Singular problems on the half–line*

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Abstract

The paper investigates singular nonlinear problems arising in hydrodynamics. In particular, it deals with the problem on the half–line of the form

$$(p(t)u'(t))' = p(t)f(u(t)),$$
$$u'(0) = 0, \quad u(\infty) = L.$$

The existence of a strictly increasing solution (a homoclinic solution) of this problem is proved by the dynamical systems approach and the lower and upper functions method.

Mathematics Subject Classification 2000: 34B16, 34B40.

Key words: Singular ordinary differential equation of the second order, lower and upper functions, time singularities, unbounded domain, homoclinic solution.

1 Introduction

In the Cahn–Hilliard theory used in hydrodynamics to study the behaviour of nonhomogenous fluids the following system of PDE's was derived

$$\rho_t + \operatorname{div}(\rho v) = 0, \qquad \frac{dv}{dt} + \nabla(\mu(\rho) - \gamma\triangle\rho) = 0$$

with the density ρ and the velocity v of the fluid, μ is its chemical potential, γ is a constant. In the simplest model, this system can be reduced into the boundary value problem for the ODE of the second order (see [5] or [7])

$$(t^k u')' = 4\lambda^2 t^k (u+1)u(u-\xi), \quad t \in (0,\infty),$$

$$u'(0) = 0, \quad u(\infty) = \xi,$$

*Supported by the Council of Czech Government MSM 6198959214.

where $k \in \mathbb{N}$, $\xi \in (0,1)$, $\lambda \in (0,\infty)$ are parameters. The function $u(t) \equiv \xi$ is a solution of this problem and it corresponds to the case of homogenous fluid (without bubbles). But only the existence of a strictly increasing solution of this problem and the solution itself has a great physical significance. We call it a homoclinic solution. We refer to [1] and [2], where an equivalent problem was investigated. The numerical treatment was done in papers [5], [7].

Here, we study the generalized problem

$$(p(t)u'(t))' = p(t)f(u(t)), \tag{1}$$

$$u'(0) = 0, \quad u(\infty) = L, \tag{2}$$

where $L > 0$.

2 Autonomous equation

The investigation of autonomous equations corresponding to (1) turned out to be quite useful, because some solutions of the perturbed autonomous equation (14) can serve as an upper functions to (1).

Let $h : \mathbb{R} \to \mathbb{R}$ and x_1, x_2, $x_3 \in \mathbb{R}$ be such that $x_1 < x_2 < x_3$ and

$$h \text{ is lipschitzian on } [x_1, x_3], \tag{3}$$

$$h(x_i) = 0 \qquad \text{for } i = 1, 2, 3, \tag{4}$$

$$\left.\begin{array}{l}\text{there exists } \delta > 0 \text{ such that } h \in C^1((x_2 - \delta, x_2)) \\ \text{and } \lim\limits_{x \to x_2^-} h'(x) = h'_-(x_2) < 0,\end{array}\right\} \tag{5}$$

$$(x - x_2)h(x) < 0 \qquad \text{for } x \in (x_1, x_3) \setminus \{x_2\}, \tag{6}$$

$$H(x_1) > H(x_3), \tag{7}$$

where

$$H(x) = -\int_{x_2}^{x} h(z)\,\mathrm{d}z \qquad \text{for } x \in \mathbb{R}.$$

Moreover we will assume that

$$\left\{\begin{array}{lll} h(x) = 0 & \text{for} & x \le x_1, \\ h(x) = x - x_3 & \text{for} & x \ge x_3. \end{array}\right. \tag{8}$$

Let us consider equation

$$u'' = h(u) \tag{9}$$

and the initial condition

$$u(0) = B, \ u'(0) = 0 \tag{10}$$

for $B \in (x_1, x_2)$. Equation (9) is equivalent with the gradient system

$$u_1' = u_2,\ u_2' = h(u_1). \tag{11}$$

An energy function of the system (11) has the form

$$E(u_1, u_2) = \frac{u_2^2}{2} + H(u_1), \quad u_1, u_2 \in \mathbb{R}.$$

Lemma 1 *Let* (3), (4), (6), (7) *be satisfied. The function* H *has following properties*

1. $H(x) > 0$ *for* $x \in [x_1, x_2) \cup (x_2, x_3]$,
2. H *is decreasing on* (x_1, x_2) *and increasing on* (x_2, x_3),
3. *there exists unique* $\bar{B} \in (x_1, x_2)$ *such that*

$$H(\bar{B}) = H(x_3),$$

4. *if* (8) *is satisfied, then*

$$\begin{cases} H(x) = H(x_1) & \text{for} \quad x \le x_1, \\ H(x) = H(x_3) - (x - x_3)^2/2 & \text{for} \quad x \ge x_3. \end{cases}$$

Proof. The first two properties follow from the definition of H and (6). The third property is a consequence of (6) and (7). The fourth one can be obtained by simple computation. □

Lemma 2 *Let* (3), (4), (6) – (8) *be satisfied. Let* (v_1, v_2) *be a solution of problem* (11),

$$u_1(0) = B, u_2(0) = 0, \tag{12}$$

where $B \in (x_1, \bar{B})$, $\bar{B}$ *is from Lemma* 1. *Then there exists* $b > 0$ *such that*

$$v_1(b) = x_3$$

and

$$0 < v_2(t) \le \sqrt{2H(x_1)}$$

for $t \in (0, b]$.

Proof. It is well known that the level sets of the energy function E consist of the orbits of the second–order conservative system (11), in particular, the orbit $\gamma((B, 0))$ of system (11) passing the point $(B, 0)$ in the phase plane is a subset of

$$\{(u_1, u_2) \in \mathbb{R}^2 :\ u_2 = \pm\sqrt{2(H(B) - H(u_1))} \wedge H(u_1) \le H(B)\}.$$

From the properties of the function H, we can see that this set can be expressed in the form

$$\{(u_1, u_2) \in \mathbb{R}^2 : u_2 = \pm\sqrt{2(H(B) - H(u_1))} \wedge u_1 \ge B\}.$$

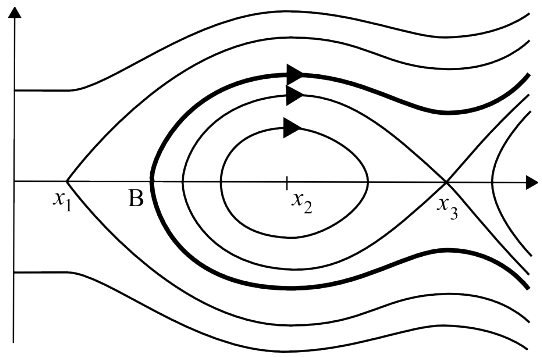

Figure 1: The escape orbit.

