# Impulsive BVPs with nonlinear boundary conditions for the second order differential equations without growth restrictions*

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**Abstract**

The paper deals with the impulsive nonlinear boundary value problem

$$u''(t) = f(t, u(t), u'(t)),$$

$$\left.\begin{array}{rcl} g_1(u(a), u(b)) & = & 0, \\ g_2(u'(a), u'(b)) & = & 0, \end{array}\right\}$$

$$\left.\begin{array}{rcl} u(t_j+) & = & I_j(u(t_j)), \quad j = 1, \dots, p, \\ u'(t_j+) & = & M_j(u'(t_j)), \quad j = 1, \dots, p, \end{array}\right\}$$

where $J = [a, b]$, $f \in Car(J \times \mathrm{R}^2)$, $g_1$, $g_2 \in C(\mathrm{R}^2)$, $I_j$, $M_j \in C(\mathrm{R})$. We prove the existence of a solution to this problem under the assumption that there exist lower and upper functions associated with the problem. Our proofs are based on the Schauder fixed point theorem and on the method of a priori estimates. No growth restrictions are imposed on $f$, $g_1$, $g_2$, $I_j$, $M_j$.

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## 1 Introduction

Let $J = [a, b] \subset \mathrm{R}$. For a real valued function $u$ defined a. e. on $J$, we put

$$\|u\|_\infty = \sup_{t \in J} \mathrm{ess}\, |u(t)| \quad \text{and} \quad \|u\|_1 = \int_a^b |u(s)|\, \mathrm{d}s.$$

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For $k \in \mathrm{N}$ and a given set $B \subset \mathrm{R}^k$, let $C(B)$ denote the set of real valued functions which are continuous on $B$. Furthermore, let $C^1(J)$ be the set of functions having continuous first derivatives on $J$ and $L(J)$ be the set of functions which are Lebesgue integrable on $J$.
Let $p \in \mathrm{N}$ and

$$a = t_0 < t_1 < \ldots < t_p < t_{p+1} = b$$

be a division of the interval $J$. We denote $D = \{t_1, \ldots, t_p\}$ and define $C^1_D$ (or $C_D$) as the set of functions $u : J \to \mathrm{R}$,

$$u(t) = \begin{cases} u_{(0)}(t) & \text{for} \quad t \in [a, t_1], \\ u_{(1)}(t) & \text{for} \quad t \in (t_1, t_2], \\ & \ldots \\ u_{(p)}(t) & \text{for} \quad t \in (t_p, b], \end{cases}$$

where $u_{(j)} \in C^1[t_j, t_{j+1}]$ (or $u_{(j)} \in C[t_j, t_{j+1}]$) for $j = 0, \ldots, p$. Moreover $AC^1_D$ (or $AC_D$) stands for the set of functions $u \in C^1_D$ (or $u \in C_D$) having first derivatives absolutely continuous (or which are absolutely continuous) on each subinterval $(t_j, t_{j+1})$, $j = 0, \ldots, p$. For $u \in C^1_D$ and $j = 1, \ldots, p+1$ we write

$$u'(t_j) = u'(t_j-) = \lim_{t \to t_j-} u'(t), \quad u'(a) = u'(a+) = \lim_{t \to a+} u'(t) \tag{1}$$

and

$$\|u\|_D = \|u\|_\infty + \|u'\|_\infty.$$

Note that the set $C^1_D$ becomes a Banach space when equipped with the norm $\|\cdot\|_D$ and with the usual algebraic operations.

Let $k \in \mathrm{N}$. We say that $f : J \times S \to \mathrm{R}$, $S \subset \mathrm{R}^k$ satisfies the *Carathéodory conditions* on $J \times S$ if $f$ has the following properties: (i) for each $x \in S$ the function $f(\cdot, x)$ is measurable on $J$; (ii) for almost each $t \in J$ the function $f(t, \cdot)$ is continuous on $S$; (iii) for each compact set $K \subset S$ there exists a function $m_K(t) \in L(J)$ such that $|f(t, x)| \le m_K(t)$ for a. e. $t \in J$ and for all $x \in K$. For the set of functions satisfying the *Carathéodory conditions* on $J \times S$ we write $Car(J \times S)$. For a subset $\Omega$ of a Banach space, $\mathrm{cl}(\Omega)$ stands for the closure of $\Omega$.

We study the following boundary value problem with nonlinear boundary conditions

$$u''(t) = f(t, u(t), u'(t)), \tag{2}$$

$$\left.\begin{aligned} g_1(u(a), u(b)) &= 0, \\ g_2(u'(a), u'(b)) &= 0, \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} u(t_j+) &= I_j(u(t_j)), \quad j = 1, \ldots, p, \\ u'(t_j+) &= M_j(u'(t_j)), \quad j = 1, \ldots, p, \end{aligned}\right\} \tag{4}$$

where $f \in Car(J \times \mathrm{R}^2)$, $g_1, g_2 \in C(\mathrm{R}^2)$, $I_j, M_j \in C(\mathrm{R})$ and $u'(t_j)$ are understood in the sense of (1) for $j = 1, \ldots, p$.

**Definition 1** A function $u \in AC^1_D$, which satisfies equation (2) for a. e. $t \in J$ and fulfils conditions (3) and (4) is called a solution of the problem (2)–(4).

In the present paper we provide conditions which are sufficient for the solvability of problem (2)–(4). Our main assumption is the existence of lower and upper functions $\sigma_1$ and $\sigma_2$ of the problem (2)–(4).

