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TOPOLOGICAL DEGREE METHOD IN FUNCTIONAL BOUNDARY VALUE PROBLEMS AT RESONANCE†

IRENA RACHŮNKOVÁ and SVATOSLAV STANĚK

Department of Mathematics, Palacký University, Tomkova 40, 77900 Olomouc, Czech Republic

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1. INTRODUCTION, NOTATION

Let $\mathbf{X}$ be the Banach space of C^0-functions on $J = [0, 1]$ with the sup norm $\|\cdot\|$. Denote by $\mathfrak{D}$ the set of all operators $K: \mathbf{X} \to \mathbf{X}$ which are continuous and bounded (i.e. $K(\Omega)$ is bounded for any bounded $\Omega \subset \mathbf{X}$).

In the paper we study boundary value problems at resonance for the second order functional differential equation

$$x''(t) = f(t, x(t), (Fx)(t), x'(t), (Hx')(t)), \qquad t \in J, \tag{1}$$

where $f: J \times \mathbb{R}^4 \to \mathbb{R}$ and $F, H \in \mathfrak{D}$. We will consider both the classical and the Carathéodory case, i.e. f is supposed to be continuous on $J \times \mathbb{R}^4$ and a solution of (1) is found in $C^2(J)$ or f satisfies the local Carathéodory conditions on $J \times \mathbb{R}^4$ ($f \in \mathrm{Car}(J \times \mathbb{R}^4)$ for short) and a solution of (1) is a function $x \in AC^1(J)$ (having the absolutely continuous first derivative on J) satisfying (1) a.e. on J.

The special case of (1) is the differential equation

$$x'' = h(t, x, x'), \tag{2}$$

where $h \in C^0(J \times \mathbb{R}^2)$ or $h \in \mathrm{Car}(J \times \mathbb{R}^2)$.

We show sufficient conditions for the existence of solutions of (1) satisfying one of the following boundary conditions

$$x'(0) = 0, \qquad x'(1) = 0, \qquad \text{(Neumann conditions)}, \tag{3}$$

or

$$x(0) = x(1), \qquad x'(0) = x'(1), \qquad \text{(periodic conditions)}. \tag{4}$$

We prove the existence results provided f satisfies only sign conditions. Let us note that the existence results with strict sign conditions for the periodic problem were proved also in [1], but there h was continuous. Here, moreover, under an appropriate combination of sign conditions we get multiplicity results as well.

This paper is a continuation of the authors paper [2] and it has been motivated by the recent paper [3], in which, by the topologial transversality method (see, e.g. [4]) the author considered the differential equation (q): $x'' = q(t, x, x')$, $q \in C^0(J \times \mathbb{R}^2)$ together with the Neumann conditions. His existence result is formulated only by sign conditions in the following theorem.

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THEOREM [3, theorem 5.1]. Let there exist $M, L_j \in \mathbb{R}$ $(j = 1, \dots, 4)$ such that $M \geq 0$, $L_2 > L_1 \geq M$, $-M \geq L_4 > L_3$ and

(i) $xq(t, x, 0) > 0$ for $|x| > M$,

(ii) $q(t, x, y)$ does not change its sign for $(t, x, y) \in J \times [-M, M] \times [L_1, L_2]$ and for $(t, x, y) \in J \times [-M, M] \times [L_3, L_4]$.

Then BVP (q), (3) has at least one solution in $C^2(J)$.

We shall generalize this result in the following directions:

(a) sign condition (i) is replaced by a weaker sign condition (24);

(b) "intervals" in sign condition (ii) for the variable y are replaced by "points" (see (25));

(c) there are considered the Carathéodory solutions;

(d) nonlinearity f depends also on the continuous bounded operators which are applicated to a solution and its derivative.

Moreover, our existence results include also the case of sign condition (i) with the inverse sign of inequality (see theorems 2, 4 and corollaries 2, 4).

The proofs of our results are based on the Mawhin continuation theorem. (See, e.g. [5] or [6].)

Let $\mathbf{Y}$, $\mathbf{Z}$ be real Banach spaces, L: dom $L \subset \mathbf{Y} \to \mathbf{Z}$ a Fredholm map of index zero and $P: \mathbf{Y} \to \mathbf{Y}$, $Q: \mathbf{Z} \to \mathbf{Z}$ continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$ and $\mathbf{Y} = \operatorname{Ker} L \oplus \operatorname{Ker} P$, $\mathbf{Z} = \operatorname{Im} L \oplus \operatorname{Im} Q$. Denote by $L_P: \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{dom} L$ the generalized inverse (to L) and $\mathfrak{J}: \operatorname{Im} Q \to \operatorname{Ker} L$ an isomorphism of $\operatorname{Im} Q$ onto $\operatorname{Ker} L$.

THEOREM (continuation theorem [5, p. 40]). Let $\Omega \subset \mathbf{Y}$ be an open bounded set and $N: \mathbf{Y} \to \mathbf{Z}$ be a continuous operator which is L-compact on $\bar{\Omega}$ (i.e. $QN: \bar{\Omega} \to \mathbf{Z}$ and $K_P(I - Q)N: \bar{\Omega} \to \mathbf{Y}$ are compact). Assume

(I) for each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$,

(II) $QNx \neq 0$ for each $x \in \operatorname{Ker} L \cap \partial\Omega$,

(III) the Brouwer degree $d[\mathfrak{J}QN, \Omega \cap \operatorname{Ker} L, 0] \neq 0$.

Then the operator equation $Lx = Nx$ has at least one solution in dom $L \cap \bar{\Omega}$.

Notation. For each constants $r_1, r_2 \in \mathbb{R}$, $r_1 \leq r_2$, operator $F \in \mathfrak{D}$, nonnegative Lebesgue integrable (on J) function φ and bounded set $\Omega \subset \mathbf{X}$ we set

$$\rho(F, \Omega) = \sup\{\|Fx\| \mid x \in \Omega\}$$

$$(r_1, r_2)_X = \{x \mid x \in X,\ r_1 \leq x(t) \leq r_2 \text{ for } t \in J\},$$

$$(r_1, r_2; F)_2 = \{(u, w) \mid (u, w) \in \mathbb{R}^2,\ |u| \leq \rho(F, (r_1, r_2)_X)\},$$

$$(r_1, r_2; F)_4 = \{(x, u, v, w) \mid (x, u, v, w) \in \mathbb{R}^4,\ r_1 \leq x \leq r_2,\ |u| \leq \rho(F, (r_1, r_2)_X\}$$

and for each $a, b, L_1, L_2 \in \mathbb{R}$, $a \leq b$, $L_1 \leq 0 \leq L_2$, and $F, H \in \mathfrak{D}$ we set

$$(a, b, L_1, L_2; F, H)_2 = \{(u, w) \mid (u, w) \in \mathbb{R}^2,\ |u| \leq \rho(F, (a, b)_X),\ |w| \leq \rho(H, (L_1, L_2)_X)\}$$

$$(a, b, L_1, L_2; F, H)_3 = \{(x, u, w) \mid (x, u, w) \in \mathbb{R}^3,\ a \leq x \leq b,\ |u| \leq \rho(F, (a, b)_X),$$
$$|w| \leq \rho(H, (L_1, L_2)_X)\}.$$

