Lower and upper functions in singular Dirichlet problem with ϕ-Laplacian

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1. NOTATION

$[a,b] \subset \mathbb{R}$; $J \subset \mathbb{R}$; $\mathcal{M} \subset \mathbb{R}^2$; meas J - the Lebesgue measure of J;

$C[a,b]$ - the Banach space of functions continuous on interval $[a,b]$ with the norm $\|f\|_{C[a,b]} = \max\{|f(t)|\colon t \in [a,b]\}$;

$C^1[a,b]$ - the Banach space of functions having continuous first derivatives on $[a,b]$ with the norm $\|f\|_{C^1[a,b]} = \|f\|_{C[a,b]} + \|f'\|_{C[a,b]}$;

$AC[a,b]$ - the set of absolutely continuous functions on $[a,b]$;

$AC_{loc}(J)$ - the set of functions $f \in AC\,[c,d]$ for each $[c,d] \subset J$;

$L[a,b]$ - the Banach space of functions Lebesgue integrable on $[a,b]$ with the norm $\|f\|_{L[a,b]} = \int_a^b |f(t)|\mathrm{d}t$;

$L_{loc}(J)$ - the set of functions $f \in L\,[c,d]$ for each $[c,d] \subset J$;

$Car([a,b] \times \mathcal{M})$ - the set of functions $f\colon [a,b] \times \mathcal{M} \to \mathbb{R}$ satisfying the Carathéodory conditions on $[a,b] \times \mathcal{M}$, i.e.
$f(\cdot,x,y)\colon [a,b] \to \mathbb{R}$ is measurable for all $(x,y) \in \mathcal{M}$;
$f(t,\cdot,\cdot)\colon \mathcal{M} \to \mathbb{R}$ is continuous for a.e. $t \in [a,b]$;
for each compact set $\mathcal{K} \subset \mathcal{M}$ there is a function $m_{\mathcal{K}} \in L[a,b]$ such that

$$|f(t,x,y)| \le m_{\mathcal{K}}(t) \text{ for a.e. } t \in [a,b] \text{ and all } (x,y) \in \mathcal{K}\ .$$

$Car((a,b) \times \mathcal{M})$ - the set of function $f \in Car([c,d] \times \mathcal{M})$ for each $[c,d] \subset (a,b)$.

2. INTRODUCTION

We will study the existence of a solution of singular Dirichlet problem

$$(\phi(u'))' + f(t,u,u') = 0\ , \quad u(0) = u(T) = 0\ , \tag{2.1}$$ {eq1}

where ϕ is an increasing odd homeomorphism with $\phi(\mathbb{R}) = \mathbb{R}$, $T \in (0,\infty)$ and where f can have singularities in all its variables.

In particular, we assume that $\mathcal{A}_1, \mathcal{A}_2 \subset \mathbb{R}$ are closed intervals containing 0 and

$$\begin{cases} f \in Car((0,T) \times \mathcal{D})\ , \text{ where } \mathcal{D} = (\mathcal{A}_1 \setminus \{0\}) \times (\mathcal{A}_2 \setminus \{0\})\ , \\ f \text{ may have time singularities at } t = 0 \text{ and at } t = T\ , \\ f \text{ may have space singularities at } x = 0 \text{ and at } y = 0\ . \end{cases} \tag{2.2}$$ {f1}

Definition 2.1. A function f has *a time singularity* at $t = 0$ resp. $t = T$ if there exists $(x,y) \in \mathcal{D}$ such that

$$\int_0^{\varepsilon} |f(t,x,y)|\mathrm{d}t = \infty \text{ resp. } \int_{T-\varepsilon}^{T} |f(t,x,y)|\mathrm{d}t = \infty$$

for any sufficiently small $\varepsilon > 0$.

Definition 2.2. A function f has *a space singularity* at $x = 0$ resp. $y = 0$ if there exists a set $J \subset [0,T]$ with a positive Lebesgue measure such that the condition

$$\limsup_{x \to 0} |f(t,x,y)| = \infty \text{ resp. } \limsup_{y \to 0} |f(t,x,y)| = \infty$$

holds for a.e. $t \in J$ and some $y \in \mathcal{A}_2$ resp. $x \in \mathcal{A}_1$.

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{def3}

Definition 2.3. A function $u \in C^1[0,T]$ with $\phi(u') \in AC[0,T]$ is a *solution* of problem (2.1) if u satisfies

$$(\phi(u'(t)))' + f(t,u(t),u'(t)) = 0 \text{ for a.e. } t \in [0,T] \tag{2.3}$$ {eq2}

and fulfils the boundary conditions $u(0) = u(T) = 0$.

Now we bring out the definition of upper and lower function and auxiliary theorems, which we will use in proofs.