This set contains no equilibrium point and hence it is the orbit $\gamma((B,0))$. Consider the function

$$u_2 = \Phi(u_1) = \sqrt{2(H(B) - H(u_1))} \qquad \text{for } u_1 \geq B.$$

Simple computation yields

$$0 < \Phi(u_1) \leq \Phi(x_2) \qquad \text{for } u_1 \in (B, x_3].$$

Therefore the orbit $\gamma((B,0))$ belonging to the solution (v_1, v_2) of (11), (12) has the form on the Figure 1. The direction of the flow on $\gamma((B,0))$ is determined by the equalities

$$v_1'(0) = v_2(0) = 0 \qquad \text{and} \qquad v_2'(0) = h(v_1(0)) > 0,$$

see Figure 1.

Hence there exists $b > 0$ such that

$$(v_1(b), v_2(b)) = (x_3, \Phi(x_3)) = (x_3, \sqrt{2(H(B) - H(x_3))})$$

and

$$0 < v_2(t) \leq \Phi(x_2) \leq \sqrt{2H(x_1)} \qquad \text{for } t \in (0, b].$$

The proof is complete. □

As an immediate consequence of Lemma 2 we get Lemma 3.

Lemma 3 *(On escape solution) Let* (3), (4), (6) – (8) *be satisfied and* u *be a solution of problem* (9), (10) *with* $B \in (x_1, \bar{B})$. *Then there exists* $b > 0$ *such that*

$$u(b) = x_3, \quad u'(t) > 0 \qquad \textit{for } t \in (0, b]. \tag{13}$$

Choose $\epsilon > 0$ and consider the perturbed equation

$$u'' = h(u) - \epsilon. \tag{14}$$

Lemma 4 *(On the perturbed equation) Let* (3) – (8) *be satisfied. There exists* $\epsilon_0 > 0$ *such that for* $\epsilon \in (0, \epsilon_0)$ *the function* $h - \epsilon$ *has roots* $x_i(\epsilon)$ *for* $i = 1, 2, 3$, *such that*

$$h - \epsilon \text{ is lipschitzian on } [x_1(\epsilon), x_3(\epsilon)], \tag{15}$$

$$h(x_i(\epsilon)) = \epsilon \qquad \text{for } i = 1, 2, 3, \tag{16}$$

$$\left.\begin{array}{l} \text{there exists } \delta > 0 \text{ such that } h - \epsilon \in C^1((x_2(\epsilon) - \delta, x_2(\epsilon))) \\ \text{and } \lim\limits_{x \to x_2(\epsilon)^-} (h - \epsilon)'(x) = (h - \epsilon)'_-(x_2(\epsilon)) < 0, \end{array}\right\} \tag{17}$$

$$(x - x_2(\epsilon))(h(x) - \epsilon) < 0 \qquad \text{for } x \in (x_1(\epsilon), x_3(\epsilon)) \setminus \{x_2(\epsilon)\}, \tag{18}$$

$$H_\epsilon(x_1(\epsilon)) > H_\epsilon(x_3(\epsilon)), \tag{19}$$

where

$$H_\epsilon(x) = -\int_{x_2(\epsilon)}^{x} (h(z) - \epsilon)\,\mathrm{d}z$$

for $x \in \mathbb{R}$.

Proof. From (4), (5), (6) and the Implicit function theorem, it follows that there exists $\bar{\epsilon}_0 > 0$ and a continuous function $x_2 : [0, \bar{\epsilon}_0) \to (x_1, x_2]$ such that

$$h(x_2(\epsilon)) = \epsilon \text{ for } \epsilon \in [0, \bar{\epsilon}_0),\ x_2(\epsilon) \text{ is decreasing},\ x_2(0) = x_2. \tag{20}$$

We define

$$x_1(\epsilon) = \sup\{x \in [x_1, x_2(\bar{\epsilon}_0)] : h(x) \leq \epsilon\} \tag{21}$$

for $\epsilon \in (0, \bar{\epsilon}_0)$. From the continuity of the function h, the definition of $x_1(\epsilon)$ and the supremum it follows that

$$x_1(\epsilon) \in [x_1, x_2(\bar{\epsilon}_0)) \qquad \text{for } \epsilon \in (0, \bar{\epsilon}_0)$$

and

$$h(x_1(\epsilon)) = \epsilon, \quad \epsilon \in (0, \bar{\epsilon}_0). \tag{22}$$

We will prove that

$$\lim_{\epsilon \to 0+} x_1(\epsilon) = x_1, \tag{23}$$

by contradiction. If (23) does not hold, then there exists a decreasing sequence $\{\epsilon_n\}$, $\epsilon_n \to 0$ such that $x_1(\epsilon_n) \to \bar{x}_1 \in (x_1, x_2(\bar{\epsilon}_0)]$ as $n \to \infty$. From (20) it follows that

$$h(x_1(\epsilon_n)) = \epsilon_n \to 0.$$

From the continuity of h and (4), (6), we get a contradiction.
We put

$$x_3(\epsilon) = x_3 + \epsilon, \quad \epsilon \in (0, \bar{\epsilon}_0).$$

Then, for $\epsilon \in (0, \bar{\epsilon}_0)$, relations (15) – (18) are satisfied. For $\epsilon \in (0, \bar{\epsilon}_0)$, $x \in [x_1, x_1 + \epsilon_0]$ it is valid

$$H_\epsilon(x) = -\int_{x_2(\epsilon)}^{x} (h(z) - \epsilon)\, \mathrm{d}z = -\int_{x_2(\epsilon)}^{x} h(z)\, \mathrm{d}z + \epsilon(x - x_2(\epsilon))$$
$$= H(x) + \int_{x_2}^{x_2(\epsilon)} h(z)\, \mathrm{d}z + \epsilon(x - x_2(\epsilon)).$$

Then

$$|H_\epsilon(x) - H(x)| \le |x_2(\epsilon) - x_2| \max\{|h(z)| : z \in [x_1, x_3 + \bar{\epsilon}_0]\} + \epsilon |x_3 + \bar{\epsilon}_0 - x_1|$$

for $\epsilon \in (0, \bar{\epsilon}_0)$ and $x \in [x_1, x_3 + \bar{\epsilon}_0]$. Since the terms on the right–hand side of the inequality converges to zero as $\epsilon \to 0+$ independently on x, we can write

$$H_\epsilon(x) \rightrightarrows H(x) \qquad \text{on } [x_1, x_3 + \bar{\epsilon}_0] \text{ as } \epsilon \to 0+ .$$

From this fact and the relations

$$\lim_{\epsilon \to 0+} x_i(\epsilon) = x_i \qquad \text{for } i = 1, 3,$$

it follows that

$$\lim_{\epsilon \to 0+} H_\epsilon(x_i(\epsilon)) = H(x_i) \qquad \text{for } i = 1, 3.$$