**Definition 2** A function $\sigma_k \in AC^1_D$ is called a lower (upper) function of the problem (2)–(4) provided the conditions

$$[\sigma_k''(t) - f(t, \sigma_k(t), \sigma_k'(t))](-1)^k \le 0 \quad \text{for a. e. } t \in J, \tag{5}$$

$$\left.\begin{array}{rcl} g_1(\sigma_k(a), \sigma_k(b)) & = & 0, \\ g_2(\sigma_k'(a), \sigma_k'(b))(-1)^k & \le & 0, \end{array}\right\} \tag{6}$$

$$\left.\begin{array}{c} \sigma_k(t_j+) = I_j(\sigma_k(t_j)), \quad j = 1, \dots, p, \\ [\sigma_k'(t_j+) - M_j(\sigma_k'(t_j))](-1)^k \le 0, \quad j = 1, \dots, p, \end{array}\right\} \tag{7}$$

where $k = 1$ $(k = 2)$, are satisfied.

Throughout the paper we assume:

$$\left.\begin{array}{l} \sigma_1 \text{ and } \sigma_2 \text{ are respectively lower and upper functions} \\ \text{of the problem (2)–(4) and } \sigma_1(t) \le \sigma_2(t) \text{ for } t \in J, \end{array}\right\} \tag{8}$$

$$\left.\begin{array}{lll} g_1(\sigma_1(a), \sigma_1(b)) \ne g_1(x, \sigma_1(b)) & \text{if} & x > \sigma_1(a), \\ g_1(\sigma_2(a), \sigma_2(b)) \ne g_1(x, \sigma_2(b)) & \text{if} & x < \sigma_2(a), \end{array}\right\} \tag{9}$$

$$\left.\begin{array}{lll} g_1(\sigma_1(a), \sigma_1(b)) \le g_1(\sigma_1(a), y) & \text{if} & \sigma_1(b) \le y, \\ g_1(\sigma_2(a), \sigma_2(b)) \ge g_1(\sigma_2(a), y) & \text{if} & \sigma_2(b) \ge y, \end{array}\right\} \tag{10}$$

$$\left.\begin{array}{llll} g_2(\sigma_1'(a), \sigma_1'(b)) \le g_2(x, y) & \text{if} & x \ge \sigma_1'(a), & y \le \sigma_1'(b), \\ g_2(\sigma_2'(a), \sigma_2'(b)) \ge g_2(x, y) & \text{if} & x \le \sigma_2'(a), & y \ge \sigma_2'(b), \end{array}\right\} \tag{11}$$

$$I_j(\sigma_1(t_j)) \le I_j(x) \le I_j(\sigma_2(t_j)), \quad \text{if} \quad \sigma_1(t_j) \le x \le \sigma_2(t_j), \tag{12}$$

for $j = 1, \dots, p$,

$$\left.\begin{array}{l} \text{there exist } \varphi_1, \varphi_2 \in AC_D, \quad \varphi_1(t) \le \sigma_i'(t) \le \varphi_2(t) \;\; \text{for} \;\; t \in J, \\ \varphi_1'(t) > f(t, x, \varphi_1(t)), \;\; \varphi_2'(t) < f(t, x, \varphi_2(t)), \\ \text{for a. e. } t \in J, \;\; x \in [\sigma_1(t), \sigma_2(t)] \end{array}\right\} \tag{13}$$

$$g_2(x, \varphi_i(b))(-1)^i < 0 \quad \text{for} \;\; x \in [\varphi_1(a), \varphi_2(a)], \tag{14}$$

$i = 1, 2,$

$$M_j(\varphi_1(t_j)) \le \varphi_1(t_j+), \quad M_j(\varphi_2(t_j)) \ge \varphi_2(t_j+), \tag{15}$$

and

$$M_j(x) \text{ is nondecreasing for } \; x \in [\varphi_1(t_j), \varphi_2(t_j)] \tag{16}$$

for $j = 1, \dots, p$.

**Remark 3** If we put for $x,\, y \in \mathrm{R}$

$$g_1(x,y) = y - x, \quad g_2(x,y) = x - y, \tag{17}$$

then (3) reduces to the periodic conditions

$$u(a) = u(b), \quad u'(a) = u'(b). \tag{18}$$

By virtue of (17) we see that $g_1$ is one-to-one in $x$ which implies that $g_1$ satisfies (9). Moreover $g_1$ fulfils (10) because $g_1$ is increasing in $y$. Similarly, since $g_2$ is increasing in $x$ and decreasing in $y$, we have that $g_2$ satisfies (11). If $\varphi_1(a) > \varphi_1(b)$ and $\varphi_2(a) < \varphi_2(b)$, then $g_2$ fulfils (14), as well.

Existence results for problem (2), (3) (without impulses) can be found for example in [5], [6], [8], [13], [14], but the methods of their proofs cannot be applied on the impulsive problem (2)–(4). Therefore we have developed a different approach here. It is based on the method of lower and upper functions providing the construction of a proper auxiliary problem (problem (29)–(31)) and on the method of a priori estimates for solutions of the auxiliary problem (Proposition 8). Similar problems with different kinds of nonlinear boundary condition and with a continuous right-hand side $f$ have been solved in [4]. The impulsive problem (2), (4) with the periodic conditions (which are a special case of conditions (4), by Remark 3) has been already studied by means of the lower and upper functions method in [1] – [3], [7], [9], [11], [12], [17], [18], [19]. The most of these works impose the Nagumo growth conditions on the right-hand side $f$ of equation (2) (see [1], [3], [4], [7], [9], [11], [17], [19]). Other works (see [2], [12]) assume that $f$ does not depend on the first derivative of solutions. The existence proofs in [2], [12], [18] are based on the monotone iterative technique which makes demands on the monotonous behaviour of the right-hand side $f$ as well as of the impulse function $I_j$, $M_j$. In contrast to all the works citied above we prove the existence result for the equation (2) with $f$ satisfying conditions of the sign type with respect to the third variable of $f$ (conditions (13)), which means that we impose no growth restrictions on $f$. Moreover, we do not require the monotonicity of the impulse functions $I_j$ and use the weaker conditions (12). No growth restrictions are imposed on $g_1$, $g_2$, $I_j$, $M_j$, as well. Let us note that the corresponding first order impulsive problem

$$u'(t) = f(t, u(t)), \quad g(u(a), u(b)) = 0, \quad u(t_j+) = I_j(u(t_j)), \quad j = 1, \dots, p$$

has been solved in [10] and [15] for the scalar case and in [16] for the vector case.