2. EXISTENCE RESULTS FOR BOUNDED NONLINEARITY f

First we shall prove the existence of solutions for BVP (1), (3) or BVP (1), (4) (in what follows only (1), (i), $i \in \{3, 4\}$, for short) with $f \in \mathrm{Car}(J \times \mathbb{R}^4)$ bounded by a Lebesgue integrable function φ. We shall assume that f fulfils:

(A_1) $f \in \mathrm{Car}(J \times \mathbb{R}^4)$ and there exist $r_1, r_2 \in \mathbb{R}$ and $\varphi \in L_1(J)$ such that $r_1 \le r_2$ and

$$f(t, r_1, u, 0, w) \le 0 \le f(t, r_2, u, 0, w)$$

for a.e. $t \in J$ and for each $(u, w) \in (r_1, r_2; F)_2$,

$$|f(t, x, u, v, w)| \le \varphi(t)$$

for a.e. $t \in J$ and for each $(x, u, v, w) \in (r_1, r_2; F)_4$.

To obtain a priori estimates for BVP (1), (i), $i \in \{3, 4\}$, we define the functions $f_n \in \mathrm{Car}(J \times \mathbb{R}^4)$ for each $n \in \mathbb{N}$ in the following way

$$f_n(t, x, u, v, w) = \begin{cases} f(t, r_2, \bar{u}, 0, w) + \dfrac{x - r_2 - 1/n}{x - r_2 + 1} & \text{for } x > r_2 + 1/n \\ f(t, r_2, \bar{u}, v, w) + p_n(r_2, x, u, v, w) & \text{for } r_2 < x \le r_2 + 1/n \\ f(t, x, \bar{u}, v, w) & \text{for } r_1 \le x \le r_2 \\ f(t, r_1, \bar{u}, v, w) - p_n(r_1, x, u, v, w) & \text{for } r_1 - 1/n \le x < r_1 \\ f(t, r_1, \bar{u}, 0, w) + \dfrac{x - r_1 + 1/n}{r_1 - x + 1} & \text{for } x < r_1 - 1/n, \end{cases} \tag{5}$$

where

$$p_n(r_j, x, u, v, w) = (f(t, r_j, \bar{u}, 0, w) - f(t, r_j, \bar{u}, v, w))(x - r_j)n, \qquad j = 1, 2,$$

and

$$\bar{u} = \begin{cases} u & \text{for } |u| \le \rho(F, (r_1, r_2)_X) \\ \rho(F, (r_1, r_2)_X) \operatorname{sign} u & \text{for } |u| > \rho(F, (r_1, r_2)_X). \end{cases}$$

Consider the differential equation

$$x''(t) = \lambda f_n(t, x(t), (Fx)(t), x'(t), (Hx')(t)), \qquad \lambda \in [0, 1]. \tag{6_λ}_n$$

LEMMA 1 (a priori estimates). Let f satisfy (A_1) and let BVP $(6_\lambda)_n$, (i) have a solution u for some $\lambda \in (0, 1]$, $i \in \{3, 4\}$ and $n \in \mathbb{N}$. Then the estimates

$$r_1 - 1/n \le u(t) \le r_2 + 1/n, \qquad |u'(t)| \le \int_0^1 \varphi(s)\, ds \tag{7}$$

are fulfilled for each $t \in J$.

Proof. Assume $r_2 + 1/n < \max\{u(t) \mid t \in J\} = u(t_0)$ for a $t_0 \in J$. Then $u'(t_0) = 0$ which is clear for $t_0 \in (0, 1)$ and follows from boundary conditions (3) or (4) for $t_0 \in \{0, 1\}$. With a little work one can show that there is an interval $(\alpha, \beta) \subset J$ such that $u(t) > r_2 + 1/n$ for $t \in (\alpha, \beta)$ and

$$\int_\alpha^\beta u''(s)\,\mathrm{d}s \le 0. \tag{8}$$

On the other hand, by (A_1) and (5), we get

$$\begin{aligned}\int_\alpha^\beta u''(s)\,\mathrm{d}s &= \lambda \int_\alpha^\beta f_n(s, u(s), (Fu)(s), u'(s), (Hu')(s))\,\mathrm{d}s\\ &= \lambda \int_\alpha^\beta \left[f(s, r_2, \overline{(Fu)(s)}, 0, (Hu')(s)) + \frac{u(s) - r_2 - 1/n}{u(s) - r_2 + 1}\right] \mathrm{d}s > 0,\end{aligned}$$

which contradicts (8). Similarly, for $\min\{u(t) \mid t \in J\} < r_1 - 1/n$. Thus, we have proved the first estimate in (7).

By (A_1), (5) and the first estimate in (7), we can verify $|f_n(t, u(t), (Fu)(t), u'(t), (Hu')(t))| \le \varphi(t)$ for a.e. $t \in J$. Since $u'(t_1) = 0$ for a $t_1 \in J$, integrating $(6_\lambda)_n$ (with $x = u$) from t_1 to t, we obtain the second estimate in (7). ■

For using the Continuation Theorem (CT for short), we denote by $\mathbf{Y} = C^1(J)$, $\mathbf{Z} = L_1(J)$ the Banach spaces with the usual norms and set for $n \in \mathbb{N}$, $i \in \{3, 4\}$

$$L_i\colon \operatorname{dom} L_i \to \mathbf{Z}, \qquad x \mapsto x'',$$

$$N\colon \mathbf{Y} \to \mathbf{Z}, \qquad x \mapsto f_n(\cdot, x(\cdot), (Fx)(\cdot), x'(\cdot), (Hx')(\cdot)),$$

where $\operatorname{dom} L_i = \{x \mid x \in AC^1(J), x$ satisfies boundary conditions (i)$\} \subset \mathbf{Y}$. Then BVP $(6_\lambda)_n$, (i) can be written in the operator form

$$L_i(x) = \lambda N(x), \qquad \lambda \in [0, 1].$$

LEMMA 2. *L_i is a Fredholm map of index 0 and N is L_i-compact on $\bar{\Omega}$ for any open bounded set $\Omega \subset \mathbf{Y}$ and each $i \in \{3, 4\}$.*

Proof. Fix $i \in \{3, 4\}$. Evidently, $\operatorname{Ker} L_i = \{x \mid x \in \mathbf{Y},\ x = k,\ k \in \mathbb{R}\}$, $\operatorname{Im} L_i = \{y \mid y \in \mathbf{Z}, \int_0^1 y(s)\,\mathrm{d}s = 0\}$ is closed in $\mathbf{Z}$ and $\dim \operatorname{Ker} L_i = \operatorname{codim} \operatorname{Im} L_i = 1$. Hence, L_i is a Fredholm map of index 0. Consider the continuous projectors

$$P\colon \mathbf{Y} \to \mathbf{Y}, \qquad x \mapsto x(0),$$

$$Q\colon \mathbf{Z} \to \mathbf{Z}, \qquad y \mapsto \int_0^1 y(s)\,\mathrm{d}s.$$

