0.1025968934	−0.1025968934
$\widehat{\lambda}_2^{[1]}$	0.01354245262	0.01349550897	0.01349550899	0.01349550899
$\widehat{\lambda}_1^{[2]}$	−0.08216614009	−0.08152274765	−0.08152274798	−0.08152274798
$\widehat{\lambda}_2^{[2]}$	−0.007555388281	−0.007578636417	−0.007578636385	−0.007578636385
$\widehat{\lambda}_1^{[3]}$	−0.03123212422	−0.03063507088	−0.03063507121	−0.03063507121
$\widehat{\lambda}_2^{[3]}$	−0.05848940415	−0.05846631319	−0.05846631315	−0.05846631315
$\widehat{\lambda}_1^{[4]}$	0.04593622419	0.04641955097	0.04641955065	0.04641955065
$\widehat{\lambda}_2^{[4]}$	−0.1356577526	−0.1355209350	−0.1355209350	−0.1355209350

In addition, computations show that condition (8.6) with $p = 3$ holds for $m = 3$. Consequently, the function

$$\widehat{u}_3(t) = \begin{cases} X_3(t) & \text{if} \quad t \in [0, \widehat{\tau}_1]\,, \\ Y_3^{[1]}(t) & \text{if} \quad t \in [\widehat{\tau}_1, \widehat{\tau}_2]\,, \\ Y_3^{[2]}(t) & \text{if} \quad t \in [\widehat{\tau}_2, \widehat{\tau}_3]\,, \\ Y_3^{[3]}(t) & \text{if} \quad t \in [\widehat{\tau}_3, 0.5]\,, \end{cases} \tag{9.19}$$

is the third approximation of a solution to problem (9.1), (9.14), (9.5).

The graph and its orthogonal projection of the third approximation $\widehat{u}_3$ of a solution to problem (9.1), (9.14), (9.5) is on Fig. 3 while Fig. 4 shows the graph of barrier (9.13) and the points where it is intersected by the graph of $\widehat{u}_3$.

Substituting approximation (9.19) into system (9.1), we obtain a residual estimated as follows:

$$\begin{aligned}
\max_{t\in[a,\widehat{\tau}_1]} \left|X'_{31}(t) - X^2_{32}(t) + X^2_{31}(t) - t\right| &= 2\times 10^{-9},\\
\max_{t\in[0,\widehat{\tau}_1]} \left|X'_{32}(t) - X^2_{31}(t) + X^2_{32}(t) + t\right| &= 2\times 10^{-9},
\end{aligned}$$

$$\begin{aligned}
\max_{t\in[\widehat{\tau}_1,\widehat{\tau}_2]} \left|\frac{dY^{[1]}_{31}(t)}{dt} - {Y^{[1]}_{32}}^2(t) + {Y^{[1]}_{31}}^2(t) - t\right| &= 8\times 10^{-7},\\
\max_{t\in[\widehat{\tau}_1,\widehat{\tau}_2]} \left|\frac{dY^{[1]}_{32}(t)}{dt} - {Y^{[1]}_{31}}^2(t) + {Y^{[1]}_{32}}^2(t) + t\right| &= 8\times 10^{-7},
\end{aligned}$$

$$\begin{aligned}
\max_{t\in[\widehat{\tau}_2,\widehat{\tau}_3]} \left|\frac{dY^{[2]}_{31}(t)}{dt} - {Y^{[2]}_{32}}^2(t) + {Y^{[2]}_{31}}^2(t) - t\right| &= 9\times 10^{-7},\\
\max_{t\in[\widehat{\tau}_2,\widehat{\tau}_3]} \left|\frac{dY^{[2]}_{32}(t)}{dt} - {Y^{[2]}_{31}}^2(t) + {Y^{[2]}_{32}}^2(t) + t\right| &= 9\times 10^{-7},
\end{aligned}$$

$$\begin{aligned}
\max_{t\in[\widehat{\tau}_3,b]} \left|\frac{dY^{[3]}_{31}(t)}{dt} - {Y^{[3]}_{32}}^2(t) + {Y^{[3]}_{31}}^2(t) - t\right| &= 1.5\times 10^{-6},\\
\max_{t\in[\widehat{\tau}_3,b]} \left|\frac{dY^{[3]}_{32}(t)}{dt} - {Y^{[3]}_{31}}^2(t) + {Y^{[3]}_{32}}^2(t) + t\right| &= 1.5\times 10^{-6}.
\end{aligned}$$

Example 3 *Two jumps.* We apply our technique to the same system (9.1) on the interval $[0, 0.5]$ with the same boundary conditions (9.5) but with a different barrier and two jumps.

So, put $p = 2$, consider the barrier

$$G = \left\{(t, x) \in [0, 0.5] \times \mathbb{R}^2 : x_1^2 + x_2^2 - 0.125\, t = 0\right\}, \tag{9.20}$$

and the state-dependent impulse conditions at two unknown points τ_1 and τ_2

$$\left.\begin{aligned}
u_1(\tau_1+) - u_1(\tau_1) = -0.015625, \qquad & u_2(\tau_1+) - u_2(\tau_1) = 0.015625,\\
u_1(\tau_2+) - u_1(\tau_2) = 0.140625, \qquad & u_2(\tau_2+) - u_2(\tau_2) = -0.140625,
\end{aligned}\right\} \tag{9.21}$$

where, by (1.5), τ_1 and τ_2 have to satisfy

$$\left.\begin{aligned}
& u_1^2(\tau_k) + u_2^2(\tau_k) - 0.125\, \tau_k = 0, \quad k = 1, 2,\\
& u_1^2(t) + u_2^2(t) - 0.125\, t \neq 0, \quad t \in [0, 0.5] \setminus \{\tau_1, \tau_2\}.
\end{aligned}\right\} \tag{9.22}$$