## 2 Auxiliary problem

This section is devoted to the construction and the study of certain auxiliary problem. In the construction we will use functions

$$\omega_i(t, \epsilon) = \sup\{|f(t, \sigma_i(t), \sigma_i'(t)) - f(t, \sigma_i(t), y)| : |\sigma_i'(t) - y| \le \epsilon\}, \tag{19}$$

for a. e. $t \in J$, and for $\epsilon \in [0,1]$, $i = 1, 2$.

**Lemma 4** *The functions* $\omega_i$ *defined in* (19) *fulfil the Carathéodory conditions on* $J \times [0,1]$, *for* $i = 1, 2$.

On purpose of proving this lemma, we need two following lemmas.

**Lemma 5** *Let* $h \in Car(J \times S)$, $S \subset \mathrm{R}^k$, $k \in \mathrm{N}$. *Then for every compact set* $K \subset S$ *the function*

$$\psi_K(t) = \sup_{x \in K} |h(t,x)|$$

*is Lebesgue integrable on* $J$.

*Proof.* Let $K \subset S$ be a compact set. First, we will prove that $\psi_K$ is measurable on $J$. There exists a countable set $L \subset K$ such that

$$\mathrm{cl}(L) = K. \tag{20}$$

We write $L = \{q_n\}$, where $\{q_n\}$ is a sequence in $\mathrm{R}^k$ and get the sequence of measurable functions

$$\{|h(\cdot, q_n)| : n \in \mathrm{N}\}.$$

Let us define a function

$$\psi_L(t) = \sup_{x \in L} |h(t,x)| = \sup_{n \in \mathrm{N}} \{|h(t,q_n)|\} \quad \text{for a. e. } t \in J.$$

From the third *Carathéodory* condition of the function $h$ we get that there is $m_K \in L(J)$ such that $\psi_L \le m_K$ a. e. on $J$, and so $\psi_L$ is measurable on $J$. It remains to prove that

$$\psi_K = \psi_L \quad \text{ a. e. on } J. \tag{21}$$

Let us take such $t \in J$, for which $h(t, \cdot)$ is continuous on $S$. Then there exists $x_0 \in K$ such that

$$|h(t,x_0)| = \max_{x \in K} |h(t,x)| = \sup_{x \in K} |h(t,x)|.$$

From (20) it follows that there exists

$$\{x_n\} \subset L \qquad \text{and} \qquad x_n \to x_0 \ \text{ in } \mathrm{R}^k.$$

Since $h(t, \cdot)$ is continuous on $K$ it follows

$$\lim_{n \to \infty} |h(t,x_n)| = |h(t,x_0)| = \psi_K(t).$$

Obviously, $\psi_L(t) \ge \lim_{n \to \infty} |h(t,x_n)|$ for a. e. $t \in J$, i. e. $\psi_L \ge \psi_K$ a. e. on $J$. From the definitions $\psi_L$ and $\psi_K$ we also get the inverse inequality a. e. on $J$. Thus, (21) is valid. □

**Lemma 6** *Let* $f \in C[0, \eta]$, *where* $\eta > 0$. *Then the function*

$$g(y) = \max_{0 \le x \le y} f(x), \qquad y \in [0, \eta]$$

*is continuous on* $[0, \eta]$.

*Proof.* Let $\epsilon > 0$ be an arbitrary real number. Let $q \in [0, \eta)$. Since $f \in C[0, \eta]$, it follows that there exists $\delta_1 > 0$ such that $(q, q + \delta_1) \subset (0, \eta)$ and

$$|f(x) - f(q)| < \epsilon \tag{22}$$

for every $x \in (q, q + \delta_1)$. Let $y \in (q, q + \delta_1)$. Then we can write

$$g(y) = \max\Big(g(q), \max_{q \le x \le y} f(x)\Big).$$

Obviously, if $g(y) > g(q)$, then $g(y) = \max_{q \le x \le y} f(x)$. There exists $\xi \in [q, y]$ such that $g(y) = f(\xi)$ and consequently from (22) we get

$$g(y) - g(q) = f(\xi) - g(q) \le f(\xi) - f(q) < \epsilon.$$

Let $q \in (0, \eta]$. Then there exists $\delta_2$ such that (22) is valid for $x \in (q - \delta_2, q)$. Let $y \in (q - \delta_2, q)$. We can write

$$g(q) = \max\Big(g(y), \max_{y \le x \le q} f(x)\Big).$$

If $g(q) > g(y)$, then $g(q) = f(\theta)$, where $\theta \in [y, q]$. Thus,

$$g(q) - g(y) \le f(\theta) - f(y) = f(\theta) - f(q) + f(q) - f(y) < 2\epsilon.$$