Definition 2.4. A function $\sigma \in C[0,T]$ is called an *upper function* of problem (2.1) if there exists a finite set $\Sigma \subset (0,T)$ such that

$$\phi(\sigma') \in AC_{loc}([0,T] \setminus \Sigma) , \quad \sigma'(\tau+) := \lim_{t \to \tau+} \sigma'(t) \in \mathbb{R} ,$$

$$\sigma'(\tau-) := \lim_{t \to \tau-} \sigma'(t) \in \mathbb{R} \text{ for each } \tau \in \Sigma ,$$

$$\begin{gathered}(\phi(\sigma'(t)))' + g(t,\sigma(t),\sigma'(t)) \le 0 \text{ for a.e. } t \in [0,T] , \\ \sigma(0) \ge 0 , \quad \sigma(T) \ge 0 , \quad \sigma'(\tau-) > \sigma'(\tau+) \text{ for each } \tau \in \Sigma .\end{gathered} \tag{2.4}$$ {s1}

If the inequalities in (2.4) are reversed, then σ is called a *lower function* of problem (2.1).

Theorem 2.5 (Lower and upper functions method, [20]**).** *Consider a problem* {t25}

$$(\phi(u'))' + g(t,u,u') = 0 , \quad u(0) = u(T) = 0 , \tag{2.5}$$ {eq3}

where $g \in Car([0,T] \times \mathbb{R}^2)$. Let σ_1 and σ_2 be a lower function and an upper function of problem (2.5) *and $\sigma_1(t) \le \sigma_2(t)$ for $t \in [0,T]$. Assume that there exists a function $m \in L[0,T]$ such that*

$$|g(t,x,y)| \le m(t) \text{ for a.e. } t \in [0,T] \text{ and all } x \in [\sigma_1(t), \sigma_2(t)] ,\ y \in \mathbb{R} .$$

Then problem (2.5) *has a solution u such that*

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \text{ for } t \in [0,T] .$$

A systematic study of the solvability of Dirichlet problems having both time and space singularities was initiated by Taliaferro [25]. Now, we can find a large group of works which focused their attention on the existence of w-solutions, that is on the existence of functions u satisfying (2.3) and $u(0) = u(T) = 0$ but do not belonging to $C^1[0,T]$. We can refer to the papers [1]–[4], [10]–[16]. There exists a less number of works which provide also conditions for the existence of solutions in the sense of Definition 2.3, e.g. [5], [7], [9], [17]–[22], [25]–[27]. All the above works deal with differential equations where the nonlinearity $f(t,x,y)$ has a space singularity at $x = 0$ and/or time singularities at $t = 0, t = T$. The first existence result for the Dirichlet problem where $f(t,x,y)$ has singularities at both variables x and y was reached by Staněk [21]. He assumed that f is strictly positive and its behaviour on a right neighbourhood of the singular point $x = 0$ is controlled by a function $\omega_0(x)$ which is integrable. Then we say that f has a weak space singularity at $x = 0$.

In this paper we generalize and extend the existence results for the Dirichlet problem (2.1), which has been studied in the papers [1], [10], [19], [20] and [22]. Our methods of proofs are similar to those in [19] and [20]. In [19] we study the Dirichlet problem without ϕ-Laplacian. The function $f(t,x,y)$ can have a strong or weak singularity in x and a weak singularity in y. Note that f has a strong space singularity at $x = 0$ if it is controlled near the point $x = 0$ by a nonintegrable function $\omega_0(x)$. Similarly for $y = 0$. Moreover f can have a sublinear growth in x, y or a linear growth with small coefficients. In [20] we have the Dirichlet problem with ϕ-Laplacian, with singularities in t, x (weak or strong) and in y (only weak). The function f can have a quadratic growth in variable y and an arbitrary growth in x.

In this paper we also solve the Dirichlet problem with ϕ-Laplacian. We modify an existence principle of [20]. By means of this modified principle (Theorem 3.1) we prove Theorem 4.1 which yields the existence of a solution of (2.1) with $f(t,x,y)$, which can have time singularities in $t=0$ and $t=T$ and weak or strong singularities in x and y. In addition, the function f can have an arbitrary growth in x and y.

Let us add some other recent results for singular Dirichlet problems. Extremal solutions for the equation $u''=p(t)(f(t,u,u')-r(t))$ have been investigated in [23]. Variational methods leadinag to the existence of one or two positive solutions of problems with the equation $-u''=\lambda f(t,u)$ have been used in [24] and for $\lambda=1$ in [6]. By means of the fixed point theorem on cones the paper [8] has got multiplicity results for problems with the equation $u''+q(t)f(t,u)+e(t)=0$. Note that conditions which guarantee the existence of infinitely many solutions can be found in [24].

3. EXISTENCE PRINCIPLE

We define a sequence of auxiliary regular problems:

$$(\phi(u'))'+f_n(t,u,u')=0\ ,\quad u(0)=u(T)=0\ , \tag{3.1}$$ {eq4}

where $f_n \in Car([0,T]\times\mathbb{R}^2)$.

