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TOPOLOGICAL DEGREE METHOD IN FUNCTIONAL BOUNDARY VALUE PROBLEMS

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1. INTRODUCTION, NOTATION

Let $\mathbf{X}$ be the Banach space of C^0-functions on $J = [0,1]$ endowed with the sup norm $\|\cdot\|$. Denote by $\mathscr{D}$ the set of all operators $K : \mathbf{X} \to \mathbf{X}$ which are continuous and bounded (i.e. $K(\Omega)$ is bounded for any bounded $\Omega \subset \mathbf{X}$) and $\mathscr{A}$ the set of all functionals $\gamma : \mathbf{X} \to \mathbb{R}$ which are linear bounded and increasing (i.e. $x, y \in \mathbf{X}, x(t) < y(t)$ on $J \Rightarrow \gamma(x) < \gamma(y)$).

In the paper we consider the second order differential equation

$$x''(t) = f(t, x(t), (Fx)(t), x'(t), (Hx')(t)), \qquad t \in J, \tag{1}$$

where $f : J \times \mathbb{R}^4 \to \mathbb{R}$ satisfies the local Carathéodory conditions on $J \times \mathbb{R}^4$ ($f \in \operatorname{Car}(J \times \mathbb{R}^4)$ for short) and F, $H \in \mathscr{D}$.

The special case of (1) is the differential equation

$$x'' = h(t, x, x'), \tag{2}$$

where $h \in \operatorname{Car}(J \times \mathbb{R}^2)$.

We find sufficient conditions for the existence of solutions of (1) satisfying one of the following boundary conditions ($A, B \in \mathbb{R}$, $\alpha \in \mathscr{A}$):

$$\alpha(x) = A, \quad x'(1) = B, \tag{3}$$

$$\alpha(x) = A, \qquad x'(0) = B, \tag{4}$$

$$x(0) = A, \quad x(1) = B. \tag{5}$$

Example 1. Let $0 \le a < b \le 1$, $a_k (k = 1, 2, \dots, n)$ be positive constants, $0 \le t_1 < t_2 < \dots < t_n \le 1$. Then the functionals α, β, where

$$\alpha(x) = \sum_{k=1}^{n} a_k x(t_k) \qquad \text{and} \qquad \beta(x) = \int_a^b x(s)\,\mathrm{d}s,$$

belong to the set $\mathscr{A}$.

If we put $n=1$, $t_1=0$ or $t_1=1$ in the first formula, then (3) and (4) have the form of the mixed boundary conditions from [2].

Example 2. Let $\varphi : J \to J$ be a continuous function and $g \in C^0(\mathbb{R})$. The operators belonging to the set $\mathscr{D}$ can be given like this

$$\max\{x(s), 0 \le s \le \varphi(t)\}, \qquad \min\{x(s), 0 \le s \le \varphi(t)\}, \qquad x(\varphi(t)),$$

$$\int_0^{\varphi(t)} g(x(s))\,\mathrm{d}s, \qquad \int_0^{\varphi(t)} \int_0^s g(x(\tau))\,\mathrm{d}\tau\,\mathrm{d}s.$$

Remark 1. Let $\alpha \in \mathscr{A}$ and $\alpha(x)=0$ for an $x \in \mathbf{X}$. Then there exists a $\xi \in J$ such that $x(\xi)=0$ (see e.g. [1]).

This paper was motivated by the recent paper by Kelevedjiev [2], where using the topological transversality method (see e.g. [3]) the author considered the boundary value problems for the equation $x''=q(t,x,x')$, $q \in C^0(J \times \mathbb{R}^2)$ with the Neumann, Dirichlet or mixed boundary conditions. The sufficient conditions for the existence of solutions are formulated only in the terms of sign conditions. The typical result, e.g. for the Dirichlet boundary problem, is the following one.

THEOREM ([2], theorem 4.1). Let $q \in C^0(J \times \mathbb{R}^2)$. Suppose there are constants L_i, $i=1,\ldots,8$, such that $L_2 > L_1 \ge C$, $L_4 > L_3 \ge C$, $L_5 < L_6 \le C$, $L_7 < L_8 \le C$ where $C = B - A$ and

$$q(t,x,y) \ge 0 \qquad \text{for } (t,x,y) \in J \times \mathbb{R} \times ([L_1,L_2] \cup [L_5,L_6]),$$

$$q(t,x,y) \le 0 \qquad \text{for } (t,x,y) \in J \times \mathbb{R} \times ([L_3,L_4] \cup [L_7,L_8]).$$

Then BVP $x''=q(t,x,x')$, (5) has at least one solution in $C^2(J)$.

In our paper, using the topological degree method (see e.g. [4] or [5]), we generalize the results of [2] for the Dirichlet and mixed boundary value problem in the following directions:

(i) "intervals" in the sign conditions for the variable which corresponds to the derivative of a solution are replaced by "points";

(ii) they are considered the Carathéodory solutions;

(iii) in the nonlinearity f there are moreover continuous bounded operators which are applicated to a solution and its derivative;

(iv) the boundary conditions corresponding to the mixed problem have a functional form.

Remark 2. The existence results for the Neumann problem will be proved in our following paper.

For other existence results without growth conditions see, e.g. the papers [6], [7] or [8].