□

*Proof of Lemma 4.* Let $i \in \{1, 2\}$. We denote

$$k_i(t, y) = f(t, \sigma_i(t), \sigma_i'(t) - y) - f(t, \sigma_i(t), \sigma_i'(t)) \tag{23}$$

for a. e. $t \in J$ and $y \in [-1, 1]$. Let $\epsilon \in [0, 1]$. Obviously, $k_i(t, y) \in Car(J \times [-\epsilon, \epsilon])$ and $\omega_i(t, \epsilon) = \sup\{|k_i(t, y)| : |y| \le \epsilon\}$. From Lemma 5, it follows that $\omega_i(\cdot, \epsilon)$ is measurable on $J$. Since

$$\omega_i(t, \epsilon) \le \omega_i(t, 1) \quad \text{for a. e. } t \in J, \quad \text{all} \quad \epsilon \in [0, 1],$$

and $\omega_i(\cdot, 1)$ is Lebesgue integrable it follows that $\omega_i$ fulfils the third Carathéodory condition.
It remains to prove the continuity of the function $\omega_i(t, \cdot)$ for a. e. $t \in J$. Let us take $t \in J$ such that $t \neq t_j$ for $j = 0, \dots, p+1$ and such that $f(t, \cdot)$ is continuous on $\mathrm{R}^2$. According to (23), we have

$$\omega_i(t, \epsilon) = \max\Big(\max_{0 \le y \le \epsilon} |k_i(t, y)|, \max_{0 \le y \le \epsilon} |k_i(t, -y)|\Big) \quad \text{for each } \epsilon \in [0, 1].$$

In view of Lemma 6, the proof is complete. □

We define functions

$$\tilde{f}(t, x, y) = \begin{cases} f(t, \sigma_1(t), y) - \omega_1\Big(t, \frac{\sigma_1(t) - x}{\sigma_1(t) - x + 1}\Big) - \frac{\sigma_1(t) - x}{\sigma_1(t) - x + 1} & \text{for} \quad x < \sigma_1(t), \\ f(t, x, y) & \text{for} \quad \sigma_1(t) \le x \le \sigma_2(t), \\ f(t, \sigma_2(t), y) + \omega_2\Big(t, \frac{x - \sigma_2(t)}{x - \sigma_2(t) + 1}\Big) + \frac{x - \sigma_2(t)}{x - \sigma_2(t) + 1} & \text{for} \quad \sigma_2(t) < x, \end{cases} \tag{24}$$

for a. e. $t \in J$ and all $x, y \in \mathrm{R}$,

$$\varphi(t,y) = \begin{cases} \varphi_1(t) & \text{for} \quad y < \varphi_1(t), \\ y & \text{for} \quad \varphi_1(t) \le y \le \varphi_2(t), \\ \varphi_2(t) & \text{for} \quad \varphi_2(t) < y, \end{cases} \quad \text{for all } t \in J,\ x \in \mathrm{R}, \tag{25}$$

$$f^*(t,x,y) = \tilde{f}(t,x,\varphi(t,y)) \qquad \text{for a. e.} \quad t \in J \text{ and all } \ x,y \in \mathrm{R}, \tag{26}$$

and

$$g_2^*(x,y) = g_2(\varphi(a,x),\varphi(b,y)) \qquad \text{for all} \ \ x,y \in \mathrm{R}. \tag{27}$$

By virtue of Lemma 4 we have $f^* \in Car(J \times \mathrm{R}^2)$. Finally, put

$$\sigma(t,x) = \begin{cases} \sigma_1(t) & \text{for} \quad x < \sigma_1(t), \\ x & \text{for} \quad \sigma_1(t) \le x \le \sigma_2(t), \\ \sigma_2(t) & \text{for} \quad \sigma_2(t) < x, \end{cases} \tag{28}$$

for all $t \in J$, $x \in \mathrm{R}$.

Now, we define the auxiliary problem

$$u''(t) = f^*(t,u(t),u'(t)), \tag{29}$$

$$\left.\begin{array}{rcl} u(a) & = & \sigma(a, u(a) + g_1(u(a),u(b))), \\ u(b) & = & \sigma(b, u(b) + g_2^*(u'(a),u'(b))), \end{array}\right\} \tag{30}$$

$$\left.\begin{array}{rcl} u(t_j+) - u(t_j) & = & I_j(\sigma(t_j,u(t_j))) - \sigma(t_j,u(t_j)), \quad j = 1,\dots,p, \\ u'(t_j+) - u'(t_j) & = & M_j(\varphi(t_j,u'(t_j))) - \varphi(t_j,u'(t_j)), \quad j = 1,\dots,p. \end{array}\right\} \tag{31}$$

**Definition 7** A function $u \in AC_D$, which satisfies differential equation (29) for a. e. $t \in J$ and fulfils conditions (30), (31) is called a solution of the problem (29)–(31).

**Proposition 8** *Let the conditions* (8)–(16) *and* (24)–(28) *hold. Let $u$ be a solution of the problem* (29)–(31). *Then*

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \qquad \textit{for all } t \in J, \tag{32}$$

$$\varphi_1(t) \le u'(t) \le \varphi_2(t) \qquad \textit{for all } t \in J \tag{33}$$

*and $u$ is a solution of the problem* (2)–(4).