Theorem 3.1 (Existence principle). *Assume* (2.2). *Let* $\varepsilon_n>0$, $\eta_n>0$ *for* $n\in\mathbb{N}$ *and assume that* {t31} {a311}

1. $$f_n(t,x,y)=f(t,x,y) \text{ for a.e. } t\in\Delta_n \text{ and each } (x,y)\in\mathcal{A}_1\times\mathcal{A}_2\ ,$$
$$|x|\ge\varepsilon_n,\ |y|\ge\eta_n,\ n\in\mathbb{N},\ \text{where } \Delta_n=\left[\frac{1}{n},T-\frac{1}{n}\right]\cap[0,T]\ ,$$
$$\lim_{n\to\infty}\varepsilon_n=0\ ,\quad \lim_{n\to\infty}\eta_n=0\ ;$$
{a312}
2. *there exists a bounded set* $\Omega\subset C^1[0,T]$ *such that for each* $n\in\mathbb{N}$, *problem* (3.1) *has a solution* $u_n\in\Omega$ *and* $(u_n(t),u_n'(t))\in\mathcal{A}_1\times\mathcal{A}_2$ *for* $t\in[0,T]$.

Then there exist $u\in C[0,T]$ *and a subsequence* $\{u_k\}\subset\{u_n\}$ *such that*

$$\lim_{k\to\infty}u_k(t)=u(t) \text{ uniformly on } [0,T]\ . \tag{3.2}$$ {lim1}

Assume in addition that {a313}

3. *there exists a finite set* $S=\{s_1,\cdots s_\zeta\}\subset(0,T)$ *such that on each interval* $[a,b]\subset(0,T)\setminus S$ *the sequence* $\{\phi(u_k')\}$ *is equicontinuous.*

Then $u\in C^1((0,T)\setminus S)$ *and*

$$\lim_{k\to\infty}u_k'(t)=u'(t) \text{ locally uniformly on } (0,T)\setminus S\ . \tag{3.3}$$ {lim2}

Assume moreover that {a314}

4. *the set* S *has the form* $S=\{s\in(0,T)\colon u(s)=0 \text{ or } u'(s)=0 \text{ or } u'(s) \text{ does not exist }\}$; {a315}
5. *there exist* $\eta\in\left(0,\frac{T}{2}\right)$, $\lambda_0,\mu_0,\lambda_1,\mu_1,\cdots,\lambda_{\zeta+1},\mu_{\zeta+1}\in\{-1,1\}$, $k_0\in\mathbb{N}$ *and* $\psi\in L[0,T]$ *such that*
$$\begin{gathered}\lambda_i f_k\left(t,u_k(t),u_k'(t)\right)\ge\psi(t) \text{ for a.e. } t\in(s_i-\eta,s_i)\cap(0,T)\ ,\\ \mu_i f_k(t,u_k(t),u_k'(t))\ge\psi(t) \text{ for a.e. } t\in(s_i,s_i+\eta)\cap(0,T)\ ,\\ \text{for all } i\in\{0,\cdots,\zeta+1\}\ ,\quad k\in\mathbb{N}\ ,\quad k\ge k_0\ .\end{gathered} \tag{3.4}$$ {e34}

Here $s_0=0$ *and* $s_{\zeta+1}=T$.

Then $\phi(u')\in AC[0,T]$ *and* u *is a solution of* (2.1) *satisfying* $(u(t),u'(t))\in\mathcal{A}_1\times\mathcal{A}_2$ *for* $t\in[0,T]$.

Proof. By assumption 2, there exists $r > 0$ and a sequence $\{u_n\}$ of solutions of (3.1) such that

$$\|u_n\|_{C^1[0,T]} \le r \text{ for each } n \in \mathbb{N} . \qquad (3.5) \quad \{e35\}$$

Therefore the sequence $\{u_n\}$ is bounded in $C[0,T]$. Moreover, the Lagrange mean value theorem yields that the sequence $\{u_n\}$ is equicontinuous on $[0,T]$. By the Arzelà - Ascoli theorem we can choose a subsequence $\{u_\ell\}$ such that

$$\lim_{\ell\to\infty} u_\ell(t) = u(t) \text{ uniformly on } [0,T] , \quad u \in C[0,T] . \qquad (3.6) \quad \{lim3\}$$

Now choose an arbitrary interval $[a,b] \subset [0,T] \setminus S$. Then, by assumption 3, the sequence $\{\phi(u'_\ell)\}$ is equicontinuous on $[a,b]$. By (3.5) the sequence $\{u'_\ell\}$ is bounded in $C[a,b]$. Since ϕ is homeomorphism, the sequence $\{\phi(u'_\ell)\}$ is bounded in $C[a,b]$ too. The Arzelà - Ascoli theorem guarantees that we can choose a subsequence $\{\phi(u_k)\} \subset \{\phi(u_\ell\})$ such that

$$\lim_{k\to\infty} \phi(u'_k(t)) = \phi(u'(t)) \text{ uniformly on } [a,b]$$

and consequently we get

$$\lim_{k\to\infty} u'_k(t) = u'(t) \text{ uniformly on } [a,b] .$$

By virtue of (3.6) the sequence $\{u_k\}$ satisfies (3.2). Using the diagonalization method we can choose such sequence $\{u_k\}$ that

$$\lim_{k\to\infty} u'_k(t) = u'(t) \text{ locally uniformly on } (0,T) \setminus S \qquad (3.7) \quad \{lim4\}$$

holds, as well. Therefore $u \in C^1((0,T) \setminus S)$. For $k \in \mathbb{N}$ it holds $u_k(0) = u_k(T) = 0$ and, by (3.2), u satisfies $u(0) = u(T) = 0$.