Then the generalized inverse (to L_i) $K_{iP}\colon \operatorname{Im} L_i \mapsto \operatorname{Ker} P \cap \operatorname{dom} L_i$ has the form

$$K_{3P}(y) = \int_0^t \int_0^s y(\tau)\,\mathrm{d}\tau\,\mathrm{d}s,$$

$$K_{4P}(y) = -t \int_0^1 \int_0^s y(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + \int_0^t \int_0^s y(\tau)\,\mathrm{d}\tau\,\mathrm{d}s.$$

Thus

$$QN\colon \mathbf{Y} \to \mathbf{Z}, \qquad x \mapsto \int_0^1 f_n(s, x(s), (Fx)(s), x'(s), (Hx')(s))\,ds, \tag{9}$$

$$K_{3P}(I - Q)N\colon \mathbf{Y} \to \mathbf{Y}, \qquad x \mapsto \int_0^t \int_0^s f_n(\tau, x(\tau), (Fx)(\tau), x'(\tau), (Hx')(\tau))\,d\tau\,ds$$
$$-\frac{t^2}{2}\int_0^1 f_n(s, x(s), (Fx)(s), x'(s), (Hx')(s))\,ds,$$

and

$$K_{4P}(I - Q)N\colon \mathbf{Y} \to \mathbf{Y}, \qquad x \mapsto \frac{t(1-t)}{2}\int_0^1 f_n(s, x(s), (Fx)(s), x'(s), (Hx')(s))\,ds$$
$$- t\int_0^1 \int_0^s f_n(\tau, x(\tau), (Fx)(\tau), x'(\tau), (Hx')(\tau))\,d\tau\,ds$$
$$+ \int_0^t \int_0^s f_n(\tau, x(\tau), (Fx)(\tau), x'(\tau), (Hx')(\tau))\,d\tau\,ds.$$

Since $F, H \in \mathfrak{D}$ and (cf. (5), (A_1)) $|f_n(t, x, u, v, w)| \le \varphi(t) + 1$ for a.e. $t \in J$ and each $(x, u, v, w) \in \mathbb{R}^4$, QN and $K_{iP}(I - Q)N$ $(i \in \{3, 4\})$ are continuous by the Lebesgue theorem and, moreover, $QN(\bar{\Omega})$, $K_{iP}(I - Q)N(\bar{\Omega})$ $(i \in \{3, 4\})$ are relatively compact for any open bounded set $\Omega \subset \mathbf{Y}$. Hence, N is L_i-compact on $\bar{\Omega}$ for any open bounded set $\Omega \subset \mathbf{Y}$ and each $i \in \{3, 4\}$. ∎

Lemma 3. Let f satisfy (A_1). Then for each $n \in \mathbb{N}$ and $i \in \{3, 4\}$, BVP $(6_1)_n$, (i) has a solution u satisfying (7).

Proof. Fix $i \in \{3, 4\}$ and $n \in \mathbb{N}$. Let P, Q and K_{iP} be as in the proof of lemma 2 and set

$$\Omega = \left\{x \mid x \in \mathbf{Y},\ r_1 - \frac{2}{n} < x(t) < r_2 + \frac{2}{n},\ |x'(t)| < \int_0^1 \varphi(s)\,ds + 1 \text{ for } t \in J\right\}.$$

By lemma 2, N is L_i-compact on $\bar{\Omega}$ and then lemma 1 implies that assumption (I) of CT is fulfilled. Suppose that $x \in \operatorname{Ker} L_i \cap \partial\Omega$. Then $x = r_1 - 2/n$ or $x = r_2 + 2/n$ and, by (A_1), (5) and (9),

$$QN\left(r_1 - \frac{2}{n}\right) = \int_0^1 f_n\left(s, r_1 - \frac{2}{n}, \left(F\left(r_1 - \frac{2}{n}\right)\right)(s), 0, (H(0))(s)\right) ds$$
$$= \int_0^1 \left[f\left(s, r_1, \overline{\left(F\left(r_1 - \frac{2}{n}\right)\right)(s)}, 0, (H(0))(s)\right) - \frac{1}{n+2}\right] ds < 0, \tag{10}$$

$$QN\left(r_2 + \frac{2}{n}\right) = \int_0^1 f_n\left(s, r_2 + \frac{2}{n}, \left(F\left(r_2 + \frac{2}{n}\right)\right)(s), 0, (H(0))(s)\right) ds$$
$$= \int_0^1 \left[f\left(s, r_2, \overline{\left(F\left(r_2 + \frac{2}{n}\right)\right)(s)}, 0, (H(0))(s)\right) + \frac{1}{n+2}\right] ds > 0. \tag{11}$$

Hence, condition (II) of CT is realized. Let $\mathcal{J}$ be an isomorphism from $\operatorname{Im} Q = \{y \mid y \in \mathbf{Z}, y = k, k \in \mathbb{R}\}$ onto $\operatorname{Ker} L_i = \{x \mid x \in \mathbf{Y}, x = k, k \in \mathbb{R}\}$. Inequalities (10) and (11) imply $d[\mathcal{J}QN, \Omega \cap \operatorname{Ker} L_i, 0] \neq 0$ and the last condition (III) of CT is fulfilled. The assertion of our lemma follows from CT and lemma 1. ■

THEOREM 1. Let f satisfy (A_1) and $i \in \{3, 4\}$. Then BVP (1), (i) has a solution u fulfilling

$$r_1 \leq u(t) \leq r_2, \qquad |u'(t)| \leq \int_0^1 \varphi(s)\,ds \qquad \text{for } t \in J. \tag{12}$$

Proof. Fix $i \in \{3, 4\}$. For $n \in \mathbb{N}$ let us consider the sequence of BVPs $\{(6_1)_n, (i)\}$. By lemma 3, we get an appropriate sequence of solutions $\{u_n\}$ for which (7) holds (with $u = u_n$). Then, by (5) and (7),

$$|u_n''(t)| = |f_n(t, u_n(t), (Fu_n)(t), u_n'(t), (Hu_n')(t))| \leq \varphi(t)$$

for a.e. $t \in J$ and each $n \in \mathbb{N}$. Further, by the Arzelà–Ascoli theorem, there exists a subsequence $\{u_{k_n}\}$ of $\{u_n\}$ converging in $C^1(J)$ to a u. The function u satisfies (12) and, hence, (cf. (5)) it is a solution of BVP (1), (i). ■

COROLLARY 1. Let $h \in \operatorname{Car}(J \times \mathbb{R}^2)$ and let there exist $r_1, r_2 \in \mathbb{R}$ and $\varphi \in L_1(J)$ such that $r_1 \leq r_2$ and

$$h(t, r_1, 0) \leq 0 \leq h(t, r_2, 0), \qquad |h(t, x, y)| \leq \varphi(t)$$

for a.e. $t \in J$ and each $(x, y) \in [r_1, r_2] \times \mathbb{R}$. Then for each $i \in \{3, 4\}$ BVP (2), (i) has a solution u satisfying (12).