Notation. For each constants $L_1 \le 0 \le L_2$, F, $H \in \mathscr{D}$ and bounded set $\Omega \subset \mathbf{X}$ we set

$$\rho(F,\Omega) = \sup\{\|Fx\| \,|\, x \in \Omega\},$$

$$[L_1,L_2]_X = \{x \,|\, x \in X, \|x\| \le \max\{-L_1, L_2\}\},$$

$$(L_1,L_2)_X = \{x \,|\, x \in X, L_1 \le x(t) \le L_2 \qquad \text{for } t \in J\},$$

$$[L_1, L_2;\ F, H]_R = \{(x,u,w) | (x,u,w) \in \mathbb{R}^3, |x| \le \max\{-L_1, L_2\},$$
$$|u| \le \rho(F, [L_1, L_2]_X), |w| \le \rho(H, (L_1, L_2)_X)\},$$
$$(L_1, L_2;\ F, H)_R = \{(x,u,w) | (x,u,w) \in \mathbb{R}^3,\ L_1 \le x \le L_2,$$
$$|u| \le \rho(F, (L_1, L_2)_X), |w| \le \rho(H, (L_1, L_2)_X)\},$$

and for each A, B, L, $M \in \mathbb{R}$, $L \le M$, $\alpha \in \mathscr{A}$ and F, $H \in \mathscr{D}$ we set

$$[A, B, L, M, \alpha;\ F, H]_R = \{(x,u,v) | (x,u,v) \in \mathbb{R}^3, |x| \le \max\{|L-B|, |M-B|\} + |A|/\alpha(1) + |B|,$$
$$|u| \le \rho(F, [0, \max\{|L-B|, |M-B|\} + |A|/\alpha(1) + |B|]_X), |w| \le \rho(H, (L, M)_X)\},$$
$$(A, B, L, M;\ F, H)_R = \{(x,u,w) | (x,u,w) \in \mathbb{R}^3,\ L + \min\{2A - B, A\} \le x \le M + \max(2A -$$
$$|u| \le \rho(F, (L + \min\{2A - B, A\}, M + \max\{2A - B, A\})_X), |w| \le \rho(H, (L, M)_X)\}.$$

2. HOMOGENEOUS CASE A = B = 0

(a) Problem (1), (3)

In this part we assume that f fulfills the following condition.

(A_1) There exist real numbers $L_1 \le 0 \le L_2$ such that

$$f(t, x, u, L_1, w) \le 0 \le f(t, x, u, L_2, w)$$

for a.e. $t \in J$ and for each $(x, u, w) \in [L_1, L_2;\ F, H]_R$.

For using topological degree arguments to prove existence results to BVP (1), (3) with $A = B = 0$, we need a priori estimates for auxiliary problems defined below.

Let us put

$$g(t, x, u, v, w) = f(t, \hat{x}, u, \bar{v}, \bar{\bar{w}})$$

for $(t, x, u, v, w) \in J \times \mathbb{R}^4$ where (see Notation)

$$\hat{x} = \begin{cases} x & \text{for } |x| \le L \\ L \operatorname{sign} x & \text{for } |x| > L,\ L = \max\{-L_1, L_2\}, \end{cases}$$

$$\tilde{u} = \begin{cases} u & \text{for } |u| \le \rho(F, [L_1, L_2]_X) \\ \rho(F, [L_1, L_2]_X) \operatorname{sign} u & \text{for } |u| > \rho(F, [L_1, L_2]_X), \end{cases}$$

$$\bar{v} = \begin{cases} L_2 & \text{for } v > L_2 \\ v & \text{for } L_1 \le v \le L_2 \\ L_1 & \text{for } v < L_1 \end{cases}$$

and

$$\bar{\bar{w}} = \begin{cases} w & \text{for } |w| \le \rho(H, (L_1, L_2)_X) \\ \rho(H, (L_1, L_2)_X) \operatorname{sign} w & \text{for } |w| > \rho(H, (L_1, L_2)_X). \end{cases}$$

Consider BVP

$$x''(t) = \lambda f^*(t, x(t), (Fx)(t), x'(t), (Hx')(t), \lambda), \qquad \lambda \in [0,1], \tag{6_λ}$$

$$\alpha(x) = 0, \qquad x'(1) = 0, \tag{7}$$

where

$$f^*(t, x, u, v, w, \lambda) = \lambda g(t, x, u, v, w) + (1-\lambda)(v - L_2). \tag{8}$$

LEMMA 1. (A priori estimates). Let f satisfy (A_1). Assume u is a solution of BVP (6_λ), (7) for a $\lambda \in (0,1)$. Then the estimates

$$\|u\| \le \max\{-L_1, L_2\}, \qquad L_1 \le u'(t) \le L_2 \tag{9}$$

are fulfilled for each $t \in J$.

Proof. Fixed $n \in \mathbb{N}$. Suppose $\max\{u'(t) | t \in J\} = u'(t_0) > L_2 + 1/n$. Then $t_0 \neq 1$ and there exists $\delta > 0$ such that

$$L_2 < u'(t) \le u'(t_0)$$

for $t \in [t_0, t_0 + \delta] \subset J$. Hence

$$\int_{t_0}^{t_0+\delta} u''(s)\,\mathrm{d}s = u'(t_0 + \delta) - u'(t_0) \le 0.$$

On the other hand, by (A_1) and the definition of f^*,

$$\int_{t_0}^{t_0+\delta} u''(s)\,\mathrm{d}s = \lambda \int_{t_0}^{t_0+\delta} [g(s, u(s), (Fu)(s), u'(s), (Hu')(s)) + (1-\lambda)(u'(s) - L_2)]\,\mathrm{d}s > 0,$$

a contradiction.

