Singular Nonlinear Problem for Ordinary Differential Equation of the Second–Order on the Half–Line

Irena Rachunková and Jan Tomeček

Department of Mathematical Analysis and Applications of Mathematics, Faculty of Science, Palacký University,
Tomkova 40, 779 00 Olomouc, Czech Republic,
e-mail: rachunko@inf.upol.cz, tomecek@inf.upol.cz

Abstract. The paper investigates singular nonlinear problems arising in hydrodynamics. In particular, it deals with the problem on the half–line of the form

$$(p(t)u'(t))' = p(t)f(u(t)), \quad u'(0) = 0,\ u(\infty) = L.$$

The existence of a strictly increasing solution (a homoclinic solution) of this problem is proved by the dynamical systems approach and the lower and upper functions method.

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1. INTRODUCTION

In the Cahn–Hillard theory used in hydrodynamics to study the behaviour of nonhomogenous fluids the following system of PDE's was derived

$$\rho_t + \operatorname{div}(\rho v) = 0, \qquad \frac{dv}{dt} + \nabla(\mu(\rho) - \gamma \triangle \rho) = 0$$

with the density ρ and the velocity v of the fluid, μ is its chemical potential, γ is a constant. In the simplest model, this system can be reduced into the boundary value problem for the ODE of the second order (see [5] or [7])

$$(t^k u')' = 4\lambda^2 t^k (u+1)u(u-\xi), \quad u'(0) = 0,\ u(\infty) = \xi,$$

where $k \in \mathbb{N}$, $\xi \in (0,1)$, $\lambda \in (0,\infty)$ are parameters. The function $u(t) \equiv \xi$ is a solution of this problem and it corresponds to the case of homogenous fluid (without bubbles). But only the existence of a strictly increasing solution of this problem and the solution itself has a great physical significance. We refer to [1] and [2], where an equivalent problem was investigated. The numerical treatment was done in papers [5], [7].

Here, for $L > 0$, we study the generalized problem

$$(p(t)u'(t))' = p(t)f(u(t)), \tag{1.1}$$

$$u'(0) = 0, \quad u(\infty) = L. \tag{1.2}$$

2. AUTONOMOUS EQUATION

The investigation of autonomous equations corresponding to (1.1) turned out to be quite useful, because some solutions of the perturbed autonomous equation (2.10) can serve as an upper functions to (1.1).

Let $h:\mathbb{R}\to\mathbb{R}$ and $x_1, x_2, x_3 \in \mathbb{R}$ be such that $x_1 < x_2 < x_3$ and

$$h \text{ is lipschitzian on } [x_1,x_3], \quad h(x_i) = 0 \text{ for } i = 1,2,3, \tag{2.1}$$

$$\left.\begin{array}{l} \text{there exists } \delta > 0 \text{ such that } h \in C^1((x_2-\delta,x_2)) \\ \text{and } \lim\limits_{x\to x_2^-} h'(x) = h'_-(x_2) < 0, \end{array}\right\} \tag{2.2}$$

$$(x-x_2)h(x) < 0 \text{ for } x \in (x_1,x_3)\setminus\{x_2\}, \tag{2.3}$$

$$H(x_1) > H(x_3), \quad H(x) = -\int_{x_2}^{x} h(z)\,\mathrm{d}z \text{ for } x \in \mathbb{R}. \tag{2.4}$$

Moreover we will assume that

$$\left\{\begin{array}{lll} h(x) = 0 & \text{for} & x \le x_1, \\ h(x) = x - x_3 & \text{for} & x \ge x_3. \end{array}\right. \tag{2.5}$$

For $B \in (x_1,x_2)$, let us consider the initial problem

$$u'' = h(u), \tag{2.6}$$

$$u(0) = B,\ u'(0) = 0. \tag{2.7}$$

Equation (2.6) is equivalent with the gradient system

$$u_1' = u_2,\ u_2' = h(u_1). \tag{2.8}$$

An energy function of the system (2.8) has the form

$$E(u_1,u_2) = \frac{u_2^2}{2} + H(u_1), \quad u_1,u_2 \in \mathbb{R}.$$

Lemma 2.1. *Let* (2.1) – (2.4) *be satisfied. The function H has following properties*

1. $H(x) > 0$ *for* $x \in [x_1,x_2)\cup(x_2,x_3]$,
2. H *is decreasing on* (x_1,x_2) *and increasing on* (x_2,x_3),
3. *there exists unique* $\bar{B} \in (x_1,x_2)$ *such that* $H(\bar{B}) = H(x_3)$.