Define sets

$$V = \{t \in (0,T) \colon f(t,\cdot,\cdot) \colon \mathcal{D} \to \mathbb{R} \text{ is not continuous}\} ,$$

$$U = (0,T) \setminus (S \cup V) .$$

We see that

$$\text{meas}(S \cup V) = 0 . \qquad (3.8) \quad \{e38\}$$

Choose an arbitrary $t \in U$. Then there exists $k_0 \in \mathbb{N}$, such that for each $k \in \mathbb{N}$, $k \ge k_0$:

$$t \in \Delta_k , \quad |u_k(t)| > \varepsilon_k , \quad |u'_k(t)| > \eta_k .$$

By assumption 1,

$$f_k(t,u_k(t),u'_k(t)) = f(t,u_k(t),u'_k(t)) \text{ for a.e. } t \in \Delta_k .$$

Therefore by (3.2), (3.7) and (3.8) we get

$$\lim_{k\to\infty} f_k(t,u_k(t),u'_k(t)) = f(t,u(t),u'(t)) \text{ a.e. on } [0,T] . \qquad (3.9) \quad \{lim5\}$$

Since u_k is a solution of (3.1), we get

$$-(\phi(u'_k(t)))' = f_k(t,u_k(t),u'_k(t)) \text{ for a.e. } t \in [0,T] . \qquad (3.10) \quad \{e310\}$$

Now choose an arbitrary interval $[a,b] \subset (0,T) \setminus S$ and integrate equation (3.10). We get

$$-\phi(u'_k(t)) + \phi(u'_k(a)) = \int_a^t f_k(s,u_k(s),u'_k(s))\mathrm{d}s \text{ for each } t \in [a,b] . \qquad (3.11) \quad \{e311\}$$

Moreover there exists $k^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq k^*$

$$|f_k(t, u_k(t), u_k'(t))| \leq m(t) \text{ for a.e. } t \in [a, b] \ ,$$

where

$$m(t) = \sup \{|f(t, x, y)| \colon \varepsilon_{k^*} \leq |x| \leq r; \ \eta_{k^*} \leq |y| \leq r; \ x \in \mathcal{A}_1; \ y \in \mathcal{A}_2\} \in L[a, b].$$

Since $m \in L[a, b]$ we can apply the Lebesgue dominated convergence theorem on $[a, b]$ and get $f(\cdot, u(\cdot), u'(\cdot)) \in L[a, b]$. Moreover

$$\lim_{k\to\infty} \int_a^b f_k(s, u_k(s), u_k'(s)) \mathrm{d}s = \int_a^b f(s, u(s), u'(s)) \mathrm{d}s \ . \tag{3.12}$$ {lim6}

It holds by (3.2), (3.7), (3.11) and (3.12)

$$-\phi(u'(t)) + \phi(u'(a)) = \int_a^t f(s, u(s), u'(s)) \mathrm{d}s \text{ for each } t \in [a, b] \ . \tag{3.13}$$ {e313}

Since $[a, b]$ is an arbitrary interval in $(0, T) \setminus S$, we get that $\phi(u') \in AC_{loc}((0, T) \setminus S)$, u satisfies (2.3) and the boundary conditions $u(0) = u(T) = 0$.

It remains to prove that $\phi(u') \in AC[0, T]$. Choose $i \in \{0, \cdots, \zeta + 1\}$ and denote $(c_i, d_i) = (s_i - \eta, s_i) \cap (0, T)$. For $k \in \mathbb{N}$ and for a.e. $t \in (c_i, d_i)$ we denote

$$h_k(t) = \lambda_i f_k(t, u_k(t), u_k'(t)) + |\psi(t)| \ , \quad h(t) = \lambda_i f(t, u(t), u'(t)) + |\psi(t)| \ .$$

Then $h_k \in L[c_i, d_i]$ and according to (3.9) we have

$$\lim_{k\to\infty} h_k(t) = h(t) \text{ for a.e. } t \in [c_i, d_i] \ .$$

Integrating (3.10) on $[c_i, d_i]$ we get

$$-\phi(u_k'(d_i)) + \phi(u_k'(c_i)) = \int_{c_i}^{d_i} f_k(s, u_k(s), u_k'(s)) \mathrm{d}s \ .$$