From these facts and (7) it follows that there exists $\epsilon_0 \in (0, \bar{\epsilon}_0)$ such that (19) is valid for $\epsilon \in (0, \epsilon_0)$, together with (15) – (18), as well. □

Lemma 5 *Let* (3) – (8) *be satisfied. Let* $\epsilon \in (0, \epsilon_0)$, *where* ϵ_0 *is from Lemma* 4. *Then there exist* $B \in (x_1, x_2)$ *and* $b > 0$ *such that the corresponding solution* u *of problem* (14), (10) *satisfies* (13) *and*

$$0 \le u'(t) \le \sqrt{2H(x_1)} \qquad \text{for } t \in [0, b]. \tag{24}$$

Proof. Let ϵ_0 be from Lemma 4 and $\epsilon \in (0, \epsilon_0)$ be arbitrary. Then relations (15) – (19) hold. From Lemma 1 (with H_ϵ in place of H) it follows that there exists the unique $\bar{B}(\epsilon) \in (x_1(\epsilon), x_2(\epsilon))$ such that $H_\epsilon(\bar{B}(\epsilon)) = H_\epsilon(x_3(\epsilon))$. Let $B(\epsilon) \in (x_1(\epsilon), \bar{B}(\epsilon))$ and u be the solution of problem (14), (10) with $B = B(\epsilon)$. According to Lemma 3 there exists $b(\epsilon) > 0$ such that

$$u(b(\epsilon)) = x_3(\epsilon) \quad \text{and} \quad u' > 0 \qquad \text{on } (0, b(\epsilon)]. \tag{25}$$

In particular, $u(t) \in (x_1(\epsilon), x_3(\epsilon)]$ for every $t \in [0, b(\epsilon)]$. Multiplying the perturbed equation (14) by u' and integrating it over interval $(0, t)$ for $t \in [0, b(\epsilon)]$, we get

$$\frac{u'^2(t)}{2} - \frac{u'^2(0)}{2} = -H_\epsilon(u(t)) + H_\epsilon(u(0)),$$

that is

$$u'(t) = \sqrt{2(H_\epsilon(B(\epsilon)) - H_\epsilon(u(t)))}$$

for $t \in [0, b(\epsilon)]$. Since $H_\epsilon(x_1(\epsilon))$ is the maximum of the function H_ϵ in $[x_1(\epsilon), x_3(\epsilon)]$ and H_ϵ is nonnegative, we get

$$u'(t) \le \sqrt{2H_\epsilon(x_1(\epsilon))}$$

for $t \in [0, b(\epsilon)]$. In view of the fact

$$H_\epsilon(x_1(\epsilon)) = \int_{x_1(\epsilon)}^{x_2(\epsilon)} (h(z) - \epsilon)\,\mathrm{d}z \le \int_{x_1(\epsilon)}^{x_2(\epsilon)} h(z)\,\mathrm{d}z \le \int_{x_1}^{x_2} h(z)\,\mathrm{d}z = H(x_1)$$

and (25), it follows that

$$0 \le u'(t) \le \sqrt{2H(x_1)}$$

for $t \in [0, b(\epsilon)]$. By $B(\epsilon) < x_3 < x_3(\epsilon)$ and (25), there exists $b \in (0, b(\epsilon))$ such that (13) and (24) are valid. □

3 Nonautonomous equation

Let us consider equation (1), where

$$f \text{ is locally lipschitzian on } \mathbb{R}, \tag{26}$$

$$\text{there exist } L_0 < 0 < L \text{ such that } f(L_0) = f(0) = f(L) = 0, \tag{27}$$

$$\left.\begin{array}{l} \text{there exists } \delta > 0 \text{ such that } f \in C^1((-\delta, 0)) \\ \text{and } \lim\limits_{x \to 0^-} f'(x) = f'_-(0) < 0, \end{array}\right\} \tag{28}$$

$$xf(x) < 0 \qquad \text{for } x \in (L_0, L) \setminus \{0\}, \tag{29}$$

$$F(L_0) > F(L), \tag{30}$$

where

$$F(x) = -\int_0^x f(z)\,\mathrm{d}z, \qquad x \in \mathbb{R}.$$

Further we assume that

$$p \in C^2((0, \infty)) \cap C([0, \infty)), \tag{31}$$

$$p(0) = 0, \quad p'(t) > 0 \quad \text{for } t \in (0, \infty), \tag{32}$$

$$\lim_{t \to \infty} \frac{p'(t)}{p(t)} = 0, \tag{33}$$

$$\lim_{t \to \infty} \frac{p''(t)}{p(t)} = 0. \tag{34}$$

Moreover, in some lemmas, we will assume that

$$f(x) = 0 \qquad \text{for } x \in (-\infty, L_0] \cup [L, \infty). \tag{35}$$

If (35) is valid, then

$$\begin{cases} F(x) = F(L_0) & \text{for} \quad x \le L_0, \\ F(x) = F(L) & \text{for} \quad x \ge L. \end{cases}$$

The following classical result for non–singular initial problems will be useful in the proofs.

Lemma 6 *Let* (26), (31), (32), (35) *be satisfied,* $a > 0$*,* B_0*,* $B_1 \in \mathbb{R}$*. Then there exists the unique solution on* $[a, \infty)$ *of the initial value problem* (1)*,*

$$u(a) = B_0, \; u'(a) = B_1. \tag{36}$$

Proof. It is well known that the problem (1), (36) is equivalent to the IVP

$$\begin{cases} u_1' = \frac{u_2}{p(t)}, \; u_2' = p(t)f(u_1), \\ u_1(a) = B_0, \; u_2(a) = B_1. \end{cases}$$

From (26), (31), (32) it follows the unique solvability of this problem and of the problem (1), (36), as well. □

We will study the singular initial value problem (1),

$$u(0) = B, \quad u'(0) = 0 \tag{37}$$

with $B \in (L_0, 0)$.

Definition 7 Let $[a, c) \subset [0, \infty)$. A function $u \in C^1([a, c)) \cap C^2((a, c))$ satisfying equation (1) on $[a, c)$ and fulfilling conditions (37) is a solution of problem (1), (37) on $[a, c)$.

First we state several lemmas.