*Proof.* Let $u$ be a solution of the problem (29)–(31).
Step 1. We will prove the inequality (32). Let us consider a function

$$v(t) = u(t) - \sigma_2(t) \qquad \text{for } t \in J.$$

Suppose, that there exist $j \in \{0,\dots,p\}$ and $\tau \in (t_j,t_{j+1})$ such that

$$\max_{t \in (t_j,t_{j+1}]} v(t) = v(\tau) > 0. \tag{34}$$

Then

$$v'(\tau) = 0,$$

which together with (34) implies that there exists $\gamma > 0$ such that

$$v(t) > 0 \quad \text{and} \quad |v'(t)| < \frac{v(t)}{v(t)+1} < 1, \tag{35}$$

for $t \in (\tau, \tau+\gamma) \subset (t_j, t_{j+1})$. Then

$$\begin{aligned} v''(t) &= u''(t) - \sigma_2''(t) \geq \tilde{f}(t, u(t), \varphi(t, u'(t))) - f(t, \sigma_2(t), \sigma_2'(t)) = \\ &= f(t, \sigma_2(t), \varphi(t, u'(t))) - f(t, \sigma_2(t), \sigma_2'(t)) + \omega_2\Big(t, \frac{v(t)}{v(t)+1}\Big) + \frac{v(t)}{v(t)+1} \end{aligned}$$

for a. e. $t \in (\tau, \tau+\gamma)$. Note, that

$$|\varphi(t, u'(t)) - \sigma_2'(t)| \leq |v'(t)| \quad \text{for} \quad t \in (t_j, t_{j+1}). \tag{36}$$

By virtue of (19), (35) and (36), we get

$$v''(t) \geq -\omega_2(t, |v'(t)|) + \omega_2\Big(t, \frac{v(t)}{v(t)+1}\Big) + \frac{v(t)}{v(t)+1} \geq \frac{v(t)}{v(t)+1} > 0 \tag{37}$$

for a. e. $t \in (\tau, \tau+\gamma)$.
Thus

$$0 < \int_\tau^t v''(s)\,\mathrm{d}s = v'(t) - v'(\tau) = v'(t), \quad \text{for } t \in (\tau, \tau+\gamma),$$

which contradicts (34). So, we have proved that

$$\left.\begin{array}{l}\text{the function } v \text{ cannot have any positive maximum inside of the} \\ \text{interval } (t_j, t_{j+1}), \text{ for } j = 0, \dots, p.\end{array}\right\} \tag{38}$$

Now, from (30) it follows that $v(a) \leq 0$. Let us suppose that there exists $q \in (a, t_1)$ such that $v(q) > 0$. According to (38) we have

$$\max_{t \in [a, t_1]} v(t) = v(t_1) > 0, \tag{39}$$

i. e. $u(t_1) > \sigma_2(t_1)$. We get $\sigma(t_1, u(t_1)) = \sigma_2(t_1)$ and from the first equality in (31) it follows that

$$u(t_1+) = I_1(\sigma_2(t_1)) - \sigma_2(t_1) + u(t_1) > I_1(\sigma_2(t_1)).$$

Using (7) we get $u(t_1+) > \sigma_2(t_1+)$, which means $v(t_1+) > 0$. From (38) and (39) we get

$$v'(t_1) \geq 0. \tag{40}$$

Let us suppose that

$$v'(t_1+) < 0. \tag{41}$$

In view of (40), (7), (13), (16) and (25), we have

$$M_1(\varphi(t_1, u'(t_1))) \geq M_1(\varphi(t_1, \sigma_2'(t_1))) = M_1(\sigma_2'(t_1)) \geq \sigma_2'(t_1+).$$

Applying it to (31), we get

$$u'(t_1+) - \sigma_2'(t_1+) \geq u'(t_1) - \varphi(t_1, u'(t_1)).$$

Due to (41), we have the inequality $u'(t_1) < \varphi(t_1, u'(t_1))$, i. e. $u'(t_1) < \varphi_1(t_1)$. Using this and (40) we have $\sigma_2'(t_1) < \varphi_1(t_1)$, which contradicts (13).
Therefore $v'(t_1+) \geq 0$. If $v'(t_1+) = 0$ and $v$ is nonincreasing on some interval $(t_1, t_1 + \gamma) \subset (t_1, t_2)$, where $\gamma > 0$, then (35) is valid for all $t \in (t_1, t_1 + \gamma_1)$, $0 < \gamma_1 \leq \gamma$. Hence, the relation (37) is satisfied for a. e. $t \in (t_1, t_1 + \gamma_1)$. We get

$$0 < \int_{t_1}^{t} v''(s)\,\mathrm{d}s = v'(t) - v'(t_1+) = v'(t), \qquad \text{for } t \in (t_1, t_1 + \gamma_1),$$

which contradits the assumption of monotony of the function $v$.
In view of (38) we get

$$0 < v(t_1+) < v(t_2) \qquad \text{and} \qquad v'(t_2) \geq 0$$

in all other cases. Then we use the preceding procedure and deduce by induction that

$$v(t_j) > 0, \qquad \text{for } j = 1, \ldots, p+1,$$

i. e. $v(b) > 0$, contrary to (30). This means that (39) is not valid, which together with (38) gives $v \leq 0$ on $[a, t_1]$, i. e. $u(t) \leq \sigma_2(t)$ for $t \in [a, t_1]$.
To prove that $u(t) \geq \sigma_1(t)$ for $t \in [a, t_1]$, we argue similarly. Therefore we get $\sigma_1(t) \leq u(t) \leq \sigma_2(t)$ for $t \in [a, t_1]$. Particularly

$$\sigma_1(t_1) \leq u(t_1) \leq \sigma_2(t_1)$$

and in view of (12) we have

$$I_1(\sigma_1(t_1)) \leq I_1(u(t_1)) \leq I_1(\sigma_2(t_1)). \tag{42}$$

Further, due to the first equality in (31) we get

$$u(t_1+) = I_1(u(t_1)).$$

Therefore, according to (7) and (42) we have

$$\sigma_1(t_1+) \leq u(t_1+) \leq \sigma_2(t_1+).$$

In such a way we argue on each interval $[t_j, t_{j+1}]$, $j = 1, \ldots, p$, and get (32).
Step 2. We will prove that