Therefore, by (3.4) and (3.5)

$$\int_{c_i}^{d_i} |h_k(s)| \mathrm{d}s = \int_{c_i}^{d_i} h_k(s) \mathrm{d}s = \lambda_i \int_{c_i}^{d_i} f_k(s, u_k(s), u_k'(s)) \mathrm{d}s$$

$$+ \int_{c_i}^{d_i} |\psi(s)| \mathrm{d}s \leq |\phi(u_k'(d_i))| + |\phi(u_k'(c_i))| + \int_{c_i}^{d_i} |\psi(s)| \mathrm{d}s \leq c \ ,$$

where $c = 2\phi(r) + \|\psi\|_{L[0,T]}$. The Fatou lemma implies that $h \in L[c_i, d_i]$ and $f(\cdot, u(\cdot), u'(\cdot)) \in L[c_i, d_i]$. If $(c_i, d_i) = (s_i, s_i + \eta) \cap (0, T)$ we argue similarly. Hence $f(\cdot, u(\cdot), u'(\cdot)) \in L[0, T]$ and the equality in (3.13) is fulfilled for each $t \in [0, T]$ and $\phi(u') \in AC[0, T]$. Consequently $u' \in C[0, T]$. We have proved that u is a solution of (2.1). According to assumption 2 and (3.2), (3.3), we get $(u(t), u'(t)) \in \mathcal{A}_1 \times \mathcal{A}_2$ for $t \in [0, T]$. □

4. EXISTENCE THEOREM

Theorem 4.1 (Existence theorem). *Let* $\nu \in \left(0, \frac{T}{2}\right), \varepsilon \in \left(0, \frac{\phi(\nu)}{\nu}\right)$, $c_1, c_2 \in (\nu, \infty)$. *Let assumption* (2.2) *hold with* $\mathcal{A}_1 = [0, \infty)$, $\mathcal{A}_2 = [-c_1, c_2]$. *Denote* $\sigma_2(t) = \min\{c_2 t; c_1(T - t)\}$ *for* $t \in [0, T]$ *and assume that* {t41}

$$f(t, \sigma_2(t), \sigma_2'(t)) = 0 \textit{ for a.e. } t \in [0, T] \ , \tag{4.1}$$ {a411}

$$0 \le f(t,x,y) \text{ for a.e. } t \in [0,T],\ \forall x \in (0,\sigma_2(t)],\ y \in [-c_1,c_2] \setminus \{0\}\ , \tag{4.2}$$ {a412}

$$\varepsilon \le f(t,x,y) \text{ for a.e. } t \in [0,T],\ \forall x \in (0,\sigma_2(t)],\ y \in [-\nu,\nu] \setminus \{0\}\ . \tag{4.3}$$ {a413}

Then problem (2.1) *has solution* u *which fulfils*

$$0 < u(t) \le \sigma_2(t)\ ; \quad -c_1 \le u'(t) \le c_2 \text{ for } t \in (0,T)\ . \tag{4.4}$$ {e44}

Proof. *Step* 1. *Construction of an auxiliary problem.*

Let $n \in \mathbb{N}$, $\frac{1}{n} < \nu$, $n > \frac{2}{T}$. Choose $\sigma_1(t) \equiv 0$ on $[0,T]$. Put $\varepsilon_n = \min\left\{\sigma_2\left(\frac{1}{n}\right); \sigma_2\left(T-\frac{1}{n}\right)\right\}$, $\eta_n = \frac{1}{n}$. For $x, y \in \mathbb{R}$ we define

$$\alpha_n(x) = \begin{cases} x & \text{for } \varepsilon_n \le x\ , \\ \varepsilon_n & \text{for } x < \varepsilon_n\ , \end{cases}$$

$$\beta(y) = \begin{cases} c_2 & \text{for } y > c_2\ , \\ y & \text{for } -c_1 \le y \le c_2\ , \\ -c_1 & \text{for } y < -c_1\ , \end{cases}$$

$$\gamma(y) = \begin{cases} \varepsilon & \text{for } |y| \le \nu\ , \\ 0 & \text{for } y \le -c_1 \text{ or } y \ge c_2\ , \\ \varepsilon\frac{c_2-y}{c_2-\nu} & \text{for } \nu < y < c_2\ , \\ \varepsilon\frac{c_1+y}{c_1-\nu} & \text{for } -c_1 < y < -\nu\ . \end{cases}$$

For a.e. $t \in [0,T]$, $\forall x, y \in \mathbb{R}$ we define auxiliary functions

$$\widetilde{f_n}(t,x,y) = \begin{cases} \gamma(y) & \text{for } t \in \left[0,\frac{1}{n}\right) \cap \left(T-\frac{1}{n},T\right]\ , \\ f(t,\alpha_n(x),\beta(y)) & \text{for } t \in \left[\frac{1}{n}, T-\frac{1}{n}\right]\ , \end{cases}$$

$$f_n(t,x,y) = \begin{cases} \widetilde{f_n}(t,x,y) & \text{for } |y| \ge \frac{1}{n}\ , \\ \frac{n}{2}\left(\widetilde{f_n}\left(t,x,\frac{1}{n}\right)\left(y+\frac{1}{n}\right) - \widetilde{f_n}\left(t,x,-\frac{1}{n}\right)\left(y-\frac{1}{n}\right)\right) & \text{for } |y| < \frac{1}{n}\ . \end{cases}$$

Function $f \in Car((0,T)\times\mathcal{D})$ and so $f_n \in Car([0,T]\times\mathbb{R}^2)$. We get a sequence of auxiliary problems

$$(\phi(u'))' + f_n(t,u,u') = 0\ , \quad u(0) = u(T) = 0\ , \tag{4.5}$$ {eq5}

$n \in \mathbb{N}$, $n > \frac{2}{T}$.