Lemma 8 *Let us assume that* (26) − (29), (31) − (34) *be satisfied. Let* u *be a solution of the initial value problem* (1)*,*

$$u(a) = B, \quad u'(a) = 0 \tag{38}$$

on $[a, \infty)$*, where* $a \ge 0$ *and* $B \in (L_0, 0)$*. Then there exists* $\theta > a$ *such that*

$$u(\theta) = 0 \quad \textit{and} \quad u'(t) > 0 \textit{ for } t \in (a, \theta]. \tag{39}$$

Moreover, for every $b > \theta$ *satisfying*

$$u(b) \in (0, L) \quad \textit{and} \quad u'(t) > 0 \textit{ for } t \in [\theta, b), \tag{40}$$

there exist $\alpha \in (a, \theta)$*,* $\beta \in (\theta, b)$ *such that*

$$p^2(b)u'^2(b) = 2[p^2(\alpha)F(B) - p^2(\beta)F(u(b))]. \tag{41}$$

Proof. Let u be a solution of problem (1), (38), where $a \geq 0$ and $B \in (L_0, 0)$. From (1) and (29) it follows that there exists $\xi > a$ such that $u(t) \in (L_0, 0)$ and $u'(t) > 0$ for $t \in (a, \xi)$. Let us assume that $\xi = \infty$. Then there exists $l \in (B, 0]$ such that

$$\lim_{t \to \infty} u(t) = l. \tag{42}$$

From (1) and (38), it follows that

$$\frac{u'^2(t)}{2} + \int_a^t \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s = F(B) - F(u(t)). \tag{43}$$

Since the right–hand side of the equation (43) has a finite nonnegative limit $F(B) - F(l)$ as $t \to \infty$ and the function $\int_a^t \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s$ is positive and monotone, it follows that there exists finite nonnegative limit $\lim\limits_{t \to \infty} u'^2(t)/2$. Since $u' > 0$ on $(0, \infty)$, there exists nonnegative $\lim\limits_{t \to \infty} u'(t)$. If $\lim\limits_{t \to \infty} u'(t) > 0$, then $\lim\limits_{t \to \infty} u(t) = \infty$, which contradicts (42). Consequently,

$$\lim_{t \to \infty} u'(t) = 0. \tag{44}$$

From (1) it follows that

$$u''(t) = -\frac{p'(t)}{p(t)} u'(t) + f(u(t)) \quad \text{for } t \in (0, \infty).$$

This, together with (42), (44), (26) and (33) implies

$$\lim_{t \to \infty} u''(t) = f(l).$$

Using (44), (27) and (29) we can check that $l = 0$.
We define a function

$$v(t) = \sqrt{p(t)}\, u(t) \quad \text{for } t \in [0, \infty).$$

By virtue of (31) and (32) we see that v is well defined, negative and there exist finite derivatives

$$v'(t) = \frac{p'(t) u(t)}{2\sqrt{p(t)}} + \sqrt{p(t)}\, u'(t)$$

and

$$v''(t) = v(t) \left[\frac{1}{2} \frac{p''(t)}{p(t)} - \frac{1}{4} \left(\frac{p'(t)}{p(t)} \right)^2 + \frac{f(u(t))}{u(t)} \right]$$

for $t > a$. In view of (33), (34), from the fact that $\lim_{t \to \infty} u(t) = 0$, u is negative and from (28), it follows that there exist $\omega > 0$ and $R > 0$ such that

$$\frac{1}{2} \frac{p''(t)}{p(t)} - \frac{1}{4} \left(\frac{p'(t)}{p(t)} \right)^2 + \frac{f(u(t))}{u(t)} < -\omega \quad \text{for } t \geq R.$$

Then

$$v''(t) > -\omega v(t) > 0 \quad \text{for } t \geq R. \tag{45}$$

Thus, v' is increasing on $[R, \infty)$ and has the limit

$$\lim_{t \to \infty} v'(t) = V.$$

If $V > 0$, then $\lim_{t\to\infty} v(t) = +\infty$, which contradicts the negativity of v. If $V \leq 0$, then $v'(t) < 0$ for every $t \in (R, \infty)$ and therefore

$$0 > v(R) \geq v(t) \quad \text{for } t \geq R.$$

In view of (45) we can see that

$$0 < -\omega v(R) \leq -\omega v(t) < v''(t) \qquad \text{for } t \geq R.$$

We get $\lim_{t\to\infty} v'(t) = \infty$, which implies $\lim_{t\to\infty} v(t) = \infty$, again. These contradictions imply the existence of $\theta > a$ such that $u(\theta) = 0$ and $u'(t) > 0$ for $t \in (a, \theta)$. Let us assume that $u'(\theta) = 0$. Since $u(\theta) = 0$ we get from Lemma 6, (1) and (27) that $u(t) = 0$ for $t \in (0, \infty)$, which is a contradiction. Thus (39) holds.

Let us consider $b > \theta$ such that (40) is satisfied. Multiplying equation (1) by pu' and integrating it over (a, θ) and (θ, b) we get

$$(pu')^2(\theta) - (pu')^2(a) = 2\int_a^\theta p^2(s) f(u(s)) u'(s) \, \mathrm{d}s,$$

$$(pu')^2(b) - (pu')^2(\theta) = 2\int_\theta^b p^2(s) f(u(s)) u'(s) \, \mathrm{d}s.$$

Using the Mean value theorem, we get $\alpha \in (a, \theta)$ and $\beta \in (\theta, b)$ such that

$$(pu')^2(\theta) = 2p^2(\alpha) \int_a^\theta f(u(s)) u'(s) \, \mathrm{d}s,$$

$$(pu')^2(b) - (pu')^2(\theta) = 2p^2(\beta) \int_\theta^b f(u(s)) u'(s) \, \mathrm{d}s$$

and substituing $\tau = u(s)$ we get

$$(pu')^2(\theta) = 2p^2(\alpha)(F(u(a)) - F(u(\theta))),$$

$$(pu')^2(b) - (pu')^2(\theta) = 2p^2(\beta)(F(u(\theta)) - F(u(b))).$$

From these two equations, using the fact that $F(u(\theta)) = 0$, we have (41). □

Lemma 9 *Let us assume that* (26) – (34) *be satisfied. Let* u *be a solution of the initial value problem* (1), (37) *on* $[0, \infty)$ *and let* $b > 0$, $\bar{L} \in (0, L)$ *be such that*

$$u(b) = \bar{L}, \quad u'(b) = 0. \tag{46}$$

Then there exists $\theta > b$ such that

$$u(\theta) = 0 \quad \textit{and} \quad u'(t) < 0 \qquad \textit{for } t \in (b, \theta]. \tag{47}$$

Moreover, for every $c > \theta$ satisfying

$$u(c) \in (L_0, 0) \quad \textit{and} \quad u'(t) < 0 \quad \textit{for } t \in (\theta, c), \tag{48}$$

there exist $\alpha \in (b, \theta)$ and $\beta \in (\theta, c)$ such that

$$(pu')^2(c) = 2[p^2(\alpha)F(\bar{L}) - p^2(\beta)F(u(c))]. \tag{49}$$

Proof. First of all we will prove the existence of θ satisfying (47). By (29) and (46) there exists $b_1 > b$ such that $f(u(t)) < 0$ for $t \in (b, b_1)$. Thus $p(t)u'(t)$ and $u'(t)$ are decreasing and negative on (b, b_1) and $u(t)$ is decreasing and positive on (b, b_1). Assume that $\theta > b$ satisfying (47) does not exist. Then $b_1 = \infty$ and $\lim_{t\to\infty} u(t) \in [0, \bar{L})$. On the other hand, $\lim_{t\to\infty} u'(t) < 0$, which gives $\lim_{t\to\infty} u(t) = -\infty$.
Let us consider $c > \theta$ such that (48) is satisfied. Multiplying equation (1) by pu' and integrating it over (b, θ) and (θ, c) we get $\alpha \in (b, \theta)$ and $\beta \in (\theta, c)$ such that

$$(pu')^2(\theta) - (pu')^2(b) = 2p^2(\alpha)(F(u(b)) - F(u(\theta))),$$
$$(pu')^2(c) - (pu')^2(\theta) = 2p^2(\beta)(F(u(\theta)) - F(u(c))).$$