$$g_1(u(a), u(b)) = 0, \quad g_2^*(u'(a), u'(b)) = 0. \tag{43}$$

To this aim we will show that

$$\sigma_1(a) \leq u(a) + g_1(u(a), u(b)) \leq \sigma_2(a) \tag{44}$$

and

$$\sigma_1(b) \leq u(b) + g_2^*(u'(a), u'(b)) \leq \sigma_2(b). \tag{45}$$

Let us suppose that the first inequality in (44) is not true. Then

$$\sigma_1(a) > u(a) + g_1(u(a), u(b)).$$

In view to (30) we have $u(a) = \sigma_1(a)$, thus it follows from (10) and (32) that

$$0 > g_1(\sigma_1(a), u(b)) \geq g_1(\sigma_1(a), \sigma_1(b)),$$

which contradicts (6). We prove the second inequality in (44) similarly. Let us suppose that the first inequality in (45) is not valid, i. e. let

$$\sigma_1(b) > u(b) + g_2^*(u'(a), u'(b)). \tag{46}$$

It follows from (30) that

$$u(b) = \sigma_1(b) \tag{47}$$

and $0 > g_2^*(u'(a), u'(b))$. Further, by virtue of (6), (30) and (44), we have

$$g_1(\sigma_1(a), \sigma_1(b)) = 0 = g_1(u(a), u(b)) = g_1(u(a), \sigma_1(b)).$$

In view of (9), we get

$$u(a) = \sigma_1(a). \tag{48}$$

It follows from (32), (47) and (48) that $\sigma_1'(b) \geq u'(b)$ and $u'(a) \geq \sigma_1'(a)$. Finally, by (11), we get the inequalities

$$0 > g_2^*(u'(a), u'(b)) \geq g_2(\sigma_1'(a), \sigma_1'(b)),$$

contrary to (6). The second inequality in (45) can be proved by a similar argument. Due to (30), the conditions (44) and (45) imply (43).
Step 3. We will prove (33). According to (32), we have

$$f^*(t, u(t), u'(t)) = \tilde{f}(t, u(t), \varphi(t, u'(t))) = f(t, u(t), \varphi(t, u'(t))) \tag{49}$$

for a. e. $t \in J$. We define $z = u' - \varphi_2$ on $J$ and suppose that there exists $q \in [a, t_1)$ such that

$$\max_{t \in [a, t_1]} z(t) = z(q) > 0. \tag{50}$$

Then there exists $\delta > 0$, such that $z(t) > 0$, i. e. $u'(t) > \varphi_2(t)$ for $t \in (q, q + \delta)$. From (13) we get

$$z'(t) = u''(t) - \varphi_2'(t) = f(t, u(t), \varphi(t, u'(t)) - \varphi_2'(t) > 0$$

for a. e. $t \in (q, q + \delta)$. This implies that

$$0 < \int_q^t z'(s)\, \mathrm{d}s = z(t) - z(q)$$

for all $t \in (q, q + \delta)$, which contradicts (50). Let us suppose that (50) is valid for $q = t_1$. From (25), we get $\varphi(t_1, u'(t_1)) = \varphi_2(t_1)$ and from (31) we have

$$u'(t_1+) - M_1(\varphi_2(t_1)) = u'(t_1) - \varphi_2(t_1).$$

In view of (50) for $q = t_1$ and by (15) we have

$$u'(t_1+) > M_1(\varphi_2(t_1)) \geq \varphi_2(t_1+),$$

i. e. $z(t_1+) > 0$. We can apply the preceding procedure on $(t_j, t_{j+1}]$, for $j = 1, \ldots, p$ and get $z(t_2) > 0, \ldots, z(b) > 0$. From the last inequality we have $\varphi(b, u'(b)) = \varphi_2(b)$ and therefore (14) and (27) lead to

$$g_2^*(u'(a), u'(b)) = g_2(\varphi(a, u'(a)), \varphi_2(b)) < 0.$$

According to (43) we get a contradiction. The second inequality in (33) can be derived similarly.
Step 4. To summarize, we have proved that an arbitrary solution $u$ of the problem (29) - (31) satisfies (32), (33) and (43). This implies, by (24) - (26) and (28), that $u$ satisfies the conditions (4) and $u$ fulfils the equation (2) for a. e. $t \in J$. Moreover, due to (27), $u$ satisfies (3). This completes the proof. □

# 3 Main result

**Theorem 9** *Let the conditions* (8) - (16) *hold. Then there exists a solution* $u$ *of the problem* (2) - (4) *such that*

$$\sigma_1 \le u \le \sigma_2 \quad \text{and} \quad \varphi_1 \le u' \le \varphi_2 \quad \text{on } J. \tag{51}$$

*Proof.* Let $f^*$ be defined by (24) and (26). Since $f^* \in Car(J \times \mathrm{R}^2)$, it follows that there exists $h \in L(J)$ such that

$$|f^*(t, x, y)| \le h(t) \qquad \text{for a. e. } \; t \in J \quad \text{and all } \; x, y \in \mathrm{R}. \tag{52}$$

Let us consider the Green function

$$G(t,s) = \begin{cases} \frac{(a-s)(b-t)}{b-a} & \text{for } \; a \le s < t \le b, \\ \frac{(a-t)(b-s)}{b-a} & \text{for } \; a \le t \le s \le b, \end{cases}$$

and a function $G_1 : J \times J \to \mathrm{R}$ defined by

$$G_1(t,s) = \begin{cases} \frac{b-t}{b-a} & \text{for } \; a \le s < t \le b, \\ \frac{a-t}{b-a} & \text{for } \; a \le t \le s \le b. \end{cases}$$