Similarly $\min\{u'(t) | t \in J\} < L_1 - 1/n$ leads to a contradiction. Therefore

$$L_1 - 1/n \le u'(t) \le L_2 + 1/n \qquad \text{for } t \in J. \tag{10}$$

Since $\alpha(u) = 0$, there exists a $c \in J$ such that $u(c) = 0$ (see remark 1). So integrating the inequality (10) on $[0, c]$ and $[c, 1]$, we get

$$|u(t)| \le \max\{-L_1, L_2\} + 1/n \qquad \text{for } t \in J. \tag{11}$$

Since n is an arbitrary positive integer, (10), (11) imply (9). ■

Now, let us suppose that $\mathbf{Y}$, $\mathbf{Z}$ be Banach spaces, $\Omega \subset \mathbf{Y}$ an open bounded set

$$D : \operatorname{dom} D \subset \mathbf{Y} \to \mathbf{Z} \qquad \text{a linear operator,}$$

$$N : \mathbf{Y} \times [0,1] \to \mathbf{Z} \qquad \text{a continuous operator}$$

and

$$D^{-1}N : \overline{\Omega} \times [0,1] \to \mathbf{Y} \qquad \text{a compact operator.}$$

LEMMA 2. Let $\operatorname{Ker} D = \{0\}$, $0 \in \Omega$ and

$$Dx - \lambda N(x, \lambda) \neq 0$$

for each $(x, \lambda) \in (\text{dom } D \cap \partial\Omega) \times (0,1)$. Then equation

$$Dx = N(x, 1)$$

has at least one solution in dom $D \cap \overline{\Omega}$.

Proof. Lemma 2 follows from the corollary by Gaines and Mawhin in ([4], corollary IV.1, p. 29).

LEMMA 3. Let f satisfy (A_1). Then BVP (6_1), (7) has a solution u satisfying (9).

Proof. Fix $n \in \mathbb{N}$. Let $\mathbf{Y} = C^1(J)$ and $\mathbf{Z} = L_1(J)$ be the Banach spaces with the norms as usual. Let

$$\text{dom } D = \{x \mid x \in AC^1(J),\ \alpha(x) = 0,\ x'(1) = 0\}, \qquad D: \text{dom } D \to \mathbf{Z},\ x \mapsto x'',$$

$$N: \mathbf{Y} \times [0,1] \to \mathbf{Z},\ (x, \lambda) \mapsto f^*(\cdot, x(\cdot), (Fx)(\cdot), x'(\cdot), (Hx')(\cdot), \lambda).$$

Then BVP (6_λ), (7) can be written in the operator form $Dx = \lambda N(x, \lambda)$. We see (cf. remark 1) that Ker $D = \{0\}$, Im $D = \mathbf{Z}$ and $D^{-1}N: \mathbf{Y} \times [0,1] \to \mathbf{Y}$ is a completely continuous (i.e. compact on each bounded set in $\mathbf{Y} \times [0,1]$). Set

$$\Omega = \{x \mid x \in \mathbf{Y}, \|x\| < \max\{-L_1, L_2\} + 1/n, L_1 - 1/n < x'(t) < L_2 + 1/n \text{ for } t \in J\}.$$

Lemma 1 implies that for each $\lambda \in (0,1)$ no solution x of $Dx = \lambda N(x, \lambda)$ can belong to $\partial\Omega$, i.e. $Dx - \lambda N(x, \lambda) \neq 0$ for each $(x, \lambda) \in (\text{dom } D \cap \partial\Omega) \times (0,1)$. By lemma 2, BVP (6_1), (7) has a solution $u \in \overline{\Omega}$. Since n is arbitrary, u satisfies (9). ■

THEOREM 1 (existence theorem). Let f satisfy (A_1). Then BVP (1), (7) has a solution u satisfying (9).

Proof. By lemma 3, there exists a solution u of BVP (6_1), (7) satisfying (9). From the definition of f^* it follows that u is also a solution of (1). ■

The next existence result for BVP (2), (7) immediately follows from theorem 1.

COROLLARY 1. Let there exist real numbers $L_1 \le 0 \le L_2$ such that

$$h(t, x, L_1) \le 0 \le h(t, x, L_2)$$

for a.e. $t \in J$ and each $|x| \le \max\{-L_1, L_2\}$. Then BVP (2), (7) has a solution u satisfying (9).

(b) Problem (1), (4)

Consider BVP (1), (4) for $A = B = 0$, i.e. BVP (1), (12) where

$$\alpha(x) = 0, \qquad x'(0) = 0. \tag{12}$$

Using the substitution

$$t = 1 - s, \qquad x(t) = u(s), \tag{13}$$

we can write BVP (1), (12) in the form

$$u''(s) = f(1-s, u(s), (F^*u)(s), -u'(s), (H^*(-u'))(s)), \tag{14}$$

$$\alpha^*(u) = 0, \qquad u'(1) = 0, \tag{15}$$

with $\alpha^* : \mathbf{X} \to \mathbb{R}$, F^*, $H^* : \mathbf{X} \to \mathbf{X}$ defined by $\alpha^*(x) = \alpha(x^*)$, $(F^*x)(t) = (Fx^*)(1-t)$, $(H^*x)(t) = (Hx^*)(1-t)$ where $x^*(t) = x(1-t)$ for $t \in J$. Obviously, $\alpha^* \in \mathscr{A}$, F^*, $H^* \in \mathscr{D}$ and $[L_1, L_2;\ F, H]_R = [L_1, L_2;\ F^*, H^*]_R$.

If we apply theorem 1 to BVP (14), (15) and use again substitution (13), we get for BVP (1), (12) the following result.

THEOREM 2. Assume that f satisfies the assumption:

(A_2) There exist real numbers $L_1 \le 0 \le L_2$ such that

$$f(t, x, u, L_2, w) \le 0 \le f(t, x, u, L_1, w)$$

for a.e. $t \in J$ and for each $(x, u, w) \in [L_1, L_2;\ F, H)_R$.

Then BVP (1), (12) has solution u satisfying (9).

COROLLARY 2. Assume that there are real numbers $L_1 \le 0 \le L_2$ such that

$$h(t, x, L_2) \le 0 \le h(t, x, L_1)$$

for a.e. $t \in J$ and each $|x| \le \max\{-L_1, L_2\}$. Then BVP (2), (12) has a solution u satisfying (9).