From these two equations we get (49). □

Lemma 10 *(On three types of solutions) Let* (26) – (35) *be satisfied, $B \in (L_0, 0)$. Then there exists a unique solution u of problem* (1), (37) *and it is defined on* $[0, \infty)$*. There are just three types of solutions:*

- an escape solution *if there exists $b > 0$ such that $u(b) = L$ and $u' > 0$ on* $(0, b]$*,*
- a homoclinic solution *if $u' > 0$ on* $(0, \infty)$ *and* $\lim_{t\to\infty} u(t) = L$*,*
- an oscillatory solution *if u has infinitely many roots and $u(t) \in (B, L)$ for* $t \in (0, \infty)$*.*

Moreover, for $t \in (0, \infty)$ it is valid

$$|u'(t)| \le \max_{L_0 \le x \le L} |f(x)| \cdot t, \quad |u(t)| \le L_0 + \max_{L_0 \le x \le L} |f(x)| \cdot \frac{t^2}{2}.$$

Proof. Step 1. (On the existence of a solution on some neighbourhood of $t = 0$) From (26) and (35) it follows that there exists $\bar{L} > 0$ such that

$$|f(x_1) - f(x_2)| \le \bar{L}|x_1 - x_2| \tag{50}$$

for $x_1,\, x_2 \in \mathbb{R}$. Let us take $\eta > 0$ such that

$$\frac{\bar{L}\eta^2}{2} < 1. \tag{51}$$

Consider the Banach space $C([0,\eta])$ with the maximum norm $\|\cdot\|_\infty$ and using (32), define an operator $\mathcal{F} : C([0,\eta]) \to C([0,\eta])$

$$(\mathcal{F}u)(t) = B + \int_0^t \frac{1}{p(s)} \int_0^s p(\tau) f(u(\tau))\, \mathrm{d}\tau\, \mathrm{d}s.$$

From (50), (32) it follows that for $u_1,\, u_2 \in C([0,\eta])$, $t \in [0,\eta]$

$$\begin{aligned} |(\mathcal{F}u_1)(t) - (\mathcal{F}u_2)(t)| &\le \left| \int_0^t \frac{1}{p(s)} \int_0^s p(\tau)(f(u_1(\tau)) - f(u_2(\tau)))\, \mathrm{d}\tau\, \mathrm{d}s \right| \\ &\le \bar{L}\|u_1 - u_2\|_\infty \int_0^t \frac{1}{p(s)} \int_0^s p(\tau)\, \mathrm{d}\tau\, \mathrm{d}s \\ &\le \bar{L}\|u_1 - u_2\|_\infty \int_0^t \int_0^s \mathrm{d}\tau\, \mathrm{d}s \le \frac{\bar{L}\eta^2}{2}\|u_1 - u_2\|_\infty. \end{aligned}$$

Inequality (51) implies that $\mathcal{F}$ is a contraction. From the Banach fixed point theorem it follows that there exists a unique fixed point u of the operator $\mathcal{F}$. Then

$$u(t) = B + \int_0^t \frac{1}{p(s)} \int_0^s p(\tau) f(u(\tau))\, \mathrm{d}\tau\, \mathrm{d}s \quad \text{for } t \in [0,\eta].$$

We have $u(0) = B$ and deriving the equality we get

$$u'(t) = \frac{1}{p(t)} \int_0^t p(s) f(u(s))\, \mathrm{d}s, \quad \text{for } t \in (0,\eta). \tag{52}$$

From (52), (26), (35) and (32) we have

$$|u'(t)| \le \max_{L_0 \le x \le L} |f(x)| \frac{1}{p(t)} \int_0^t p(s)\, \mathrm{d}s \le \max_{L_0 \le x \le L} |f(x)| \cdot t, \quad \text{for } t \in (0,\eta).$$

This fact implies $u'(0) = 0$. Moreover, multiplying equation (52) by $p(t)$ and deriving it we get (1). So, the fixed point u is a solution of problem (1), (37). Analogously, every solution of (1),(37) defined on $[0,\eta]$ is a fixed point of the operator $\mathcal{F}$. We conclude that there exists a unique solution of problem (1), (37).

Step 2. (Global solution) From Lemma 6 it follows, that the solution u can be extended onto every interval, where it is bounded. Lemma 8 gives $\theta > 0$ such that

$$u(\theta) = 0 \quad \text{and} \quad u'(t) > 0 \quad \text{for } (0,\theta]. \tag{53}$$

If u is defined on $[0,\omega)$, where $\omega \in (\theta,\infty]$, then

$$u'(t) = \frac{p(\theta)}{p(t)} u'(\theta) + \frac{1}{p(t)} \int_\theta^t p(s) f(u(s))\, \mathrm{d}s$$

for $t \in (\theta, \omega)$. From (29), (53) and the last equation we get three possibilities:
CASE A. There exists $b > \theta$ such that

$$u(b) = L \qquad \text{and} \qquad u'(t) > 0 \quad \text{for } t \in [\theta, b).$$

CASE B. For $t \in (\theta, \infty)$ it is valid $u(t) \in (0, L)$ and $u'(t) > 0$.
CASE C. There exists $b > \theta$ such that

$$u'(b) = 0, \quad u(b) \in (0, L) \qquad \text{and} \qquad u'(t) > 0 \quad \text{for } t \in (\theta, b). \tag{54}$$

Let us consider CASE A. Since $\tilde{u} \equiv L$ is the solution of the equation (1) and it satisfies $\tilde{u}(b) = L$, $\tilde{u}'(b) = 0$, then from Lemma 6 we get

$$u'(b) > 0.$$

It follows that there exists $\delta > 0$ such that

$$u'(t) > 0 \quad \text{and} \quad u(t) > L \quad \text{for } t \in (b, b + \delta).$$