Step 2. *Existence of a solution of problem* (4.5).

We define

$$m_n(t) = \sup\{f_n(t,x,y) \colon x \in [0,\sigma_2(t)]\ ; \ \ y \in \mathbb{R}\} \text{ for a.e. } t \in [0,T]\ .$$

Then $m_n \in L[0,T]$ and $|f_n(t,x,y)| \le m_n(t)$ for a.e. $t \in [0,T]$, $\forall x \in [0,\sigma_2(t)]$, $\forall y \in \mathbb{R}$.

In order to use Theorem 2.5, we must prove that σ_1, σ_2 are lower and upper functions of problem (4.5). We have

$$\begin{aligned} (\phi(\sigma_1'(t)))' + f_n(t,\sigma_1(t),\sigma_1'(t)) &= f_n(t,0,0) \\ &= \frac{n}{2}\left[\widetilde{f_n}\left(t,0,\frac{1}{n}\right)\frac{1}{n} - \widetilde{f_n}\left(t,0,-\frac{1}{n}\right)\left(-\frac{1}{n}\right)\right] \\ &= \frac{1}{2}\left[\widetilde{f_n}\left(t,0,\frac{1}{n}\right) + \widetilde{f_n}\left(t,0,-\frac{1}{n}\right)\right] \end{aligned}$$

$$= \begin{cases} \varepsilon > 0 & \text{for } t \in \left[0, \frac{1}{n}\right) \cup \left(T - \frac{1}{n}, T\right] \ , \\ \frac{1}{2}\left[f\left(t, \varepsilon_n, \frac{1}{n}\right) + f\left(t, \varepsilon_n, -\frac{1}{n}\right)\right] \geq 0 & \text{for a.e. } t \in \left[\frac{1}{n}, T - \frac{1}{n}\right] \ , \end{cases}$$

and so $\sigma_1 \equiv 0$ is a lower function of problem (4.5). Further $\alpha_n(\sigma_2(t)) = \sigma_2(t)$ for $t \in \left[\frac{1}{n}, T - \frac{1}{n}\right]$. Since $\sigma_2'(t) = -c_1$ or c_2, we have $(\phi(\sigma_2'(t)))' = 0$ on $[0, T]$ and, by (4.1),

$$(\phi(\sigma_2'(t)))' + f_n(t, \sigma_2(t), \sigma_2'(t)) = f_n(t, \sigma_2(t), \sigma_2'(t))$$

$$= \begin{cases} \gamma(\sigma_2'(t)) = 0 & \text{for } t \in \left[0, \frac{1}{n}\right) \cup \left(T - \frac{1}{n}, T\right] \ , \\ f(t, \sigma_2(t), \sigma_2'(t)) = 0 & \text{for a.e. } t \in \left[\frac{1}{n}, T - \frac{1}{n}\right] \ . \end{cases}$$

We see that $\sigma_2(t)$ is an upper function of problem (4.5). Functions $f_n, \sigma_1, \sigma_2, m_n$ satisfy assumptions of Theorem 2.5 and so there exists a solution u_n of problem (4.5) satisfying $0 \leq u_n(t) \leq \sigma_2(t)$ for $t \in [0, T]$.

Step 3. *Estimates of a solution of problem* (4.5).

By (4.2) and the construction of f_n we get $(\phi(u_n'))' \leq 0$ for a.e. $t \in [0, T]$ and so $\phi(u_n')$ is nonincreasing. Since ϕ is increasing homeomorphism, the function u_n' is nonincreasing. Therefore $u_n'(0) \leq c_2$ implies $u_n'(t) \leq c_2$ for $t \in [0, T]$. Further $u_n'(T) \geq -c_1$ and we get $u_n'(t) \geq -c_1$ on $[0, T]$. Hence

$$-c_1 \leq u_n'(t) \leq c_2 \text{ for } t \in [0, T]. \tag{4.6}$$ {e46}

Let $t_n \in (0, T)$ be a point of maximum of u_n. Then $u_n'(t_n) = 0$ and $u_n'(t) \geq 0$ for $t \in [0, t_n]$, $u_n'(t) \leq 0$ for $t \in [t_n, T]$.