(c) Problem (1), (5)

BVP (1), (5) will be first solved also in the case of $A = B = 0$, i.e. for the boundary conditions

$$x(0) = 0, \qquad x(1) = 0. \tag{16}$$

Let $L_1 \le 0 \le L_2$, $L_3 \le 0 \le L_4$, $L_3 < L_1$, $L_4 > L_2$ and $n_0 \in \mathbb{N}$ be a positive integer such that $L_2 + 2/n_0 < L_4$, $L_1 - 2/n_0 > L_3$. To obtain a priori estimates for BVP (1), (16) we define the function $h_n \in \mathrm{Car}(J \times \mathbb{R}^4)$ for each $n \ge n_0$ in the following way

$$h_n(t, x, u, v, w) = \begin{cases} f(t, \hat{x}, \tilde{u}, L_4, \overline{w}) & \text{for } L_4 < v, \\ f(t, \hat{x}, \tilde{u}, v, \overline{w}) & \text{for } L_2 + 2/n \le v \le L_4, \\ f(t, \hat{x}, \tilde{u}, L_2 + 2/n, \overline{w}) & \text{for } L_2 + 1/n < v < L_2 + 2/n, \\ \quad +g(L_2, 2/n, v) & \\ f(t, \hat{x}, \tilde{u}, L_2, \overline{w}) & \text{for } L_2 < v \le L_2 + 1/n, \\ f(t, \hat{x}, \tilde{u}, v, \overline{w}) & \text{for } L_1 \le v \le L_2, \\ f(t, \hat{x}, \tilde{u}, L_1, \overline{w}) & \text{for } L_1 - 1/n \le v < L_1, \\ f(t, \hat{x}, \tilde{u}, L_1 - 2/n, \overline{w}) & \text{for } L_1 - 2/n < v < L_1 - 1/n, \\ \quad -g(L_1 - 2/n, v) & \\ f(t, \hat{x}, \tilde{u}, v, \overline{w}) & \text{for } L_3 < v \le L_1 - 2/n, \\ f(t, \hat{x}, \tilde{u}, L_3, \overline{w}) & \text{for } v < L_3, \end{cases} \tag{17}$$

where

$$g(L_i,k,v)=\big(f\big(t,\hat{x},\tilde{u},L_i,\overline{w}\big)-f\big(t,\hat{x},\tilde{u},L_i+k,\overline{w}\big)\big)(L_i+k-v)n, \qquad i=1,2$$

and (see notation)

$$\hat{x}=\begin{cases} L_4 & \text{for } x>L_4 \\ x & \text{for } L_3\le x\le L_4, \\ L_3 & \text{for } x<L_3, \end{cases}$$

$$\tilde{u}=\begin{cases} u & \text{for } |u|\le \rho(F,(L_3,L_4)_X), \\ \rho(F,(L_3,L_4)_X)\operatorname{sign}u & \text{for } |u|>\rho(F,(L_3,L_4)_X), \end{cases}$$

$$\overline{w}=\begin{cases} w & \text{for } |w|\le \rho(H,(L_3,L_4)_X), \\ \rho(H,(L_3,L_4)_X)\operatorname{sign}w & \text{for } |w|>\rho(H,(L_3,L_4)_X). \end{cases}$$

We will again consider auxiliary BVP $(18_\lambda)_n$, (16), where

$$x''(t)=\lambda f_n^*(t,x(t),(Fx)(t),x'(t),(Hx')(t),\lambda), \qquad \lambda\in[0,1],\ n\ge n_0 \qquad (18_\lambda)_n$$

and

$$f_n^*(t,x,u,v,w,\lambda)=\lambda h_n(t,x,u,v,w)+(1-\lambda)p(v), \qquad (19)$$

$p:\mathbb{R}\to\mathbb{R}$ is continuous function with the property

$$p(v)\ge 1 \qquad \text{for } v\in[L_3-1/n_0,L_3]\cup[L_2,L_2+1/n_0],$$

$$p(v)\le -1 \qquad \text{for } v\in[L_1-1/n_0,L_1]\cup[L_4,L_4+1/n_0].$$

Here, the main condition for f has the following form.

(A_3) There exist constants $L_1\le 0\le L_2$, $L_3\le 0\le L_4$ such that

$$f(t,x,u,L_1,w)\le 0\le f(t,x,u,L_2,w),$$

$$f(t,x,u,L_4,w)\le 0\le f(t,x,u,L_3,w)$$

for a.e. $t\in J$ and each $(x,u,w)\in(L,M;\ F,H)_R$, where $L=\min\{L_1,L_3\}$, $M=\max\{L_2,L_4\}$.

LEMMA 4. (A priori estimates). Let f satisfy (A_3) with $L_3<L_1$, $L_4>L_2$ and BVP $(18_\lambda)_n$, (16) has a solution u for some $\lambda\in(0,1)$ and $n\ge n_0$. Then the estimates

$$L_3-1/n\le u(t)\le L_4+1/n, \qquad L_3-1/n\le u'(t)\le L_4+1/n$$

are fulfilled for each $t\in J$.

Proof. By (16), we can find an $a\in(0,1)$ such that $u'(a)=0$. Assume $\max\{u'(t)|\ t\in[0,a)\}=u'(t_0)>L_2+1/n$. Then there exists an interval $[\gamma,\delta]\subset(t_0,a)$ such that $L_2\le u'(t)\le L_2+1/n$ on $[\gamma,\delta]$ and $u'(\gamma)=L_2+1/n$, $u'(\delta)=L_2$; hence

$$\int_\gamma^\delta u''(s)\,\mathrm{d}s=u'(\delta)-u'(\gamma)=-1/n<0.$$

On the other hand we get by (A_3), (17) and (19),

$$\int_\gamma^\delta u''(s)\,\mathrm{d}s = \lambda \int_\gamma^\delta [\lambda h_n(s,u(s),(Fu)(s),u'(s),(Hu')(s)) + (1-\lambda)p(u'(s))]\,\mathrm{d}s$$

$$\geq \lambda(1-\lambda)\int_\gamma^\delta p(u'(s))\,\mathrm{d}s \geq \lambda(1-\lambda)(\delta-\gamma) > 0,$$

a contradiction.