Now, we shall prove analogous results as above under the inequalities which are inverse to that in (A_1). We shall assume:

(A_2) $f \in C^0(J \times \mathbb{R}^4)$ and there are $r_1, r_2, K \in \mathbb{R}$ such that $r_1 < r_2$, $K > 0$ and

$$\begin{aligned} f(t, x, u, 0, w) &\geq 0 \qquad \text{for } (t, x, u, w) \in J \times [r_1 - K, r_1] \times (r_1 - K, r_2 + K; F)_2, \\ f(t, x, u, 0, w) &\leq 0 \qquad \text{for } (t, x, u, w) \in J \times [r_2, r_2 + K] \times (r_1 - K, r_2 + K; F)_2, \\ |f(t, x, u, v, w)| &\leq K \qquad \text{for } (t, x, u, v, w) \in J \times (r_1 - K, r_2 + K; F)_4. \end{aligned}$$

Assume $f \in C^0(J \times \mathbb{R}^4)$ and define $f^* \in C^0(J \times \mathbb{R}^4)$ by

$$f^*(t, x, u, v, w) = f(t, \tilde{x}, \bar{u}, v, w), \tag{13}$$

where

$$\tilde{x} = \begin{cases} r_2 + K & \text{for } x > r_2 + K \\ x & \text{for } r_1 - K \leq x \leq r_2 + K \\ r_1 - K & \text{for } x < r_1 - K, \end{cases}$$

$$\bar{u} = \begin{cases} u & \text{for } |u| \leq \rho(F; (r_1 - K, r_2 + K)_X) \\ \rho(F; (r_1 - K, r_2 + K)_X) \operatorname{sign} u & \text{for } |u| > \rho(F; (r_1 - K, r_2 + K)_X). \end{cases}$$

Let ε be a positive constant, $\varepsilon < r_2 - r_1$, $c \in [0, 1)$ and consider the differential equation

$$x''(t) = \lambda\left(cf^*(t, x(t), (Fx)(t), x'(t), (Hx')(t)) + (1 - c)\frac{K(r_2 - x(t) - \varepsilon)}{|r_2| + |x(t)| + \varepsilon}\right), \qquad \lambda \in [0, 1] \tag{14_λ}_c$$

LEMMA 4 (a priori estimates). Let f satisfy (A_2) and let BVP $(14_\lambda)_c$, (i) have a solution u for some $\lambda \in (0, 1]$, $c \in [0, 1)$ and $i \in \{3, 4\}$. Then the estimates

$$r_1 - K < u(t) < r_2 + K, \qquad |u'(t)| < K \qquad \text{for } t \in J \tag{15}$$

hold and

$$r_1 < u(a_u) < r_2 \tag{16}$$

for an $a_u \in J$.

Proof. Assume $r_2 \leq \min\{u(t) \mid t \in J\} = u(t_0)$ for a $t_0 \in J$. Then $u'(t_0) = 0$ (see the first part of the proof of lemma 1) and $u''(t_0) \geq 0$. Since

$$\begin{aligned} u''(t_0) &= \lambda\left(cf^*(t_0, u(t_0), (Fu)(t_0), 0, (Hu')(t_0)) + (1 - c)\frac{K(r_2 - u(t_0) - \varepsilon)}{|r_2| + |u(t_0)| + \varepsilon}\right) \\ &\leq \lambda(1 - c)\frac{K(r_2 - u(t_0) - \varepsilon)}{|r_2| + |u(t_0)| + \varepsilon} < 0, \end{aligned}$$

we have a contradiction. Assume $r_1 \geq \max\{u(t) \mid t \in J\} = u(t_1)$ for a $t_1 \in J$. Then $u'(t_1) = 0$, $u''(t_1) \leq 0$ and since

$$\begin{aligned} u''(t_1) &= \lambda\left(cf^*(t_1, u(t_1), (Fu)(t_1), 0, (Hu')(t_1)) + (1 - c)\frac{K(r_2 - u(t_1) - \varepsilon)}{|r_2| + |u(t_1)| + \varepsilon}\right) \\ &\geq \lambda(1 - c)\frac{K(r_2 - u(t_1) - \varepsilon)}{|r_2| + |u(t_1)| + \varepsilon} > 0, \end{aligned}$$

we have a contradiction.

Hence, there exists an $a_u \in J$ such that $u(a_u) \in (r_1, r_2)$, so, (16) is valid. Since u satisfies boundary conditions (i), there exists a $b \in J$ such that $u'(b) = 0$. Integrating $(14_\lambda)_c$ (with $x = u$) from b to t and using the inequality

$$\left|\left(cf^*(t, u(t), (Fu)(t), u'(t), (Hu')(t)) + (1 - c)\frac{K(r_2 - u(t) - \varepsilon)}{|r_2| + |u(t)| + \varepsilon}\right)\right| < K \qquad \text{for } t \in J,$$

we get

$$|u'(t)| \leq \left|\int_b^t u''(s)\,\mathrm{d}s\right| < K \qquad \text{for } t \in J.$$

Then

$$u(t) = u(a_u) + \int_{a_u}^t u'(s)\,\mathrm{d}s < r_2 + K,$$

$$u(t) = u(a_u) + \int_{a_u}^t u'(s)\,\mathrm{d}s > r_1 - K$$

on J; hence, (15) is proved. ■

LEMMA 5. Let f satisfy (A_2). Then for each $i \in \{3, 4\}$ and $c \in [0, 1)$ BVP $(14_1)_c$, (i) has a solution u satisfying (15) and (16) with an $a_u \in J$.

Proof. Fix $i \in \{3, 4\}$ and $c \in [0, 1)$. Let L_i, P, Q and K_{iP} be as in the proof of lemma 2 with $\mathbf{Y} = C^2(J)$, $\mathbf{Z} = C^0(J)$. Set

$$N_c : \mathbf{Y} \to \mathbf{Z}, \qquad x \mapsto cf^*(\cdot, x(\cdot), (Fx)(\cdot), x'(\cdot), (Hx')(\cdot)) + (1 - c)\frac{K(r_2 - u(\cdot) - \varepsilon)}{|r_2| + |u(\cdot)| + \varepsilon}$$

and

$$\Omega = \{x \mid x \in \mathbf{Y},\ r_1 - K < x(t) < r_2 + K,\ |x'(t)| < K \text{ for } t \in J\}.$$

Let us write problem $(14_\lambda)_c$, (i) in the form $L_i x = \lambda N_c x$ and apply CT. By the same consideration as in the proof of lemma 2 we get that N_c is L_i-compact on $\bar{\Omega}$. From lemma 4 it follows that assumption (I) of CT is fulfilled. Assume that $x \in \operatorname{Ker} L_i \cap \partial\Omega$. Then $x = r_1 - K$ or $x = r_2 + K$ and, by (A_2), (13) and (9)

$$QN_c(r_1 - K) = \int_0^1 \Bigg[cf(s, r_1 - K, \overline{(F(r_1 - K))(s)}, 0, (H(0))(s)) + (1 - c)\frac{K(r_2 - r_1 + K - \varepsilon)}{|r_2| + |r_1 - K| + \varepsilon}\Bigg]\, ds > 0, \tag{17}$$

$$QN_c(r_2 + K) = \int_0^1 \Bigg[cf(s, r_2 + K, \overline{(F(r_2 + K))(s)}, 0, (H(0))(s)) + (1 - c)\frac{K(-K - \varepsilon)}{|r_2| + |r_2 + K| + \varepsilon}\Bigg]\, ds < 0. \tag{18}$$

Hence, condition (II) of CT is realized. Moreover, inequalities (17) and (18) imply $d[\mathcal{J}QN_c, \Omega \cap \operatorname{Ker} L_i, 0] \neq 0$ and the last condition (III) of CT is fulfilled. By CT, there exists a solution u of BVP $(14_1)_c$, (i). By lemma 4, u satisfies (15) and (16) with an $a_u \in J$. ■

THEOREM 2. Let f satisfy (A_2) and $i \in \{3, 4\}$. Then BVP (1), (i) has a solution u satisfying

$$r_1 - K \le u(t) \le r_2 + K, \qquad |u'(t)| \le K \qquad \text{for } t \in J \tag{19}$$

and

$$r_1 \le u(a_u) \le r_2 \tag{20}$$

for an $a_u \in J$.