Let us denote

$$L = \sup\left\{ |G(t,s)| + |G_1(t,s)| + \left|\frac{\partial G(t,s)}{\partial t}\right| + \left|\frac{\partial G_1(t,s)}{\partial t}\right| : (t,s) \in J \times J \right\}$$

and

$$\begin{aligned} K = {} & 2 \max\{1, \frac{1}{b-a}\}(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty) + L\Big[\int_a^b h(s)\,\mathrm{d}s \\ + {} & p\,(\|\varphi_1\|_\infty + \|\varphi_2\|_\infty + \|\sigma_1\|_\infty + \|\sigma_2\|_\infty) \\ + {} & \sum_{k=1}^{p} \Big( \max_{\varphi_1(t_k) \le x \le \varphi_2(t_k)} |M_k(x)| + \max_{\sigma_1(t_k) \le x \le \sigma_2(t_k)} |I_k(x)| \Big) \Big]. \end{aligned}$$

On purpose of proving the existence of a solution of the problem (29)–(31) we consider an operator $T : \Omega \subset C^1_D \to C^1_D$, where

$$\Omega = \{u \in C^1_D : \|u\|_D \le K\}.$$

We define the operator $T$ by

$$T = A + B,$$

where

$$Au(t) = \int_a^b G(t,s) f^*(s, u(s), u'(s))\,\mathrm{d}s \tag{53}$$

and

$$\begin{aligned} Bu(t) = & \frac{b-t}{b-a}\sigma(a, u(a) + g_1(u(a), u(b)))+ \\ & \frac{t-a}{b-a}\sigma(b, u(b) + g_2^*(u'(a), u'(b))) + \\ & \sum_{k=1}^{p} G(t,t_k)[M_k(\varphi(t_k, u'(t_k))) - \varphi(t_k, u'(t_k))] + \\ & \sum_{k=1}^{p} G_1(t,t_k)[I_k(\sigma(t_k, u(t_k))) - \sigma(t_k, u(t_k))], \end{aligned} \tag{54}$$

for each $u \in C^1_D$ and each $t \in J$. Here $\varphi$, $g_2^*$ and $\sigma$ are given by (25), (27) and (28), respectively. Obviously, $T(\Omega) \subset \Omega$.
We will use the Schauder fixed point theorem to prove the existence of a fixed point of the operator $T$. The set $\Omega$ is a nonempty, closed, convex and bounded subset of $C^1_D$. The only thing that left to prove is the absolute continuity of $T$. From the Lebesgue dominated convergence theorem and the continuity of the functions $\sigma$, $g_1$, $g_2$, $\varphi$, and $I_j$, $M_j$, for $j = 1, \dots, p$, it follows that $A$ and $B$ are continuous. From (52) and the Arzelà-Ascoli theorem, it follows that the operator $A : \Omega \to C^1(J)$ is absolutely continuous. Since $B$ maps the set $\Omega$ into the subspace of the finite dimension of $C^1_D$, with the base $\{1, t, G(t,t_j), G_1(t,t_j),\ j = 1, \dots, p\}$ and $B$ is a bounded, continuous operator, it follows that $B$ is also absolutely continuous.
Thus, there exists the fixed point $u$ of the mapping $T$, i. e.

$$u = Au + Bu.$$

The definition (53) implies that $Au \in AC^1(J)$ and by (54) we have $Bu \in AC^1_D$. Therefore $u \in AC^1_D$.
It is valid that

$$(Au)''(t) = f^*(t, u(t), u'(t)) \quad \text{and} \quad (Bu)''(t) = 0 \quad \text{for a. e. } t \in J,$$

which means that $u$ satisfies (29). Further,

$$\begin{aligned} (Au)(a) &= (Au)(b) = 0, \\ (Bu)(a) &= \sigma(a, u(a) + g_1(u(a), u(b))) \\ (Bu)(b) &= \sigma(b, u(b) + g_2^*(u'(a), u'(b))) \end{aligned}$$

hence (30) is valid. Finally

$$(Au)^{(i)}(t_j+) = (Au)^{(i)}(t_j), \qquad i = 0, 1,$$

and

$$\begin{aligned}(Bu)(t_j+) - (Bu)(t_j) &= I_j(\sigma(t_j, u(t_j))) - \sigma(t_j, u(t_j)),\\ (Bu)'(t_j+) - (Bu)'(t_j) &= M_j(\varphi(t_j, u'(t_j))) - \varphi(t_j, u'(t_j))\end{aligned}$$

for $j = 1, \ldots, p$. Thus, $u$ is a solution of the problem (29) - (31) and in view of Proposition 8 it is a solution of the problem (2) - (4), as well. □

In the following theorem we assume weaker conditions than (14). Let us note that the conditions (6), (13) and (14) imply

$$\varphi_1(b) < \sigma_2'(b) \quad \text{and} \quad \sigma_1'(b) < \varphi_2(b). \tag{55}$$

**Theorem 10** *Let the conditions* (8) – (13), (55), (15), (16) *and*

$$g_2(x, \varphi_i(b))(-1)^i \le 0 \quad \textit{for } x \in [\varphi_1(a), \varphi_2(a)], \tag{56}$$

*hold. Then there exists a solution $u$ of the problem* (2)–(4) *such that*

$$\sigma_1 \le u \le \sigma_2 \quad \textit{and} \quad \varphi_1 \le u' \le \varphi_2 \quad \textit{on } J.$$