Similarly for $\min\{u'(t)|\ t\in[0,a)\} < L_1 - 1/n$.

Now, let $\max\{u'(t)|t\in (a,1]\} = u'(t_1) > L_4 + 1/n$. Then there exists an interval $[\varepsilon,\nu]\subset (a,t_1)$ such that $L_4 \leq u'(t) \leq L_4 + 1/n$ for $t\in[\varepsilon,\nu]$ and $u'(\varepsilon) = L_4$, $u'(\nu) = L_4 + 1/n$. Then

$$\int_\varepsilon^\nu u''(s)\,\mathrm{d}s = u'(\nu) - u'(\varepsilon) = 1/n > 0$$

and by (A_3), (17) and (19) we can prove $\int_\varepsilon^\nu u''(s)\,\mathrm{d}s < 0$ by the same arguments like above, a contradiction. Similarly for $\min\{u'(t)|t\in (a,1]\} < L_3 - 1/n$. Therefore, using the inequalities $L_3 < L_1$, $L_4 > L_2$, we obtain

$$L_3 - 1/n \leq u'(t) \leq L_4 + 1/n \qquad \text{for } t\in J.$$

Integrating the last inequality from 0 to t, we obtain the estimates for u. ■

COROLLARY 3. Let f satisfy (A_3) with $L_3\neq L_1$, $L_4\neq L_2$ and $|L_3-L_1|>2/n_0$, $|L_4-L_2|>2/n_0$ for a positive integer n_0. Let BVP $(18_\lambda)_n$, (16) has a solution u for some $\lambda\in(0,1)$ and $n\geq n_0$. Then the estimates

$$L - 1/n \leq u(t) \leq M + 1/n, \qquad L - 1/n \leq u'(t) \leq M + 1/n \tag{20}$$

are fulfilled on J.

Proof. If $L_3 < L_1$, $L_4 > L_2$, the assertion follows from lemma 4. Let $L_3 > L_1$, $L_4 > L_2$. If we replace L_3 and L_1 in (17) and in the formulae for $\hat{x}$, $\tilde{u}$, $\bar{w}$, then by the same arguments as in the proof of lemma 4 we prove

$$L_1 - 1/n \leq u(t) \leq L_4 + 1/n, \qquad L_1 - 1/n \leq u'(t) \leq L_4 + 1/n$$

for $t\in J$. Similarly for $L_4 < L_2$. ■

LEMMA 5. Let f satisfy (A_3). Then for each sufficiently large $n\in\mathbb{N}$ BVP $(18_1)_n$, (16) has a solution u satisfying (20).

Proof. For $L_3\neq L_1$, $L_4\neq L_2$ the proof is similar to that of lemma 3, only we put $\operatorname{dom} D = \{x|x\in AC^1(J),\ x(0)=0,\ x(1)=0\}$ and

$$\Omega = \{x|x\in \mathbf{Y},\ L - 2/n < x(t) < M + 2/n,\ L - 2/n \leq x'(t) \leq M + 2/n \text{ for } t\in J\}$$

and use corollary 3. If $L_1 = L_3$ or $L_2 = L_4$, BVP $(18_1)_n$, (16) has the trivial solution. ■

THEOREM 3. Let f satisfy (A_3). Then BVP (1), (16) has a solution u satisfying.

$$L \leq u(t) \leq M, \qquad L \leq u'(t) \leq M \qquad \text{for } t\in J. \tag{21}$$

Proof. By lemma 5, there exists a solution u_n of BVP $(18_1)_n$, (16) satisfying (20) (with $u = u_n$) for each sufficiently large $n \in \mathbb{N}$. Since $f \in \mathrm{Car}(J \times \mathbb{R}^4)$, we can find $\phi \in L_1(J)$ such that

$$|f(t,x,u,v,w)| \le \phi(t) \qquad \text{for a.e. } t \in J \text{ and each } L \le x \le M,$$
$$|u| \le \rho(F,(L,M)_x),\ L \le v \le M \qquad \text{and} \qquad |w| \le \rho(H,(L,M)_x).$$

Then (cf. (19))

$$\begin{aligned} |u_n'(t_2) - u_n'(t_1)| &= \left| \int_{t_1}^{t_2} f_n^*(s, u_n(s), (Fu_n)(s), u_n'(s), (Hu_n')(s), 1) \mathrm{d}s \right| \\ &= \left| \int_{t_1}^{t_2} h_n(s, u_n(s), (Fu_n)(s), u_n'(s), (Hu_n')(s)) \mathrm{d}s \right| \\ &\le \left| \int_{t_1}^{t_2} \phi(s) \, \mathrm{d}s \right| \end{aligned}$$

for t_1, $t_2 \in J$. Thus (cf. (20) with $u = u_n$) $\{u_n(t)\}$, $\{u_n'(t)\}$ are equibounded and equicontinuous on J and, by the Arzelà–Ascoli theorem, we can choose a subsequence $\{u_{k_n}(t)\}$ converging in the Banach space $C^1(J)$ to u. One can see that u fulfils (16) and (21) and so, by (17) and (19), it is a solution of (1). ■

COROLLARY 4. Let there exist constants $L_1 \le 0 \le L_2$, $L_3 \le 0 \le L_4$ such that

$$h(t,x,L_1) \le 0 \le h(t,x,L_2), \qquad h(t,x,L_4) \le 0 \le h(t,x,L_3)$$

for a.e. $t \in J$ and each $x \in [L, M]$, where $L = \min\{L_1, L_3\}$, $M = \max\{L_2, L_4\}$. Then BVP (2), (16) has a solution u satisfying (21).