Proof. Fix $i \in \{3, 4\}$. Let $\{c_n\} \subset (0, 1)$ be a convergent sequence $\lim_{n\to\infty} c_n = 1$. By lemma 5, there exists a solution u_n of BVP $(14_1)_{c_n}$, (i) for each $n \in \mathbf{N}$ satisfying (15) (with $u = u_n$) and

$$r_1 < u_n(a_n) < r_2, \qquad n \in \mathbf{N}$$

for an $a_n \in J$. Evidently, by the Arzelà–Ascoli theorem and the Bolzano–Weierstrass theorem, we can assume that $\lim_{n\to\infty} u_n = u$ in $C^1(J)$ and $\lim_{n\to\infty} a_n = a$. Then u is a solution of BVP (1), (i) satisfying (19) and (20) with $a_u = a$. ■

Note. Clearly, if f satisfy (A_2) with $r_1 = r_2$, the constant function $u(t) \equiv r_1$ is a solution of (1), (i), $i \in \{3, 4\}$.

Corollary 2. Let $h \in C^0(J \times \mathbb{R}^2)$ and there exist $r_1, r_2, K \in \mathbb{R}$ such that $r_1 \le r_2$, $K > 0$ and

$$h(t, x, 0) \ge 0 \qquad \text{for } (t, x) \in J \times [r_1 - K, r_1],$$

$$h(t, x, 0) \le 0 \qquad \text{for } (t, x) \in J \times [r_2, r_2 + K],$$

$$|h(t, x, y)| \le K \qquad \text{for } (t, x, y) \in J \times [r_1 - K, r_2 + K] \times \mathbb{R}.$$

Then for each $i \in \{3, 4\}$ BVP (2), (i) has a solution u satisfying (19) and (20) with an $a_u \in J$.

3. EXISTENCE RESULTS FOR GENERALLY UNBOUNDED NONLINEARITY f, MAIN RESULTS

In this section we shall assume that f satisfies some of the following assumptions:

(H_1) $f \in \mathrm{Car}(J \times \mathbb{R}^4)$, there exist $r_1, r_2, L_1, L_2 \in \mathbb{R}$ and $\mu, \nu \in \{-1, 1\}$ such that $r_1 \le r_2$, $L_1 \le 0 \le L_2$ and

$$f(t, r_1, u, 0, w) \le 0 \le f(t, r_2, u, 0, w)$$

for a.e. $t \in J$ and each $(u, w) \in (r_1, r_2, L_1, L_2; F, H)_2$,

$$\nu f(t, x, u, L_1, w) \le 0 \le \mu f(t, x, u, L_2, w)$$

for a.e. $t \in J$ and each $(x, u, w) \in (r_1, r_2, L_1, L_2; F, H)_3$.

(H_2) $f \in C^0(J \times \mathbb{R}^4)$, there exist $r_1, r_2, L_1, L_2 \in \mathbb{R}$ and $\mu, \nu \in \{-1, 1\}$ such that $r_1 \le r_2$, $L_1 \le 0 \le L_2$ and

$$f(t, x, u, 0, w) \ge 0 \qquad \text{for } (t, x, u, w) \in J \times [r_1 + L_1, r_1] \times (r_1 + L_1, r_2 + L_2, L_1, L_2; F, H)_2,$$

$$f(t, x, u, 0, w) \le 0 \qquad \text{for } (t, x, u, w) \in J \times [r_2, r_2 + L_2] \times (r_1 + L_1, r_2 + L_2, L_1, L_2; F, H)_2,$$

$$\nu f(t, x, u, L_1, w) \le 0 \le \mu f(t, x, u, L_2, w)$$

for $(t, x, u, w) \in J \times (r_1 + L_1, r_2 + L_2, L_1, L_2; F, H)_3$.

Theorem 3. Let f satisfy (H_1) and $i \in \{3, 4\}$. Then BVP (1), (i) has a solution u with

$$r_1 \le u(t) \le r_2, \qquad L_1 \le u'(t) \le L_2 \qquad \text{for } t \in J. \tag{21}$$

Proof. Define the function $\bar{f}_{\mu\nu} \in \mathrm{Car}(J \times \mathbb{R}^4)$ by f in the following way

$$\bar{f}_{\mu\nu}(t, x, u, v, w) = \begin{cases} f(t, x, u, L_2, \bar{w}) + \mu \dfrac{v - L_2}{v - L_2 + 1} & \text{for } v > L_2 \\ f(t, x, u, v, \bar{w}) & \text{for } L_1 \le v \le L_2 \\ f(t, x, u, L_1, \bar{w}) + \nu \dfrac{v - L_1}{L_1 - v + 1} & \text{for } v < L_1, \end{cases} \tag{22}$$

where

$$\bar{w} = \begin{cases} w & \text{for } |w| \le \rho(H; (L_1, L_2)_X) \\ \rho(H; (L_1, L_2)_X) \operatorname{sign} w & \text{for } |w| > \rho(H; (L_1, L_2)_X). \end{cases}$$

Then $\bar{f}_{\mu\nu}$ fulfils assumption (A_1) with $\varphi(t) = 1 + \sup\{|f(t, x, u, v, w)| \mid (x, u, v, w) \in \mathbb{R}^4, r_1 \le x \le r_2, |u| \le \rho(F; (r_1, r_2)_X), L_1 \le v \le L_2, |w| \le \rho(H; (L_1, L_2)_X)\}$. So, by theorem 1, BVP (23), (i) ($i = 3, 4$) has a solution u satisfying (12), where

$$x''(t) = \bar{f}_{\mu\nu}(t, x(t), (Fx)(t), x'(t), (Hx')(t)), \qquad t \in J. \tag{23}$$

Let us prove that u fulfils the second inequality in (21). Assume, on the contrary, $\max\{u'(t) \mid t \in J\} = u'(t_0) > L_2$. Boundary conditions (3) (resp. (4)) imply $t_0 \in (0, 1)$ (resp. $t_0 \in J$). Let $t_0 \in (0, 1)$. Then there is a $\delta > 0$ such that $L_2 < u'(t) \le u'(t_0)$ for each t belonging to the interval with the end points t_0 and $t_0 + \mu\delta$ and, consequently,

$$\int_{t_0}^{t_0+\mu\delta} u''(s)\,\mathrm{d}s = u'(t_0 + \mu\delta) - u'(t_0) \le 0.$$