Proof. The first two properties follow from the definition of H and (2.3). The third property is a consequence of (2.3) and (2.4). □

It is well known that the level sets of the energy function E consist of the orbits of the second–order conservative system (2.8). As an immediate consequence of the phase portrait of system (2.8) and of the equivalence of (2.8) and (2.6), we get Lemma 2.2.

Lemma 2.2. *(On escape solution) Let* (2.1) – (2.5) *be satisfied and u be a solution of problem* (2.6), (2.7) *with $B \in (x_1, \bar{B})$. Then there exists $b > 0$ such that*

$$u(b) = x_3, \quad u'(t) > 0 \qquad \textit{for } t \in (0, b]. \tag{2.9}$$

Choose $\varepsilon > 0$ and consider the perturbed equation

$$u'' = h(u) - \varepsilon. \tag{2.10}$$

Lemma 2.3. *(On the perturbed equation) Let* (2.1) – (2.5) *be satisfied. There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ the function $h - \varepsilon$ has roots $x_i(\varepsilon)$ for $i = 1, 2, 3$, such that*

$$h - \varepsilon \textit{ is lipschitzian on } [x_1(\varepsilon), x_3(\varepsilon)], \quad h(x_i(\varepsilon)) = \varepsilon \textit{ for } i = 1, 2, 3, \tag{2.11}$$

$$\left.\begin{array}{l} \textit{there exists } \delta > 0 \textit{ such that } h - \varepsilon \in C^1((x_2(\varepsilon) - \delta, x_2(\varepsilon))) \\ \textit{and } \lim\limits_{x \to x_2(\varepsilon)^-} (h(x) - \varepsilon)' = (h - \varepsilon)'_-(x_2(\varepsilon)) < 0, \end{array}\right\} \tag{2.12}$$

$$(x - x_2(\varepsilon))(h(x) - \varepsilon) < 0 \qquad \textit{for } x \in (x_1(\varepsilon), x_3(\varepsilon)) \setminus \{x_2(\varepsilon)\}, \tag{2.13}$$

$$H_\varepsilon(x_1(\varepsilon)) > H_\varepsilon(x_3(\varepsilon)), \quad H_\varepsilon(x) = -\int_{x_2(\varepsilon)}^{x} (h(z) - \varepsilon)\,\mathrm{d}z \textit{ for } x \in \mathbb{R}. \tag{2.14}$$

Proof. The assertion follows from (2.1) – (2.5) and the Implicit function theorem. □

Lemma 2.4. *Let* (2.1) – (2.5) *be satisfied. Let $\varepsilon \in (0, \varepsilon_0)$, where ε_0 is from Lemma* 2.3. *Then there exist $B \in (x_1, x_2)$ and $b > 0$ such that the corresponding solution u of problem* (2.10), (2.7) *satisfies* (2.9) *and*

$$0 \le u'(t) \le \sqrt{2H(x_1)} \qquad \textit{for } t \in [0, b]. \tag{2.15}$$

Proof. Let ε_0 be from Lemma 2.3 and $\varepsilon \in (0, \varepsilon_0)$ be arbitrary. Then relations (2.11) – (2.14) hold. From Lemma 2.1 (with H_ε in place of H) it follows that there exists the unique $\bar{B}(\varepsilon) \in (x_1(\varepsilon), x_2(\varepsilon))$ such that $H_\varepsilon(\bar{B}(\varepsilon)) = H_\varepsilon(x_3(\varepsilon))$. Let $B(\varepsilon) \in (x_1(\varepsilon), \bar{B}(\varepsilon))$ and u be the solution of problem (2.10), (2.7) with $B = B(\varepsilon)$. According to Lemma 2.2 there exists $b(\varepsilon) > 0$ such that

$$u(b(\varepsilon)) = x_3(\varepsilon) \quad \text{and} \quad u' > 0 \qquad \text{on } (0, b(\varepsilon)]. \tag{2.16}$$

In particular, $u(t) \in (x_1(\varepsilon), x_3(\varepsilon)]$ for every $t \in [0, b(\varepsilon)]$. Multiplying the perturbed equation (2.10) by u' and integrating it over interval $(0, t)$ for $t \in [0, b(\varepsilon)]$, we get $u'^2(t)/2 - u'^2(0)/2 = -H_\varepsilon(u(t)) + H_\varepsilon(u(0))$, that is $u'(t) = \sqrt{2(H_\varepsilon(B(\varepsilon)) - H_\varepsilon(u(t)))}$ for $t \in [0, b(\varepsilon)]$. Since $H_\varepsilon(x_1(\varepsilon))$ is the maximum of the function H_ε in $[x_1(\varepsilon), x_3(\varepsilon)]$ and H_ε is nonnegative, we get $u'(t) \le \sqrt{2H_\varepsilon(x_1(\varepsilon))}$ for $t \in [0, b(\varepsilon)]$. In view of