Calculation of approximate roots of the corresponding determining system (7.1) yields for $m = 3$ the frozen parameters to problem (9.1), (9.21), (9.5)

$$\begin{aligned}
\widehat{\tau}_1 &= 0.4300565098 & \widehat{\tau}_2 &= 0.4516205829\\
\widehat{\xi}_1 &= -0.2868788395 & \widehat{\xi}_2 &= 0.01090944068\\
\widehat{\lambda}^{[1]}_1 &= -0.2265214409 & \widehat{\lambda}^{[1]}_2 &= -0.04944795790\\
\widehat{\lambda}^{[2]}_1 &= -0.2338309782 & \widehat{\lambda}^{[2]}_2 &= -0.04213842058\\
\widehat{\lambda}^{[3]}_1 &= -0.06866993025 & \widehat{\lambda}^{[3]}_2 &= -0.2072994685.
\end{aligned}$$

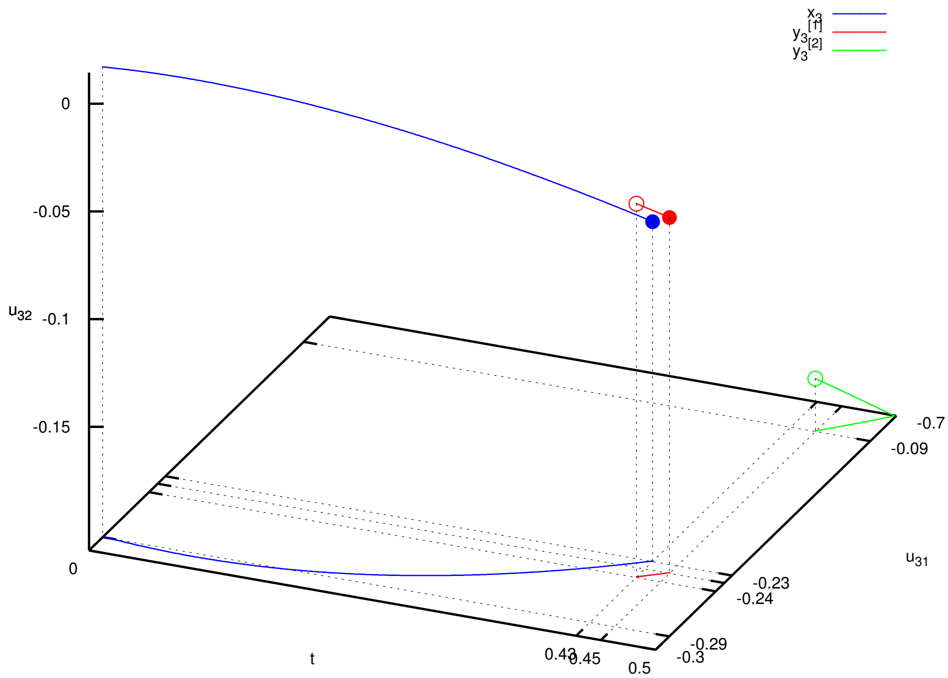

Figure 5: Third approximation $\widehat{u}_3$ of a solution to problem (9.1), (9.21), (9.5)

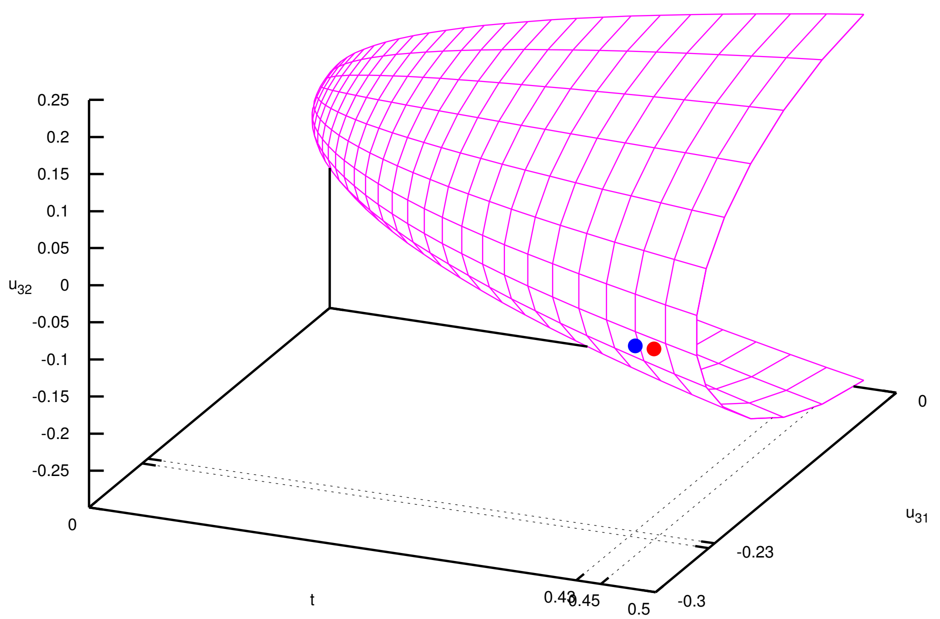

Figure 6: Barrier (9.20)

The graph and its orthogonal projection of the third approximation $\widehat{u}_3$ of a solution to problem (9.1), (9.21), (9.5) is shown on Fig. 5 while Fig. 6 shows the graph of barrier (9.20) and the points where it is intersected by the graph of $\widehat{u}_3$.

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