On the other hand (cf. (22)),

$$\begin{aligned} \int_{t_0}^{t_0+\mu\delta} u''(s)\,\mathrm{d}s &= \int_{t_0}^{t_0+\mu\delta} \bar{f}_{\mu\nu}(s, u(s), (Fu)(s), u'(s), (Hu')(s))\,\mathrm{d}s \\ &= \mu \int_{t_0}^{t_0+\mu\delta} \left[\mu f(s, u(s), (Fu)(s), L_2, \overline{(Hu')(s)}) + \frac{u'(s) - L_2}{u'(s) - L_2 + 1} \right] \mathrm{d}s > 0, \end{aligned}$$

a contradiction. Let $t_0 \in \{0, 1\}$. Then necessarily u satisfies boundary conditions (4). Set $\tau_\mu = \frac{1}{2}(1 - \operatorname{sign}\mu)$. Since $u'(\tau_\mu) = \max\{u'(t) \mid t \in J\}$, there is an $\varepsilon > 0$ such that $u'(\tau_\mu) \ge u'(t) > L_2$ on the interval with the end points τ_μ and $\tau_\mu + \mu\varepsilon$. Then

$$\int_{\tau_\mu}^{\tau_\mu+\mu\varepsilon} u''(s)\,\mathrm{d}s = u'(\tau_\mu + \mu\varepsilon) - u'(\tau_\mu) \le 0.$$

On the other hand

$$\int_{\tau_\mu}^{\tau_\mu+\mu\varepsilon} u''(s)\,\mathrm{d}s = \mu \int_{\tau_\mu}^{\tau_\mu+\mu\varepsilon} \left[\mu f(s, u(s), (Fu)(s), L_2, \overline{(Hu')(s)}) + \frac{u'(s) - L_2}{u'(s) - L_2 + 1} \right] \mathrm{d}s > 0,$$

a contradiction. Hence, $u'(t) \le L_2$ on J.

Similarly, $u'(t) \ge L_1$ on J. Hence, (cf. (12)) u satisfies (21) and then (cf. (22)) u is a solution of BVP (1), (i), $i \in \{3, 4\}$. ■

Corollary 3. Let $h \in \operatorname{Car}(J \times \mathbb{R}^2)$ and let there exist $r_1, r_2, L_1, L_2 \in \mathbb{R}$ such that $r_1 \le r_2$, $L_1 \le 0 \le L_2$,

$$h(t, r_1, 0) \le 0 \le h(t, r_2, 0) \qquad \text{for a.e. } t \in J \tag{24}$$

and

$$\begin{gathered} h(t, x, L_1),\ h(t, x, L_2) \text{ do not change their signs for} \\ \text{a.e. } t \in J \text{ and each } x \in [r_1, r_2]. \end{gathered} \tag{25}$$

Then for each $i \in \{3, 4\}$ BVP (2), (i) has a solution u satisfying (21).

THEOREM 4. Let f satisfy (H_2) and $i \in \{3, 4\}$. Then BVP (1), (i) has a solution u satisfying

$$r_1 + L_1 \leq u(t) \leq r_2 + L_2, \qquad L_1 \leq u'(t) \leq L_2 \qquad \text{for } t \in J. \tag{26}$$

Proof. Define the function $f^*_{\mu\nu} \in C^0(J \times \mathbb{R}^4)$ by f as follows

$$f^*_{\mu\nu}(t, x, u, v, w) = \begin{cases} f(t, \tilde{x}, \hat{u}, L_2, \bar{w}) + \mu \dfrac{v - L_2}{v - L_2 + 1} & \text{for } v > L_2 \\ f(t, \tilde{x}, \hat{u}, v, \bar{w}) & \text{for } L_1 \leq v \leq L_2 \\ f(t, \tilde{x}, \hat{u}, L_1, \bar{w}) + \nu \dfrac{v - L_1}{L_1 - v + 1} & \text{for } v < L_1, \end{cases} \tag{27}$$

where

$$\tilde{x} = \begin{cases} r_2 + L_2 & \text{for } x > r_2 + L_2 \\ x & \text{for } r_1 + L_1 \leq x \leq r_2 + L_2 \\ r_1 + L_1 & \text{for } x < r_1 + L_1, \end{cases}$$

$$\hat{u} = \begin{cases} u & \text{for } |u| \leq \rho(F; (r_1 + L_1, r_2 + L_2)_X) \\ \rho(F; (r_1 + L_1, r_2 + L_2)_X) \operatorname{sign} u & \text{for } |u| > \rho(F; (r_1 + L_1, r_2 + L_2)_X), \end{cases}$$

and

$$\bar{w} = \begin{cases} w & \text{for } |w| \leq \rho(H; (L_1, L_2)_X) \\ \rho(H; (L_1, L_2)_X) \operatorname{sign} w & \text{for } |w| > \rho(H; (L_1, L_2)_X). \end{cases}$$

Then $f^*_{\mu\nu}$ fulfils assumption (A_2) (with $f = f^*_{\mu\nu}$ and $K = 1 + \max\{|f(t, x, u, v, w)| \mid (t, x, u, v, w) \in J \times \mathbb{R}^4, r_1 + L_1 \leq x \leq r_2 + L_2, |u| \leq \rho(F; (r_1 + L_1, r_2 + L_2)_X), L_1 \leq v \leq L_2, |w| \leq \rho(H; (L_1, L_2)_X)\}$. By theorem 2, BVP (28), (i), $i \in \{3, 4\}$, has a solution u satisfying (19) and (20), where

$$x''(t) = f^*_{\mu\nu}(t, x(t), (Fx)(t), x'(t), (Hx')(t)), \qquad t \in J. \tag{28}$$

By the same arguments as in the proof of theorem 3 we can prove that u fulfils also the second inequality in (26). Then (cf. (20)) u satisfies the first inequality of (26); hence, (cf. (27)) u is a solution of BVP (1), (i) ($i \in \{3, 4\}$). ■

Example 1. Consider the differential equation

$$x''(t) = -x(t) + \lambda \operatorname{arctg} x'(t) + p(t) + \beta \sin(x(\alpha(t))), \tag{29}$$

where $p \in C^0(J)$, $\alpha: J \to J$ is continuous and λ, μ are parameters. Let L be an arbitrary but fixed positive constant. Applying theorem 4 (with $-r_1 = r_2 = \|p\| + |\mu|$, $-L_1 = L_2 = L$ and $Fx = x \circ \alpha$) we can verify that for each $\lambda, \mu \in \mathbb{R}$ such that

$$|\lambda| > \frac{2\|p\| + 2|\mu| + L}{\operatorname{arctg} L},$$

there exists a solution u of BVP (29), (i), $i \in \{3, 4\}$ and, moreover,

$$\|u\| \le \|p\| + |\mu| + L, \qquad \|u'\| \le L.$$

COROLLARY 4. Let $h \in C^0(J \times \mathbb{R}^2)$ and let $r_1, r_2, L_1, L_2 \in \mathbb{R}$ be such that $r_1 \le r_2$, $L_1 \le 0 \le L_2$,

$$h(t, x, 0) \le 0 \qquad \text{for } (t, x) \in J \times [r_2, r_2 + L_2],$$

$$h(t, x, 0) \ge 0 \qquad \text{for } (t, x) \in J \times [L_1 + r_1, r_1]$$

and $h(t, x, L_1)$, $h(t, x, L_2)$ do not change their signs for $(t, x) \in J \times [r_1 + L_1, r_2 + L_2]$. Then for each $i \in \{3, 4\}$ BVP (2), (i) has a solution u satisfying (26).