1. Let $t_n - \nu \geq 0$. Then there exists $a_n \in [0, t_n)$ such that $u_n'(t) \leq \nu$ for $t \in [a_n, t_n]$. Assuming $a_n \leq t_n - \nu$ and integrating (4.3), we get

$$\varepsilon(t_n - t) \leq \phi(u_n'(t)) \text{ for } t \in [t_n - \nu, t_n] \ . \tag{4.7}$$ {e47}

If $a_n > t_n - \nu$ and $u_n'(t) > \nu$ for $t \in [0, a_n)$, then similarly

$$\varepsilon(t_n - t) \leq \phi(u_n'(t)) \text{ for } t \in [a_n, t_n] \ .$$

Since $\varepsilon \in \left(0, \frac{\phi(\nu)}{\nu}\right)$, the inequalities $\phi(u_n'(t)) > \phi(\nu) > \varepsilon\nu \geq \varepsilon(t_n - t)$ hold for $t \in [t_n - \nu, a_n]$, and we get estimate (4.7) again. Integration of (4.7) over $[t_n - \nu, t_n]$ yields the estimate

$$u_n(t_n) \geq \int_0^\nu \phi^{-1}(\varepsilon s) \mathrm{d}s = \nu_0 > 0 \ . \tag{4.8}$$ {e48}

2. Let $t_n - \nu \leq 0$. Then $t_n + \nu \leq T$ and there exists $b_n \in (t_n, T]$ such that $-u_n'(t) \leq \nu$ for $t \in [t_n, b_n]$. Assuming $b_n \geq t_n + \nu$ and integrating (4.3), we obtain

$$\varepsilon(t - t_n) \leq -\phi(u_n'(t)) \text{ for } t \in [t_n, t_n + \nu] \ . \tag{4.9}$$ {e49}

If $b_n < t_n + \nu$ and $-u_n'(t) > \nu$ for $t \in (b_n, T]$, then similarly

$$\varepsilon(t - t_n) \leq -\phi(u_n'(t)) \text{ for } t \in [t_n, b_n] \ .$$

Since -$\phi(u_n'(t)) > \phi(\nu) > \varepsilon\nu \geq \varepsilon(t - t_n)$ for $t \in [b_n, t_n + \nu]$, we get inequality (4.9) again. Integration of (4.9) over $[t_n, t_n + \nu]$ yields estimate (4.8).

Using this estimate and the fact that u_n' is nonincreasing on $[0,T]$ we conclude that

$$\alpha_n^*(t) \le u_n(t) \le \sigma_2(t) \text{ for } t \in [0,T] \ ,$$

where

$$\alpha_n^*(t) = \begin{cases} \frac{\nu_0}{T} t & \text{for } t \in [0,t_n] \ , \\ \frac{\nu_0}{T}(T-t) & \text{for } t \in (t_n,T] \ . \end{cases}$$

Step 4. *Existence of a solution of singular problem* (2.1).

Consider the sequence of solutions $\{u_n\}$, $n > \frac{2}{T}$. Define

$$\Omega = \left\{ v \in C^1[0,T] \colon 0 \le v(t) \le \sigma_2(t);\ -c_1 \le v'(t) \le c_2 \text{ on } [0,T] \right\} \ .$$

We see that ε_n, η_n and f_n fulfil condition 1 of Theorem 3.1. Since also condition 2 of Theorem 3.1 is valid, we can choose a subsequence $\{u_n\}$ which is uniformly converging on $[0,T]$ to a function $u \in C[0,T]$. By estimates (4.6) and (4.8) we get

$$0 < \frac{\nu_0}{c_2} \le t_n \ , \quad t_n \le T - \frac{\nu_0}{c_1} < T \text{ for } n \in \mathbb{N} \ .$$

So, we can choose a subsequence $\{u_k\}$ in such way that $\lim_{k\to\infty} t_k = t_u \in (0,T)$ and

$$\alpha_u^*(t) \le u(t) \le \sigma_2(t) \text{ for } t \in [0,T] \ , \tag{4.10}$$ {e410}

where

$$\alpha_u^*(t) = \begin{cases} \frac{\nu_0}{T} t & \text{for } t \in [0,t_u] \ , \\ \frac{\nu_0}{T}(T-t) & \text{for } t \in (t_u,T] \ . \end{cases}$$

Put $S = \{t_u\}$ and choose $[a,b] \subset (0,t_u)$. Then there exists $k_0 \in \mathbb{N}$ such that for $k \ge k_0$ we have

$$|t_k - t_u| \le \frac{t_u - b}{2} \ , \quad [a,b] \subset \left(\frac{1}{k}, t_k\right) \ ,$$

$$u_k(t) \ge \frac{\nu_0 a}{t} =: m_0 \ , \quad \phi(u_k'(t)) \ge \frac{\varepsilon}{2}(t_u - b) =: m_1 \ , \quad t \in [a,b] \ .$$

Thus, for a.e. $t \in [a,b]$

$$|f_k(t, u_k(t), u_k'(t))| \le h(t) \in L[a,b] \ ,$$

where $h(t) = \sup\{|f(t,x,y)| \colon m_0 \le x \le \sigma_2(t) \ ; \quad \phi^{-1}(m_1) \le y \le c_2\}$. If we choose $[a,b] \subset (t_u,T)$, we argue similarly and obtain also a Lebesgue integrable majorant for f_k, $k \ge k_0$, on $[a,b]$. So, we have proved that condition 3 of Theorem 3.1 holds. Hence, we get $u \in C^1((0,T)\setminus S)$ and $\lim_{k\to\infty} u_k'(t) = u'(t)$ locally uniformly on $(0,T)\setminus S$.