3. NONHOMOGENEOUS CASE

Here, we show theorems for nonhomogeneous BVP (1), (i) with $i = 3$, 4 and 5.

Remark 3. One can easily verify that the function $\varphi(t) = (1/\alpha(1))(A - B\alpha(f)) + Bt$, $t \in J$ satisfies boundary conditions both (3) and (4).

Remark 4. Since $0 \le f \le 1$ for $f \in J$ and $\alpha \in \mathscr{A}$, $0 \le \alpha(f) \le \alpha(1)$. Hence $|t - \alpha(f)/\alpha(1)| \le 1$ for $t \in J$.

THEOREM 4. Suppose f satisfies the following assumption.

(H_1) There exist A, B, L_1, $L_2 \in \mathbb{R}$ such that $L_1 \le B \le L_2$ and

$$f(t,x,u,L_1,w) \le 0 \le f(t,x,u,L_2,w)$$

for a.e. $t \in J$ and each $(x,u,w) \in [A,B,L_1,L_2,\alpha;\ F,H]_R$.

Then BVP (1), (3) has a solution u satisfying

$$\|u\| \le \max\{B - L_1, L_2 - B\} + |A|/\alpha(1) + |B|, \qquad L_1 \le u'(t) \le L_2 \qquad \text{for } t \in J. \qquad (22)$$

Proof. Let $\varphi(t) = (1/\alpha(1))(A - B\alpha(f)) + Bt$. By remark 3, φ satisfies boundary conditions (3) and (cf. remark 4)

$$\|\varphi\| \leq \frac{|A|}{\alpha(1)} + |B|, \qquad \varphi'(t) = B \quad \text{for } t \in J. \tag{23}$$

Using the substitution

$$x(t) = z(t) + \varphi(t) \tag{24}$$

we can write BVP (1), (3) in the form

$$z''(t) = g(t, z(t), (F^*z)(t), z'(t), (H^*z')(t)), \qquad \alpha(z) = 0,\ z'(1) = 0, \tag{25}$$

where $g(t, x, u, v, w) = f(t, x + \varphi(t), u, v + B, w)$ for a.e. $t \in J$ and each $(x, u, v, w) \in \mathbb{R}^4$ and $F^*(u) = F(u + \varphi)$, $H^*(u) = H(u + B)$ for $u \in X$. By theorem 1 there is a solution z of BVP (25) such that $\|z\| \leq \max\{B - L_1, L_2 - B\}$, $L_1 - B \leq z'(t) \leq L_2 - B$ for $t \in J$, hence $x(t) = z(t) + \varphi(t)$ is a solution of BVP (1), (3) satisfying (22). ■

COROLLARY 5. Let A, B, L_1, $L_2 \in \mathbb{R}$ be such that $L_1 \leq B \leq L_2$ and

$$h(t, x, L_1) \leq 0 \leq h(t, x, L_2)$$

for a.e. $t \in J$ and each $|x| \leq \max\{B - L_1, L_2 - B\} + |A|/\alpha(1) + |B|$. Then BVP (2), (3) has a solution u satisfying (22).

THEOREM 5. Suppose f satisfies the following assumption.

(H_2) There exist A, B, L_1, $L_2 \in \mathbb{R}$ such that $L_1 \leq B \leq L_2$ and

$$f(t, x, u, L_2, w) \leq 0 \leq f(t, x, u, L_1, w)$$

for a.e. $t \in J$ and each $(x, u, w) \in [A, B, L_1, L_2, \alpha;\ F, H]_R$.

Then BVP (1), (4) has a solution u satisfying (22).

Proof. We proceed as in the proof of theorem 4, only we apply theorem 2 instead of theorem 1. ■

COROLLARY 6. Let A, B, L_1, $L_2 \in \mathbb{R}$ be such that $L_1 \leq B \leq L_2$ and

$$h(t, x, L_2) \leq 0 \leq h(t, x, L_1)$$

for a.e. $t \in J$ and each $|x| \leq \max\{B - L_1, L_2 - B\} + (|A|/\alpha(1)) + |B|$. Then BVP (2), (4) has a solution u satisfying (22).

For BVP (1), (5) respectively (2), (5) we get the following results.

THEOREM 6. Suppose f satisfies the assumption:

(H_3) There exist A, B, L_1, L_2, L_3, $L_4 \in \mathbb{R}$ such that $L_1 \leq B - A \leq L_2$, $L_3 \leq B - A \leq L_4$, and

$$f(t, x, u, L_1, w) \leq 0 \leq f(t, x, u, L_2, w),$$

$$f(t, x, u, L_4, w) \leq 0 \leq f(t, x, u, L_3, w)$$

for a.e. $t \in J$ and each $(x, u, w) \in (A, B, L, M; F, H)_R$, where $L = \min\{L_1, L_3\}$, $M = \max\{L_2, L_4\}$.