$$H_\varepsilon(x_1(\varepsilon)) = \int_{x_1(\varepsilon)}^{x_2(\varepsilon)} (h(z) - \varepsilon)\,\mathrm{d}z \le \int_{x_1(\varepsilon)}^{x_2(\varepsilon)} h(z)\,\mathrm{d}z \le \int_{x_1}^{x_2} h(z)\,\mathrm{d}z = H(x_1)$$

and (2.16), it follows that $0 \le u'(t) \le \sqrt{2H(x_1)}$ for $t \in [0, b(\varepsilon)]$. By $B(\varepsilon) < x_3 < x_3(\varepsilon)$ and (2.16), there exists $b \in (0, b(\varepsilon))$ such that (2.9) and (2.15) are valid. □

3. NONAUTONOMOUS EQUATION

Let us consider equation (1.1), where

$$f \text{ is locally lipschitzian on } \mathbb{R}, \tag{3.1}$$

$$\text{there exist } L_0 < 0 < L \text{ such that } f(L_0) = f(0) = f(L) = 0, \tag{3.2}$$

$$\left.\begin{array}{l} \text{there exists } \delta > 0 \text{ such that } f \in C^1((-\delta, 0)) \\ \text{and } \lim\limits_{x \to x_2^-} f'(x) = f'_-(x_2) < 0, \end{array}\right\} \tag{3.3}$$

$$x f(x) < 0 \qquad \text{for } x \in (L_0, L) \setminus \{0\}, \tag{3.4}$$

$$F(L_0) > F(L), \quad F(x) = -\int_0^x f(z)\,\mathrm{d}z \text{ for } x \in \mathbb{R}. \tag{3.5}$$

Further we assume that

$$p \in C^2((0,\infty)) \cap C([0,\infty)), \tag{3.6}$$

$$p(0) = 0, \quad p'(t) > 0 \quad \text{for } t \in (0,\infty), \tag{3.7}$$

$$\lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0, \quad \lim_{t\to\infty} \frac{p''(t)}{p(t)} = 0. \tag{3.8}$$

The following classical result for non–singular initial problems will be useful in the proofs.

Lemma 3.1. *Assume that* $a > 0$, $B_0, B_1 \in \mathbb{R}$. *Let* (3.1), (3.6), (3.7) *and*

$$f(x) = 0 \qquad \textit{for } x \in (-\infty, L_0] \cup [L, \infty) \tag{3.9}$$

be satisfied. Then there exists a unique solution on $[a,\infty)$ *of the initial value problem* (1.1),

$$u(a) = B_0,\ u'(a) = B_1. \tag{3.10}$$

We will study the singular initial value problem (1.1),

$$u(0) = B, \quad u'(0) = 0 \tag{3.11}$$

with $B \in (L_0, 0)$. For this purpose we state several lemmas.

Lemma 3.2. *Let us assume that* (3.1) – (3.4), (3.6) – (3.8) *are satisfied. Let* u *be a solution of the initial value problem* (1.1), (3.11) *on* $[0,\infty)$. *Then there exists* $\theta > 0$ *such that*

$$u(\theta) = 0 \quad \textit{and} \quad u'(t) > 0 \textit{ for } t \in (0,\theta]. \tag{3.12}$$

Moreover, for every $b > \theta$ *satisfying*

$$u(b) \in (0, L) \quad \textit{and} \quad u'(t) > 0 \textit{ for } t \in [\theta, b), \tag{3.13}$$

there exist $\alpha \in (0,\theta)$, $\beta \in (\theta, b)$ *such that*

$$p^2(b)u'^2(b) = 2[p^2(\alpha)F(B) - p^2(\beta)F(u(b))]. \tag{3.14}$$

Proof. Let u be a solution of problem (1.1), (3.11). From (1.1) and (3.4) it follows that there exists $\xi \geq 0$ such that $u(t) \in (L_0, 0)$ and $u'(t) > 0$ for $t \in (0, \xi)$. Let us assume that $\xi = \infty$. Then there exists $l \in (B, 0]$ such that $\lim_{t\to\infty} u(t) = l$. From (1.1) and (3.11), it follows that

$$\frac{u'^2(t)}{2} + \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s = F(B) - F(u(t)). \tag{3.15}$$

Consequently, $\lim_{t\to\infty} u'(t) = 0$. Then (1.1) together with (3.8) implies $\lim_{t\to\infty} u''(t) = f(l)$. By (3.2) and (3.4), $l = 0$.