In view of (35) the solution u satisfies

$$(p(t)u'(t))' = 0 \qquad \text{for } t \in (b, b + \delta)$$

and consequently

$$u'(t) = \frac{p(b)u'(b)}{p(t)} > 0 \quad \text{and} \quad u(t) = L + p(b)u'(b) \int_b^t \frac{\mathrm{d}s}{p(s)},$$

for $t \in (b, b+\delta)$. From (31) and (32) it follows that u can be extended on $[0, \infty)$. This solution is an escape solution.
Let us consider CASE B. The monotonicity of u implies the existence of $\tilde{L} \in (0, L]$ such that

$$\lim_{t \to \infty} u(t) = \tilde{L}. \tag{55}$$

We will prove that $\tilde{L} = L$. Since $f(u(t)) < 0$ for $t > \theta$, from (1) it follows, that pu' is decreasing on (θ, ∞). The inequality $u'(t) > 0$ for $t \in (\theta, \infty)$ implies that $u'' < 0$ and hence u' is decreasing on (θ, ∞). That yields the existence of $\lim_{t\to\infty} u'$. Since u is bounded, necessarily

$$\lim_{t \to \infty} u'(t) = 0.$$

From (1) it follows that

$$u''(t) = -\frac{p'(t)}{p(t)} u'(t) + f(u(t))$$

for $t \in (0, \infty)$. In view of (33) we get

$$\lim_{t \to \infty} u''(t) = f(\tilde{L}).$$

According to (27) and (29) we get $\tilde{L} = L$. This solution satisfies the conditions (2) and so it is a homoclinic solution.
Let us consider Case C. From the second part of Lemma 8 we get $\alpha \in (0, \theta)$ and $\beta \in (\theta, b)$ such that (41) holds. In view of (54) we get

$$F(u(b)) = \left(\frac{p(\alpha)}{p(\beta)}\right)^2 F(B). \tag{56}$$

Using Lemma 9 we get the existence of $\theta_1 > b$ such that $u(\theta_1) = 0$ and $u'(t) < 0$ for $t \in (b, \theta_1]$. Let us suppose that there exists $\bar{b}_1 \in (\theta_1, \infty)$ such that

$$u(\bar{b}_1) = B \qquad \text{and} \qquad u'(t) < 0, \quad \text{for } t \in [\theta_1, \bar{b}_1).$$

Using the second part of Lemma 9, we get $\bar{\alpha}_1 \in (b, \theta_1)$ and $\bar{\beta}_1 \in (\theta_1, \bar{b}_1)$ such that

$$(pu')^2(\bar{b}_1) = 2[p^2(\bar{\alpha}_1)F(u(b)) - p^2(\bar{\beta}_1)F(B)],$$

and together with (56) we obtain

$$\begin{aligned}(pu')^2(\bar{b}_1) &= 2F(B)\left[p^2(\bar{\alpha}_1)\left(\frac{p(\alpha)}{p(\beta)}\right)^2 - p^2(\bar{\beta}_1)\right] \\ &= 2F(B)p^2(\bar{\beta}_1)\left[\left(\frac{p(\bar{\alpha}_1)p(\alpha)}{p(\bar{\beta}_1)p(\beta)}\right)^2 - 1\right] < 0.\end{aligned}$$

This is a contradiction. Hence by Lemma 8 there exists $b_1 > \theta_1$ such that

$$u(b_1) \in (B, 0), \quad u'(b_1) = 0 \quad \text{and} \quad u'(t) < 0 \quad \text{for } t \in (\theta_1, b_1).$$

From the second part of Lemma 9 we get $\alpha_1 \in (b, \theta_1)$ and $\beta_1 \in (\theta_1, b_1)$ such that

$$0 = 2[p^2(\alpha_1)F(u(b)) - p^2(\beta_1)F(u(b_1))].$$

By (56), we get

$$F(u(b_1)) = \left(\frac{p(\alpha_1)}{p(\beta_1)}\right)^2 F(u(b)) = \left(\frac{p(\alpha_1)p(\alpha)}{p(\beta_1)p(\beta)}\right)^2 F(B). \tag{57}$$

Using Lemma 8 we get $\theta_2 > b_1$ such that $u(\theta_2) = 0$ and $u'(t) > 0$ for $t \in (b_1, \theta_2]$. Let us suppose that there exists $\bar{b}_2 \in (\theta_2, \infty)$ such that

$$u(\bar{b}_2) = u(b) \quad \text{and} \quad u'(t) > 0 \quad \text{for } t \in [\theta_2, \bar{b}_2).$$

By virtue of the second part of Lemma 8, we can find $\bar{\alpha}_2 \in (b_1, \theta_2)$ and $\bar{\beta}_2 \in (\theta_2, \bar{b}_2)$ such that

$$(pu')^2(\bar{b}_2) = 2[p^2(\bar{\alpha}_2)F(u(b_1)) - p^2(\bar{\beta}_2)F(u(b))],$$

and together with (57) we obtain

$$(pu')^2(\bar{b}_2) = 2F(u(b))p^2(\bar{\beta}_2)\left[\left(\frac{p(\bar{\alpha}_2)p(\alpha_1)}{p(\bar{\beta}_2)p(\beta_1)}\right)^2 - 1\right] < 0$$

a contradiction. Hence there exists $b_2 > \theta_2$ such that

$$u(b_2) \in (0, u(b_1)), \quad u'(b_2) = 0 \quad \text{and} \quad u'(t) < 0 \quad \text{for } (\theta_2, b_2).$$

Repeating this procedure we get a sequence $\{\theta_n\}_{n=1}^{\infty}$ of roots of the solution u and a sequence $\{b_n\}_{n=1}^{\infty}$ of roots of the derivative u' such that $\{|u(b_n)|\}_{n=1}^{\infty}$ is decreasing. This solution corresponds to an oscillatory solution.
STEP 3. (Estimations) Let u be a solution of problem (1), (37) with $B \in (L_0, 0)$. Then from (1) it follows that

$$u'(t) = \frac{1}{p(t)} \int_0^t p(s) f(u(s))\,\mathrm{d}s, \quad \text{for } t \in (0, \infty). \tag{58}$$

Then, in view of (26) and (35)

$$|u'(t)| \le \max_{L_0 \le x \le L} |f(x)| \cdot \int_0^t \mathrm{d}s = \max_{L_0 \le x \le L} |f(x)| \cdot t \quad \text{for } t \in (0, \infty).$$

Integrating (58) we get

$$|u(t)| \le |u(0)| + \left| \int_0^t \frac{1}{p(s)} \int_0^s p(\tau) f(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s \right| \le B + \max_{L_0 \le x \le L} |f(x)| \cdot \frac{t^2}{2}.$$

The proof is complete. □

Lemma 11 *(On oscillatory solutions) Let* (26) – (34) *be satisfied,* $B \in (L_0, 0)$ *be such that*

$$F(B) < F(L). \tag{59}$$

Then the corresponding solution of problem (1), (37) *is oscillatory.*

Proof. Let u be a solution of problem (1), (37) with $B \in (L_0, 0)$ satisfying (59).
STEP 1. Let us assume that u is an escape solution. Then there exist $b > 0$, $\theta \in (0, b)$ such that

$$u(\theta) = 0, \quad u(b) = L \quad \text{and} \quad u'(t) > 0 \quad \text{for } t \in (0, b].$$