Then BVP (1), (5) has a solution u satisfying

$$L + \min\{2A - B, A\} \le u(t) \le M + \max\{2A - B, A\}, \qquad L \le u'(t) \le M \qquad \text{for } t \in J. \quad (26)$$

Proof. Let $\varphi(t) = A(1 - t) + Bt$ for $t \in J$. Using substitution (24) we can see that x is a solution of BVP (1), (5) if and only if z is a solution of BVP

$$z''(t) = q(t, z(t), (F^* z)(t), z'(t), (H^* z')(t)), \qquad z(0) = 0, \qquad z(1) = 0 \qquad (27)$$

where

$$q(t, x, u, v, w) = f(t, x + \varphi(t), u, v + B - A, w) \qquad \text{for a.e. } t \in J \text{ and each } (x, u, v, w) \in \mathbb{R}^4$$

and $F^*(u) = F(u + \varphi)$, $H^*(u) = H(u + B - A)$ for $u \in \mathbf{X}$. By theorem 3, there is a solution z of BVP (27) such that $L - B + A \le z(t) \le M - B + A$, $L - B + A \le z'(t) \le M - B + A$ for $t \in J$. Then $x(t) = z(t) + \varphi(t)$ is a solution of BVP (1), (5) satisfying (26). ■

Corollary 7. Let $A, B, L_1, L_2, L_3, L_4 \in \mathbb{R}$ be such that $L_1 \le B - A \le L_2$, $L_3 \le B - A \le L_4$, and

$$h(t, x, L_1) \le 0 \le h(t, x, L_2), \qquad h(t, x, L_4) \le 0 \le h(t, x, L_3)$$

for a.e. $t \in J$ and each $L + \min\{2A - B, A\} \le x \le M + \max\{2A - B, A\}$ where $L = \min\{L_1, L_3\}$, $M = \max\{L_2, L_4\}$. Then BVP (2), (5) has a solution u satisfying (26).

Example 3. Consider the differential equation

$$x'' = P_m(x'), \qquad m \in \mathbb{N} - \{1\} \qquad (28)$$

with the mixed boundary conditions

$$x'(0) = A, \qquad x(1) = B \qquad (29)$$

or the Dirichlet conditions (5), where P_m is a polynomial of the degree m.

In [2] the author shows that provided the polynomial P_m has a simple zero bigger and a simple zero smaller than A (in the case of BVP (28), (29)), or than $B - A$ (in the case of BVP (28), (5)) the considered problem is solvable. Here, using corollary 6 for BVP (28), (29) or corollary 7 for BVP (28), (5), we obtain the solvability of these problems even in the case of arbitrary multiplicity of zeros of P_m as well.

Example 4. Consider the functional differential equation

$$x''(t) = t + d(t) + \max\{x'(s) | 0 \le s \le t^2\}$$
$$\int_0^t x^2(s)\,\mathrm{d}s + 10\pi^3(e^t + x^{2n}(t))\sin x'(t), \qquad t \in J, \qquad (30)$$

where $d(t)$ is the Dirichlet function and $n \in \mathbb{N}$. Equation (30) has the form (1) with $f(t, x, u, v, w) = t + d(t) + uw + 10\pi^3(e^t + x^{2n})\sin v$, $(Fx)(t) = \int_0^t x^2(s)\,\mathrm{d}s$, $(Hx)(t) = \max\{x'(s) | 0 \le s \le t^2\}$. One can verify that:

(a) theorem 4 can be applied to BVP (30), (3) with $A \in [-2\pi\alpha(1), 2\pi\alpha(1)]$, $B \in [-\pi/2, \pi/2]$ setting $L_1 = -\pi/2$ and $L_2 = \pi/2$;

(b) theorem 5 can be applied to BVP (30), (4) with $A \in [-\pi\alpha(1)/2, \pi\alpha(1)/2]$, $B \in [\pi/2, \pi]$ setting $L_1 = \pi/2$ and $L_2 = 3\pi/2$;
(c) theorem 6 can be applied to BVP (30), (5) with $A \in [-\pi/2, \pi/2]$ and $B \in [A - \pi/2, A + \pi/2]$ setting $L_1 = -\pi/2$, $L_2 = \pi/2$, $L_3 = -3\pi/2$ and $L_4 = 3\pi/2$.

4. APPLICATIONS

In this section we give some applications of the above results to BVPs for third and fourth order differential equations.

First, denote by $\mathscr{E}$ the set of all continuous increasing (not necessarily linear) functionals $\gamma : \mathbf{X} \to \mathbb{R}$ with $\gamma(0) = 0$. If $\gamma \in \mathscr{E}$ we can easily verify that $\gamma(x) = 0$ for an $x \in \mathbf{X}$ implies $x(\xi) = 0$ with a $\xi \in J$.

LEMMA 6. Let α, $\beta \in \mathscr{E}$, $v \in \mathbf{X}$. Then BVP

$$x'' = v(t), \qquad \alpha(x) = 0, \quad \beta(x') = 0 \tag{31}$$

and

$$x'' = v(t), \qquad \beta(x) = 0, \qquad x'(0) = 0 \tag{32}$$

have a unique solution x_1 and x_2, respectively. Moreover,

$$\|x_i\| \le \|v\|/2, \qquad \|x_i'\| \le \|v\|, \qquad \text{for } i = 1, 2. \tag{33}$$

Proof. Since $x(t) = c_1 + c_2 t + \int_0^t \int_0^s v(\tau)\,d\tau\,ds$ is the general solution of the equation $x'' = v(t)$ and $\alpha(x) = 0$, $\beta(x') = 0$ imply $x(\xi) = 0$, $x'(\varepsilon) = 0$ for some ξ, $\varepsilon \in J$, we see that $x(t) = \int_\xi^t \int_\varepsilon^s v(\tau)\,d\tau\,ds$ is the unique solution of BVP (31) and $\|x\| \le \|v\|/2$, $\|x'\| \le \|v\|$. Similarly for BVP (32). ■

Consider BVP

$$x^{(4)} = p(t, x, x', x'', x'''), \tag{34}$$

$$\alpha(x) = 0, \qquad \beta(x') = 0, \qquad x''(0) = 0, \qquad x''(1) = 0, \tag{35}$$

where $p \in \mathrm{Car}(J \times \mathbb{R}^4)$ and α, $\beta \in \mathscr{E}$.