Since u_k' is nonincreasing on $[0,T]$ for $k \ge k_0$, u' is nonincreasing on $(0,t_u)$ and on (t_u,T). Therefore,

$$\begin{cases} 0 \le u'(t) \le c_2 & \text{for } t \in [0,t_u) \ , \\ -c_1 \le u'(t) \le 0 & \text{for } t \in (t_u,T] \ , \end{cases} \tag{4.11}$$ {e411}

and the limits $\lim_{t\to t_u-} u'(t)$ and $\lim_{t\to t_u+} u'(t)$ exist.

1. Let $\lim_{t\to t_u-} u'(t) = 0$. Assume that there exists $t^* \in (0,t_u)$ such that $u'(t^*) = 0$. Then $u'(t) = 0$ for $t \in [t^*, t_u]$. On the other hand, by (4.3), we get

   $$0 < \phi^{-1}(\varepsilon(t_u - t)) \le u'(t) \text{ for } t \in (t^*, t_u] \ ,$$

   a contradiction. Similarly for $\lim_{t\to t_u+} u'(t) = 0$.

2. Let $\lim\limits_{t\to t_u-} u'(t) > 0$. Since u' is nonincreasing, we have $u'(t) > 0$ for $t \in (0, t_u]$. Similarly for $\lim\limits_{t\to t_u+} u'(t) < 0$.

Hence, t_u is the unique point in $[0, T]$ where either $u'(t_u) = 0$ or $u'(t_u)$ does not exist. By estimate (4.10), u is positive in $(0, T)$. Therefore S has the form as in condition 4 of Theorem 3.1. Finally , by (4.2) and the definition of f_k, we have $f_k(t, u_k(t), u_k'(t)) \geq 0$ for a.e. $t \in [0, T]$, $k \in \mathbb{N}$, $k \geq k_0$. Hence, assumption 5 of Theorem 3.1 is fulfiled and u is a solution of problem (2.1). Estimates (4.4) follow from (4.10) and (4.11). $\square$

Example 4.2. Assume that $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0, \infty)$, and functions $h_i \in L_{loc}(0, \infty)$ are nonnegative, $i = 1, 2, 3, 4$. Let us put

$$\begin{aligned} f(t,x,y) = (1-y^2)\Bigg(&\frac{1}{2t(T-t)} + h_1(t)x^{\alpha_1} \\ &+ h_2(t)|y|^{\alpha_2} + h_3(t)\frac{1}{x^{\beta_1}} + h_4(t)\frac{1}{|y|^{\beta_2}}\Bigg) \end{aligned} \tag{4.12}$$ {e412}

for a.e. $t \in [0, T]$ and all $x \in (0, \infty)$, $y \in \mathbb{R} \setminus \{0\}$. Then function f fulfils the assumptions of Theorem 4.1 with $c_1 = c_2 = 1$, $\nu = \min\left\{\frac{T}{4}; \frac{1}{2}\right\}$, $\mathcal{A}_1 = [0, \infty)$ and $\mathcal{A}_2 = [-1, 1]$.

Really, we see that $f \in Car((0, T) \times \mathcal{D})$, where $\mathcal{D} = (0, \infty) \times ([-1, 1] \setminus \{0\})$ and that $f(t, x, y)$ has singularities at $t = 0$, $t = T$, $x = 0$, $y = 0$. Consequently (2.2) holds. If we put $\sigma_2(t) = \min\{t; (T - t)\}$ for $t \in [0, T]$ we get $|\sigma_2'(t)| = 1$ for a.e. $t \in [0, T]$ and (4.1), (4.2) are valid. Further, for a.e. $t \in [0, T]$ and all $x \in (0, \sigma_2(t))$, $|y| \in (0, \nu]$ we have

$$f(t,x,y) \geq \frac{1-\nu^2}{2t(T-t)} \geq \frac{2(1-\nu^2)}{T^2}\ .$$

Therefore if we choose a positive $\varepsilon < \min\left\{\frac{2(1-\nu^2)}{T^2}; \frac{\phi(\nu)}{\nu}\right\}$ we see that (4.3) holds as well. Theorem 4.1 guarantees the existence of a solution u of problem (2.1) with f given by (4.12). Moreover u fulfils $0 < u(t) \leq \sigma_2(t)$, $-1 \leq u'(t) \leq 1$ for $t \in (0, T)$.

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