From Lemma 8 we get $\alpha \in (0, \theta)$, $\beta \in (\theta, b)$ such that (41) holds. Then

$$p^2(b) u'^2(b) = 2F(L) p^2(\beta) \left[\left(\frac{p(\alpha)}{p(\beta)} \right)^2 \frac{F(B)}{F(L)} - 1 \right] < 0.$$

This contradicts the fact that $u'(b) > 0$.
STEP 2. Let us assume that u is a homoclinic solution. Let $\theta > 0$ be the root of u and $b > \theta$ be arbitrary. Then, by Lemma 8, there exist $\alpha \in (0, \theta)$, $\beta \in (\theta, b)$ such that (41) holds. From (41), the fact $(pu')^2(b) > 0$ and (32) we get

$$F(B) > \left(\frac{p(\beta)}{p(\alpha)} \right)^2 F(u(b)) > F(u(b)).$$

Letting $b \to \infty$ we get $F(B) \geq F(L)$, which contradicts (59). $\square$

Actually, the homoclinic solution is the desired strictly increasing solution of the problem (1), (2). In order to prove the existence of such solution we need the lower and upper functions method for the singular mixed problem

$$(p(t)u')' = p(t)f(u), \qquad u'(a) = 0,\ u(b) = L, \tag{60}$$

where $a,\, b \in \mathbb{R}$, $a \geq 0$, $b > a$.

Definition 12 A function $\sigma \in C([a,b])$ is called a lower function of problem (60), if there exists a finite set $\Sigma \subset (a,b)$ such that $\sigma \in C^2((a,b] \setminus \Sigma)$, $\sigma'(\tau^+)$, $\sigma'(\tau^-) \in \mathbb{R}$ for $\tau \in \Sigma$,

$$(p(t)\sigma'(t))' \geq p(t)f(\sigma(t)) \qquad \text{for } t \in (a,b] \setminus \Sigma,$$

$$\sigma'(a^+) \geq 0,\ \sigma(b) \leq L,\ \sigma'(\tau^-) < \sigma'(\tau^+) \quad \text{for } \tau \in \Sigma.$$

If all inequalities are reversed, then σ is called an upper function of problem (60).

Note that $\sigma'(a^+)$ need not be bounded if $a = 0$.

Theorem 13 *Let p satisfy* (31), (32), *$f \in C(\mathbb{R})$, σ_1 and σ_2 be a lower function and an upper function of problem* (60) *and let $\sigma_1(t) \leq \sigma_2(t)$ for $t \in [a,b]$. Then problem* (60) *has a solution $u \in C^1([a,b]) \cap C^2((a,b])$ such that $\sigma_1(t) \leq u(t) \leq \sigma_2(t)$ for $t \in [a,b]$.*

Proof. See [8] Theorem 2.3 for $a = 0$. For $a > 0$ problem (60) is regular and therefore we can use simplified form of the proof in [8].

The next assertion is based on Lemma 4 and Theorem 13.

Lemma 14 *(On escape solutions) Let* (26) – (35) *be satisfied. There exist $B_* \in (L_0, 0)$ and $c_* \in (0, \infty)$ such that a solution u_* of problem* (1), (37) *with $B = B_*$ satisfies the condition*

$$u_*(c_*) = L, \qquad u_*'(t) > 0 \quad \textit{on } (0, c_*].$$

Proof. Let us put

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for} \quad x \leq L, \\ x - L & \text{for} \quad x \geq L. \end{cases} \tag{61}$$

Let $\epsilon_0 \in \mathbb{R}$ be from Lemma 4 for L_0, 0, L, $\tilde{f}$, $\tilde{F}$ in place of x_1, x_2, x_3, h, H, respectively. Here, $\tilde{F}(x) = -\int_0^x \tilde{f}(z)\,\mathrm{d}z$, $x \in \mathbb{R}$. The assumptions of Lemma 4 are satisfied due to (26) – (30), (61). Consider the perturbed equation

$$u'' = \tilde{f}(u) - \epsilon \tag{62}$$

with $\epsilon \in (0, \epsilon_0)$. From Lemma 5 it follows that there exists $B_L \in (L_0, 0)$ such that for the corresponding solution u_L of problem (62), (37) with $B = B_L$, there exists $b > 0$ such that $u_L(b) = L$ and

$$0 < u_L'(t) \le \sqrt{2\tilde{F}(L_0)} \quad \text{for } t \in [0, b]. \tag{63}$$

From (33) it follows that there exists $a > 0$ such that

$$\frac{p'(t)}{p(t)} < \frac{\epsilon}{\sqrt{2\tilde{F}(L_0)}} \qquad \text{for } t > a.$$

Put $v(t) = u_L(t - a)$ for $t \in [a, a+b]$. Then v satisfies equation (62) on $[a, a+b]$ and fulfils the initial conditions

$$v(a) = B_L, \quad v'(a) = 0.$$

Moreover, $v(a+b) = L$, $\tilde{f}(v(t)) = f(v(t))$ and

$$0 < \frac{p'(t)}{p(t)} v'(t) < \frac{\epsilon}{\sqrt{2F(L_0)}} \sqrt{2F(L_0)} = \epsilon$$

for $t \in [a, a+b]$. Therefore

$$v''(t) = f(v(t)) - \epsilon < f(v(t)) - \frac{p'(t)}{p(t)} v'(t)$$

for $t \in (a, a+b]$. We can see that v is an upper function of the problem

$$u'' + \frac{p'(t)}{p(t)} u' = f(u), \qquad u'(a) = 0, \; u(a+b) = L. \tag{64}$$

Since L_0 is a lower function of problem (64), by Theorem 13 and Lemma 6 there exists a solution u_0 of (64) such that

$$L_0 < u_0(t) \le v(t) \qquad \text{for } t \in [a, a+b]. \tag{65}$$

By (63), (64), (65) we have $v'(a+b) > 0$, $u_0(a+b) = v(a+b)$ and $u_0(t) \le v(t)$ for $t \in [a, a+b]$. Therefore

$$u_0'(a+b) > 0. \tag{66}$$

Since $u_0''(a) = f(u_0(a)) > 0$ there exists a minimal $a_0 \in [0, a)$ such that $u_0'(t) < 0$ for $t \in (a_0, a)$ and $u_0(t) < 0$ for $t \in (a_0, a]$. There are two possibilities.
(i) $a_0 > 0$, $u_0(a_0) = 0$,
(ii) $a_0 = 0$, $u_0(t) \le 0$ for $t \in [0, a]$.
Assume that (i) holds. Then we put

$$\beta(t) = \begin{cases} 0 & \text{for} \quad t \in [0, a_0], \\ u_0(t) & \text{for} \quad t \in (a_0, a+b]. \end{cases}$$