We define a function $v(t) = \sqrt{p(t)}u(t)$ for $t \in [0, \infty)$. From (3.6) and (3.7) we see that v is well defined and

$$v''(t) = v(t)\left[\frac{1}{2}\frac{p''(t)}{p(t)} - \frac{1}{4}\left(\frac{p'(t)}{p(t)}\right)^2 + \frac{f(u(t))}{u(t)}\right]$$

for $t > 0$. In view of (3.8), from the fact that $\lim_{t\to\infty} u(t) = 0$, u is negative and from (3.3), it follows that there exist $\omega > 0$ and $R > 0$ such that

$$\frac{1}{2}\frac{p''(t)}{p(t)} - \frac{1}{4}\left(\frac{p'(t)}{p(t)}\right)^2 + \frac{f(u(t))}{u(t)} < -\omega, \qquad t \geq R.$$

Then

$$v''(t) > -\omega v(t) \text{ for } t \geq R. \tag{3.16}$$

Thus, v' is increasing on $[R, \infty)$ and has the limit $\lim_{t\to\infty} v'(t) = V$. If $V > 0$, then $\lim_{t\to\infty} v(t) = +\infty$, which contradicts the boundedness of v. If $V \leq 0$, then $v'(t) < 0$ for every $t \in (R, \infty)$ and therefore $0 > v(R) \geq v(t)$ for $t \geq R$. In view of (3.16) we can see that $0 < -\omega v(R) \leq -\omega v(t) < v''(t)$ for $t \geq R$. We get $\lim_{t\to\infty} v'(t) = \infty$, which implies $\lim_{t\to\infty} v(t) = \infty$, again. These contradictions imply the existence of $\theta > 0$ satisfying (3.12). Let us consider $b > \theta$ such that (3.13) is satisfied. Multiplying equation (1.1) by pu', integrating it over $(0, \theta)$ and (θ, b) and using the Mean value theorem, we get (3.14). □

Lemma 3.3. *Let us assume that* (3.1) – (3.8) *be satisfied. Let u be a solution of the initial value problem* (1.1)*,* (3.11) *on* $[0, \infty)$ *and let* $b > 0$, $\bar{L} \in (0, L)$ *be such that*

$$u(b) = \bar{L}, \quad u'(b) = 0. \tag{3.17}$$

Then there exists $\theta > b$ *such that*

$$u(\theta) = 0 \quad \text{and} \quad u'(t) < 0 \qquad \text{for } t \in (b, \theta]. \tag{3.18}$$

Moreover, for every $c > \theta$ *satisfying*

$$u(c) \in (L_0, 0) \quad \text{and} \quad u'(t) < 0 \quad \text{for } t \in (\theta, c), \tag{3.19}$$

there exist $\alpha \in (b, \theta)$ *and* $\beta \in (\theta, c)$ *such that*

$$(pu')^2(c) = 2[p^2(\alpha)F(\bar{L}) - p^2(\beta)F(u(c))]. \tag{3.20}$$

Proof. First of all we will prove the existence of θ satisfying (3.18). By (3.4) and (3.17) there exists $b_1 > b$ such that $f(u(t)) < 0$ for $t \in (b, b_1)$. Thus $p(t)u'(t)$ and $u'(t)$ are decreasing and negative on (b, b_1) and $u(t)$ is decreasing and positive on (b, b_1). Assume that $\theta > b$ satisfying (3.18) does not exist. Then $b_1 = \infty$ and $\lim_{t\to\infty} u(t) \in [0, \bar{L})$. On the other hand, $\lim_{t\to\infty} u'(t) < 0$, which gives $\lim_{t\to\infty} u(t) = -\infty$. The rest of the proof is similar to the previous one. □

Lemma 3.4. *(On three types of solutions) Let* (3.1) – (3.9) *be satisfied,* $B \in (L_0, 0)$. *Then there exists a unique solution* u *of problem* (1.1), (3.11) *and it is defined on* $[0, \infty)$. *There are just three types of solutions:*

- an escape solution *if there exists* $b > 0$ *such that* $u(b) = L$ *and* $u' > 0$ *on* $(0, b]$,
- a homoclinic solution *if* $u' > 0$ *on* $(0, \infty)$ *and* $\lim_{t\to\infty} u(t) = L$,
- an oscillatory solution *if* u *has infinitely many roots and* $u(t) \in (B, L)$ *for* $t \in (0, \infty)$.