Example 2. Consider the differential equation

$$x'' = h(x) + p(x') + s(t), \tag{30}$$

where $h, p \in C^0(\mathbb{R})$, $s \in C^0(J)$, $\lim_{x \to \varepsilon\infty} h(x) = -\varepsilon\infty$ for each $\varepsilon \in \{-1, 1\}$ and

$$\limsup_{|x| \to \infty} \left| \frac{p(x)}{h^*(k|x|)} \right| =: \alpha > 1 \qquad \text{with a constant } k > 1 \text{ and}$$

$$h^*(x) := \max\{|h(u)|; -x \le u \le x\} \qquad \text{for } x \in [0, \infty).$$

Then for each $i \in \{3, 4\}$, BVP (30), (i) has a solution.

To verify this fact set $S = \|s\|$ and suppose that r is a positive constant such that $h(x) \ge S - p(0)$ for $x \le -r$ and $h(x) \le -S - p(0)$ for $x \ge r$. Let L be a positive constant such that $L \ge r/(1 - k)$, $h^*(L) \ge 2S/(\alpha - 1)$ and $|p(\pm L)/h^*(kL)| \ge (1 + \alpha)/2$. Then

$$h(x) + p(0) + s(t) \le 0 \text{ for } x \ge r, \qquad h(x) + p(0) + s(t) \ge 0 \text{ for } x \le -r,$$

and

$$|p(\pm L)| \ge h^*(kL) + (\alpha - 1)h^*(kL)/2 \ge h^*(L + (k - 1)L) + (\alpha - 1)h^*(L)/2 \ge h^*(L + r) + S.$$

If $p(\tau L) > 0$ for a $\tau \in \{-1, 1\}$, then

$$h(x) + p(\tau L) + s(t) \ge h(x) + h^*(L + r) + S + s(t) \ge 0 \qquad \text{for } x \in [-L - r, L + r]$$

and if $p(\tau L) < 0$ for a $\tau \in \{-1, 1\}$, then

$$h(x) + p(\tau L) + s(t) \le h(x) - h^*(L + r) - S + s(t) \le 0 \qquad \text{for } x \in [-L - r, L + r].$$

By corollary 4 (with $-r_1 = r_2 = r$, $-L_1 = L_2 = L$), BVP (30), (i), $i \in \{3, 4\}$, has a solution u satisfying

$$-r - L \le u(t) \le r + L, \qquad -L \le u'(t) \le L \qquad \text{for each } t \in J.$$

For example functions $h(x) = -x^{2n-1} + \sum_{k=0}^{2n-2} a_k x^k$, $n \in \mathbb{N}$, $n \ge 1$, $p(x) = \sin x \cdot e^{|x|}$ satisfy the above conditions.

4. MULTIPLICITY RESULTS

Here, combining the previous results, we get the existence of at least two or three solutions of BVP (1), (i), $i \in \{3, 4\}$.

Using theorem 3 two times, we obtain the following theorem.

THEOREM 5 (two solutions). Assume that

(H_3) $f \in \mathrm{Car}(J \times \mathbb{R}^4)$ and there exist $r_1, r_2, r_3, r_4, L_1, L_2 \in \mathbb{R}$ and $\mu, \nu \in \{-1, 1\}$ such that $r_1 \le r_2 < r_3 \le r_4$, $L_1 \le 0 \le L_2$ and

$$f(t, r_j, u, 0, w) \le 0 \le f(t, r_k, u, 0, w)$$

for a.e. $t \in J$ and each $(u, w) \in (r_1, r_4, L_1, L_2; F, H)_2$, $j = 1, 3$, $k = 2, 4$, and

$$\nu f(t, x, u, L_1, w) \le 0 \le \mu f(t, x, u, L_2, w)$$

for a.e. $t \in J$ and each $(x, u, w) \in (r_1, r_4, L_1, L_2; F, H)_3$.

Then for each $i \in \{3, 4\}$ BVP (1), (i) has at least two different solutions u_1, u_2 and

$$r_1 \le u_1(t) \le r_2, \qquad r_3 \le u_2(t) \le r_4, \qquad L_1 \le u_k'(t) \le L_2 \qquad \text{for } t \in J,\ k = 1, 2. \tag{31}$$

Proof. Fix $i \in \{3, 4\}$. By theorem 3, there exists a solution u_1 of BVP (1), (i) satisfying (21) (with $u = u_1$) and by the same theorem there exists a solution u_2 of BVP (1), (i) satisfying $r_3 \le u_2(t) \le r_4$, $L_1 \le u_2'(t) \le L_2$ on J. Since $r_2 < r_3$, we get $u_1 \ne u_2$. ■

COROLLARY 5. Let $h \in \mathrm{Car}(J \times \mathbb{R}^2)$ and there exist $r_1, r_2, r_3, r_4, L_1, L_2 \in \mathbb{R}$ such that $r_1 \le r_2 < r_3 \le r_4$, $L_1 \le 0 \le L_2$ and

$$h(t, r_j, 0) \le 0 \le h(t, r_k, 0) \qquad \text{for a.e. } t \in J, \qquad \text{where } j = 1, 3,\ k = 2, 4, \tag{32}$$

and

$$\begin{gathered} h(t, x, L_1),\ h(t, x, L_2) \text{ do not change their signs} \\ \text{for a.e. } t \in J \text{ and each } x \in [r_1, r_4]. \end{gathered} \tag{33}$$

Then for each $i \in \{3, 4\}$ BVP (2), (i) has at least two different solutions u_1, u_2 satisfying (31).

Suppose $f \in C^0(J \times \mathbb{R}^4)$. Then we can use theorem 3 together with theorem 4 and get various multiplicity results. For example if the distance between r_2 and r_3 is long enough we can obtain a theorem which guarantees three different solutions.