Assume that (ii) holds. Then $u_0''(t) > 0$ for $t \in [0, a]$ and

$$\lim_{t\to 0+} u_0'(t) < 0$$

and we put

$$\beta(t) = u_0(t) \quad \text{for } t \in [0, a+b].$$

Denote $c_* = a+b$. In both cases (i) and (ii) the function β is an upper function of the problem

$$u'' + \frac{p'(t)}{p(t)}u' = f(u), \qquad u'(0) = 0, \quad u(c_*) = L. \tag{67}$$

Since the constant L_0 is a lower function of problem (67), then by Theorem 13 and Lemma 6 there exists a solution u_* of the problem (67) such that

$$L_0 < u_*(t) \le \beta(t) \qquad \text{for } t \in [0, c_*]. \tag{68}$$

We put $B_* = u_*(0)$. Then u_* is a solution of (1), (37) with $B = B_*$. Finally, by (64) and (66) we have

$$\beta(c_*) = L, \quad \beta'(c_*) > 0.$$

This, together with (68) gives $u_*'(c_*) > 0$. Hence by Lemma 10, $u_*'(t) > 0$ for $t \in (0, c_*]$. □

Theorem 15 *(On homoclinic solutions) Let* (26) – (34) *be satisfied. Then there exists at least one strictly increasing solution of problem* (1), (2).

Proof. First, we will assume that (35) is satisfied. Let us define

$$\mathcal{M} = \{B_0 \in (L_0, 0) : \text{ each solution of } (1), (37) \text{ with } B \in [B_0, 0) \text{ is oscillatory}\},$$

and $\tilde{B} = \inf \mathcal{M}$. Lemma 11 guarantees that $\mathcal{M} \ne \emptyset$ and from Lemma 14 it follows that $\tilde{B} > L_0$. We will prove that there exists $B_{\text{hom}} \in (L_0, \tilde{B}]$ such that the corresponding solution of the problem (1), (37) with $B = B_{\text{hom}}$ is a homoclinic solution. Assume that B_{hom} does not exist.

Case A. Let $\tilde{u}$ be an oscillatory solution of (1), (37) with $B = \tilde{B}$. Then, according to the definition of $\tilde{B}$, we can find a sequence $\{B_n\} \subset (L_0, \tilde{B})$ such that $\lim_{n\to\infty} B_n = \tilde{B}$ and the corresponding solutions u_n of (1), (37) with $B = B_n$ are escape solutions. Let θ_1 be the second zero of $\tilde{u}$, that is, θ_1 fulfils

$$\tilde{u}(\theta_1) = 0, \quad \tilde{u}'(\theta_1) < 0.$$

From Lemma 10 we can see that

$$|u_n(t)| \le L_0 + \frac{\theta_1^2}{2} \max_{L_0 \le x \le L} |f(x)|, \quad |u_n'(t)| \le \theta_1 \cdot \max_{L_0 \le x \le L} |f(x)|$$

for $t \in [0, \theta_1]$, $n \in \mathbb{N}$. Hence the sequence $\{u_n\}$ is bounded and equicontinuous on $[0, \theta_1]$. Therefore we can choose a subsequence $\{u_m\}$, which is uniformly convergent on $[0, \theta_1]$ to a function $v \in C([0, \theta_1])$. Obviously,

$$u_m(t) = B_m + \int_0^t \frac{1}{p(s)} \int_0^s p(\tau) f(u_m(\tau))\, \mathrm{d}\tau\, \mathrm{d}s$$

for $t \in [0, \theta_1]$, $m \in \mathbb{N}$, and consequently

$$v(t) = \tilde{B} + \int_0^t \frac{1}{p(s)} \int_0^s p(\tau) f(v(\tau))\, \mathrm{d}\tau\, \mathrm{d}s$$

for $t \in [0, \theta_1]$. We can check that v is a solution of problem (1), (37) and therefore

$$v = \tilde{u} \qquad \text{on } [0, \theta_1].$$

Since u_m are increasing, it follows that v is nondecreasing on $[0, \theta_1]$. This contradicts the fact that $v'(\theta_1) < 0$.

Case B. Let $\tilde{u}$ be an escape solution of (1), (37) with $B = \tilde{B}$. Then there exists $b > 0$ such that

$$\tilde{u}(b) = L, \quad \tilde{u}'(t) > 0 \quad \text{for } t \in (0, \infty). \tag{69}$$

From the definition of $\tilde{B}$ we get a sequence $\{B_n\} \subset (\tilde{B}, 0)$ such that $\lim_{n\to\infty} B_n = \tilde{B}$ and the corresponding solutions u_n of (1), (37), with $B = B_n$, are oscillatory. Therefore

$$L_0 \le u_n(t) \le L, \qquad |u_n'(t)| \le t \cdot \max_{L_0 \le x \le L} |f(x)| \quad \text{for } t \in [0, \infty), n \in \mathbb{N},$$

and there exist $b_n > 0$ such that $u_n(b_n) = L_n \in (0, L)$, $u_n'(b_n) = 0$ for $n \in \mathbb{N}$. Then there exist $\theta_n > b_n$ such that

$$u_n(\theta_n) = 0, \quad u_n'(\theta_n) < 0, \;\; n \in \mathbb{N}. \tag{70}$$

The sequence $\{u_n\}$ is bounded and equicontinuous on every $[0, K] \subset [0, \infty)$ and so we can choose a subsequence $\{u_m\}$ which is uniformly convergent on $[0, K]$ to a function $w \in C([0, K])$. As in Case A we conclude that $w = \tilde{u}$ on $[0, K]$. Now, we have two possibilities.

(i) Let $\lim_{m\to\infty} \theta_m = \theta_0 < \infty$. Put $K = \max\{\theta_0, b\} + 1$. By (70), each u_m is decreasing at a neighbourhood of θ_m and hence $\tilde{u}$ is nonincreasing at θ_0, which contradicts (69).

(ii) Let $\lim_{m\to\infty} \theta_m = \infty$. Put $K = b + 1$. Since $u_m(b+1) < L$ for $m \in \mathbb{N}$, it follows that $\tilde{u}(b+1) \le L$, which is a contradiction.

We have proved that the function $\tilde{u}$ can be neither an escape solution nor an oscillatory solution. Lemma 10 yields that $\tilde{u}$ is a homoclinic solution of problem (1), (2). Since $\tilde{u}(t) \in [L_0, L]$ for $t \in [0, \infty)$ we see that assumption (35) can be omitted. □

Acknowledgements

The authors were supported by the Council of Czech Government MSM 6198959214.

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