Moreover, for $t \in (0, \infty)$ *it is valid*

$$|u'(t)| \le \max_{L_0 \le x \le L} |f(x)| \cdot t, \quad |u(t)| \le L_0 + \max_{L_0 \le x \le L} |f(x)| \cdot \frac{t^2}{2}. \tag{3.21}$$

Proof. From (3.1) and (3.9) it follows that there exists $\bar{L} > 0$ such that $|f(x_1) - f(x_2)| \le \bar{L}|x_1 - x_2|$ for $x_1, x_2 \in \mathbb{R}$. Let us take $\eta > 0$ such that $\bar{L}\eta^2/2 < 1$ and consider the Banach space $C([0, \eta])$ with the maximum norm and an operator

$$(\mathscr{F}u)(t) = B + \int_0^t \frac{1}{p(s)} \int_0^s p(\tau) f(u(\tau))\, \mathrm{d}\tau\, \mathrm{d}s,$$

$\mathscr{F} : C([0, \eta]) \to C([0, \eta])$. Then $\mathscr{F}$ is a contraction and the Banach fixed point theorem yields a unique fixed point u of the operator $\mathscr{F}$. Therefore

$$u'(t) = \frac{1}{p(t)} \int_0^t p(s) f(u(s))\, \mathrm{d}s \quad \text{for } t \in (0, \eta). \tag{3.22}$$

Using (3.1), (3.7), (3.9) and (3.22) we derive that the fixed point u is a unique solution of problem (1.1), (3.11). From Lemma 3.1 it follows, that the solution u can be extended onto every interval, where it is bounded. Lemma 3.2 gives $\theta > 0$ satisfying (3.12). Now, we get three possibilities:

CASE A. There exists $b > \theta$ such that $u(b) = L$ and $u'(t) > 0$ for $t \in [\theta, b]$. From (3.6), (3.7) and (3.9) it follows that u can be extended on $[0, \infty)$. This solution is an escape solution.

CASE B. For $t \in (\theta, \infty)$ it is valid $u(t) \in (0, L)$ and $u'(t) > 0$. The monotonicity implies the existence of $\tilde{L} \in (0, L]$ such that

$$\lim_{t\to\infty} u(t) = \tilde{L}. \tag{3.23}$$

Since $f(u(t)) < 0$ for $t > \theta$, from (1.1) it follows, that pu' and u' are decreasing on (θ, ∞). Since u is bounded, necessarily $\lim_{t\to\infty} u'(t) = 0$. From (1.1) and (3.8) we get

$\lim_{t\to\infty} u''(t) = f(\tilde{L})$. According to (3.2) and (3.4) we get $\tilde{L} = L$. This solution satisfies conditions (1.2) and so it is a solution with homoclinic orbit.

CASE C. There exists $b > \theta$ such that

$$u'(b) = 0, \quad u(b) \in (0,L) \qquad \text{and} \qquad u'(t) > 0 \quad \text{for } t \in (\theta, b). \tag{3.24}$$

From the second part of Lemma 3.2 we get $\alpha \in (0,\theta)$ and $\beta \in (\theta, b)$ such that (3.14) holds. In view of (3.24) we get

$$F(u(b)) = \left(\frac{p(\alpha)}{p(\beta)}\right)^2 F(B). \tag{3.25}$$

Using Lemma 3.3 we get the existence of $\theta_1 > b$ such that $u(\theta_1) = 0$ and $u'(t) < 0$ for $t \in (b, \theta_1]$. Let us suppose that there exists $\bar{b}_1 \in (\theta_1, \infty)$ such that $u(\bar{b}_1) = B$ and $u'(t) < 0$ for $t \in [\theta_1, \bar{b}_1)$. Using the second part of Lemma 3.3, we get $\bar{\alpha}_1 \in (b, \theta_1)$ and $\bar{\beta}_1 \in (\theta_1, \bar{b}_1)$ such that $(pu')^2(\bar{b}_1) = 2[p^2(\bar{\alpha}_1)F(u(b)) - p^2(\bar{\beta}_1)F(B)]$. This together with (3.25) yield a contradiction. Hence there exists $b_1 > \theta_1$ such that $u(b_1) \in (B, 0)$, $u'(b_1) = 0$ and $u'(t) < 0$ for $t \in (\theta_1, b_1)$. Repeating this procedure we get a sequence $\{\theta_n\}_{n=1}^{\infty}$ of roots of the solution u and a sequence $\{b_n\}_{n=1}^{\infty}$ of roots of the derivative u' such that $\{|u(b_n)|\}_{n=1}^{\infty}$ is decreasing. This solution corresponds to oscillatory solution. Estimations (3.21) can be reached from (1.1) by a direct computation. □