THEOREM 6 (three solutions). Assume that

(H_4) $f \in C^0(J \times \mathbb{R}^4)$ and there exist $r_1, r_2, r_3, r_4, L_1, L_2 \in \mathbb{R}$ and $\mu, \nu \in \{-1, 1\}$ such that

$$r_1 \le r_2, \qquad r_2 - L_1 + L_2 < r_3 \le r_4, \qquad L_1 \le 0 \le L_2 \tag{34}$$

and for each $(t, u, w) \in J \times (r_1, r_4, L_1, L_2; F, H)_2$ the following inequalities are fulfilled

$$f(t, r_1, u, 0, w) \le 0 \le f(t, r_4, u, 0, w),$$

$$f(t, x, u, 0, w) > 0 \qquad \text{for } x \in (r_2, r_2 - L_1],$$

$$f(t, x, u, 0, w) < 0 \qquad \text{for } x \in [r_3 - L_2, r_3),$$

$$\nu f(t, x, u, L_1, w) \le 0 \le \mu f(t, x, u, L_2, w) \qquad \text{for } x \in [r_1, r_4].$$

Then for each $i \in \{3, 4\}$, BVP (1), (i) has at least three different solutions u_1, u_2, u_3 fulfilling for each $t \in J$

$$r_1 \le u_1(t) \le r_2, \quad r_2 < u_2(t) < r_3, \quad r_3 \le u_3(t) \le r_4, \quad L_1 \le u_k'(t) \le L_2, \quad k = 1, 2, 3. \tag{35}$$

Proof. Fix $i \in \{3, 4\}$. Since $f \in C^0(J \times \mathbb{R}^4)$, there exists $\varepsilon > 0$ such that $r_2 - L_1 + L_2 + 2\varepsilon < r_3$, and for each $(t, u, w) \in J \times (r_1, r_4, L_1, L_2; F, H)_2$ the inequalities $f(t, x, u, 0, w) \geq 0$ for $x \in [r_2, r_2 - L_1 + \varepsilon]$, and $f(t, x, u, 0, w) \leq 0$ for $x \in [r_3 - L_2 - \varepsilon, r_3]$ are valid. By theorem 3, there exists a solution u_1 of BVP (1), (i) satisfying (21) (with $u = u_1$). Further, by theorem 4, there exists a solution u_2 of BVP (1), (i) satisfying $r_2 + \varepsilon \leq u_2(t) \leq r_3 - \varepsilon$, $L_1 \leq u_2'(t) \leq L_2$ for $t \in J$, and finally, by theorem 3, there exists a solution u_3 of BVP (1), (i) satisfying $r_3 \leq u_3(t) \leq r_4$, $L_1 \leq u_3'(t) \leq L_2$ for $t \in J$. Clearly, $u_1 \neq u_2 \neq u_3$. ■

COROLLARY 6. Let $h \in C^0(J \times \mathbb{R}^2)$ and there exist $r_1, r_2, r_3, r_4, L_1, L_2 \in \mathbb{R}$ such that the conditions (33), (34) and the inequalities

$$h(t, r_1, 0) \leq 0 \leq h(t, r_4, 0) \qquad \text{for each } t \in J,$$

$$h(t, x, 0) > 0 \qquad \text{for each } (t, x) \in J \times (r_2, r_2 - L_1],$$

$$h(t, x, 0) < 0 \qquad \text{for each } (t, x) \in J \times [r_3 - L_2, r_3),$$

are satisfied.

Then for each $i \in \{3, 4\}$, BVP (1), (i) has at least three different solutions u_1, u_2, u_3 fulfilling (35).

Example 3. Consider a polynomial

$$p_n \colon \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \sum_{i=0}^{n} a_i x^i$$

and a continuous function $g \colon \mathbb{R} \to \mathbb{R}$. Suppose that $g(0) = 0$ and p_n has k different real zeros $x_1, x_2, \ldots, x_k$, $k \in \mathbb{N}$. Then it is clear that equation $x'' = p_n(x) + g(x')$ has k different constant solutions which clearly fulfil (3) or (4) (cf. [2, example 6.4]).

Example 4. Consider the nonautonomous equation

$$x'' = p_n(x) + g(t, x'), \tag{36}$$

where $g \in C^0(J \times \mathbb{R})$.

Denote $M = \max\{g(t, 0) \mid t \in J\}$, $m = \min\{g(t, 0) \mid t \in J\}$.

(1) Let p_n have a simple zero $x_1 \in \mathbb{R}$ and

(a) p_n is increasing in x_1. Then if $p_n(x) \geq M$ for some $x > x_1$ and $p_n(x) \leq m$ for some $x < x_1$, we can choose $r_1, r_2 \in \mathbb{R}$ such that (24) is fulfilled. Further, let

$$\limsup_{|x| \to \infty} |g(t, x)| = \infty \qquad \text{on } J. \tag{37}$$

Then there exist L_1, L_2, $L_1 \leq 0 \leq L_2$ such that (25) is satisfied. Therefore, by corollary 3, problem (36), (i), $i \in \{3, 4\}$ has at least one solution.

(b) p_n is decreasing in x_1. Then the connection between p_n and g has to be closer. Let $[a_1, a_2] \subset (-\infty, x_1)$, $[b_1, b_2] \subset (x_1, \infty)$ be such that

$$p_n(x) \geq M \text{ for each } x \in [a_1, a_2], \qquad p_n(x) \leq m \text{ for each } x \in [b_1, b_2], \tag{38}$$

and let on $J \times [a_1, b_2]$

$$|p_n(x) + g(t, L_j)| > 0 \qquad \text{for } j = 1, 2 \text{ and for some } L_1 \in [a_1 - a_2, 0),\ L_2 \in (0, b_2 - b_1]. \tag{39}$$

Then we can set $r_1 = a_2$, $r_2 = b_1$ and we see that all conditions of corollary 4 are fulfilled, hence, BVP (36), (i), $i \in \{3, 4\}$, has at least one solution.

(2) Let p_n have two simple zeros $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$. Then we can apply corollary 3 for p_n increasing as well as for p_n decreasing in x_1. It is also possible to combine corollary 3 and corollary 4 and get two solutions. This technique will be shown more precisely for the case of three different solutions.

(3) Let p_n have three different simple zeros $x_1, x_2, x_3 \in \mathbb{R}$, $x_1 < x_2 < x_3$. Let p_n increase in x_1.

(a) Suppose that $p_n(x) \geq M$ for some $x \in (x_1, x_2)$ and some $x > x_3$ and $p_n(x) \leq m$ for some $x \in (x_2, x_3)$ and some $x < x_1$. Further, let condition (37) be fulfilled. Then we can choose $r_1 < x_1$, $r_2 \in (x_1, x_2)$, $r_3 \in (x_2, x_3)$, $r_4 > x_3$ and $L_1 \leq 0 \leq L_2$ such that all conditions of corollary 5 are fulfilled and problem (34), (i), $i \in \{3, 4\}$ has at least two different solutions.

(b) Let $r_1 \in (-\infty, x_1)$, $r_4 \in (x_3, \infty)$, $[a_1, a_2] \subset [x_1, x_2]$ and $[b_1, b_2] \subset [x_2, x_3]$ be such that $p_n(r_1) \leq m$, $p_n(r_4) \geq M$ and (38) is satisfied. Further, let (39) be fulfilled on $J \times [r_1, r_4]$. Then we can set $r_2 = a_1$ and $r_3 = b_2$ and by corollary 6 our problem has at least three different solutions.

This occurs, e.g. for $p_3(x) = x^3 - 3x$ and $g(t, v) = 5v^3 + \sin 2\pi t$. Then we have $x_1 = -\sqrt{3}$, $x_2 = 0$, $x_3 = \sqrt{3}$, $M = 1$, $m = -1$, and we can set $r_1 = -2$, $r_2 = a_1 = -3/2$, $r_3 = b_2 = 3/2$, $r_4 = 2$, $L_1 = -1$, $L_2 = 1$, $a_2 = -1/2$, $b_1 = 1/2$.

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