*Proof.* We define a function $\psi : \mathrm{R} \to [-1, 1]$

$$\psi(y) = \begin{cases} 1 & \text{for } \ y \le \varphi_1(b), \\ \frac{L-y}{L-\varphi_1(b)} & \text{for } \ \varphi_1(b) < y < L, \\ 0 & \text{for } \ L \le y \le U, \\ \frac{U-y}{\varphi_2(b)-U} & \text{for } \ U < y < \varphi_2(b), \\ -1 & \text{for } \ \varphi_2(b) \le y, \end{cases}$$

where $\varphi_1(b) < L \le U < \varphi_2(b)$. If $\varphi_1(b) < \sigma_1'(b)$ and $\sigma_2'(b) < \varphi_2(b)$, then we put $L = \min(\sigma_1'(b), \sigma_2'(b))$ and $U = \max(\sigma_1'(b), \sigma_2'(b))$. In the case if $\varphi_1(b) = \sigma_1'(b)$ and $\sigma_2'(b) < \varphi_2(b)$ we put $L = U = \sigma_2'(b)$ and similarly, if $\varphi_1(b) < \sigma_1'(b)$ and $\sigma_2'(b) = \varphi_2(b)$, then $L = U = \sigma_1'(b)$. Otherwise, we can take $U$ and $L$ arbitrarily. We define functions

$$g_{2,n}(x, y) = g_2(x, y) + \frac{1}{n}\psi(y),$$

for all $n \in \mathrm{N}$, $x, y \in \mathrm{R}$. Consider the sequence of problems (2),

$$\left.\begin{aligned} g_1(u(a), u(b)) &= 0, \\ g_{2,n}(u'(a), u'(b)) &= 0, \end{aligned}\right\} \tag{57}$$

(4), for all $n \in \mathrm{N}$. It is easy to see, that the conditions (8) – (16) for the problems (2), (57), (4) are satisfied. Applying Theorem 9 we get the sequence $\{u_n\}$ of solutions of problems (2), (57), (4). In view of proof of Theorem 9 we can see that the function $u_n$ satisfies relation

$$T_n u_n = u_n,$$

where $T_n = A + B_n$ is the operator representation of the auxiliary problems to the problems (2), (57), (4). Since $A$ (defined in (53)) is a compact operator and $\{B_n u_n\}$ ($B_n$ are defined in (54), where $g_{2,n}$ are in place of $g_2$) is a bounded sequence in a subspace of finite dimension, it follows that there exists a convergent subsequence of $\{u_n\}$. Without any loss of generality we can assume, that $\{u_n\}$ is such a sequence and $u \in C_D^1$ is its limit. We will show, that $u$ is a solution of the problem (2)–(4). Consider the operator representation $T = A + B$ (defined by (53) and (54)) of the auxiliary problem of (2)–(4). We have

$$\|Tu - u\|_D \leq \|Tu - Tu_n\|_D + \frac{1}{n(b-a)} + \frac{1}{n} + \|T_n u_n - u\|_D.$$

Since the right side of this inequality approaches zero as $n \to \infty$, it follows that $u$ is a fixed point of $T$ and consequently $u \in AC_D^1$.

From the uniform convergence of $\{u_n\}$,$\{u_n'\}$ and $\{g_{2,n}\}$, we get (3) and (4). It remains to prove that $u$ satisfies the differential equation (2). We have

$$u_n''(t) = f(t, u_n(t), u_n'(t)), \quad \text{for a. e. } t \in J.$$

Let $j \in \{0, \ldots, p\}$ and $t \in (t_j, t_{j+1})$. Then

$$u_n'(t) - u_n'(t_j) = \int_{t_j}^{t} f(s, u_n(s), u_n'(s))\,\mathrm{d}s,$$

for all $n \in \mathrm{N}$. From the fact that $f \in Car(J \times \mathrm{R}^2)$, $u_n \to u$ in $C_D^1$ and from the Lebesgue bounded theorem we have

$$u'(t) - u'(t_j) = \int_{t_j}^{t} f(s, u(s), u'(s))\,\mathrm{d}s,$$

for each $t \in (t_j, t_{j+1})$. The proof is complete. □

**Remark 11** In Remark 3 we have shown that if $g_1$ and $g_2$ are defined by (17), they fulfil (9) – (11). For the validity of (56) it suffices to assume that $\varphi_1(a) \geq \varphi_1(b)$ and $\varphi_2(a) \leq \varphi_2(b)$ instead of the strict inequalities which are necessary for (14). Then $\varphi_1$ and $\varphi_2$ can be constant functions. The existence result for constant lower and upper functions $\sigma_1(t) = r_1$, $\sigma_2(t) = r_2$ for $t \in J$ and constant functions $\varphi_1(t) = c_2$, $\varphi_2(t) = c_2$ for $t \in J$ follows from Theorem 10 and is presented in the next corollary.

**Corollary 12** Let $r_1, r_2 \in \mathrm{R}$ be such that $r_1 \leq r_2$,

$$f(t, r_1, 0) \leq 0, \quad f(t, r_2, 0) \geq 0 \quad \text{for a. e. } t \in J$$

and let

$$I_j(r_k) = r_k, \qquad I_j(x) \in (r_1, r_2) \quad \text{if } x \in (r_1, r_2)$$

for $j = 1, \ldots, p$, $k = 1, 2$. Further, let $c_1, c_2 \in \mathrm{R}$ be such that $c_1 < 0 < c_2$,

$$f(t, x, c_1) < 0, \quad f(t, x, c_2) > 0 \quad \text{for a. e. } t \in J \text{ and for } x \in [r_1, r_2],$$

and let

$$M_j(0) = 0, \quad M_j(c_k) = c_k, \quad M_j(x) \text{ is nondecreasing on } [c_1, c_2]$$

for $j = 1, \ldots, p$, $k = 1, 2$. Then the periodic impulsive problem (2), (4), (18) has a solution $u$ and that

$$r_1 \leq u \leq r_2 \qquad \text{and} \qquad c_1 \leq u' \leq c_2 \quad \text{on } J.$$

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