Lemma 3.5. *(On oscillatory solutions) Let* (3.1) – (3.8) *be satisfied,* $B \in (L_0, 0)$ *be such that*

$$F(B) < F(L). \tag{3.26}$$

Then the corresponding solution of problem (1.1), (3.11) *is oscillatory.*

Proof. Let u be a solution of problem (1.1), (3.11) with $B \in (L_0, 0)$ satisfying (3.26). Let us assume that u is an escape solution. Then there exist $b > 0$, $\theta \in (0, b)$ such that $u(\theta) = 0$, $u(b) = L$ and $u'(t) > 0$ for $t \in (0, b]$. From Lemma 3.2 we get $\alpha \in (0, \theta)$, $\beta \in (\theta, b)$ such that (3.14) holds. Then

$$(pu')^2(b) = 2F(L)p^2(\beta)\left[\left(\frac{p(\alpha)}{p(\beta)}\right)^2 \frac{F(B)}{F(L)} - 1\right] \leq 0.$$

This contradicts the fact that $u'(b) > 0$. Let us assume that u is a homoclinic solution. Let $\theta > 0$ be the root of u and $b > \theta$ be arbitrary. Then by Lemma 3.2 there exist $\alpha \in (0, \theta)$, $\beta \in (\theta, b)$ such that (3.14) holds. From (3.14), the fact $(pu')^2(b) > 0$ and (3.7) we get

$$F(B) > \left(\frac{p(\beta)}{p(\alpha)}\right)^2 F(u(b)) > F(u(b)).$$

Letting $b \to \infty$ we get $F(B) \geq F(L)$, which contradicts (3.26). □

Actually, the homoclinic solution is the desired strictly increasing solution of problem (1.1), (1.2). In order to prove the existence of such solution we need the lower and upper functions method for the singular mixed problem

$$(p(t)u')' = p(t)f(u), \qquad u'(a) = 0,\ u(b) = L, \tag{3.27}$$

where $a,\, b \in \mathbb{R}$, $a \geq 0$, $b > a$.

Definition 3.6. A function $\sigma \in C([a,b])$ is called a lower function of problem (3.27), if there exists a finite set $\Sigma \subset (a,b)$ such that $\sigma \in C^2((a,b] \setminus \Sigma)$, $\sigma'(\tau^+)$, $\sigma'(\tau^-) \in \mathbb{R}$ for $\tau \in \Sigma$,

$$(p(t)\sigma'(t))' \geq p(t)f(\sigma(t)) \qquad \text{for } t \in (a,b] \setminus \Sigma,$$

$$\sigma'(a^+) \geq 0,\ \sigma(b) \leq L,\ \sigma'(\tau^-) < \sigma'(\tau^+) \quad \text{for } \tau \in \Sigma.$$

If all inequalities are reversed, then σ is called an upper function of problem (3.27).

Note that $\sigma'(a^+)$ need not be bounded if $a = 0$.

Theorem 3.7. *Let p satisfy* (3.6), (3.7)*, $f \in C(\mathbb{R})$, σ_1 and σ_2 be a lower function and an upper function of problem* (3.27) *and let $\sigma_1(t) \leq \sigma_2(t)$ for $t \in [a,b]$. Then problem* (3.27) *has a solution $u \in C^1([a,b]) \cap C^2((a,b])$ such that $\sigma_1(t) \leq u(t) \leq \sigma_2(t)$ for $t \in [a,b]$.*

Proof. See [8] or [9]. □

The next assertion is based on Lemma 2.3 and Theorem 3.7.

Lemma 3.8. *(On escape solutions) Let* (3.1) – (3.9) *be satisfied. There exist $B_* \in (L_0, 0)$ and $c_* \in (0, \infty)$ such that a solution u_* of problem* (1.1), (3.11) *with $B = B_*$ satisfies the condition $u_*(c_*) = L$, $u_*'(t) > 0$ on $(0, c_*]$.*

Proof. Let us put $\tilde{f}(x) = f(x)$ for $x \leq L$ and $\tilde{f}(x) = x - L$ for $x > L$. Let $\varepsilon_0 \in \mathbb{R}$ be from Lemma 2.3 for L_0, 0, L, $\tilde{f}$, $\tilde{F}$ in place of x_1, x_2, x_3, h, H, respectively. Here, $\tilde{F}(x) = -\int_0^x \tilde{f}(z)\,\mathrm{d}z$ for $x \in \mathbb{R}$. Consider

$$u'' = \tilde{f}(u) - \varepsilon, \tag{3.28}$$

with $\varepsilon \in (0, \varepsilon_0)$. From Lemma 2.4 it follows that there exists $B_L \in (L_0, 0)$ such that for the corresponding solution u_L of problem (3.28), (3.11) with $B = B_L$, there exists $b > 0$ such that $u_L(b) = L$ and $0 < u_L'(t) \leq \sqrt{2\tilde{F}(L_0)}$ for $t \in [0,b]$. From (3.8) it follows that there exists $a > 0$ such that

$$\frac{p'(t)}{p(t)} < \frac{\varepsilon}{\sqrt{2\tilde{F}(L_0)}} \qquad \text{for } t > a.$$