THEOREM 7. Let $L_1 \le 0 \le L_2$, $L_3 \le 0 \le L_4$ be constants such that

$$p(t, x, u, v, L_1) \le 0 \le p(t, x, u, v, L_2),$$
$$p(t, x, u, v, L_3) \le 0 \le p(t, x, u, v, L_4)$$

are satisfied for a.e. $t \in J$ and each $(x, u, v) \in [-L/2, L/2] \times [-L, L] \times [-L, L]$, $L = \max\{-L_1, -L_3, L_2, L_4\}$. Then BVP (34), (35) has at least one solution x satisfying

$$\|x\| \le L/2, \qquad \|x'\| \le L, \qquad \|x''\| \le L, \qquad \min\{L_1, L_3\} \le x'''(t) \le \max\{L_2, L_4\}, \qquad t \in J. \tag{36}$$

Proof. Define the operators F, $H : \mathbf{X} \to \mathbf{X}$ by

$$Fv = x_1, \qquad Hv = x_2,$$

where x_1 and x_2 are the unique solutions of BVP (31) and (32), respectively (see lemma 6). Inequalities (33) imply that F and H are bounded and $\rho(F;[0,L]_X) \leq L/2$, $\rho(H;[0,L]_X) \leq L/2$. We shall prove that F is continuous. Let $\{v_n\} \subset \mathbf{X}$ be a convergent sequence, $v_n \to v$ as $n \to \infty$. Then there are bounded sequences $\{a_n\}$, $\{b_n\} \subset \mathbb{R}$ and a, $b \in \mathbb{R}$ such that

$$x_n(t) = a_n + b_n t + \int_0^t \int_0^s v_n(\tau)\,\mathrm{d}\tau\,\mathrm{d}s,\ x(t) = a + bt + \int_0^t \int_0^s v(\tau)\,\mathrm{d}\tau\,\mathrm{d}s, \qquad t \in J,\ n \in \mathbb{N}.$$

If $\lim_{n\to\infty} b_n \neq b$, then there is a convergent subsequence $\{b_{k_n}\}$ of $\{b_n\}$, $\lim_{n\to\infty} b_{k_n} = b^*$, $b^* \neq b$. Taking the limit as $k_n \to \infty$ in the equalities

$$\left(\beta(x'_{k_n}) =\right) \beta\left(b_{k_n} + \int_0^t v_{k_n}(s)\,\mathrm{d}s\right) = 0,$$

we obtain $\beta(b^* + \int_0^t v(s)\,\mathrm{d}s) = 0$. Since $\beta(x') = \beta(b + \int_0^t v(s)\,\mathrm{d}s) = 0$ and β is increasing, $b = b^*$, and consequently $\lim_{n\to\infty} b_n = b$. Assume, on the contrary, $\lim_{n\to\infty} a_n \neq a$. Then $\lim_{n\to\infty} a_{l_n} = a^* \neq a$ for a subsequence $\{a_{l_n}\}$ of $\{a_n\}$ and taking the limit as $l_n \to \infty$ in the equalities

$$\left(\alpha(x_{l_n}) =\right) \alpha\left(a_{l_n} + b_{l_n} t + \int_0^t \int_0^s v_{l_n}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\right) = 0,$$

we get $\alpha(a^* + bt + \int_0^t\int_0^s v(\tau)\,\mathrm{d}\tau\,\mathrm{d}s) = 0$. Since $\alpha(a + bt + \int_0^t\int_0^s v(\tau)\,\mathrm{d}\tau\,\mathrm{d}s) = 0$ and α is increasing, $a = a^*$, and consequently $\lim_{n\to\infty} a_n = a$. Hence $\lim_{n\to\infty} x_n = x$ which proves that F is continuous and therefore $F \in \mathscr{D}$. Similarly, $H \in \mathscr{D}$.

Using the substitution $u = x''$ we see that BVP (34), (35) can be written in the form

$$u''(t) = p(t, (Fu)(t), (Hu')(t), u(t), u'(t)),\ u(0) = 0,\ u(1) = 0. \tag{37}$$

Set $f(t,x,u,v,w) = p(t,u,w,x,v)$ for $(t,x,u,v,w) \in J \times \mathbb{R}^4$. Then f satisfies the assumptions of theorem 3, and consequently there exists a solution u of BVP (37) such that $\|u\| \leq L$, $\min\{L_1, L_3\} \leq u'(t) \leq \max\{L_2, L_4\}$ for $t \in J$. Obviously there exists a unique $x \in AC^3(J)$ satisfying $\alpha(x) = 0$, $\beta(x') = 0$ (which implies $x(\xi) = 0$, $x'(\epsilon) = 0$ for some ξ, $\epsilon \in J$) and $x''(t) = u(t)$ for $t \in J$. This function x is a solution of BVP (34), (35) for which (36) holds. ∎

Analogously we can prove for BVP

$$x''' = q(t, x, x', x''), \tag{38}$$

$$\alpha(x) = 0, \qquad x'(0) = 0, \qquad x'(1) = 0, \tag{39}$$

where $q \in \mathrm{Car}(J \times \mathbb{R}^3)$ and $\alpha \in \mathscr{E}$, the following theorem.

THEOREM 8. Let $L_1 \leq 0 \leq L_2$, $L_3 \leq 0 \leq L_4$ be constants such that

$$q(t,x,u,L_1) \leq 0 \leq q(t,x,u,L_2), \qquad q(t,x,u,L_3) \leq 0 \leq q(t,x,u,L_4)$$

are satisfied for a.e. $t \in J$ and each $(x,u) \in [-L,L] \times [-L,L]$, $L = \max\{-L_1, -L_3, L_2, L_4\}$. Then BVP (38), (39) has at least one solution x satisfying

$$\|x\| \leq L, \qquad \|x'\| \leq L, \qquad \min\{L_1, L_3\} \leq x''(t) \leq \max\{L_2, L_4\}, \qquad t \in J.$$

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