Put $v(t) = u_L(t - a)$ for $t \in [a, a+b]$. Then $\tilde{f}(v(t) = f(v(t))$ on $[a, a+b]$ and we can check that v is an upper function of the problem

$$u'' + \frac{p'(t)}{p(t)}u' = f(u), \qquad u'(a) = 0,\ u(a+b) = L. \tag{3.29}$$

Since L_0 is a lower function of problem (3.29), by Theorem 3.7 there exists a solution u_0 of problem (3.29) such that

$$L_0 < u_0(t) < v(t) \qquad \text{for } t \in (a, a+b),\ u_0'(a+b) > 0. \tag{3.30}$$

If there exists $a_0 > 0$ such that $u_0(a_0) = 0$, we put

$$\beta(t) = \begin{cases} 0 & \text{for} \quad t \in [0, a_0], \\ u_0(t) & \text{for} \quad t \in (a_0, a+b]. \end{cases}$$

If $u_0(t) \le 0$ for $t \in [0, a]$, we put $\beta(t) = u_0(t)$ for $t \in [0, a+b]$. Denote $c_* = a+b$. In both cases the function β is an upper function of the problem

$$u'' + \frac{p'(t)}{p(t)} u' = f(u), \qquad u'(0) = 0, \quad u(c_*) = L. \tag{3.31}$$

Since the constant L_0 is a lower function of problem (3.31), there exists a solution u_* of problem (3.31) such that

$$L_0 < u_*(t) < \beta(t) \qquad \text{for } t \in (0, c_*). \tag{3.32}$$

We put $B_* = u_*(0)$. Then u_* is a solution of (1.1), (3.11) with $B = B_*$. Finally, by (3.29), (3.30) we have $\beta(c_*) = L$, $\beta'(c_*) > 0$. This, together with the inequality in (3.30) gives $u_*'(c_*) > 0$. Hence by Lemma 3.4, $u'(t) > 0$ for $t \in (0, c_*]$. □

Theorem 3.9. *(On homoclinic solutions) Let* (3.1) – (3.8) *be satisfied. Then there exists at least one strictly increasing solution of problem* (1.1), (1.2).

Proof. First, we will assume that (3.9) is satisfied. Let us define

$$\mathscr{M} = \{B_0 \in (L_0, 0) : \text{ each solution of } (1.1), (3.11) \text{ with } B \in [B_0, 0) \text{ is oscillatory}\},$$

and $\tilde{B} = \inf \mathscr{M}$. Lemma 3.5 guarantees that $\mathscr{M} \neq \emptyset$ and from Lemma 3.8 it follows that $\tilde{B} > L_0$. We will prove that there exists $B_{\text{hom}} \in (L_0, \tilde{B}]$ such that the corresponding solution of problem (1.1), (3.11) with $B = B_{\text{hom}}$ is a homoclinic solution. Assume that B_{hom} does not exist.

CASE A. Let $\tilde{u}$ be an oscillatory solution of (1.1), (3.11) with $B = \tilde{B}$. Then we can find a sequence $\{B_n\} \subset (L_0, \tilde{B})$ such that $\lim_{n\to\infty} B_n = \tilde{B}$ and the corresponding solutions u_n of (1.1), (3.11) with $B = B_n$ are escape solutions. Let θ_1 be the second zero of $\tilde{u}$, that is, θ_1 fulfils $\tilde{u}(\theta_1) = 0$, $\tilde{u}'(\theta_1) < 0$. By Lemma 3.4, the sequence $\{u_n\}$ is bounded and equicontinuous on $[0, \theta_1]$. Therefore we can choose a subsequence $\{u_m\}$, which is uniformly convergent on $[0, \theta_1]$ to a function $v \in C([0, \theta_1])$.

We can check that v is a solution of problem (1.1), (3.11) and therefore $v = \tilde{u}$ on $[0, \theta_1]$. Since u_m are increasing, it follows that v is nondecreasing on $[0, \theta_1]$. This contradicts the fact that $v'(\theta_1) < 0$.

CASE B. Let $\tilde{u}$ be an escape solution of (1.1), (3.11) with $B = \tilde{B}$. Then there exists $b > 0$ such that

$$\tilde{u}(b) = L, \quad \tilde{u}'(t) > 0 \quad \text{for } t \in (0, \infty). \tag{3.33}$$

From the definition of $\tilde{B}$ we get a sequence $\{B_n\} \subset (\tilde{B}, 0)$ such that $\lim_{n\to\infty} B_n = \tilde{B}$ and the corresponding solutions u_n of (1.1), (3.11), with $B = B_n$, are oscillatory. Therefore

$$L_0 \le u_n(t) \le L, \qquad |u_n'(t)| \le t \cdot \max_{L_0 \le x \le L} |f(x)|, \quad t \in [0, \infty),\ n \in \mathbb{N},$$

and there exist $b_n > 0$ such that $u_n(b_n) = L_n \in (0, L)$, $u_n'(b_n) = 0$ for $n \in \mathbb{N}$. Then there exist $\theta_n > b_n$ such that

$$u_n(\theta_n) = 0, \quad u_n'(\theta_n) < 0, \ n \in \mathbb{N}. \tag{3.34}$$

The sequence $\{u_n\}$ is bounded and equicontinuous on every $[0, K] \subset [0, \infty)$ and so we can choose a subsequence $\{u_m\}$ which is uniformly convergent on $[0, K]$ to a function $w \in C([0, K])$. As in CASE A we conclude that $w = \tilde{u}$ on $[0, K]$. Let $\lim_{m\to\infty} \theta_m = \theta_0 < \infty$. Put $K = \max\{\theta_0, b\} + 1$. By (3.34), each u_m is decreasing at a neighbourhood of θ_m and $\tilde{u}$ is nonincreasing at θ_0, which contradicts (3.33). Let $\lim_{m\to\infty} \theta_m = \infty$. Put $K = b + 1$. Since $u_m(b+1) < L$ for $m \in \mathbb{N}$, it follows that $\tilde{u}(b+1) \le L$, which is a contradiction. The function $\tilde{u}$ can be neither an escape solution nor an oscillatory solution. Lemma 3.4 yields that $\tilde{u}$ is a homoclinic solution of problem (1.1), (1.2). Since $\tilde{u}(t) \in [L_0, L]$ for $t \in [0, \infty)$ we see that assumption (3.9) can be omitted. □

For more details in the proofs see [10].

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REFERENCES

1. H. Berestycki, P. L. Lions and L. A. Peletier, An ODE approach to the existence of positive solutions for semilinear problems in $\mathbb{R}^N$, *Indiana University Mathematics Journal* **30**, 1, 141–157 (1981).
2. D. Bonheure, J. M. Gomes and L. Sanchez, Positive solutions of a second–order singular ordinary differential equation, *Nonlinear Analysis* **61**, 1383–1399 (2005).
3. F. Dell'Isola, H. Gouin and G. Rotoli, Nucleation of spherical shell-like interfaces by second gradient theory: numerical simulations, *Eur. J. Mech B/Fluids* **15**, 545–568 (1996).
4. H. Gouin and G. Rotoli, An analytical approximation of density profile and surface tension of microscopic bubbles for Van der Waals fluids, *Mech. Research Communic.* **24**, 255–260 (1997).
5. G. Kitzhofer, O. Koch, P. Lima and E. Weinmüller, Efficient numerical solution of the density profile equation in hydrodynamics, *J. Sci. Comput.* **32**, 3, 411–424 (2007).
6. O. Koch, P. Kofler, E. Weinmüller, Initial value problems for systems of ordinary first and second order differential equations with a singularity of the first kind, *Analysis* **21**, 373–389 (2001).
7. P.M. Lima, N.V.Chemetov, N.B. Konyukhova and A.I. Sukov, Analytical–numerical investigation of bubble-type solutions of nonlinear singular problems, *J. Comp. Appl. Math.* **189**, 260–273 (2006).
8. I. Rachunková, O. Koch, G. Pulverer, E. Weinmüller, On a singular boundary value problem arising in the theory of shallow membrane caps, *J. Math. Anal. Appl.* **332**, 532–541 (2007).
9. I. Rachunková, S. Staněk and M. Tvrdý, "Singularities and Laplacians in Boundary Value Problems for Nonlinear Ordinary Differential Equations," in *Handbook of Differential Equations. Ordinary Differential Equations, vol. 3*, edited by A. Cañada, P. Drábek, A. Fonda, Elsevier, 2006, pp. 607–723.
10. I. Rachunková, J. Tomeček, Singular problems on the half–line, preprint 17/2008, [online: http://mant.upol.cz/cs/preprinty.asp/p17-08.pdf].